

# The worst visibility walk in a random Delaunay triangulation is $O(\sqrt{n})^*$

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**ABSTRACT.** We show that the memoryless routing algorithms Greedy Walk, Compass Walk, and all variants of visibility walk based on orientation predicates are asymptotically optimal in the average case on the Delaunay triangulation. More specifically, we consider the Delaunay triangulation of an unbounded Poisson point process of unit rate and demonstrate that, for any pair of vertices  $(s, t)$  inside  $[0, n]^2$ , the ratio between the longest and shortest visibility walks between  $s$  and  $t$  is bounded by a constant with probability converging to one (as long as the vertices are sufficiently far apart). As a corollary, it follows that the worst-case path has  $O(\sqrt{n})$  steps in the limiting case, under the same conditions. Our results have applications in routing in mobile networks and also settle a long-standing conjecture in point location using walking algorithms. Our proofs use techniques from percolation theory and stochastic geometry.

## 1 Introduction

Given a graph  $G = (V, E)$  embedded in  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ , we define *graph navigation* to be the problem of finding a path in  $G$  (*resp.* in the dual of  $G$ ) from a source node  $z \in V$  to the node that is the nearest neighbour of a query point  $q \in \mathbb{R}^d$  (*resp.* the face of  $G$  containing  $q$ ) using only information gained from the vertices visited by the algorithm. This problem is highly related to the problems of geometric routing in networks and also point location in geometric data structures [9]. For brevity, we call graph navigation algorithms *walking algorithms*, and we say a walking algorithm is *memoryless* if every successive step can be computed without knowledge of the history of the walk. In this paper, we focus on the case of the Delaunay triangulation in  $\mathbb{R}^2$ . In the context of geometric routing in networks,<sup>1</sup> the Delaunay triangulation has been proposed as a network topology because the shortest path distances in the Delaunay triangulation approximate the Euclidean distances and because it can be constructed locally [2, 4, 11, 14]. In this context, a simple way to perform routing in a geometric network with Delaunay triangulation topology is to use walking algorithms. Walking algorithms are also used to efficiently perform point location in Delaunay triangulations by computational libraries such as CGAL [7].

Worst-case complexity bounds for walking algorithms tend to be pessimistic, since examples can easily be constructed in which algorithms visit every vertex in the graph at

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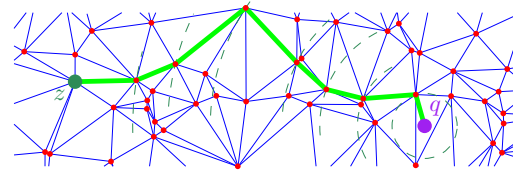
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<sup>1</sup>For example Mobile Ad-hoc NETworks or ‘MANETS’

least once. Despite this, one might expect the algorithms to require ‘approximately’  $O(\sqrt{n})$  steps to terminate in a triangulation of  $n$  points in  $\mathbb{R}^2$  under some normality conditions since this is the result obtained for regular grids. Such a result was conjectured in the very first papers using walking algorithms for point location [5, 12, 15], however it has only been proved formally for the particular *average-case* setting of  $n$  random points in a square for the algorithms Straight Walk [10], which visits every triangle intersecting the line segment between the initial point and the destination, and Cone Walk [6], which successively chooses points directed towards the destination. Up until now, the algorithms which are most used in practice have remained unanalysed, often because they are memoryless and so exhibit strong dependence between steps. In this paper, we resolve this situation by proving that the stated conjecture is true for all of most commonly used memoryless walking algorithms given the Delaunay triangulation of a Poisson point process observed in a window of area  $n$ . The bounds we achieve are in fact much stronger than the original conjecture, since we show that *no asymptotically sub-optimal path exists* for  $n$  large enough. Unfortunately, the constants we have in our complexity are quite overestimated. We are confident that the methods we employ may be recycled to achieve similar bounds on a far greater variety of geometric structures and walking algorithms. The particular algorithms we focus on in this paper are outlined below. Some walks are walks in  $G$  i.e. on vertices, while some walks are in the dual graph i.e. on triangles.

### Greedy Walk (walk on vertices) *Greedy Walk*

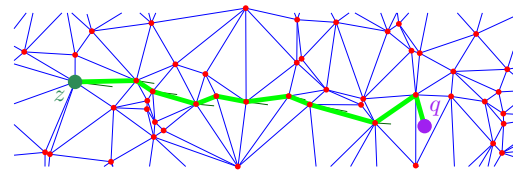
is perhaps the simplest graph navigation algorithm. The algorithm is initialised by some vertex  $z \in V$ , and then iteratively chooses the vertex that is closest to the destination among the neighbours of the current vertex at each step.



It is easy to see that Greedy Walk always succeeds on the Delaunay triangulation. Namely, assume that the walk ended at  $v$  and the disk  $D$  of centre  $q$  through  $v$  is not empty ( $v$  is not the nearest neighbour); then one can construct an empty disk tangent to  $D$  at  $v$  inside  $D$  passing through another vertex  $w$  closer to  $q$ , thus  $vw$  is a Delaunay edge giving a contradiction.

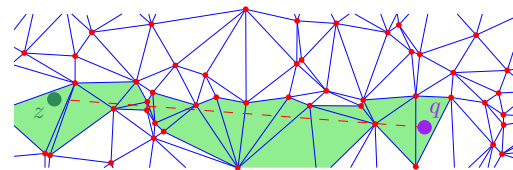
### Compass Walk (walk on vertices) *Compass Walk*

Given a current vertex  $u$  and a destination vertex  $q$ , *Compass Walk* is a memoryless graph navigation algorithm that works by choosing one of the two neighbours,  $v_1, v_2$  of  $u$  where  $uv_1v_2$  is the triangle intersected by ray  $uq$ . Often, the edge with the smallest angle with  $uq$  is preferred, but choosing randomly or based on the length of the line segment  $uv_i$ , or any other rule can lead to different behaviour on more general graphs [3]. All versions succeed on the Delaunay triangulation.



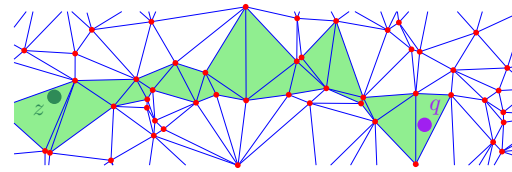
### Straight Walk (walk on triangles) *Straight Walk*

is a deterministic online graph navigation algorithm that succeeds in any triangulation. It works by visiting all triangles of the triangulation intersecting the line  $zq$ , for  $z$  the initial point and  $q$  the destination.



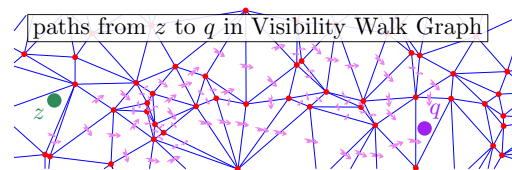
**Visibility Walks (walk on triangles)**

*Visibility walk* is the best known walking algorithm that has historically been used for point location. It was first introduced by Lawson [15], and quickly adopted by others [5, 12] to speed-up the construction of Delaunay triangulations.



As for Straight Walk the basic idea is to navigate a triangulation by moving between neighbouring triangles until the destination is reached. A visibility walk can go from a triangle  $t$  to its neighbour  $t'$  if and only if the line supporting the common edge of  $t'$  and  $t$  separates the interior of  $t$  from the query  $q$ . So the visibility walk is not unique since some triangles have two admissible successors. In such a case the successor can be chosen using any rule (e.g. at random, the first found, alternate counter clockwise and clockwise exploration...), in particular Straight Walk and all variants of Compass Walk are visibility walks.

**Visibility Walk Graph** We consider the dual graph of the triangulation, with dual edges oriented as described above for a destination  $q$ . A visibility walk corresponds to any oriented path in that graph. This graph has no cycles and a single sink: the triangle containing  $q$ .

**Previous results**

The straight walk was analysed in the average case and proved to be of expected complexity  $\Theta(\sqrt{n})$  [1, 10]. However, it uses the fact that straight walk is not memoryless since  $z$ , the origin of the walk, must be known to decide the next triangle. In straight walk, deciding whether or not a triangle belongs to the walk depends only on the segment  $zq$  and not on the rest of the walk, making the probabilistic dependencies in easily tractable.

The case of memoryless walking algorithms is more difficult. They may be treated as stochastic processes indexed by the number of steps up to some intermediate part of the walk, with random variables representing the *progress* of the walk at each step. However, the difficulty in most walking algorithms is that this sequence of random variables is neither independent and identically distributed, Markovian nor stationary, since a step in the walk may intersect a region in the process which has already been *seen* and because the distribution depends on the distance from the destination. This makes walking algorithms difficult to attack with traditional tools of probability theory. *Nearly-memoryless* walks were treated in this way by conditioning on special events to introduce independence by Broutin et al. [6]. This allowed a very accurate understanding of the walk process to be gained, with explicitly computed constants very close to what is actually observed in simulations. In addition, a  $O(\sqrt{n} \log n)$  bound for the expected number of triangles visited by Visibility Walk was claimed by Zhu [20] using induction; but it turns out that his proof incorrectly assumes that each new point visited by the walk is independent of the history of the process. Applying the kind of step-by-step reasoning to walks such as Greedy Walk and visibility walks very quickly leads to ever-increasing numbers of cases that need to be considered separately. Certain pathological configurations of points make bounding these cases particularly difficult. In

light of this, our technique is to attack the problem from another direction entirely. Instead of considering individual steps of walking algorithms, we ‘zoom out’ and focus instead on the properties of large *patches* of the point process, which may contain hundreds of individual triangles. This methodology will result in us sacrificing the fine control over the constants to obtain an asymptotic order of magnitude. However, in doing so our proofs become significantly cleaner and most importantly, tractable.

## 2 Contributions

Given a locally finite set of points,  $\mathbf{X}$  and a triangulation,  $T(\mathbf{X})$  (represented as a set of triples of points forming the triangles), we define the *visibility walk graph* with *destination*  $q$ , written  $\mathbf{W}_q(T(\mathbf{X}))$  to be the directed graph with *nodes* the triangles of  $T(\mathbf{X})$  and directed *arcs* representing the permissible steps that may be taken by visibility walks. Namely, given two adjacent triangles  $t$  and  $t'$ , the directed arc from  $t$  to  $t'$  is present if the destination  $q$  is on the same side as  $t'$  of the line supporting the common edge of  $t$  and  $t'$ . Let  $\mathbf{P}^{(q)}(T(\mathbf{X}))$  be the set of *paths* that may be constructed on the graph  $\mathbf{W}_q(T(\mathbf{X}))$ , represented as sequences of triangles. It is well known (see also Corollary 10) that, for the Delaunay triangulation denoted  $\text{Del}(\mathbf{X})$ , the visibility walk graph is *acyclic* so that every path in  $\mathbf{W}_q(\text{Del}(\mathbf{X}))$  converges on the node containing  $q$ . For  $w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}))$  we use  $w(i)$  to denote the  $i$ th triangle in the walk  $w$ ,  $|w|$  its length ( $q \in w(|w|)$ ), and  $z(t)$  the circumcenter of a triangle  $t$ . We also similarly construct the set of paths that may be taken by Greedy Walk, which we denote  $\mathbf{P}_G^{(q)}(T(\mathbf{X}))$ . In this case, the nodes in the walk graph correspond to the vertices of  $T(\mathbf{X})$ , and the set of paths forms a tree whose *root* is the nearest neighbour of  $q$  in  $\mathbf{X}$ .<sup>2</sup> This may be contrasted with visibility walks by noting that Greedy Walk is completely defined by the pair of initial and destination points, whereas visibility walks can make choices. In the sequel, the point set  $\mathbf{X}$  will be a Poisson point process of rate  $\gamma$ . In such a process, the expected number of points in a planar domain  $A$  is proportional to the area of  $A$ :  $\mathbb{E}(|X \cap A|) = \gamma \cdot \text{area}(A)$ , and what happen in two disjoint planar domains are independent. Our main results are captured in the following theorem.

**Theorem 1.** *Let  $\mathbf{X}$  be a homogeneous Poisson point process of rate 1 in  $\mathbb{R}^2$  and  $\mathbf{D} = [-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}]^2$  a domain of area  $n$ . Then for  $n$  sufficiently large,*

$$\mathbb{P} \left( \sup_{q \in \mathbf{D}} \sup_{\substack{w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_n)) \\ w(0) \in \mathbf{D}}} \frac{|w|}{\|z(w(0)) - q\| \sqrt{n} + \log^3 n} \geq c_1 \right) \leq e^{-c_2 \cdot \log^{\frac{3}{2}} n},$$

where  $c_1, c_2$  are fixed positive constants.

Theorem 1 implies that the ratio between the shortest and longest possible paths between two given points for any kind of visibility walk is bounded by a constant with high probability, as long as the distance between the destination and start points is longer than  $\log^3 n$ , in the asymptotic limit.

<sup>2</sup>If the nearest neighbour is not unique, the graph is a forest.

**Corollary 2.** *Under the same conditions as Theorem 1,*

$$\lim_{n \rightarrow \infty} \sup_{q \in \mathbf{D}} \sup_{\substack{w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{x}_n)) \\ w(0) \in \mathbf{D}}} \frac{|w|}{\sqrt{n}} \leq c_1.$$

This corollary is a direct consequence of the Borel-Cantelli Lemma.

**Corollary 3.** *Theorem 1 also holds verbatim for all variants of Compass Walk.*

**Corollary 4.** *Theorem 1 also holds for Greedy Walk with different constants.*

Corollaries 3 and 4 will be proven in Section 9.

### Sketch of proof

We first have to choose a way of measuring the progress in visibility walks. We choose to use the power of the query point with respect to the circle circumscribing the current triangle. This measure is strictly decreasing during the walk (see Section 6). This means that if we can find an event that gives ‘good’ progress in the circle power and which occurs sufficiently frequently, we can easily bound the number of steps required for a given walk. Applying this strategy *incrementally* as the walk progresses is the obvious way to proceed, however despite the authors’ best efforts, untangling the dependence between the steps proved to be consistently unsuccessful.

The solution we present here is a result of ‘rethinking’ the formulation of the problem: instead of considering the walk as a sequence of steps, we consider all possible paths simultaneously. This is done by considering a regular grid whose cells contains on average a, reasonably big, constant number of points. We say that a grid cell is good if it does not intersect a Voronoi cell that spans non adjacent grid cells and if all the triangles inside the cell induce a substantial progress in the circle power. Then, we have to tune the constants in the goodness definition so that so much cells are good that any path must use as many good cells as bad cells. This has two main advantages: the goodness of grid cells is independent if the cells are not neighbours in the grid and the number of possible paths in the grid is much smaller than in the triangulation.

In Section 4 we state our results for walks in a square of  $k \times k$  grid cells with a Poisson point process of a well-chosen intensity; this result is proved in Sections 4 to 8.

In Section 5 we study the size of paths in a grid (also called *lattice animals*) using the theory of percolation. We prove that if the badness probability is below some threshold, there is no long walk in the grid with high probability.

In Section 6 we relate the progress in the power of the query point with respect to the circle circumscribing the current triangle to the geometric parameters of the triangles.

In Section 7 we choose the grid size to correctly tune the goodness probability of a grid cell.

In Section 8 we exploit these results to actually prove that the absence of long walk in the grid translates to the absence of long walk on the Delaunay triangulation.

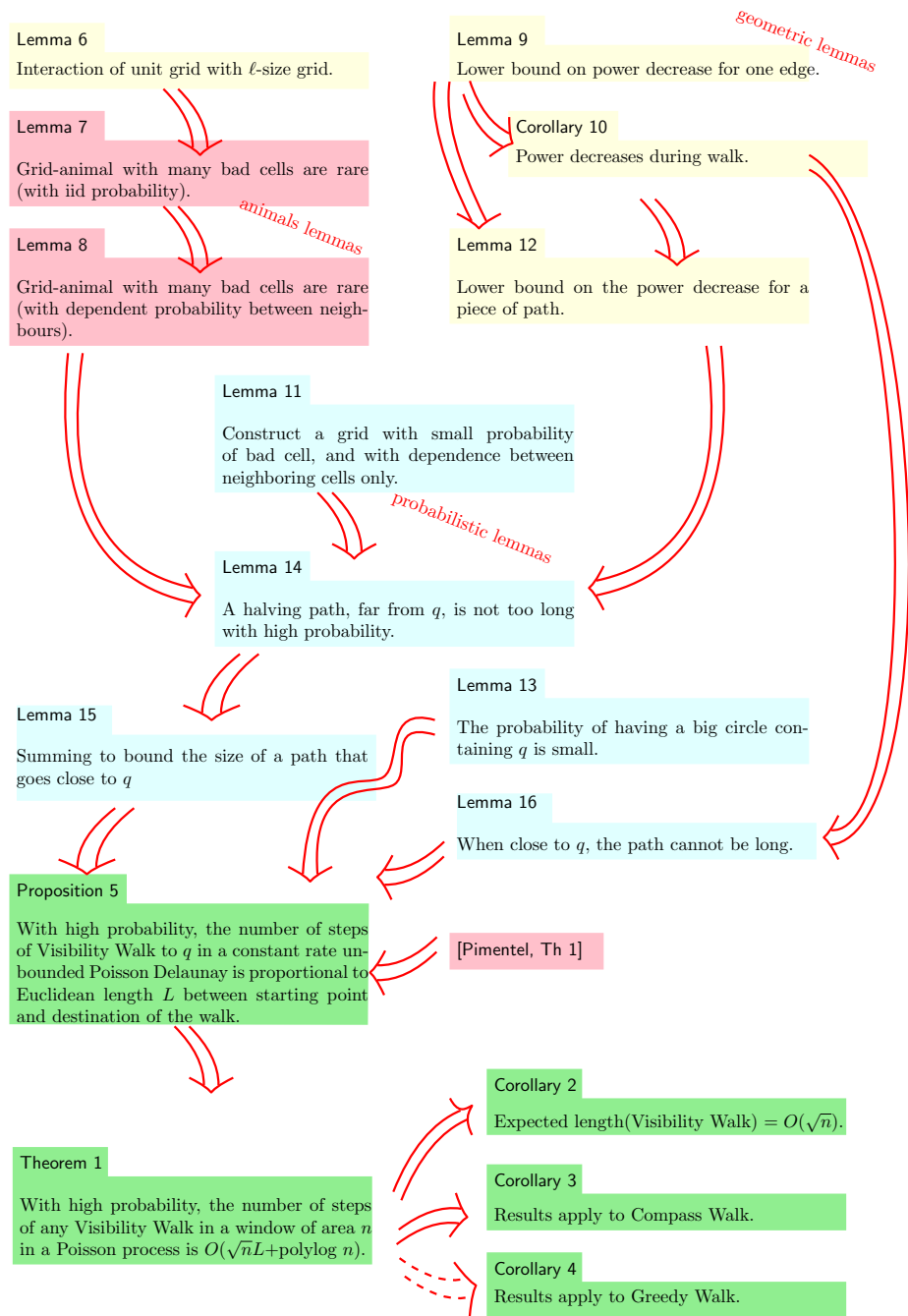


Figure 1 summarises the organisation of the proof of Theorem 1.

### 3 Preliminaries

For a triangle,  $\sigma \in \text{Del}(\mathbf{X})$ , we denote its circumcentre by  $z(\sigma)$  and its circumradius by  $r(\sigma)$ . Let  $\mathbf{G}$  be the graph whose set of vertices is  $\mathbb{Z}^2$  and with edges

$$x \leftrightarrow y \iff \|y - x\| = 1,$$

for all  $x, y \in \mathbb{Z}^2$ .  $\mathbf{G}$  is known as a *lattice* or a *grid* in the plane.  $\mathbf{G}^{(8)}$  is the graph with same vertex set as  $\mathbf{G}$  and edges

$$x \leftrightarrow y \iff \|y - x\| = 1, \quad \text{or} \quad \|y - x\| = \sqrt{2}.$$

We additionally write

$$\mathbf{B}(x, \ell) := \left\{ y \in \mathbb{Z}^2 : \|y - x\|_\infty \leq \ell \right\} \quad \text{and} \quad \partial\mathbf{B}(x, \ell) := \mathbf{B}(x, \ell) \setminus \mathbf{B}(x, \ell - 1)$$

to refer to balls of vertices in the lattice. We denote the ball in  $\mathbb{R}^2$  under the Euclidean and  $L_\infty$  norms as

$$B(x, \ell) := \left\{ y \in \mathbb{R}^2 : \|x - y\| \leq \ell \right\}, \quad B_\infty(x, \ell) := \left\{ y \in \mathbb{R}^2 : \|x - y\|_\infty \leq \ell \right\}.$$

We define:

$$\left\{ C(\{v\}) : v \in \mathbb{Z}^2 \right\}, \quad \text{where} \quad C(\mathbf{A}) := \bigcup_{v \in \mathbf{A}} B_\infty(v, \tfrac{1}{2}); \quad \mathbf{A} \subseteq \mathbb{Z}^2, \quad (1)$$

a collection of boxes covering  $\mathbb{R}^2$  which is disjoint apart from the Lebesgue measure-zero borders of each box.<sup>3</sup> Many of our arguments will follow by associating independent and identically distributed random variables with each vertex in the lattice  $\mathbf{G}$ . We denote these random variables as  $X_v$  for each  $v \in \mathbb{Z}^2$ . When  $X_v \sim \text{Bernoulli}(p)$ , this is also the standard model used in the theory of (site) percolation [13].

#### Lattice animals

A *lattice animal* is a collection of vertices  $\mathbf{A} \subset \mathbb{Z}^2$  of the grid  $\mathbf{G}$  such that for every pair of distinct vertices  $u, v \in \mathbf{A}$  there is a path in  $\mathbf{G}$  connecting  $u, v$  visiting only vertices in  $\mathbf{A}$ . Using digital geometry vocabulary, a lattice animal is a 4-connected subset of the grid, i.e. a connected set of pixels where a pixel can have 4 neighbours (horizontals or verticals). We denote by  $|\mathbf{A}|$  the size (or cardinality) of a lattice animal. A set of vertices connected in  $\mathbf{G}^{(8)}$  is called 8-connected where a pixel can have 8 neighbours (horizontals, verticals, or diagonals). The *size* of an animal is the number of vertices it contains. For  $x \in \mathbb{Z}^2$ , we shall

<sup>3</sup>We shall often abuse notation by writing  $C(v)$  to mean  $C(\{v\})$  for readability.



write  $\mathcal{A}_m^{(x)}$  to refer to the collection of all lattice animals of size  $m$  containing  $x$ . A lattice animal  $\mathbf{A} \in \mathcal{A}_m^{(x)}$  is called *greedy* if it maximises the sum over the random variables  $X_v$  for  $v \in \mathbf{A}$ ,

$$N(m) := \max_{\mathbf{A} \in \mathcal{A}_m^{(0)}} \sum_{v \in \mathbf{A}} X_v.$$

This maximum value has been studied in the literature [8] for various different distributions for the  $X_v$ , with extensions to the Poisson Voronoi tessellation by Pimentel and Rossignol [18]. Our interest in lattice animals is motivated by the following observation. We may *discretise* the path defined by some walk  $w \in \mathbf{W}_q(\mathbf{T}(\mathbf{X}))$  by considering the collection of vertices  $v \in \mathbb{Z}^2$  such that the box  $C(v)$  intersects one of the line segments connecting the circumcentres of triangles in the walk. In other words, we consider all the grid cells traversed by the path in the Voronoi diagram, visiting the Voronoi vertices dual of the triangles in  $w$ . Formally, we shall denote the sequence of Voronoi vertices visited by the walk as,  $\bar{w} := \langle z(w(0)), z(w(1)), \dots, z(w(|w|)) \rangle$ . We abuse notation by identifying  $\bar{w}$  with the piece-wise linear curve of line segments formed by  $\bar{w}(i)\bar{w}(i+1)$  for  $0 \leq i < |w|$ . We then identify a lattice animal with the walk  $w$  as follows,

$$\mathbf{A}(\bar{w}) := \left\{ v \in \mathbb{Z}^2 : C(v) \cap \bar{w} \neq \emptyset \right\}.$$

#### 4 Detailed statement of main result

Let  $\mathbf{X}_\gamma$  be a homogenous Poisson process of rate  $\gamma$  in  $\mathbb{R}^2$ . As with most applications of percolation theory, we will need to re-scale in order to fit the grid size with our needs. Instead of re-scaling the lattice, we shall re-scale the Poisson process using  $\gamma$ , since our notation will be simplified. On the process  $\mathbf{X}_\gamma$ , we construct the Delaunay triangulation,  $\text{Del}(\mathbf{X}_\gamma)$  which is almost surely in general position. In this section, we deal with the following proposition. We denote  $V_{\mathbf{X}_\gamma}(x)$  the Voronoi cell of some point  $x \in \mathbb{R}^2$ . We also introduce the following notation to denote all walks starting from the ‘ring’ of boxes at distance  $k$  in the lattice from  $q$ ,

$$\mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) := \left\{ w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : \bar{w}(0) \in C(\partial \mathbf{B}(q, k)) \right\}. \quad (2)$$

**Proposition 5.** *There exists  $\gamma$  such that for any  $k \in \mathbb{N}$  large enough,  $\mathbf{X}_\gamma$  the Poisson process on  $\mathbb{R}^2$  with rate  $\gamma$  verifies:*

1.  $\mathbb{P} \left( \exists w \in \mathbf{P}_k^{(0)}(\text{Del}(\mathbf{X}_\gamma)) : |\mathbf{A}(\bar{w})| \geq C_1 k \right) \leq e^{-C_2 \sqrt{k}},$
2.  $\mathbb{P} \left( \exists \mathbf{A} \in \mathcal{A}_m^{(0)}, m < C_1 k : \sum_{x \in \mathbf{X}_\gamma} \mathbb{1}_{V_{\mathbf{X}_\gamma}(x) \cap C(\mathbf{A}) \neq \emptyset} \geq C_3 C_1 k \right) \leq e^{-C_4 k},$
3.  $\mathbb{P} \left( \exists w \in \mathbf{P}_k^{(0)}(\text{Del}(\mathbf{X}_\gamma)) : |w| \geq C_3 C_1 k \right) \leq e^{-C_2 \sqrt{k}} + e^{-C_4 k}.$

Where  $\gamma$ ,  $C_1$  and  $C_2$ , are fixed positive constants which we explicitly compute in Lemma 11 and  $C_3, C_4$  are fixed positive constants depending only on  $\gamma$ .



Sections 5 to 8 will be devoted to the proof of Statement 1 in Proposition 5 which says that, with high probability, there is no visibility walk which induces a big lattice animal.

*Proof of Proposition 5 Parts 2 and 3.* Statement 2 says that, with high probability, the number of Voronoi cells intersecting an animal is proportional to the size of the animal. Statement 2 has been proved by Pimentel [17, Theorem 1]. Given a walk  $w \in \mathbf{P}_k^{(0)}(\text{Del}(\mathbf{X}_\gamma))$ , its animal  $\mathbf{A}(\bar{w})$  is “small” with high probability by Statement 1; for such a small animal  $C(\mathbf{A}(\bar{w}))$  intersects a number of Voronoi cells smaller than  $C_1 C_3 k$  with high probability by Statement 2; the Voronoi cells dual of the vertices of  $w$  intersects  $C(\mathbf{A}(\bar{w}))$  and thus  $|w| \leq C_1 C_3 k$  with high probability.  $\square$

*Proof of Theorem 1.* A first remark is that in the proof of Statement 3 of Proposition 5 we count all vertices in each grid cell intersected by the path. So the exact vertex we use in  $C(0)$  to begin with is not important and thus Proposition 5 apply to walks starting at any  $q \in C(0)$ .

For  $k = \log^3 n$ , Proposition 5 Part 3 gives

$$\mathbb{P}\left((\exists q \in C(0) \exists w \in \mathbf{P}_{\log^3 n}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |w| \geq C_3 C_1 \log^3 n)\right) \leq e^{-C_2 \log^{\frac{3}{2}} n}.$$

When  $k > \log^3 n$  we can condition on the distance between the first and last grid cells in the walk and get

$$\begin{aligned} & \mathbb{P}\left((\exists q \in C(0) \exists w \in \mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |w| \geq C_3 C_1 \|z(w(0)) - q\| \left\|z(w(0)) - q\right\| \in [k-1, k]\right) \\ & \leq e^{-C_2 \sqrt{k}} + e^{-C_4 k} \leq e^{-C_2 \log^{\frac{3}{2}} n}. \end{aligned}$$

Summing the last equation for  $k \in \mathbb{N}$  between  $\log^3 n$  and  $\frac{\sqrt{n}}{\gamma}$  and adding the previous equation we get all paths starting in  $C(0)$  and ending in  $[-\frac{\sqrt{n}}{\gamma}, \frac{\sqrt{n}}{\gamma}]^2$ , for  $n > 4\gamma^2$  we have:

$$\begin{aligned} & \mathbb{P}\left((\exists q \in C(0) \exists w \in \mathbf{P}_{\frac{\sqrt{n}}{\gamma}}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |w| \geq C_3 C_1 (\|z(w(0)) - q\| + \log^3 n)\right) \\ & \leq e^{-C_2 \log^{\frac{3}{2}} n} + \sum_{k=\lfloor \log^3 n \rfloor}^{\lfloor \frac{\sqrt{n}}{\gamma} \rfloor} e^{-C_2 \log^{\frac{3}{2}} n} = \frac{\sqrt{n}}{\gamma} \cdot e^{-C_2 \log^{\frac{3}{2}} n}. \end{aligned}$$

A path starting at  $q \in C(v) \subset \frac{1}{\gamma} \mathbf{D} = [-\frac{\sqrt{n}}{2\gamma}, \frac{\sqrt{n}}{2\gamma}]^2$ , and ending in  $\frac{1}{\gamma} \mathbf{D}$  is included in the square  $[-\frac{\sqrt{n}}{\gamma}, \frac{\sqrt{n}}{\gamma}]^2$  translated so that  $q$  is in the cell of the origin. Thus, if the path can start at any place in  $\frac{1}{\gamma} \mathbf{D}$  we have:

$$\begin{aligned} & \mathbb{P}\left((\exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)), w \subset \frac{1}{\gamma} \mathbf{D} : |w| \geq C_3 C_1 (\|z(w(0)) - q\| + \log^3 n)\right) \\ & \leq \sum_v \mathbb{P}\left((\exists q \in C(v) \exists w \in \mathbf{P}_{\frac{\sqrt{n}}{\gamma}}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |w| \geq C_3 C_1 (\|z(w(0)) - q\| + \log^3 n)\right) \\ & \leq \frac{n}{\gamma^2} \frac{\sqrt{n}}{\gamma} \cdot e^{-C_2 \log^{\frac{3}{2}} 4n} \leq \frac{n^{\frac{3}{2}}}{\gamma^3} \cdot e^{-2C_2 \log^{\frac{3}{2}} 4n} \end{aligned}$$

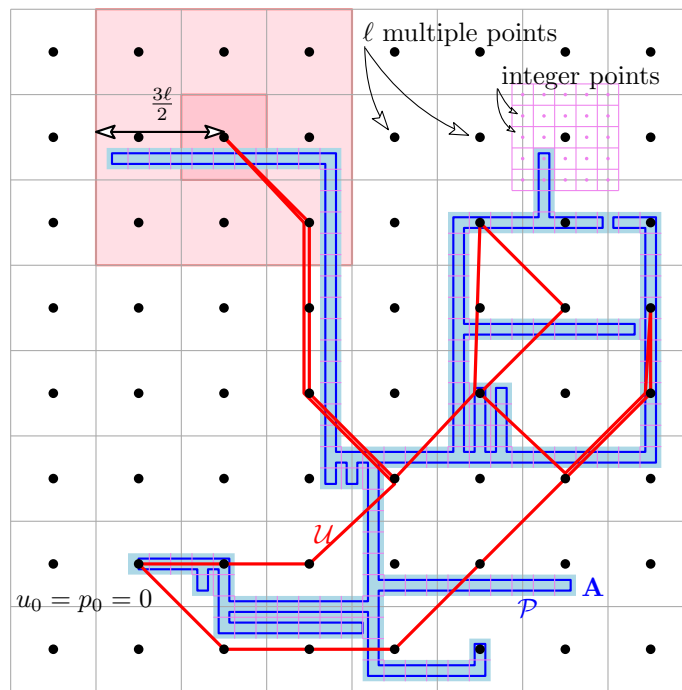


Figure 2: For the proof of Lemma 6

since  $C_2$  and  $\gamma$  are constant, the last inequality is true for  $n$  big enough. Scaling by a factor  $\gamma$  to get a Poisson point process of unit rate proves Theorem 1.  $\square$

## 5 Size of Animals

**Lemma 6.** *For any animal  $\mathbf{A} \in \mathcal{A}_m^{(0)}$  and  $1 \leq \ell \leq m$ , there exists a set of vertices  $u_0 = 0, u_1, \dots, u_r = 0$  forming a 8-connected path of  $r$  vertices in  $\mathbb{Z}^2$  such that*

$$\mathbf{A} \subset \bigcup_{0 \leq i < r} \mathbf{B}(\ell u_i, \frac{3\ell}{2}) \quad \text{and} \quad r \leq \frac{2m-2}{\ell},$$

*Proof.* Similarly to Cox et al. [8, Lemma 1] we consider first a spanning tree on  $\mathbf{A}$ . A depth first traversal of this tree gives a path  $\mathcal{P} = (p_j)$  in  $\mathbf{G}$  of length  $2m - 2$  starting and finishing at the origin.  $\mathcal{U} = (u_s)_{0 \leq s \leq r}$  is almost a discretisation of  $\mathcal{P}$  in the grid of size  $\ell$  (see Figure 2).

More precisely, we construct the path  $(u_s)$  incrementally. Assume that  $(u_s)_{0 \leq s \leq t}$  discretises  $(p_j)_{0 \leq j \leq i}$ , that is  $\forall j < i$ ;  $p_j \in \bigcup_{s=0}^t \mathbf{B}(\ell u_s, \frac{3\ell}{2})$ , and  $p_i \in \mathbf{B}(\ell u_t, \frac{\ell}{2})$ . Now consider the point  $p_k$  with smallest index  $k > i$  such that  $p_k \notin \mathbf{B}(\ell u_t, \frac{3\ell}{2})$ , then we define  $u_{t+1} = \lfloor \frac{p_{k-1}}{\ell} \rfloor$  the center of the square of side  $\ell$  containing  $p_{k-1}$  (where  $\lfloor \alpha \rfloor = \lfloor \alpha + \frac{1}{2} \rfloor$  is the nearest integer from  $\alpha \in \mathbb{R}$ ). Notice that, since  $p_{k-1} \in \mathbf{B}(\ell u_t, \frac{3\ell}{2})$ ,  $u_{t+1}u_t$  is an edge of  $\mathbf{G}^{(8)}$  and  $\mathcal{U}$  is 8-connected.

We can lower bound  $\|p_i p_{k-1}\|_\infty$ :

$$\begin{aligned} \|p_i p_k\|_\infty &> \ell && \text{since } p_i \in \mathbf{B}(\ell u_t, \frac{\ell}{2}) \text{ and } p_k \notin \mathbf{B}(\ell u_t, \frac{3\ell}{2}) \\ \|p_i p_k\|_\infty &\geq \ell + 1 \\ \|p_i p_{k-1}\|_\infty &\geq \|p_i p_k\|_\infty - \|p_k p_{k-1}\|_\infty \geq \ell + 1 - 1 = \ell \end{aligned}$$

and conclude that one step on  $\mathcal{U}$  corresponds to at least  $\ell$  steps in  $\mathcal{P}$  and thus  $|\mathcal{U}| = r \leq \left\lfloor \frac{|\mathcal{P}|}{\ell} \right\rfloor \leq \frac{2m-2}{\ell}$ .  $\square$

The following lemma has been adapted from Lee [16]. Our proof closely follows the original, though in our case we seek strong bounds for a fixed  $m$ .

**Lemma 7.** *Associate with every  $v \in \mathbf{G}$  a random variable  $X_v$  which is Bernoulli( $p$ ) distributed. Then for  $m$  sufficiently large and  $p \leq p_0 := \frac{1}{1500}$*

$$\mathbb{P} \left( \exists \mathbf{A} \in \mathcal{A}_m^{(0)} : \sum_{v \in \mathbf{A}} X_v \geq m \sqrt{\frac{p}{p_0}} \right) \leq e^{-m\sqrt{p}}.$$

*Proof.* For  $\mathbf{A} \in \mathcal{A}_m^{(0)}$ , and using Lemma 6 there exists a set of vertices  $u_0, \dots, u_r$  forming a 8-connected path of  $r \leq \frac{2m-2}{\ell}$  vertices in  $\mathbb{Z}^2$  such that

$$\mathbf{A} \subset \bigcup_{i=0}^r \mathbf{B}(\ell u_i, \frac{3\ell}{2}),$$

with  $u_0 = 0$ . In the following, we choose  $\ell := \left\lceil p^{-\frac{1}{2}} \right\rceil$ . Doing this means we can bound the number of configurations, in particular, we have at most 8 ways of choosing the successor of  $u_i$  giving  $8^r \leq 8^{2m\sqrt{p}}$  ways to choose the  $u_0, \dots, u_r$ . Furthermore, it follows that, using  $3\ell + 1 \leq 3(p^{-\frac{1}{2}} + 1) + 1 = 3p^{-\frac{1}{2}}(1 + \frac{4}{3}\sqrt{p}) \leq 3p^{-\frac{1}{2}}(1 + \frac{4}{3}\sqrt{p_0})$ :

$$\left| \bigcup_{i=0}^r \mathbf{B}(\ell u_i, \frac{3\ell}{2}) \right| \leq r(3\ell + 1)^2 \leq 2m\sqrt{p} \cdot 9p^{-1}(1 + \frac{4}{3}\sqrt{p_0})^2 \leq 18 \cdot mp^{-\frac{1}{2}}(1 + \frac{4}{3}\sqrt{p_0})^2.$$

Using the above bounds gives us,

$$\begin{aligned} & \mathbb{P} \left( \exists \mathbf{A} \in \mathcal{A}_m^{(0)} : \sum_{v \in \mathbf{A}} X_v \geq m \sqrt{\frac{p}{p_0}} \right) \\ & \leq \sum_{u_0, \dots, u_r} \mathbb{P} \left( \sum_{v \in \bigcup_{i=0}^r \mathbf{B}(\ell u_i, \frac{3\ell}{2})} X_v \geq m \sqrt{\frac{p}{p_0}} \right) \end{aligned} \quad (3)$$

$$\leq 8^{2m\sqrt{p}} \cdot \exp(-p_0^{-\frac{1}{2}} m \sqrt{p}) \cdot (1 - p + pe)^{18 \cdot mp^{-\frac{1}{2}} (1 + \frac{4}{3} \sqrt{p_0})^2} \quad (4)$$

$$\leq \exp(2 \log 8 \cdot m \sqrt{p}) \cdot \exp(-p_0^{-\frac{1}{2}} m \sqrt{p}) \cdot \exp(p(e-1))^{18 \cdot mp^{-\frac{1}{2}} (1 + \frac{4}{3} \sqrt{p_0})^2} \quad (5)$$

$$\begin{aligned} & \leq \exp \left( m \sqrt{p} (2 \log 8 - p_0^{-\frac{1}{2}} + (e-1) 18 (1 + \frac{4}{3} \sqrt{p_0})^2) \right) \\ & \leq \exp(-m \sqrt{p}). \end{aligned}$$

Where line (3) follows from the union bound, (4) follows by the Chernoff bound  $\mathbb{P}(\sum_{1 \leq i \leq m} Z_i > t) \leq e^{-t} \mathbb{E}(e^{Z_i})^n$  where  $Z_i$  are independent identically distributed random variables, and (5) uses the fact that  $\forall t, 1 + t \leq e^t$ . The value of  $p_0$  has been chosen to ensure the last step.  $\square$

Lemma 7 is the driving force behind our proof. However, we cannot apply it directly, since regions in the Delaunay triangulation are always dependent on their neighbours. To deal with this, we will allow a *band* of local dependence around each region and then apply Lemma 7 multiple times with the total sum bounded by the union bound.

**Lemma 8.** *Consider the lattice  $\mathbf{G}$  with associated random variables  $X_v$  for each  $v \in \mathbf{G}$  such that  $X_v$  and  $X_u$  are dependent only if  $\|u - v\|_\infty = 1$ , and where the marginal distribution of  $X_v$  is Bernoulli( $p$ ). Then if  $p \leq p_1 := \frac{1}{43000} = (\frac{3}{16})^2 p_0$ ,*

$$\mathbb{P} \left( \exists \mathbf{A} \in \mathcal{A}_m^{(0)}, \sum_{v \in \mathbf{A}} X_v \geq \frac{3m}{4} \right) \leq e^{-m\sqrt{p}/2},$$

for  $n$  large enough.

*Proof.* We begin by partitioning  $\mathbf{G}$  in a collection of lattices,  $\mathbf{G}_i$  for  $i = 1, \dots, 4$  whose vertex sets respectively are

$$I_i := 2\mathbb{Z}^2 + v_i, \quad \text{with } v_1 = (0, 0), v_2 = (0, 1), v_3 = (1, 0), \text{ and } v_4 = (1, 1),$$

with edges in lattice  $i$  given by,

$$x \leftrightarrow y \iff \|x - y\| = 2,$$

for  $x, y \in I_i$ . Thus the vertex sets of each grid  $\mathbf{G}_i$  are disjoint and the random variables  $X_v, X_u$  are independent for any distinct  $u, v \in I_i$ . For a given  $\mathbf{A} \in \mathcal{A}_m^{(0)}$ , we may now define

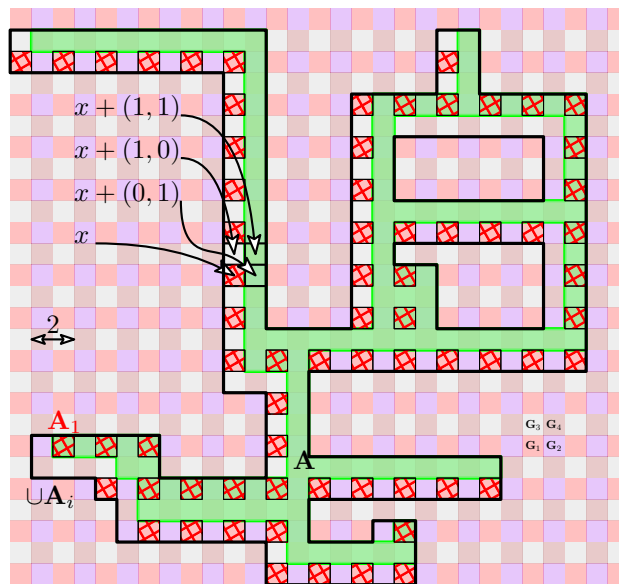


Figure 3: For the proof of Lemma 8.  $\mathbf{A}$  is green,  $\mathbf{A}_1$  is hatched red and  $\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3 \cup \mathbf{A}_4$  is shown by a black boundary. Notice that  $\mathbf{A}_1$  is connected in the grid  $\mathbf{G}_1$ .

a collection of *sub-animals* by looking at the vertices of  $\mathbf{G}_i$  that are in  $\mathbf{A}$  or have one of their three “positive” neighbours that are in  $\mathbf{A}$  (see Figure 3):

$$\mathbf{A}_i := \left\{ x \in I_i : \exists j \in [1, 4] \ x + v_j \in \mathbf{A} \right\}.$$

As the  $\left( \bigcup_{1 \leq j \leq 4} C(x + v_j) \right)_{x \in I_i}$  forms a partition of  $\mathbb{R}^2$  in squares of size 2, this definition ensures the connectivity of the sub-animals in the grid  $\mathbf{G}_i$ . Since  $\mathbf{A} \subset \bigcup \mathbf{A}_i$ , it follows that

$$\sum_{v \in \mathbf{A}} X_v \leq \sum_{i \leq 4} \sum_{v \in \mathbf{A}_i} X_i,$$

and that  $0 \leq |\mathbf{A}_i| \leq m$  for  $i = 1, \dots, 4$ . We now note that for a collection of non-negative random variables  $Z_1, \dots, Z_k$ , if  $\sum_{i=1}^k Z_i \geq t$  then, there exists an  $i$  such that  $Z_i \geq \frac{t}{k}$ . So by the union bound,

$$\mathbb{P} \left( \exists \mathbf{A} \in \mathcal{A}_m^{(0)}, \sum_{v \in \mathbf{A}} X_v \geq \frac{3m}{4} \right) \leq \sum_{i \leq 4} \mathbb{P} \left( \exists \mathbf{A} \in \mathcal{A}_m^{(0)}, \sum_{v \in \mathbf{A}_i} X_v \geq \frac{3m}{16} \right). \quad (6)$$

Note that  $|\mathbf{A}_i| \leq m$  and that the sum in the right hand side of (6) is strictly increasing in  $m$ . In particular,

$$\begin{aligned} & \left\{ \exists \mathbf{A} \in \mathcal{A}_m^{(0)}, \exists i \in [1, 4] : \sum_{v \in \mathbf{A}_i} X_v \geq \frac{3m}{16} \right\} \\ \implies & \left\{ \exists i \in [1, 4] \exists \mathbf{A} \in 2 \cdot \mathcal{A}_m^{(v_i)} : \sum_{v \in \mathbf{A}} X_v \geq \frac{3m}{16} \right\}, \end{aligned} \quad (7)$$

where  $2 \cdot \mathcal{A}_m^{(x)}$  is taken to be the set of animals living in the grids  $\mathbf{G}_i$  ( $1 \leq i \leq 4$ ). Since the scaling from  $\mathbf{G}$  to  $\mathbf{G}_i$  does not in anyway affect the bounds in Lemma 7, and since the  $X_v$  for all  $v \in \mathbf{A}_i$  are all independent and identically Bernoulli( $p$ ) distributed by assumption, applying (6) and (7) and the fact that  $p < p_1 = (\frac{3}{16})^2 p_0$  give a bound of  $4e^{-m\sqrt{p}}$  that is smaller than  $e^{-m\frac{\sqrt{p}}{2}}$  for  $m$  large enough.  $\square$

## 6 Measuring Walk Progress

To measure the progress made by the walk, we have multiple options. An obvious choice is to use the Euclidean distance to  $q$  from one of the vertices or from the circumcentre of each triangle visited to the destination. However, it is not difficult to prove that none of these distances is decreasing during a walk, leading to extra conditions to be taken care of in our proofs. We shall thus focus instead on using the *circle power*. The circle power relative to a point  $q$  of a circle centred at  $z \in \mathbb{R}^2$  and of radius  $r$  is defined to be  $\|z - q\|^2 - r^2$ . It is well known that, if a line through  $q$  cuts the circle in  $z'$  and  $z''$ , the power is also  $\overline{z'q} \cdot \overline{z''q}$  the product of the signed distance measured along the line, which is independent of the line. For a triangle  $\sigma$  and a point  $q$ , we shall say that  $\sigma$  has circle power

$$P(\sigma, q) := \|z(\sigma) - q\|^2 - r(\sigma)^2. \quad (8)$$

We remark that the power becomes negative when  $q$  is contained in  $B(z(\sigma), r(\sigma))$ . Helpfully, it turns out that when walking from a triangle  $\sigma \in \text{Del}(\mathbf{X})$  to its  $i$ th neighbour in a triangulation,  $\text{Del}(\mathbf{X})$ , the change in circle power may be very simply expressed. We first define the following properties of a triangle  $\sigma \in \text{Del}(\mathbf{X}_\gamma)$ ,

$$\begin{aligned} d(\sigma) &:= \inf \left\{ |z(\sigma) - z(\tau)| : \tau \in \text{Del}(\mathbf{X}_\gamma) \setminus \sigma \right\}, \\ \alpha_i(\sigma, q) &:= \angle(q, \sigma(i+1), \sigma(i+2)), \end{aligned} \quad (9)$$

where  $\alpha_i(\sigma, q)$  is an oriented angle and  $\sigma(i)$  is taken to be the  $(i \bmod 3)$ th vertex of  $\sigma$ , which is opposite the  $i$ th triangle neighbouring  $\sigma$  (we assume that the vertices are ordered counterclockwise.)

**Lemma 9.** *The circle power,  $\Delta_q(\sigma, i)$  relative to  $q$  when moving from a triangle  $\sigma$  to its  $i$ th neighbour,  $\tau$  decreases by exactly,*

$$\Delta_q(\sigma, i) = P(\sigma, q) - P(\tau, q) = 2 \sin \alpha_i(\sigma, q) \cdot \|z(\sigma)z(\tau)\| \cdot \|\sigma(i+1)q\|.$$

*Proof.* The proof relies on basic trigonometry. Referencing Figure 4, let  $A$  be the length of the part of line  $q\sigma(i+1)$  that overlaps the interior of the circle circumscribing  $\sigma$ , and  $B$  be the length of the part of the line overlapping the interior of the circle circumscribing  $\tau$ . If  $A$  and  $B$  do not overlap, the change in circle power when moving across the edge is now, denoting  $\sigma(i+1) = p$  for short, given by

$$P(\sigma, q) - P(\tau, q) = \|pq\|(\|pq\| + A) - \|pq\|(\|pq\| - B) = \|pq\|(A + B).$$

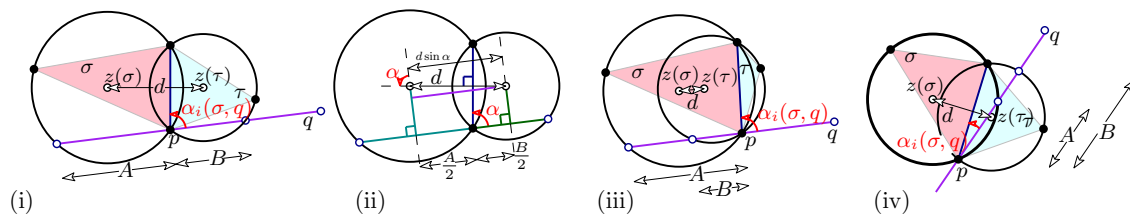


Figure 4: The change in circle power relative to  $q$  when moving between two overlapping circles is  $-2d \sin \alpha \|pq\|$ , and is thus independent of the radii.

To calculate  $A+B$ , we refer to Figure 4-(i). Basic trigonometry gives us that  $A+B = 2d \sin \alpha$  with  $d = \|z(\sigma)z(\tau)\|$ , as illustrated by Figure 4-(ii). If  $A$  and  $B$  do overlap on the side of  $p$  opposite to  $q$ , then

$$P(\sigma, q) - P(\tau, q) = \|pq\|(\|pq\| + A) - \|pq\|(\|pq\| + B) = \|pq\|(A - B).$$

again, basic trigonometry gives  $A - B = 2d \sin \alpha$  (Figure 4-(iii)). If  $A$  and  $B$  do overlap on the other side, then

$$P(\sigma, q) - P(\tau, q) = \|pq\|(\|pq\| - A) - \|pq\|(\|pq\| - B) = \|pq\|(B - A).$$

again, basic trigonometry gives  $B - A = 2d \sin \alpha$  (Figure 4-(iv)).

Notice that the position of  $q$  on the half line  $\sigma(i+1)q$  does not play any role in the proof and the result hold even if  $q$  is inside one of the circles circumscribing  $\sigma$  or  $\tau$ .  $\square$

**Corollary 10.** *The circle power is decreasing for visibility walks.*

*Proof.* It follows directly from Lemma 9 that  $\Delta_q(\sigma, i) \geq 0$  since going from  $\sigma$  to  $\tau$  match the visibility walk definition if and only if  $\alpha_i(\sigma, q) \in [0, \pi]$ .  $\square$

## 7 Getting Independence

By Lemma 9, we know that the diminution of circle power when leaving the triangle  $\sigma \in \text{Del}(\mathbf{X}_\gamma)$  is lower bounded by

$$|\Delta_q(\sigma, \cdot)| \geq 2(\|z(\sigma)q\| - r(\sigma)) \cdot d(\sigma) \cdot \min_{0 < i \leq 3} \left\{ \sin \alpha_i(\sigma, q) \right\}, \quad (10)$$

as long as  $\|z(\sigma)q\| \geq r(\sigma)$ . We may ignore the term  $(\|z(\sigma)q\| - r(\sigma))$  now, since it will be explicitly bounded by the current position in the walk later. We instead focus on the *local* properties of  $\sigma$ . We define nested boxes around  $C_v$ :  $C_\star(v) := B_\infty(v, \frac{5}{8})$  and  $C_+(v) := B_\infty(v, 0.99)$ , and the following *bad* event indexed by a vertex of  $\mathbf{G}$ ,

$$E_v := \bigcup_{\substack{\sigma \in \text{Del}(\mathbf{X}_\gamma), \\ z(\sigma) \in C_\star(v)}} \left\{ \left\{ r(\sigma) \geq C_r \right\} \cup \left\{ d(\sigma) \leq C_d \right\} \cup \left\{ \min_{0 < i \leq 3} \sin \alpha_i(\sigma, q) \leq C_\alpha \right\} \right\}, \quad (11)$$

The values  $C_r$ ,  $C_d$ ,  $C_\alpha$  will be fixed at the end of the proof of the following lemma.



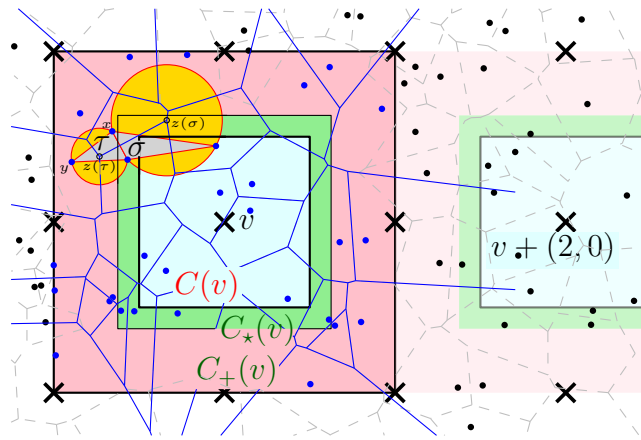


Figure 5: In blue the diagram of the points in  $C_+(v)$ , in dashed line the Voronoi diagram of all points.

**Lemma 11.** *Let  $p_2 \in (0, p_1]$  be fixed, then we can choose the constants  $C_r \leq \frac{1}{8}$ ,  $C_d \leq \frac{1}{2}C_r$ ,  $C_\alpha$  in (11) and the intensity  $\gamma$  of the process  $\mathbf{X}_\gamma$  such that the event  $E_v$  satisfies the following conditions,*

1.  $E_v$  and  $E_w$  are independent if  $\|v - w\| \geq 2$ .
2.  $\mathbb{P}(E_v) \leq p_2$
3. If event  $(E_v)^c$  occurs, then for every  $\sigma \in \text{Del}(\mathbf{X}_\gamma)$  with  $z(\sigma) \in C_*(v)$  and  $\|z(\sigma)q\| \geq C_r$ ,

$$\Delta_q(\sigma, \cdot) \geq 2C_d \cdot C_\alpha \cdot (\|z(\sigma)q\| - C_r).$$

*Proof of Lemma 11, Part (1).* Consider a Delaunay triangle  $\sigma$  of  $\text{Del}(\mathbf{X}_\gamma)$  with circumcenter  $z(\sigma)$  inside  $C_*(v)$  and a neighbouring triangle  $\tau$  of  $\sigma$ .

If  $(E_v)^c$  occurs, and using  $C_r \leq \frac{1}{8}$ ,  $C_d \leq \frac{1}{16}$ , we have that every vertex  $x$  of  $\sigma$  verifies

$$\|vx\|_\infty \leq \|vz(\sigma)\|_\infty + \|z(\sigma)x\| \leq \frac{5}{8} + C_r \leq \frac{5}{8} + \frac{1}{8} = 0.75 \leq 0.99$$

and a vertex  $y$  of  $\tau$ , with  $x$  common to  $\sigma$  and  $\tau$ , verifies

$$\begin{aligned} \|z(\tau)y\| &= \|z(\tau)x\| \leq \|z(\tau)z(\sigma)\| + \|z(\sigma)x\|, \\ \|vy\|_\infty &\leq \|vz(\sigma)\|_\infty + \|z(\sigma)z(\tau)\| + \|z(\tau)y\| \\ &\leq \frac{5}{8} + C_d + C_d + C_r \leq \frac{5}{8} + \frac{1+1+2}{16} = 0.75 \leq 0.99 \end{aligned}$$

and the vertices of  $\sigma$  and  $\tau$  are inside  $C_+(v)$  (see Figure 5). Thus  $z(\sigma)$  and  $z(\tau)$  are also Voronoi vertices of the dual of  $\text{Del}(\mathbf{X}_\gamma \cap C_+(v))$ .

So, we can decide whether or not  $E_v$  occurs given only the information in the sigma-algebra of the probability space for  $\mathbf{X}_\gamma$  restricted to events occurring in  $C_+(v)$ . This means that  $E_v, E_w$  are independent if  $\|v - w\| \geq 2$  since  $C_+(v) \cap C_+(w) = \emptyset$  and  $\mathbf{X}_\gamma$  is a Poisson point process.  $\square$

*Proof of Lemma 11, Part (2).* We begin by using the first-moment method to write the probability of the event  $E_v$  as the expectation of a sum,

$$\mathbb{P}(E_v) = \mathbb{P} \left( \sum_{\substack{\sigma \in \text{Del}(\mathbf{X}_\gamma), \\ z(\sigma) \in C_\star(v)}} \mathbb{1}_{r(\sigma) \geq C_r} + \sum_{\substack{\sigma \in \text{Del}(\mathbf{X}_\gamma), \\ z(\sigma) \in C_\star(v)}} \mathbb{1}_{r(\sigma) < C_r} \mathbb{1}_{\cup_{i \leq 3} \sin \alpha_i(\sigma) \leq C_\alpha \cup d(\sigma) \leq C_d} \geq 1 \right) \\ \leq E_v^{(r)} + E_v^{(d)} + E_v^{(\alpha)},$$

where we define

$$E_v^{(r)} := \mathbb{E} \left[ \sum_{\sigma \in \text{Del}(\mathbf{X}_\gamma)} \mathbb{1}_{z(\sigma) \in C_\star(v)} \mathbb{1}_{r(\sigma) \geq C_r} \right], \\ E_v^{(d)} := \mathbb{E} \left[ \sum_{\sigma \in \text{Del}(\mathbf{X}_\gamma)} \mathbb{1}_{z(\sigma) \in C_\star(v)} \mathbb{1}_{r(\sigma) < C_r} \mathbb{1}_{d(\sigma) \leq C_d} \right], \\ E_v^{(\alpha)} := \mathbb{E} \left[ \sum_{\sigma \in \text{Del}(\mathbf{X}_\gamma)} \mathbb{1}_{z(\sigma) \in C_\star(v)} \mathbb{1}_{r(\sigma) < C_r} \mathbb{1}_{\cup_{i \leq 3} \sin \alpha_i(\sigma) \leq C_\alpha} \right].$$

To compute these expected values we first can transform a sum on triples of points in a Poisson point process in an integral by the **Slivnyak-Mecke formula**:

$$\sum_{(x_1, x_2, x_3) \in (\mathbf{X}_\gamma)^3_{\neq}} \mathbb{1}_{f(\mathbf{X}_\gamma)} = \int_{(\mathbb{R}^2)^3} \mathbb{P} \left( f(\mathbf{X}_\gamma \cup \{x_1, x_2, x_3\}) \right) \gamma^3 dx_1 dx_2 dx_3 \quad (12)$$

(see, for example, Schneider and Weil [19, Corollary 3.2.3]). Then, the following change of variables, that goes from three points in cartesian coordinates to three points defined by their circumscribing circle (center and radius) and their positions on the circle can be applied **Blaschke-Petkanschin formula**:

$$\int_{(\mathbb{R}^2)^3} g(x_1, x_2, x_3) dx_1 dx_2 dx_3 \quad (13)$$

$$= \int_{\mathbb{R}^2} \int_0^\infty \int_{\mathbf{S}^3} g(z + ru_1, z + ru_2, z + ru_3) 2\mathcal{A}(u_1, u_2, u_3) \mu(du_1) \mu(du_2) \mu(du_3) dr dz, \quad (14)$$

where  $\mathcal{A}(u_1, u_2, u_3)$  is the area of the convex hull of  $\{u_1, u_2, u_3\}$  and  $\mathbf{S}$  is the unit 1-sphere with associated uniform (Lebesgue) measure  $\mu(\cdot)$  (see, for example, Schneider and Weil [19, Theorem 7.3.1]). Denoting by  $x_1, x_2, x_3$  the vertices of  $\sigma$ , we have,

$$\begin{aligned}
E_v^{(r)} &= \frac{1}{3!} \mathbb{E} \left[ \sum_{(x_1, x_2, x_3) \in (\mathbf{X}_\gamma)^3_{\neq}} \mathbb{1}_{B(\sigma) \cap \mathbf{X}_\gamma \setminus \{x_1, x_2, x_3\} = \emptyset} \mathbb{1}_{z(\sigma) \in C_\star(v)} \mathbb{1}_{r(\sigma) \geq C_r} \right] \\
&= \frac{1}{3!} \int_{(\mathbb{R}^2)^3} \mathbb{P} \left( B(x_1, x_2, x_3) \cap \mathbf{X}_\gamma = \emptyset \right) \mathbb{1}_{z(\sigma) \in C_\star(v)} \mathbb{1}_{r(\sigma) \geq C_r} \gamma^3 dx_1 dx_2 dx_3 \quad (15)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\gamma^3}{3!} \int_{C_\star(v)} \int_{C_r}^\infty \int_{\mathbf{S}^3} e^{-\pi\gamma r^2} \cdot r^3 2\mathcal{A}(u_1, u_2, u_3) \mu(du_1) \mu(du_2) \mu(du_3) dr dz \quad (16) \\
&= \frac{\gamma^3}{6} \cdot \left(\frac{10}{8}\right)^2 24\pi^2 \int_{C_r}^\infty r^3 \cdot e^{-\pi\gamma r^2} dr \\
&= \frac{25}{4} \pi^2 \gamma^3 \cdot \frac{1 + \pi\gamma C_r^2}{2\pi^2 \gamma^2} e^{-\pi\gamma C_r^2} = \frac{25}{8} \gamma (1 + \pi\gamma C_r^2) e^{-\pi\gamma C_r^2}
\end{aligned}$$

Equation (15) follows from the Slivnyak-Mecke formula (12), and Equation (16) uses the Blaschke-Petkanschin change of variables (13). The last steps follows from the two classical integrals:

$$\int_{\mathbf{S}^3} \mathcal{A}(u_{1:3}) \mu(du_{1:3}) = 3! \int_0^{2\pi} \int_0^{u_3} \int_0^{u_2} \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ \cos(u_1) & \cos(u_2) & \cos(u_3) \\ \sin(u_1) & \sin(u_2) & \sin(u_3) \end{vmatrix} \mu(du_1) \mu(du_2) \mu(du_3) = 12\pi^2$$

$$\int_R^\infty r^3 e^{-r^2} dr = \frac{1}{2} (R^2 + 1) e^{-R^2}.$$

To deal with  $E_v^{(d)}$ , we first remark that a Voronoi vertex  $z(\sigma)$  can have a short Voronoi edge  $z(\sigma)z(\tau)$  of length  $\leq C_d$  only if the ball  $B(z(\sigma), r(\sigma) + 2C_d)$  contains other points than the vertices of  $\sigma$ . To prove this, let's assume, w.l.o.g., that  $\tau$  has vertices  $x_1, x_2$ , and  $y$  and assuming  $\|z(\sigma)z(\tau)\| \leq C_d$ , we will prove that  $y \in B(z(\sigma), r(\sigma) + 2C_d)$  and thus  $B(z(\sigma), r(\sigma) + 2C_d) \neq \emptyset$ . We have:

$$\begin{aligned}
\|z(\sigma)y\| &\leq \|z(\sigma)z(\tau)\| + \|z(\tau)y\| = \|z(\sigma)z(\tau)\| + \|z(\tau)x_1\| \\
&\leq \|z(\sigma)z(\tau)\| + \|z(\tau)z(\sigma)\| + \|z(\sigma)x_1\| \leq C_d + C_d + r(\sigma).
\end{aligned}$$

Using the above observation, we can now bound  $E_v^{(d)}$  applying the Slivnyak-Mecke formula (12), the change of variables (13), and integrating out the constants for the angles. Denoting  $\mathcal{B}$  the annulus centered at  $z(\sigma)$  and of radii  $r(\sigma)$  and  $r(\sigma) + 2C_d$  (with  $\sigma = x_1x_2x_3$ ),

we have:

$$\begin{aligned}
E_v^{(d)} &\leq \frac{1}{3!} \int_{(\mathbb{R}^2)^3} \mathbb{P}(B(\sigma) \cap \mathbf{X}_\gamma = \emptyset) \mathbb{P}(\mathcal{B} \cap \mathbf{X}_\gamma \neq \emptyset) \mathbb{1}_{z(\sigma) \in C_\star(v)} \mathbb{1}_{r(\sigma) \leq C_r} \gamma^3 dx_1 dx_2 dx_3 \\
&\leq \frac{2 \cdot \gamma^3}{3!} \int_{C_\star(v)} \int_0^{C_r} \int_{\mathbf{S}^3} e^{-\pi\gamma r^2} (1 - e^{-4\pi\gamma r C_d}) \cdot r^3 \mathcal{A}(u_1, u_2, u_3) \mu(du_{1:3}) dr dz \\
&\leq \frac{2 \cdot \gamma^3}{3!} \left(\frac{10}{8}\right)^2 12\pi^2 \int_0^{C_r} r^3 \cdot e^{-\pi\gamma r^2} 4\pi\gamma C_d r dr \\
&= 25\pi^3 \gamma^4 C_d \int_0^{C_r} r^4 \cdot e^{-\pi\gamma r^2} dr \\
&\leq 25\pi^3 \gamma^4 C_d C_r \int_0^\infty r^3 e^{-\pi\gamma r^2} dr \\
&= 25\pi^3 \gamma^4 C_d C_r \cdot \frac{1}{2\pi^2 \gamma^2} \\
&\leq \frac{25}{2} \pi \gamma^2 C_d C_r
\end{aligned}$$

Finally, we compute bounds for  $E_v^{(\alpha)}$ . We apply (12) and (13) as before, but this time we do not sum over all permutations of the points and instead consider just the angle  $\alpha_1((x_1, x_2, x_3), q) = \angle(q, x_1, x_2)$  for each triple of points  $x_1, x_2, x_3 \in (\mathbf{X}_\gamma)_\neq^3$ . We obtain

$$\begin{aligned}
E_v^{(\alpha)} &\leq \int_{(\mathbb{R}^2)^3} \mathbb{P}(B(\sigma) \cap \mathbf{X}_\gamma = \emptyset) \mathbb{1}_{z(\sigma) \in C_\star(v)} \mathbb{1}_{r(\sigma) \leq C_r} \mathbb{1}_{\alpha_1(\sigma, q) \leq C_\alpha} \gamma^3 dx_1 dx_2 dx_3 \\
&= 2\gamma^3 \int_{C_\star(v)} \int_0^{C_r} \int_{\mathbf{S}^3} e^{-\pi\gamma r^2} \mathbb{1}_{\alpha_1(u_1, u_2, u_3, q) \leq C_\alpha} \cdot r^3 \mathcal{A}(u_1, u_2, u_3) \mu(du_{1:3}) dr dz \\
&\leq 2\gamma^3 \left(\frac{10}{8}\right)^2 \frac{3\sqrt{3}}{4} \int_0^{C_r} r^3 e^{-\pi\gamma r^2} dr \int_{\mathbf{S}^3} \mathbb{1}_{\alpha_1(u_1, u_2, u_3, q) \leq C_\alpha} \mu(du_{1:3}) \quad (17)
\end{aligned}$$

$$\begin{aligned}
&\leq 2\gamma^3 \left(\frac{10}{8}\right)^2 \frac{3\sqrt{3}}{4} \frac{1 - e^{-\pi\gamma C_r^2}}{2\pi^2 \gamma^2} \int_{\mathbf{S}} \mu(du_1) \int_{\mathbf{S}} 4C_\alpha \mu(du_2) \int_{\mathbf{S}} \mu(du_3) \quad (18) \\
&\leq \frac{75\sqrt{3}}{64\pi^2} \gamma \cdot 2\pi \cdot 8\pi C_\alpha \cdot 2\pi = \frac{75\sqrt{3}\pi}{2} \gamma C_\alpha
\end{aligned}$$

where (17) follows since  $\mathcal{A}(u_1, u_2, u_3) \leq \frac{3\sqrt{3}}{4}$  and (18) follows since to have  $\angle(q, x_1, x_2) < t$  we need  $x_2$  to be in the region denoted  $X_2$  in Figure 6. The measure of  $X_2$  is the angle from the circle center, since  $X_2$  is seen with an angle  $2t$  from a point on the circle by definition of  $t$  and  $X_2$ , basic trigonometry gives  $4t$  as measure of  $X_2$ .

Finally, we get:

$$\mathbb{P}(E_v) \leq \frac{25}{8} \gamma (1 + \pi\gamma C_r^2) e^{-\pi\gamma C_r^2} + \frac{25}{2} \pi \gamma^2 C_d C_r + \frac{75\sqrt{3}\pi}{2} \gamma C_\alpha.$$

Using the above, we have that choosing  $C_r := \gamma^{-1/3}$ ,  $C_d := \gamma^{-3}$ , and  $C_\alpha := \gamma^{-3}$  yields  $\mathbb{P}(E_v) = O(\gamma^{4/3} e^{-\pi\gamma^{1/3}})$  is asymptotically decreasing as  $\gamma \rightarrow \infty$ . Therefore it suffices to choose  $\gamma$  sufficiently large to guarantee that  $\mathbb{P}(E_v) \leq p_2$ . In the sequel,  $\gamma$ ,  $C_r$ ,  $C_d$ , and  $C_\alpha$  are fixed to the present values.  $\square$

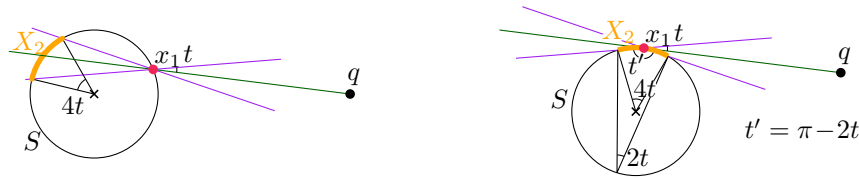


Figure 6: Bounding the measure of small angles.

*Proof of Lemma 11, Part (3).* When the good event  $(E_v)^c$  occurs, (3) is obtained by substituting the above bounds in Equation (10) for any edge with an endpoint in  $C_\star(v)$ .

$$\begin{aligned} |\Delta_q(\sigma, \cdot)| &\geq 2(\|z(\sigma)q\| - r(\sigma)) \cdot d(\sigma) \cdot \min_{0 < i \leq 3} \left\{ \sin \alpha_i(\sigma, q) \right\} \\ &\geq 2C_d \cdot C_\alpha \cdot (\|z(\sigma)q\| - C_r) \end{aligned}$$

Notice that, because  $C_d \leq \frac{1}{16}$  an edge of  $\bar{w}$  that intersects  $C(v)$  has an endpoint in  $C_\star(v)$ .  $\square$

## 8 The Longest Walk

Bounding the number of steps required by the whole walk at once is made complicated since the circle power depends on the distance to the destination. We shall thus split the walk up into a number of *stages*, each of which can be bounded separately. To do this we introduce the following notation, given a walk  $w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma))$ , we write  $\bar{w}[t]$ , for  $t \in [0, 1]$  to represent the point on the polygonal path  $\bar{w}$  at a distance  $t$  from  $\bar{w}(0)$  normalised so that  $\bar{w}[0] = \bar{w}(0)$  and  $\bar{w}[1] = \bar{w}(|\bar{w}|) = q$  and taking  $\bar{w}[a, b]$  to be a continuous part of  $\bar{w}$ .

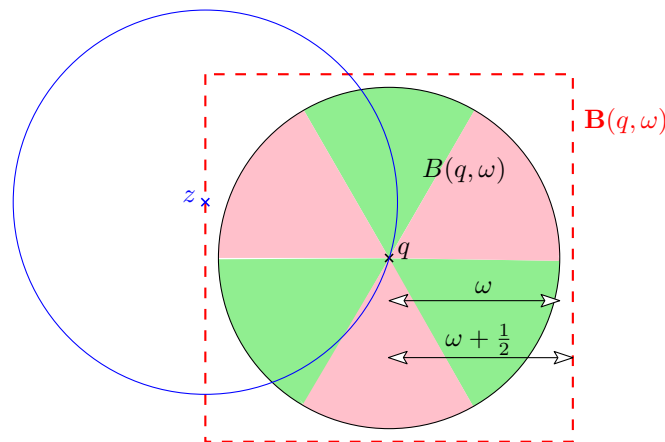
**Lemma 12.** Fix  $w \in \mathbf{P}^{(q)}(\mathbf{X}_\gamma)$  and  $S = \mathbf{A}(\bar{w}[\ell, r])$ , for some  $0 \leq \ell < r \leq 1$ . We shall define the change in circle power over the triangles in the walk  $w$  whose centres are in  $C(S)$  as,

$$\Delta_w(S) := \max_{\substack{\sigma \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)), \\ z(\sigma) \in C(S)}} P(\sigma, q) - \min_{\substack{\sigma \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)), \\ z(\sigma) \in C(S)}} P(\sigma, q).$$

Then for  $L(S, q) := \min\{\|xq\| : x \in C(S)\} > C_r$ ,

$$\left\{ \sum_{v \in S} \mathbf{1}_{E_v^c} \geq t \right\} \implies \Delta_w(S) \geq 2C_d \cdot C_\alpha \cdot (L(S, q) - C_r) \cdot \left\lceil \frac{t-2}{4} \right\rceil.$$

*Proof.* Our proof will follow by associating a *delegate* triangle,  $\sigma \in w$ , with every box  $v \in S$  such that  $E_v^c$  occurs, so that when exiting the delegate triangle, the change in circle power is bounded. Since the circle power is decreasing as the walk advances (by Corollary 10), it will suffice to count the number of unique delegates in order to lower bound the progress made in the circle power. For brevity, we define  $z_i := \bar{w}(i)$  for  $i = 0, \dots, |\bar{w}| - 1$  and  $e_i := z_i z_{i+1}$  for  $i = 0, \dots, |\bar{w}| - 2$ . Recall that for  $v \in \mathbb{Z}^2$ ,  $v$  may only be contained in  $S \subseteq \mathbf{A}(\bar{w})$  if one of the line segments,  $e_i$  intersects  $C(v)$ . Thus, for a given  $v \in S$  such that  $E_v^c$  occurs, there

Figure 7: Probability of  $F_\omega$ .

exists at least one segment,  $e_x$  for  $x \in 0, \dots, |w| - 2$  such that  $e_x \cap C(v) \neq \emptyset$ . We consider the possible configurations for  $e_x$ .

By construction of  $C_\star(v)$  and definition of  $E_v$  one of the endpoints of  $e_x$ , belongs to  $C_\star(v)$ . Thus, the progress in the circle power made when leaving the triangle dual of this endpoint is at least  $2C_d \cdot C_\alpha \cdot (L(S, q) - C_r)$ , by Lemma 11, Part (3). We select the delegate for  $v$  to be this vertex of  $\bar{w}$ .

We have thus found  $t$  delegates, assuming that  $(E_v)^c$  occurs  $t$  times for  $v \in S$ . However, we note that a single triangle  $\sigma \in \text{Del}(\mathbf{X}_\gamma)$  can be a delegate for multiple boxes,  $v \in S$ , due to overlapping between adjacent boxes. In addition, the very first and very last delegates may not be contained within  $C(S)$ , though all of the others must be since  $S$  is an animal formed from a continuous ‘sub-segment’ of  $\bar{w}$  by assumption. Finally, since one delegate may be shared by at most four boxes, it follows that at least  $\lceil (t - 2)/4 \rceil$  of the delegates must be unique and the stated result follows.  $\square$

We define for  $\omega \in \mathbb{R}^*$

$$F_\omega := \left\{ \forall z \in C(\mathbb{Z}^2 \setminus \mathbf{B}(q, \omega)) B(z, \|zq\|) \cap \mathbf{X}_\gamma \neq \emptyset \right\}$$

the fact that an empty circle centered  $\omega$  away from  $q$  does not enclose  $q$ .

**Lemma 13.**

$$\mathbb{P}(F_\omega^c) \leq 6e^{-\frac{\pi\gamma\omega^2}{6}}.$$

*Proof.* The probability of  $F_\omega^c$  can be bounded by noticing that a circle whose center is in  $C(\mathbb{Z}^2 \setminus \mathbf{B}(q, \omega)) \subset C(\mathbb{Z}^2 \setminus B(q, \omega))$  and passing through  $q$  must enclose one of the six sectors of  $B(q, \omega)$ , and one of this sector must be empty (see Figure 7). The result follows.  $\square$

For positive integers  $\ell > r$ , we now define

$$S_{\ell,r}(w) := \mathbf{A} \left( \bar{w} \left[ \inf \{ t : \bar{w}[t] \in C(\mathbf{B}(q, \ell)) \}, \inf \{ t : \bar{w}[t] \in C(\mathbf{B}(q, r)) \} \right] \right).$$

This set represents all of the boxes visited by the walk between the first time the walk intersects  $C(\partial\mathbf{B}(q, \ell))$  and the first time it intersects  $C(\partial\mathbf{B}(q, r))$ . If no such boxes exist we just have  $S_{\ell, r} = \emptyset$ . It follows that  $S_{\ell, r}(w)$  is an animal on  $\mathbf{G}$  since it is a connected subset of the animal  $\mathbf{A}(\bar{w})$ . In the next lemma we shall bound the number of boxes required to *halve* the distance remaining in the walk. Iteratively applying this bound will be the key in the remaining steps of the proof.

**Lemma 14.** *For  $k$  large enough, and  $\omega \leq \frac{k}{2}$*

$$\mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{k, k/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot k \mid F_\omega\right) \leq 8k e^{-\frac{40k\sqrt{p_2}}{C_d C_\alpha}}$$

for  $C_d, C_\alpha, p_2$  chosen in Lemma 11.

*Proof.* We define:

$$H := \bigcup_{u \in \partial\mathbf{B}(q, k)} \left\{ \forall \mathbf{A} \in \mathcal{A}_{\frac{80}{C_d C_\alpha} \cdot k}^{(u)} : \sum_{v \in \mathbf{A}} \mathbb{1}_{E_v} \leq \frac{60k}{C_d C_\alpha} \right\}, \quad (19)$$

the event that any animal of size  $\frac{80}{C_d C_\alpha} k$  starting at distance  $k$  from  $q$  does not have too many bad cells.

The probability of  $H^c$  can be bounded using Lemma 8.

$$\mathbb{P}(H^c) \leq 8k e^{-\frac{80}{C_d C_\alpha} k \frac{\sqrt{p_2}}{2}}, \quad (20)$$

We now focus on proving the following implication,

$$H \cap F_\omega \implies \left\{ \forall w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{k, k/2}(w)| \leq k \cdot \frac{80}{C_d C_\alpha} \right\}, \quad (21)$$

which will follow by contradiction. Suppose that  $H \cap F_\omega$  occurs and there exists a path,  $w^\sharp \in \mathbf{W}(q, \text{Del}(\mathbf{X}_\gamma))$  contradicting the right-hand side of (21). In which case,  $|S_{k, k/2}(w^\sharp)| > \frac{80}{C_d C_\alpha} k$ , and as a direct consequence of (19),

$$H \implies \left\{ \sum_{v \in S_{k, k/2}(w^\sharp)} \mathbb{1}_{(E_v)^c} > \frac{20k}{C_d C_\alpha} \right\}. \quad (22)$$

Applying (22) in conjunction with Lemma 12 gives us that the change in circle power is lower bounded,

$$\begin{aligned} \Delta_{w^\sharp}(S_{k, k/2}(w)) &\geq 2 \left( \frac{k-1}{2} - C_r \right) \cdot C_d \cdot C_\alpha \cdot \left\lceil \frac{\frac{20k}{C_d C_\alpha} - 2}{4} \right\rceil \\ &\geq \frac{\frac{20}{C_d C_\alpha} \cdot C_d \cdot C_\alpha \cdot k^2}{4} - \frac{C_d \cdot C_\alpha \left( \frac{20}{C_d C_\alpha} \cdot (C_r + \frac{1}{2}) + 1 \right) \cdot k}{2} \\ &\geq 5k^2 - O(k) \\ &\geq 4k^2, \end{aligned} \quad (23)$$



for  $k$  large enough. We also know that the maximum circle power of  $q$  with respect to any triangle whose centre is in  $\partial \mathbf{B}(q, k)$  is  $(\sqrt{2}k)^2 = 2k^2$  from (8). Since, in addition,  $F_\omega$  guarantees that no triangle in  $w^\sharp$  has negative circle power, it follows that

$$\Delta_{w^\sharp}(S_{k,k/2}(w)) \leq 2k^2, \quad (24)$$

which is a contradiction with (23). Therefore no such walk  $w^\sharp$  can exist, and we have proved the implication in (21). To prove the stated result, we condition as follows,

$$\begin{aligned} & \mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{k,k/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot k \mid F_\omega\right) \\ & \leq \mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{k,k/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot k \mid H \cap F_\omega\right) + \mathbb{P}(H^c) \\ & = \mathbb{P}(H^c) \end{aligned} \quad (25)$$

$$\leq 8k e^{-\frac{40k\sqrt{p_2}}{C_d C_\alpha}} \quad (26)$$

where (25) follows from (21) and (26) follows from (20).  $\square$

**Lemma 15.** For  $\omega \leq \frac{k}{2}$  large enough, we have

$$\mathbb{P}\left(\exists w \in \mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{k,2\omega}(w)| \geq \frac{160}{C_d C_\alpha} \cdot k \mid F_\omega\right) \leq e^{-\frac{30\sqrt{p_2}}{C_d C_\alpha} \omega},$$

for  $C_d, C_\alpha, p_2$  chosen in Lemma 11.

*Proof.* Let  $w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma))$  be a walk and  $\tau = \lfloor \log_2 \frac{k}{\omega} \rfloor$  we shall consider the following stages

$$S_i(w) := S_{k \cdot 2^{-i}, k \cdot 2^{-(i+1)}}(w); \quad 0 \leq i < \tau.$$

Note that these stages need *not* to be pairwise disjoint, that none of them overlap  $C(\mathbf{B}(q, \omega))$ , and that  $S_{\tau-1}(w)$  ends inside  $C(\mathbf{B}(q, 2\omega))$ . We observe that

$$\begin{aligned} & \left\{ \exists w \in \mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{k,2\omega}(w)| \geq \frac{160k}{C_d C_\alpha} \right\} \\ & \implies \left\{ \exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : \bigcup_{i < \tau} \left\{ |S_i(w)| \geq \frac{160k}{C_d C_\alpha} \cdot \frac{1}{2^{i+1}} \right\} \right\}. \end{aligned}$$

Which follows since

$$S_{k,2\omega}(w) \subset \bigcup_{0 \leq i < \tau} |S_i(w)| \quad \text{and} \quad \frac{160k}{C_d C_\alpha} \sum_{0 \leq i < \tau} 2^{-(i+1)} \leq 2 \frac{160k}{C_d C_\alpha}.$$

We can now bound the requested probability using Lemma 14.

$$\begin{aligned}
& \mathbb{P} \left( \exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{k,2\omega}(w)| \geq \frac{160k}{C_d C_\alpha} \middle| F_\omega \right) \\
& \leq \mathbb{P} \left( \exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : \exists i < \tau : |S_i(w)| \geq \frac{160k}{C_d C_\alpha} \cdot \frac{1}{2^{i+1}} \middle| F_\omega \right) \\
& \leq \sum_{0 \leq i < \tau} \mathbb{P} \left( \exists w \in \mathbf{P}^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{(k \cdot 2^{-i}), (k \cdot 2^{-i})/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot (k \cdot 2^{-i}) \middle| F_\omega \right) \\
& \leq \sum_{0 \leq i < \tau} 8 \cdot 2^{-i} k e^{-\frac{40\sqrt{p_2}}{C_d C_\alpha} 2^{-i} k} \\
& \leq 8\omega \sum_{0 \leq i < \infty} 2^i e^{-(\frac{40\sqrt{p_2}}{C_d C_\alpha} \omega) 2^i} \\
& \leq 8\omega \int_1^\infty x e^{-(\frac{40\sqrt{p_2}}{C_d C_\alpha} \omega) x} dx \\
& = \left( \left( \frac{40\sqrt{p_2}}{C_d C_\alpha} \omega \right) + \left( \frac{40\sqrt{p_2}}{C_d C_\alpha} \omega \right)^{-2} \right) e^{-\frac{40\sqrt{p_2}}{C_d C_\alpha} \omega} \leq e^{-\frac{30}{C_d C_\alpha} \omega},
\end{aligned} \tag{27}$$

for  $\omega$  large enough. Line (27) follows from Lemma 14.  $\square$

**Lemma 16.** *For  $\omega$  large enough,*

$$\mathbb{P} \left( \exists w \in \mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |S_{2\omega,0}(w)| \geq 121\omega^2 \right) \leq 20 e^{-\pi\gamma\omega^2}$$

*Proof.* Let  $\sigma$  be a triangle on the walk with center inside  $\mathbf{B}(q, 2\omega)$ , then  $P(\sigma, q) \leq \|z(\sigma)q\|^2 \leq (\sqrt{2}(2\omega + \frac{1}{2}))^2 \leq 9\omega^2$  for  $\omega$  large enough. For a triangle  $\tau$  after  $\sigma$  in the walk, we have  $P(\tau, q) \leq P(\sigma, q)$  by Corollary 10. Let's assume that  $\|z(\tau)q\| \geq 5\omega$  and that  $\|z(\tau)q\| - r(\tau) \geq 3\omega$  then  $P(\tau, q) = \|qz'\| \cdot \|qz''\| \geq 3 \cdot 5\omega^2 = 15\omega^2$  which is a contradiction (the power is computed considering  $z'$ ,  $z''$  the intersection points of line  $qz(\tau)$  and the circle of  $\tau$ , see Figure 8).

Thus, the circumscribing circle of a triangle  $\tau$  with  $\|z(\tau)q\| \geq 5\omega$  must intersect  $B(q, 3\omega)$  and, by consequence, enclose one of the 20 regions subdividing the annulus of radii  $4\omega$  and  $6\omega$  depicted in Figure 8 must be empty. Such an event can happen only with probability less than  $20 e^{-\frac{\pi\gamma\omega^2(36-16)}{20}} = 20 e^{-\pi\gamma\omega^2}$ .

If no triangle  $\tau$  with  $\|z(\tau)q\| \geq 5\omega$  exists, then the end of the walk can only use triangles with center inside  $\mathbf{B}(q, 5\omega)$  and  $|S_{2\omega,0}(w)| \leq |\mathbf{B}(q, 5\omega)| \leq (11\omega)^2$ .  $\square$

*Proof of Proposition 5, Part (1).* The proof of the this part now follows easily by combining

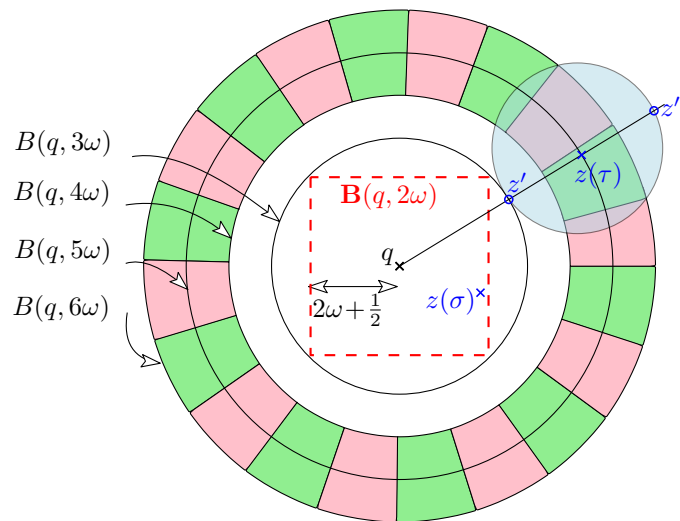


Figure 8: Last step of the walk

the results of Lemmas 13, 15, and 16, we get:

$$\begin{aligned}
 & \mathbb{P} \left( \exists w \in \mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : |\mathbf{A}(\bar{w})| \geq \frac{160}{C_d C_\alpha} \cdot k + 121\omega^2 \right) \\
 & \leq \mathbb{P} \left( \exists w \in \mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : S_{k, 2\omega}(w) \geq \frac{160}{C_d C_\alpha} k \mid F_\omega \right) + \mathbb{P}(F_\omega^c) \\
 & \quad + \mathbb{P} \left( \exists w \in \mathbf{P}_k^{(q)}(\text{Del}(\mathbf{X}_\gamma)) : S_{2\omega, 0}(w) \geq 121\omega^2 \right) \\
 & \leq e^{-\frac{30\sqrt{p_2}}{C_d C_\alpha} \omega} + 6e^{-\frac{\pi\gamma}{6} \omega^2} + 20e^{-\pi\gamma \omega^2} \leq e^{-\frac{20\sqrt{p_2}}{C_d C_\alpha} \omega}
 \end{aligned}$$

for  $k$  large enough, Statement 1 of Proposition 5 follows using  $\omega = \sqrt{k}$ . □

## 9 Compass and Greedy Walks

*Proof of Corollary 3.* The result holds because it is possible to associate a unique path in the visibility walk graph with every instance of Compass Walk.

More precisely, in Compass Walk, there are two vertices that may be taken at each step. Note that these are both vertices of a triangle which has two edges oriented ‘inwards’ and one oriented ‘outwards’ in the visibility walk graph. It is easy to see that we can always find a path to this triangle once we have entered the *star* of the point in question (see Figure 9). There is therefore always a path in the visibility walk graph associated with every Compass Walk. Since the number of triangles is an upper bound on the number of points, our bound implies a bound on Compass Walk, as stated. □

Unfortunately, there is no easy correspondence between a greedy path and the visibility graph. Let  $v$  be a Delaunay vertex,  $u$  its predecessor, and  $w$  its successor in Greedy



The main question remaining is the extension of this result to a bounded domain  $D$  considering  $\text{Del}(\mathbf{X}_\gamma \cap D)$  instead of  $\text{Del}(\mathbf{X}_\gamma)$ . This is made difficult because the visibility walk in  $\text{Del}(\mathbf{X}_\gamma \cap D)$  may have a strange behavior close to the boundary, since there are long skinny triangles that allow the walk to “jump” along the boundary. Intuition says that these jumps should shorten the walk, but unfortunately this is not always true, and it is not possible to prove something on the length of a path in  $\text{Del}(\mathbf{X}_\gamma \cap D)$  by finding an equivalent path in  $\text{Del}(\mathbf{X}_\gamma)$ .

The approach in that paper does not generalise directly since the path  $\bar{w}$  may go outside  $D$ , and we have no direct control on its length.

We therefore leave the following two open questions, proving the same result in the case of a bounded domain, and finding more reasonable constants in the complexity.

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