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# Strong Shape Derivative for the Wave Equation with Neumann Boundary Condition

Jean-Paul Zolésio\*and Lorena Bociu†

## Abstract

The paper provides shape derivative analysis for the wave equation with mixed boundary conditions on a moving domain  $\Omega_s$  in the case of non smooth neumann boundary datum. The key ideas in the paper are (i) bypassing the classical sensitivity analysis of the state by using parameter differentiability of a functional expressed in the form of Min-Max of a convex-concave Lagrangian with saddle point, and (ii) using a new regularity result on the solution of the wave problem (where the Dirichlet condition on the fixed part of the boundary is essential) to analyze the strong derivative.

## 1 Introduction

The aim of this paper is to give a full analysis of the shape differentiability for the solution to the wave equation with mixed boundary conditions on a moving domain  $\Omega_s$ . The shape derivative investigation has been solved for the wave equation with homogeneous and non-homogeneous Dirichlet boundary conditions [1, 9, 13]. The novelty and difficulty of the paper are represented by the fact that the wave equation has non-homogeneous Neumann boundary condition on part of the boundary of its geometrical domain. Moreover, the Neumann datum  $g$  is non-smooth, i.e.  $g \in H^{-1/2}(\partial\Omega_s)$ .

First, we prove existence of weak shape derivative by using parameter differentiability of a functional expressed in the form of Min-Max of a convex-concave Lagrangian with saddle point [3, 13]. This completely bypasses the classical sensitivity analysis of the state (solution) [9]. A lot of problems in shape sensitivity analysis can be expressed as a Min Max of some Lagrangian dependent on the domain  $\Omega$ . Using a velocity field of deformations  $V$  over  $\Omega$  one can build a family of perturbations  $\Omega_s$ ,  $s \geq 0$ , and then the tunic extends to situation when after some change of variable the sensitivity analysis reduces to the study of the differentiability of a Min Max Lagrangian functional with respect to the parameter  $s$  for fixed velocity fields  $V$  and domains  $\Omega$ .

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Then we use a new regularity result for the solution to the wave problem to analyze the strong derivative via a brute force estimate on the differential quotient. In particular, we study the variational solution  $y_s$  in the space  $W^{1,\infty}(0, \tau, L^2(\Omega_s)) \cap L^\infty(0, \tau, H_*^1(\Omega_s)) = \{\phi \in H^1(\Omega_s), \phi = 0 \text{ on } S\}$  by taking advantage of the Dirichlet condition on the fixed part of the boundary  $S$  and we derive sharp estimate for  $y_s$  at the boundary  $\Gamma_s$  in terms of geometrical constants, in view of controlling the differential quotient's regularity that is necessary in the proof of our main result.

The new results obtained in this paper are: (i) existence of strong material and shape derivatives for the solution to the wave problem with mixed boundary conditions, and (ii) the new wave equation that the shape derivative solves.

The rest of the paper is organized as follows. In Section 2, we provide a preliminary result on existence and uniqueness of a Galerkin solution for the classical wave equation with variable coefficients. This result will be needed in the proof of our main theorem. In Section 3, we introduce the PDE model for the wave equation on a moving domain  $\Omega_s$  and briefly recall the velocity method from shape optimization. In Section 4, we prove existence for the weak material derivatives. Finally, in Section 5 we complete the analysis by proving existence of strong material and shape derivatives, using the “extractor strategy” introduced in [5, 6] and Fourier transform techniques.

## 2 Galerkin Solution for the wave equation

Let  $D \subset \mathbb{R}^N$  be fixed (potentially included in a  $C^2$  manifold). Let  $\Omega \subset D$  be an open bounded domain, with smooth boundary  $\partial\Omega = \Gamma \cup S$ , where  $\bar{\Gamma} \cap \bar{S} = \emptyset$ . Let  $A(x) = \{a_{ij}(x)\}$  be a matrix of functions defined on  $\Omega$  with the following properties:

$$(2.1) \quad \begin{cases} a_{ij} \in L^\infty(\Omega), \text{ and} \\ \exists \alpha > 0 \text{ such that } \forall x \in \Omega, \forall \zeta = \{\zeta_i\} \in \mathbb{R}^N, a_{ij} \zeta_i \zeta_j \geq \alpha |\zeta|^2. \end{cases}$$

We associate with the matrix  $A(x)$  the following operator

$$Ay \stackrel{\text{def}}{=} -\text{div}(A(x)\nabla y).$$

Given  $\tau > 0$  and the interval  $I = [0, 2\tau]$ , consider the following wave equation problem:

$$(2.2) \quad \begin{cases} y_{tt} + A.y = f & \Omega \\ \frac{\partial y}{\partial n_A} = g & \Gamma \\ y = 0 & S \\ y(0) = y_0, \quad y_t(0) = y_1 \end{cases}$$

**Galerkin solution for (2.2) in  $L^2(I, H^1(\Omega)) \cap H^1(I, L^2(\Omega))$ .**

Let  $H_*^1(\Omega) = \{\phi \in H^1(\Omega), \phi = 0 \text{ on } S\}$  with norm  $\|\phi\|^2 = \int_\Omega |\nabla \phi|^2 dx$ .

**Proposition 2.1.** *Let  $(y_0, y_1) \in H_*^1(\Omega) \times L^2(\Omega)$  and  $f \in L^2(I \times \Omega)$  or  $f \in H(\Omega) := W^{1,\infty}(I, H^{-1}(\Omega))$ , and  $g \in G(\Gamma) := W^{1,1}(I, H^{-1/2}(\Gamma))$ . Then the solution  $y$  to (2.2) verifies  $y \in E(\Omega) := L^\infty(I, H_*^1(\Omega)) \cap W^{1,\infty}(I, L^2(\Omega))$ . Moreover when  $f \in L^2(I \times \Omega)$ , the trace  $y_\Gamma$  of  $y$  at the boundary  $\Gamma$  verifies :*

$$y_\Gamma \in F(\Gamma) := H^{1/2}(0, \tau, L^2(\Gamma)) \cap H^{-1/2}(0, \tau, H^1(\Gamma)),$$

and we have the following estimates: There exists constants  $c, k$  such that

$$(2.3) \quad \|y\|_{E(\Omega)}^2 = \|y_t\|_{L^\infty(I, L^2(\Omega))}^2 + \|y\|_{L^\infty(I, H_*^1(\Omega))}^2 \leq c \| |(f, g)| \|^2,$$

and

$$(2.4) \quad \|y\|_{F(\Gamma)}^2 = \|y\|_{H^{1/2}(0, \tau, L^2(\Gamma))}^2 + \|y\|_{H^{-1/2}(0, \tau, H^1(\Gamma))}^2 \leq k \| |(f, g)| \|^2,$$

where

$$(2.5) \quad \begin{aligned} \| |(f, g)| \|^2 &= \|g\|_{W^{1,1}(I, H^{-1/2}(\Gamma))}^2 + \|f\|^2 \\ &+ \|y_0\|_{H_*^1(\Omega)}^2 + \|y_1\|_{L^2(\Omega)}^2 + \|f(0)\|_{H^{-1}(\Omega)}^2 + \|g(0)\|_{H^{-1/2}(\Gamma)}^2, \end{aligned}$$

and

$$\|f\|^2 = \|f\|_{W^{1,1}(I, H^{-1}(\Omega))}^2 \text{ or } \|f\|_{L^2(I, L^2(\Omega))}^2.$$

### 3 The wave equation on $\Omega_s$

Let  $D \subset \mathbb{R}^N$  be fixed (potentially included in a  $C^2$  manifold). Let  $\Omega \subset D$  be an open bounded domain, with smooth boundary  $\partial\Omega = \Gamma \cup S$ , where  $\bar{\Gamma} \cap \bar{S} = \emptyset$ .

**The moving domain:** For  $s \in [0, s^*]$ , let  $V$  be a smooth vector field,

$$V \in C^0([0, s^*], C^1(D, \mathbb{R}^N)) \text{ with } V \cdot n = 0 \text{ on } \partial D, \text{ and } V = 0 \text{ on } S.$$

The flow transformation associated to  $V$  is given by:

$$T_s(V) : \bar{D} \rightarrow \bar{D}, \text{ such that } T_s(V)(S) = S.$$

Using  $T_s(V)$ , we build the family of perturbed domains  $\{\Omega_s\}_s$  as follows:  $\Omega_s = T_s(V)(\Omega)$  and  $\partial\Omega_s = S \cup \Gamma_s$ , where  $\Gamma_s = T_s(V)(\Gamma)$ . The normal component of the vector field  $V(s)$  on the boundary  $\Gamma_s$  (called the “normal speed”) is denoted by  $v(s)$ , i.e.  $v(s) = \langle V(s), n_s \rangle$ , where  $n_s = \nabla b_{\Omega_s}$  where  $b(s) = b_{\Omega_s}$  stands for the oriented distance function to  $\Omega_s$ . From [4], we know that its shape derivative verifies

$$b'(s) = -V(s) \circ p_s \text{ in a neighborhood of } \Gamma_s,$$

where  $p_s$  is the projection map onto  $\Gamma_s$ . Most of the time, the normal speed appears in the calculus evaluated at 0, hence we will use the following notation  $v = v(0)$  for it.

**The PDE model for the wave problem.** Let  $\Omega_s \subset D$  be a moving domain with boundary  $\partial\Omega_s = \Gamma_s \cup S$ , where  $\bar{\Gamma}_s \cap \bar{S} = \emptyset$ , and  $S$  is fixed with respect to the parameter  $s$ .

We consider the solution  $y^s$  to the wave equation in the cylinder  $]0, \tau[ \times \Omega_s$ , with homogeneous Dirichlet condition on the fixed part  $S$  of the boundary, and verifying a non homogeneous Neumann condition on the moving part  $\Gamma_s$ :

$$\begin{cases} y_{tt}^s - \Delta y^s = 0 & \Omega_s \\ y^s = 0 & S \\ \frac{\partial}{\partial n_s} y_s = g(s) & \Gamma_s \end{cases}$$

For each  $s$  let the boundary datum  $g$  have the following regularity  $g_s \in W^{1,1}(I, H^{-1/2}(\Gamma_s))$ . Then we consider the element  $y_s \in H_s = L^2(I, H_*^1(\Omega_s)) \cap H^1(I, L^2(\Omega_s))$  solution to

$$\forall \phi_s \in H_s, \int_0^{2\tau} \int_{\Omega_s} \left( -\frac{\partial}{\partial t} y_s \frac{\partial}{\partial t} \phi_s + \nabla y_s \cdot \nabla \phi_s \right) dx dt = \int_0^{2\tau} \int_{\Gamma_s} g_s \phi_s d\Gamma_s dt$$

As mentioned before, the goal of this paper is threefold. We want to prove existence of the material derivative  $\dot{y}(\Omega; V) = \frac{\partial}{\partial s} [y_s \circ T_s] \Big|_{\{s=0\}}$  and of the shape derivative  $y'(\Omega; V) = \dot{y}(\Omega; V) - \nabla y \cdot V(0)$ , and to render the new wave problem whose solution is the shape derivative  $y'(\Omega; V)$ . The first step consists in proving the existence of weak material derivative.

## 4 Weak material derivatives

To prove existence of material derivatives, we will take advantage of the regularity of solution for the linear wave equation and use the parameter differentiability for any functional expressed in form of a Min Max of a convex-concave Lagrangian with saddle points. The complete prove of MinMax parameter differentiability under saddle point was given in [2]. The result in case of single unique saddle point ( which is easier) is given in [3] with application to PDE problem. Of course on formal view point such results was known by ingenieurs as a "necessary expression" ( assume it is differentiable then such is the expression. The difficult part being to prove the differentiability it self)

Let  $R \in L^2(I, H^{-1}(\Omega))$  and  $q \in H^{1/2}(0, \tau, H^{-1}(\Gamma))$ . We consider the transported solution  $y^s$  in the non perturbed geometry, that is  $y^s = y_s \circ T_s(V)$ , and we set

$$j(s) = \int_0^\tau \left( \int_{\Omega} y^s R dx + \int_{\Gamma} y^s q d\Gamma \right) dt.$$

Then we obtain that

$$j(s) = \text{MinMax}_{\{(\phi; \psi) \in \times\}} \mathcal{L}(s, \phi, \psi),$$

where the Lagrangian  $\mathcal{L}$  is given by

$$\begin{aligned} \mathcal{L}(s, \phi; \psi) &= \int_0^\tau \left( \int_\Omega \phi R dx + \int_\Gamma \phi q d\Gamma \right) dt \\ &+ \int_0^\tau \int_\Omega \left( -J(s) \frac{\partial \phi}{\partial t} \frac{\partial \psi}{\partial t} + \langle A(s) \nabla \phi, \nabla \psi \rangle \right) dx dt - \int_0^\tau \int_\Gamma \omega(s) g_s \circ T_s(V) \psi d\Gamma dt, \end{aligned}$$

where  $J(s) = \det(DT_s)$  is the Jacobian of the transformation  $T_s(V)$ ,

$$A(s) = J(s)(DT_s)^{-1}(DT_s)^{-*},$$

and the density  $\omega(s)$  is given as

$$(4.1) \quad \omega(s) = \det(DT_s) |(DT_s(V))^{-*} n_s|.$$

At  $s = 0$  the unique saddle point of  $\mathcal{L}$  is  $(y, p)$ , where the co-state  $p$  is the solution to the following problem:

$$\frac{\partial^2}{\partial t^2} p - \Delta p = R, \quad \frac{\partial}{\partial n} p = q.$$

We derive that the functional  $\mathcal{L}$  is differentiable and we get the explicit expression for its derivative w.r.t  $s$  at  $s = 0$ :

$$\begin{aligned} j'(0) &= \frac{\partial}{\partial s} \mathcal{L}(0, y, p) \\ &= \int_0^\tau \int_\Omega \left( -\operatorname{div} V(0) \frac{\partial y}{\partial t} \frac{\partial p}{\partial t} + \langle [\operatorname{div} V(0) - 2\epsilon(V(0))] \nabla y, \nabla p \rangle \right) dx dt \\ &\quad - \int_0^\tau \int_\Gamma p (\dot{g}(V) + H g v) d\Gamma dt \end{aligned}$$

where  $2\epsilon(V) = DV + (DV)^*$ ,  $\dot{g}(V) = [\frac{d}{ds} g_s \circ T_s(V)]_{s=0}$ ,  $v = \langle V(0), n \rangle$ , and  $H$  is the mean curvature of the boundary.

As a conclusion we get the existence of the weak derivative of the map  $s \rightarrow y^s$  in  $L^2(0, \tau, H^1(\Omega))$ , and the weak differentiability of the trace mapping:  $s \rightarrow y^s|_\Gamma$  in  $H^{-1/2}(0, \tau, H^1(\Gamma))$ .

## 5 Strong material derivative

**Theorem 5.1.** *Assume  $g_s \in W^{1,1}(I, H^{-1/2}(\Gamma_s))$  such that there exists  $\dot{g} \in L^1(I, H^{-1/2}(\Gamma))$  verifying*

$$(5.1) \quad \frac{g_s \circ T_s(V)}{s} - \dot{g}(V) \rightarrow 0 \text{ strongly in } L^1(I, H^{-1/2}(\Gamma)).$$

*Then the solution  $y_s$  has a **strong material derivative** in the following topology:*

$$(5.2) \quad \frac{y_s \circ T_s(V) - y}{s} - (\dot{Y} - \operatorname{div} V(0) y) \rightarrow 0 \text{ in } L^\infty(0, \tau, L^2(\Omega)) \cap W^{-1, \infty}(0, \tau, H_*^1(\Omega)),$$

*where  $\dot{Y}$  is the solution to problem (5.10).*

**Corollary 5.1.1.** *Let  $g_s \in W^{2,1}(I, H^{-1/2}(\Gamma_s))$  such that there exists  $\dot{g} \in W^{1,1}(I, H^{-1/2}(\Gamma))$  verifying*

$$(5.3) \quad \frac{g_s \circ T_s(V)}{s} - \dot{g}(V) \rightarrow 0 \text{ strongly in } W^{1,1}(I, H^{-1/2}(\Gamma)).$$

*Then the solution  $y_s$  has a **strong boundary material derivative** in the following topology:*

$$(5.4) \quad \frac{y_s \circ T_s(V) - y}{s} - (\dot{Y} - \text{div}V(0) y) \rightarrow 0 \text{ in } L^\infty(0, \tau, H^{1/2}(\Gamma)),$$

where  $\dot{Y}$  is the solution to problem (5.10).

## 5.1 Shape derivative

We know that

$$\exists \mathcal{Y} \in C^0\{[0, s_*[, W^{1,\infty}(I, L^2(D)) \cap L^\infty(I, H_*^1(D))\} \cap C^1\{[0, s_*[, W^{1,\infty}(I, H^{-1}(D)) \cap L^\infty(I, L^2(D))\} \blacksquare$$

such that

$$\forall s, \quad \mathcal{Y}(s, \cdot) = y_s(\cdot) \text{ on } \Omega_s.$$

Now the term

$$y'(\Omega; V) := \left[ \frac{\partial}{\partial s} \mathcal{Y}(0, x) \right]_{\{x \in \Omega\}} \in W^{1,\infty}(I, H^{-1}(\Omega)) \cap L^\infty(I, L^2(\Omega))$$

is independent of the choice of the function  $\mathcal{Y}$  and is given by

$$y'(\Omega; V) = \dot{y}(\Omega; V) - \nabla y \cdot V(0).$$

For simplicity we write  $\dot{y}$  and  $y'$  for the material and shape derivatives.

### 5.1.1 Characterization of $y'(\Omega; V)$

**Proposition 5.1.** *The element  $y'$  is solution to the wave problem:*

$$(5.5) \quad \begin{cases} \frac{\partial^2}{\partial t^2} y' - \Delta y' = 0 \text{ in } \Omega \\ \frac{\partial}{\partial n} y' = \text{div}_\Gamma(y \nabla_\Gamma v) - v \frac{\partial^2 y}{\partial t^2} + H g v + g'_\Gamma \text{ on } \Gamma, \end{cases}$$

where  $g'_\Gamma$  stands for the **boundary shape derivative** of  $g$  given by

$$g'_\Gamma = \dot{g} - \nabla_\Gamma g \cdot V_\Gamma(0).$$

The proof is done in several steps. Let  $Y_s = \int_0^t y_s(\sigma) d\sigma$  and  $G_s = \int_0^t g_s(\sigma) d\sigma$  be the solution to

$$\forall \phi_s \in H(D), \quad \int_0^{2\tau} \int_{\Omega_s} \left( -\frac{\partial}{\partial t} Y_s \frac{\partial}{\partial t} \phi_s + \nabla Y_s \cdot \nabla \phi_s \right) dx dt = \int_0^{2\tau} \int_{\Gamma_s} G_s \phi_s d\Gamma_s dt.$$

From 5.9 we get

$$(5.6) \quad \left\| \frac{\partial}{\partial t} Y_s \right\|_{L^\infty(I, L^2(\Omega_s))}^2 + \|Y_s\|_{L^\infty(I, H_*^1(\Omega_s))}^2 \leq c (\|G_s\|_{L^\infty(I, H^{-1/2}(\Gamma_s))} + \|g_s\|_{L^1(I, H^{-1/2}(\Gamma_s))})$$

Consider the symmetrical matrix  $A(s) = J(s) DT_s(V)^{-1} \cdot DT_s(V)^{-*} \in L^\infty(D, R^{N^2})$ .  
Setting  $Y^s = J_s \circ Y_s \circ T_s(V)$  we get (with  $\phi_s = \psi \circ T_s(V)^{-1}$ )

$$\forall \psi \in H = H_0, \quad \int_0^{2\tau} \int_\Omega \left( -\frac{\partial}{\partial t} Y^s \frac{\partial}{\partial t} \psi + A(s) \cdot \nabla (J_s^{-1} Y^s) \cdot \nabla \psi \right) dx dt = \int_0^{2\tau} \int_\Gamma G^s w(s) \psi d\Gamma dt$$

Concerning the continuity of  $s \rightarrow Y^s$  let  $Z^s = Y^s - Y \in H^1(I, L^2(\Omega)) \cap L^2(I, H_*^1(\Omega))$ , setting  $\vec{m}(s) = (DT_s^{-1} \cdot DT_s^{-*} - I) \cdot \nabla_\Gamma Y^s + Y^s A \cdot \nabla (J_s^{-1}) + G^s (DT_s^{-1} \cdot DT_s^{-*} - I) \cdot n$

**Lemma 5.1.** *the term  $\frac{1}{s} \|m(s)\|_{L^\infty(0, 2\tau, L^2(\Omega))}$  remains bounded when  $s \rightarrow 0$*

Indeed from classical estimates ( see [9]) we have

$$\|DT_s^{-1} \cdot DT_s^{-*} - I\|_{L^\infty(D)^{N^2}} + \|A(s) \cdot \nabla J_s^{-1}\|_{L^\infty(D)} \leq C_V s$$

We get with 5.6

$$(5.7) \quad \frac{1}{s} \|m(s)\|_{L^\infty(0, 2\tau, L^2(\Omega))} \leq C_V c (\|G_s\|_{L^\infty(I, H^{-1/2}(\Gamma_s))} + \|g_s\|_{L^1(I, H^{-1/2}(\Gamma_s))})$$

and from the sharp regularity at the boundary 2.4 we get

**Lemma 5.2.**

$$m(s) \in H^{1/2}(0, \tau, L^2(\Gamma))$$

with the following estimate

$$(5.8) \quad \frac{1}{s} \|m(s)\|_{H^{1/2}(0, \tau, L^2(\Gamma))}^2 \leq c C_V$$

$$\|G^s w(s) - G\|_{L^\infty(\Gamma)} \leq C_V s$$

The element  $Z^s$  is solution of,  $\forall \psi \in H = H_0$

$$\begin{aligned} \int_0^{2\tau} \int_\Omega \left( -\frac{\partial}{\partial t} Z^s \frac{\partial}{\partial t} \psi + \langle \nabla Z^s, \nabla \psi \rangle \right) dx dt &= - \int_0^{2\tau} \int_\Omega \langle m(s), \nabla \psi \rangle dx dt \\ &+ \int_0^{2\tau} \int_\Gamma (G^s w(s) - G) \psi d\Gamma dt \end{aligned}$$

That is

$$\int_0^{2\tau} \int_\Omega \left( -\frac{\partial}{\partial t} Z^s \frac{\partial}{\partial t} \psi + \langle \nabla Z^s, \nabla \psi \rangle \right) dx dt = - \int_0^{2\tau} \int_\Omega \bar{f}(s) \psi dx dt$$



$$+ \int_0^{2\tau} \int_{\Gamma} \bar{g}(s) \psi d\Gamma dt$$

Where

$$\begin{aligned} \bar{f}(s) &= -\operatorname{div}(\bar{m}(s)) \rightarrow 0 \text{ in } W^{1,1}(I, H^{-1}(\Omega)) \\ \bar{g}(s) &= G^s \omega(s) - G + \langle \bar{m}(s), \bar{n} \rangle \rightarrow 0 \text{ in } W^{1,1}(I, H^{-1/2}(\Gamma)) \end{aligned}$$

From 2.5 we get:

$$\begin{aligned} \|Z_t\|_{L^\infty(I, L^2(\Omega))}^2 + \|Z\|_{L^\infty(I, H_*^1(\Omega))}^2 &\leq c(\|G^s \omega - G + \langle m(s), n \rangle\|_{L^\infty(I, H^{-1/2}(\Gamma))} \\ &+ \|g^s \omega - g + \langle \frac{\partial}{\partial t} m(s), n \rangle\|_{L^1(I, H^{-1/2}(\Gamma))}) \end{aligned}$$

(5.9)

$$+ \|\bar{f}\|_{L^\infty(I, H^{-1}(\Omega))} + \|\frac{\partial}{\partial t} \bar{f}\|_{L^1(I, H^{-1}(\Omega))} + [c \|\cdot\|^2 + \|\cdot\|^2 + \|\bar{f}(0)\|_{H^{-1}(\Omega)} + \|\bar{G}(0)\|_{H^{-1/2}(\Gamma)}]^{1/2}$$

**Proposition 5.2.** Assume that  $s \rightarrow g^s$  is continuous in  $L^1(I, H^{-1/2}(\Gamma))$ .

We consider the element  $\dot{Y} \in H_0$  solution to the problem

$$\begin{aligned} \forall \psi \in H = H_0, \int_0^{2\tau} \int_{\Omega} (-\frac{\partial}{\partial t} \dot{Y} \frac{\partial}{\partial t} \psi + \nabla \dot{Y} \cdot \nabla \psi) dx dt &= \int_0^{2\tau} \int_{\Omega} \langle \nabla(Y \operatorname{div} V(0)) - \dot{A} \cdot \nabla Y, \nabla \psi \rangle dx dt \\ &+ \int_0^{2\tau} \int_{\Gamma} (\dot{G} + GHv) \psi d\Gamma dt \\ &= \int_0^{2\tau} \langle \operatorname{div}([\dot{A} - \operatorname{div} V(0)I] \cdot \nabla Y), \psi \rangle_{H_{\Omega}^{-1}(D) \times H^1(\Omega)} dt \end{aligned}$$

$$(5.10) \quad + \int_0^{2\tau} \int_{\Gamma} (\dot{G} + GH - \langle \dot{A} \cdot n, \nabla Y \rangle + \langle \nabla(Y \operatorname{div} V(0)), n \rangle) \psi d\Gamma dt$$

This problem is relevant from the previous variational approach as the right hand side

$$\begin{aligned} f &= \operatorname{div}(\dot{A} \nabla Y - \nabla Y \operatorname{div} V(0)) = -2 \operatorname{div}(\epsilon(V(0)) \cdot \nabla Y) \in H^1(I, H^{-1}(\Omega)) \\ \bar{g} &= -\langle \dot{A} \cdot n, \nabla Y \rangle + \langle \nabla(Y \operatorname{div} V(0)), n \rangle \\ &= 2 \langle \epsilon(V) \cdot \nabla Y, n \rangle + Y \frac{\partial}{\partial n} \operatorname{div} V(0) \in H^{1/2}(I, L^2(\Gamma)) \end{aligned}$$

## 5.2 Differential quotient

We consider the elements

$$d(s) = \frac{y^s - y}{s} - \dot{y}, \quad \delta(s) = \frac{Y^s - Y}{s} - \dot{Y}$$

In order to characterise  $\delta(s)$  we introduce the following vectors functions:

$$\mathcal{M}_1(s) = \left[ \frac{DT_s^{-1} \cdot DT_s^{-*}}{s} - 2\epsilon V(0) \right] \cdot \nabla Y^s$$

$$\mathcal{M}_2(s) = 2\epsilon(V) \cdot \nabla(Y^s - Y)$$

$$\mathcal{M}_3(s) = [Y^s A \cdot \left( \frac{\nabla(J_s^{-1})}{s} - \nabla \operatorname{div} V(0) \right) + (YI - Y^s A) \cdot \nabla \operatorname{div} V(0)]$$

$$\mathcal{M}(s) = \mathcal{M}_1(s) + \mathcal{M}_2(s) + \mathcal{M}_3(s)$$

And the function on the boundary:

$$G_\delta(s) = \frac{G^s \omega(s) - G}{s} - (\dot{G} + G H v)$$

the element  $\delta(s)$  is then solution to

$$\forall \psi, \int_0^{2\tau} \int_\Omega \left( -\frac{\partial}{\partial t} \delta \frac{\partial}{\partial t} \psi + \nabla \delta \cdot \nabla \psi \right) dx dt = - \int_0^{2\tau} \int_\Omega \langle \mathcal{M}, \nabla \psi \rangle dt + \int_0^{2\tau} \int_\Gamma G_\delta(s) \psi d\Gamma dt \quad \blacksquare$$

That is

$$(5.11) \quad \int_0^{2\tau} \int_\Omega \left( -\frac{\partial}{\partial t} \delta \frac{\partial}{\partial t} \psi + \nabla \delta \cdot \nabla \psi \right) dx dt = \int_0^{2\tau} \langle f_\delta(s), \psi \rangle_{H_\Omega^{-1}(D) \times H^1(\Omega)} dt + \int_0^{2\tau} \int_\Gamma g_\delta(s) \psi d\Gamma dt \quad \blacksquare$$

Where

$$\begin{aligned} f_\delta(s) &= \operatorname{div}(\mathcal{M}) \rightarrow 0 \text{ in } H^1(I, H^{-1}(\Omega)), \quad s \rightarrow 0, \\ g_\delta(s) &= G_\delta(s) - \mathcal{M} \cdot n \rightarrow 0 \text{ in } H^1(I, H^{-1/2}(\Gamma)), \quad s \rightarrow 0. \end{aligned}$$

## 5.3 Fourier transform

From now on we assume the data,  $y_0 = y_1 = 0$  and the Neumann data  $g \in W^{1,1}(I, H^{-1/2}(\Gamma))$  being approached in this space by a smooth element  $g^m$ . Then the associated solution  $y^m$  is smoother. We consider a smooth cutting function  $0 \leq \rho(t) \leq 1$  such that  $\rho(t) = 0$  for  $|t| \geq 2\tau$  while  $\rho(t) = 1$  when  $|t| \leq \tau$ . We consider  $(y^m)^0, (g^m)^0$  the extension by zero out of  $I$  and we set

$$\tilde{y}^m(t, x) = \rho(t) (y^m)^0(t, x), \quad \tilde{g}^m(t, x) = \rho(t) (g^m)^0(t, x)$$

With

$$(5.12) \quad F^m = \rho f + 2\rho_t y_t^m + \rho_{tt} y^m \rightarrow F = \rho f + 2\rho_t y_t + \rho_{tt} y \text{ in } L^2(I, L^2(\Omega)) \text{ as } m \rightarrow \infty. \quad \blacksquare$$

We get

$$(5.13) \quad \tilde{y}_{tt}^m + A.\tilde{y}^m = F^m, \quad \frac{\partial \tilde{y}^m}{\partial n_A} = \tilde{g}^m, \quad \text{on } \Gamma, \quad \tilde{y}^m = 0 \quad \text{on } S.$$

For each  $m$  we consider the Fourier transform

$$\mathcal{F}^m(\zeta, x) = \int_{-\infty}^{+\infty} \tilde{F}^m(t, x) e^{-i\zeta t} dt, \quad z^m(\zeta, x) = \int_{-\infty}^{+\infty} \tilde{y}^m(t, x) e^{-i\zeta t} dt, \quad \mathcal{G}^m(\zeta, x) = \int_{-\infty}^{+\infty} \tilde{g}^m(t, x) e^{-i\zeta t} dt$$

$$(5.14) \quad -\zeta^2 z^m + A.z^m = \mathcal{F}^m, \quad \frac{\partial z^m}{\partial n_A} = \mathcal{G}^m, \quad \text{on } \Gamma, \quad z^m = 0 \quad \text{on } S,$$

### 5.3.1 Extractor

Given  $\mu > 0$ , consider the velocity field  $V \in C^0([0, \mu[; W^{2,\infty}(D, \mathbf{R}^N))$  and its associated flow mapping  $T_s(V)$ . Given  $s \geq 0$ , denote by  $\Omega_s = T_s(V)(\Omega)$  the perturbed domain with boundary  $\Gamma_s = T_s(V)(\Gamma)$ . Consider the functional

$$\mathcal{E}^m(s, V) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\zeta \int_{\Omega_s(V)} \left[ |\zeta| |z^m \circ T_s(V)^{-1}|^2 + \frac{1}{1+|\zeta|} | \langle A.\nabla(z^m \circ T_s(V)^{-1}), \nabla(z^m \circ T_s(V)^{-1}) \rangle | \right] dx$$

and its derivative

$$e^m \stackrel{\text{def}}{=} \left. \frac{d}{ds} \mathcal{E}^m(s, V) \right|_{s=0}$$

that will be computed in two different ways. For simplicity in the following computations we denote by  $V$  the autonomous vector field  $V(0)$ . Derivative by moving boundary results : Let

$$e_1^m \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} |\zeta| 2\mathcal{R}e\{z^m \nabla z^{\bar{m}} \cdot (-V)\} + \frac{1}{1+|\zeta|} 2\mathcal{R}e\{\langle A.\nabla(z^m), \nabla(\nabla z^{\bar{m}}(-V)) \rangle\} dx + \int_{-\infty}^{+\infty} d\zeta \left( \int_{\Gamma} \{|\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} \langle A.\nabla z^m, \nabla z^m \rangle \langle V, n \rangle d\Gamma(x) \right).$$

Consider the first integral term over  $\Omega$ :

$$a \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left( |\zeta| 2\mathcal{R}e\{z^m \nabla z^{\bar{m}} \cdot (-V)\} - \frac{1}{1+|\zeta|} 2\mathcal{R}e\{\langle A.\nabla z^m, \nabla(\nabla z^{\bar{m}}.V) \rangle\} \right) dx$$

By Stokes theorem using the fact that  $\partial z^m / \partial n_A = \mathcal{G}^m$ , we get the following expression:

$$a = \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left( |\zeta| 2\mathcal{R}e\{-z^m \nabla z^{\bar{m}} \cdot V\} + \frac{1}{1+|\zeta|} 2\mathcal{R}e\{\text{div}(A.\nabla z^m) \nabla z^{\bar{m}}.V\} \right) dx - \int_{-\infty}^{+\infty} d\zeta \int_{\Gamma} \frac{1}{1+|\zeta|} 2\mathcal{R}e\{\mathcal{G}^m \nabla z^{\bar{m}}.V\} d\Gamma$$

As we have  $-\operatorname{div}(A.\nabla z^m) = A.z^m = (\zeta^2 z^m + \mathcal{F}^m(\zeta, x))$  we get

$$\begin{aligned} a &= 2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left( -z^m \left( |\zeta| + \frac{\zeta^2}{1+|\zeta|} \right) \right) \nabla z^{\bar{m}} \cdot V dx \right\} \\ &\quad + 2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} \mathcal{F}^m \nabla z^{\bar{m}} \cdot V dx \right\} \\ &\quad - \int_{-\infty}^{+\infty} d\zeta \int_{\Gamma} \frac{1}{1+|\zeta|} 2 \operatorname{Re} \{ \mathcal{G}^m \nabla z^{\bar{m}} \cdot V \} d\Gamma. \end{aligned}$$

**Lemma 5.3.** For each  $m > 0$  we get

$$\begin{aligned} e_1^m &= \int_{-\infty}^{+\infty} d\zeta \left( \int_{\Gamma} \{ |\zeta| |z|^2 + \frac{1}{1+|\zeta|} \langle A.\nabla z^m, \nabla z^m \rangle \} \langle V, n \rangle d\Gamma(x) \right) \\ &\quad - \int_{-\infty}^{+\infty} d\zeta \int_{\Gamma} \frac{1}{1+|\zeta|} 2 \operatorname{Re} \{ \mathcal{G}^m \nabla z^{\bar{m}} \cdot V \} d\Gamma \\ &\quad + 2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left( -z^m \left( |\zeta| + \frac{\zeta^2}{1+|\zeta|} \right) \right) \nabla z^{\bar{m}} \cdot V dx \right\} \\ &\quad + 2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} \mathcal{F}^m \nabla z^{\bar{m}} \cdot V dx \right\}. \end{aligned}$$

### 5.3.2 Derivative by change of variable $T_s(V)$

Consider now the expression

$$\mathcal{E}^m(s, V) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left[ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} \langle B(s).\nabla z^m, \nabla z^m \rangle \right] j(s) dx$$

where

$$\boxed{B(s) \stackrel{\text{def}}{=} DT_s(V)^{-1} \cdot A \circ T_s(V) \cdot (DT_s(V)^{-1})^*, \quad j(s) \stackrel{\text{def}}{=} \det DT_s(V)}$$

and

$$(5.15) \quad \boxed{(B'.V) = -DV.A - A.DV^* + \nabla A.V}$$

where  $\nabla A.V$  is the matrix

$$\boxed{(\nabla A.V)_{i,j} \stackrel{\text{def}}{=} \nabla a_{i,j}.V.}$$

Then we get

**Lemma 5.4.**

$$\begin{aligned} e_2^m &= \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left[ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} \langle A.\nabla z^m, \nabla z^m \rangle \right] \operatorname{div} V dx \\ &\quad + \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} \langle (B'.V).\nabla z^m, \nabla z^m \rangle dx. \end{aligned}$$

### 5.4 Extractor identity

We now equate the two expressions,  $e_1^m = e_2^m$ , to get

**Lemma 5.5.**

$$\begin{aligned}
& \int_{-\infty}^{+\infty} d\zeta \left( \int_{\Gamma} \left\{ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} \langle A \cdot \nabla z^m, \nabla z^m \rangle \right\} \langle V, n \rangle d\Gamma(x) \right) \\
& - \int_{-\infty}^{+\infty} d\zeta \int_{\Gamma} \frac{1}{1+|\zeta|} 2 \operatorname{Re} \{ \mathcal{G}^m \nabla z^{\bar{m}} \cdot V \} d\Gamma \\
& = -2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left( -z^m \left( |\zeta| + \frac{\zeta^2}{1+|\zeta|} \right) \nabla z^{\bar{m}} \cdot V dx \right) \right\} \\
& - 2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} \mathcal{F}^m \nabla z^{\bar{m}} \cdot V dx \right\} \\
& + \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left[ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} \langle A \cdot \nabla z^m, \nabla z^m \rangle \right] \operatorname{div} V dx \\
& + \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} \langle (B' \cdot V) \cdot \nabla z^m, \nabla z^m \rangle dx.
\end{aligned}$$

The velocity vector  $V$  will be chosen in terms of the geometry of the boundary which is best handled by using the oriented distance function, (cf. [7])  $b_{\Omega} = d_{\Omega} - d_{\Omega^c}$ ,  $\Omega^c = \mathbf{R}^N \setminus \Omega$ . When  $\Omega$  is of class  $C^{1,1}$ , the unit outward normal  $n$  to the boundary  $\Gamma = \partial\Omega$  is equal to  $\nabla b_{\Omega}$ . Moreover when  $\Gamma$  is compact, there exist  $h > 0$  and an  $h$ -tubular neighborhood  $\mathcal{U}_h = \{x \in \mathbf{R}^N : |b_{\Omega}(x)| < h\}$  of  $\Gamma$  such that  $b_{\Omega} \in C^{1,1}(\mathcal{U}_h)$ . In order to work in  $\mathbf{R}^N$ , localize the oriented distance function to this neighborhood and work with a global  $C^{1,1}$  function on  $\mathbf{R}^N$ , define  $b_{\Omega}^h \stackrel{\text{def}}{=} \theta^h \circ b_{\Omega}$  for some function  $\theta^h \in C^{1,1}(\mathbf{R}; [0, 1])$  such that

$$\theta^h(t) \stackrel{\text{def}}{=} \begin{cases} 1, & |t| < h/3 \\ 0, & |t| > 2h/3. \end{cases}$$

Obviously  $b_{\Omega}^h$  is equal to  $b_{\Omega}$  in  $\mathcal{U}_{h/3}$  and  $\operatorname{supp} b_{\Omega}^h \subset \mathcal{U}_h$ , with  $b_{\Omega}^h \in C^{1,1}(\overline{\mathbf{R}})$  with the unit outward normal  $n = \nabla b_{\Omega} = \nabla b_{\Omega}^h$  on  $\Gamma$ . In order to take care of the boundary condition term  $\mathcal{G}$  in the extractor identity we have to make a specific choice for the vector field  $V$ . Choose  $V$  in the following form (here we assume the matrix  $A$  to be defined in the neighbourhood of  $\bar{\Omega}$ ):  $V \stackrel{\text{def}}{=} A \cdot \nabla b_{\Omega}^h \Rightarrow V = A \cdot n$  on  $\Gamma$ , so that  $\nabla z^m \cdot V = \langle \nabla z^m, A \cdot n \rangle = \mathcal{G}^m$ , while

$v = \langle V, n \rangle = \langle A.n, n \rangle \geq \alpha > 0$  on  $\Gamma$ . So the first term yields

$$\begin{aligned}
& \alpha \int_{-\infty}^{+\infty} d\zeta \left( \int_{\Gamma} \{ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} |\nabla z^m|^2 \} d\Gamma(x) \right) \\
& \leq \int_{-\infty}^{+\infty} d\zeta \int_{\Gamma} \frac{1}{1+|\zeta|} 2 \operatorname{Re} \{ \mathcal{G}^m \bar{\mathcal{G}}^m \} d\Gamma \dots \\
& \quad \cdot \left| -2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left( -z^m \left( |\zeta| + \frac{\zeta^2}{1+|\zeta|} \right) \right) \nabla z^{\bar{m}} \cdot V dx \right\} \right. \\
& \quad \left. - 2 \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} \mathcal{F}^m \nabla z^{\bar{m}} \cdot V dx \right\} \right. \\
& \quad \left. + \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \left[ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} \langle A.\nabla z^m, \nabla z^m \rangle \right] \operatorname{div} V dx \right. \\
& \quad \left. + \int_{-\infty}^{+\infty} d\zeta \int_{\Omega} \frac{1}{1+|\zeta|} \langle (B'.V).\nabla z^m, \nabla z^m \rangle dx \right|.
\end{aligned}$$

and we obtain the following estimate.

**Proposition 5.3.** *There exists a constant  $M > 0$  such that for all  $m > 0$*

$$\begin{aligned}
(5.16) \quad & \int_{-\infty}^{+\infty} d\zeta \left( \int_{\Gamma} \{ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} |\nabla z^m|^2 \} d\Gamma(x) \right) \\
& \leq \frac{1}{\alpha} M \left\{ \left\| \sqrt{\frac{1}{1+|\zeta|}} \mathcal{G}^m \right\|_{L^2(\mathbf{R}_\zeta, L^2(\Gamma))}^2 \right. \\
& \quad + \|V(0)\|_{L^\infty(D, \mathbf{R}^N)} \|z^m\|_{L^2(\mathbf{R}_\zeta, L^2(\Omega))} \|\nabla z^m\|_{L^2(\mathbf{R}_\zeta, L^2(\Omega, \mathbf{R}^N))} \\
& \quad + \|V(0)\|_{L^\infty(D, \mathbf{R}^N)} \left\| \sqrt{\frac{1}{1+|\zeta|}} \mathcal{F}^m \right\|_{L^2(\mathbf{R}_\zeta, L^2(\Omega))} \|\nabla z^m\|_{L^2(\mathbf{R}_\zeta, L^2(\Omega, \mathbf{R}^N))} \\
& \quad + \|\sqrt{\zeta} z^m\|_{L^2(\mathbf{R}_\zeta, L^2(\Omega))} \|\operatorname{div} V(0)\|_{L^\infty(D)} \\
& \quad \left. + \left\| \sqrt{\frac{1}{1+|\zeta|}} \nabla z^m \right\|_{L^2(\mathbf{R}_\zeta, L^2(\Omega, \mathbf{R}^N))} \|(B'.V) + A\|_{L^\infty(D, \mathbf{R}^{N^2})} \right\}
\end{aligned}$$

As  $V = A.\nabla b_\Omega^h$ , we get

$$\|\operatorname{div} V(0)\|_{L^\infty(D)} \leq M_3 ( \|A\|_{W^{1,\infty}(\Omega)} + \|\Delta b_\Omega^h\|_{L^\infty(D)} )$$

while

$$\|(B'.V)\|_{L^\infty(D)} \leq M_4 ( \|A\|_{W^{1,\infty}(\Omega)} + \|D^2 b_\Omega^h\|_{L^\infty(D)} \|A\|_{L^\infty(D, \mathbf{R}^{N^2})} )$$

Each terms depending on  $m$  in the right-hand side of (5.16) converges in the respective norms:

$$\mathcal{G}^m \rightarrow \mathcal{G}, \quad \mathcal{F}^m \rightarrow \mathcal{F}, \quad z^m \rightarrow z.$$

So that there exists a constant  $M_2 > 0$  such that, for all  $m > 0$  we have:

$$\forall m > 0, \quad \int_{-\infty}^{+\infty} d\zeta \left( \int_{\Gamma} \{ |\zeta| |z^m|^2 + \frac{1}{1+|\zeta|} |\nabla z^m|^2 \} d\Gamma(x) \right) \leq M_2.$$

Consider the following two measures on  $\mathbf{R}_\zeta$ :

$$\mu_1(\zeta) \stackrel{\text{def}}{=} \sqrt{|\zeta|} d\zeta \quad \text{and} \quad \mu_2(\zeta) \stackrel{\text{def}}{=} \sqrt{\frac{1}{1+|\zeta|}} d\zeta.$$

Then we get

$$\|z^m\|_{L^2_{\mu_1}(\mathbf{R}_\zeta, L^2(\Gamma))} \leq M_2 \quad \text{and} \quad \|\nabla z^m\|_{L^2_{\mu_2}(\mathbf{R}_\zeta, L^2(\Gamma, \mathbf{R}^N))} \leq M_2$$

and there exist elements  $\phi, \vec{\Phi}$  such that for the two associated weighted topologies we get the weak convergences:

$$\begin{aligned} z^m_\Sigma &\rightharpoonup \phi \text{ weakly in } L^2_{\mu_1}(\mathbf{R}_\zeta, L^2(\Gamma)) \\ \nabla z^m_\Sigma &\rightharpoonup \vec{\Phi} \text{ weakly in } L^2_{\mu_2}(\mathbf{R}_\zeta, L^2(\Gamma, \mathbf{R}^N)) \end{aligned}$$

Obviously  $\phi = z|_\sigma$  and  $\vec{\Phi} = \nabla \phi|_\Sigma$  then  $\vec{\Phi} = \nabla z|_\Sigma$  and the norms being weakly l.s.c. in the limit we get the estimate:

$$\int_{-\infty}^{+\infty} d\zeta \left( \int_{\Gamma} \left\{ |\zeta| |z|^2 + \frac{1}{1+|\zeta|} |\nabla z|^2 \right\} d\Gamma(x) \right) \leq M_2$$

By Plancherel isomorphism we get in the real line, we get

**Corollary 5.1.2.**

$$\begin{aligned} |\zeta|^{1/2} z \in L^2(\mathbf{R}, L^2(\Gamma)) &\iff \tilde{y} \in H^{1/2}(\mathbf{R}, L^2(\Gamma)), \\ ((1+|\zeta|)^{-1/2}) \nabla z \in L^2(\mathbf{R}, L^2(\Gamma, \mathbf{R}^N)) &\iff \nabla \tilde{y} \in H^{-1/2}(\mathbf{R}, L^2(\Gamma, \mathbf{R}^N)). \end{aligned}$$

Obviously we have:

$$\|y\|_{H^{1/2}(\mathbf{R}, L^2(\Gamma))} \leq \|\tilde{y}\|_{H^{1/2}(\mathbf{R}, L^2(\Gamma))}$$

Also:

$$\nabla y|_\Sigma = \nabla_\Gamma y + \frac{\partial y}{\partial n} n$$

But

$$\frac{\partial y}{\partial n} = g \in L^2(\Sigma)$$

So that

$$\nabla_\Gamma y \in H^{-1/2}(\mathbf{R}, L^2(\Gamma, \mathbf{R}^N))$$

Which implies

$$y|_\Sigma \in H^{-1/2}(\mathbf{R}, H^1(\Gamma))$$

from which, as  $y \in C^0([0, \tau], H^{3/5-\sigma}(\Omega))$ , we get (??).

**Proposition 5.4.** *Let  $\Gamma$  be a  $C^2$  submanifold in  $\mathbf{R}^N$  and  $g \in L^2(I; L^2(\Gamma))$ ,  $f \in L^2(I; L^2(\Omega))$ ,  $y_0 \in H^2(\Omega)$ ,  $y_1 \in L^2(\Omega)$ , and  $A \in W^{1,\infty}(D, \mathbf{R}^{N^2})$ . Then the trace at the boundary  $\Gamma$  of  $y$  verifies the regularity (2.4).*

We have:

$$\|y\|_{H^{1/2}(0,\tau,L^2(\Gamma))}^2 + \|y\|_{H^{-1/2}(0,\tau,H^1(\Gamma))}^2 \leq \|\tilde{y}\|_{H^{1/2}(0,2\tau,L^2(\Gamma))}^2 + \|\tilde{y}\|_{H^{-1/2}(0,2\tau,H^1(\Gamma))}^2 \quad \blacksquare$$

from the Plancherel isometry in the estimation 5.16, we get:

$$\begin{aligned} &\leq C \{ \|\tilde{g}\|_{H^{-1/2}(0,2\tau,L^2(\Gamma))}^2 + \|\tilde{y}\|_{H^{-1/2}(0,2\tau,H^1(\Omega))}^2 + \|\tilde{F}\|_{H^{-1/2}(0,2\tau,L^2(\Omega))}^2 \\ &\quad + \|\tilde{y}\|_{H^{1/2}(0,2\tau,L^2(\Omega))}^2 + \|\nabla \tilde{y}\|_{H^{-1/2}(0,2\tau,L^2(\Omega))}^2 \end{aligned}$$

and obviously

$$\begin{aligned} &\leq C_2 \{ \|g\|_{H^{1/2}(0,2\tau,L^2(\Gamma))}^2 + \|y\|_{H^{-1/2}(0,2\tau,H^1(\Omega))}^2 + \|F\|_{H^{-1/2}(0,2\tau,L^2(\Omega))}^2 \\ &\quad + \|y\|_{H^{1/2}(0,2\tau,L^2(\Omega))}^2 + \|\nabla y\|_{H^{-1/2}(0,2\tau,L^2(\Omega)^N)}^2 \end{aligned}$$

#### 5.4.1 Boundedness of $F$

From 5.12 there exists a constant  $c_\rho > 0$  such that

$$\|F\|_{H^{-1/2}(0,2\tau,L^2(\Omega))} \leq \|F\|_{L^2(0,2\tau,L^2(\Omega))} \leq c_\rho \{ \|y_t\|_{L^2(I \times \Omega)}^2 + \|y\|_{L^2(I \times \Omega)}^2 + \|f\|_{L^2(I \times \Omega)}^2 \} \quad \blacksquare$$

and from 5.9 we get

$$\|F\|_{H^{-1/2}(0,2\tau,L^2(\Omega))} \leq \|F\|_{L^2(0,2\tau,L^2(\Omega))} \leq c (\|f\|_{L^2(I \times \Omega)} + \|g(t)\|_{L^\infty(I, H^{-1/2}(\Gamma))} + \|\frac{\partial}{\partial t} g(t)\|_{L^1(I, H^{-1/2}(\Gamma))}) \quad \blacksquare$$

(5.17)

$$+ \|f\|_{L^\infty(I, H^{-1}(\Omega))} + \|\frac{\partial}{\partial t} f\|_{L^1(I, H^{-1}(\Omega))} + [c E(0) + \|f(0)\|_{H^{-1}(\Omega)} + \|g(0)\|_{H^{-1/2}(\Gamma)}]^{1/2} \quad \blacksquare$$

**Theorem 5.2.** *Let  $g \in W^{1,1}(0, 2\tau, H^{-1/2}(\Gamma)) \cap H^{1/2}(0, 2\tau, L^2(\Gamma))$ ,  $f \in L^2(I \times \Omega) \cap W^{1,1}(I, H^{-1}(\Omega))$ . Then there exist a constant  $k$  depending on the domain  $\Omega$ , on the  $L^\infty(I, W^{1,\infty}(\Omega))$ -norm of the coefficients matrix  $A$ , on the cutting function  $\rho$  and on the trace of the linear trace operator (restriction to  $\Gamma$  in the norm of  $\mathcal{L}(H_*^1(\Omega), H^{1/2}(\Gamma))$ ) such that:*

$$\|y\|_{H^{1/2}(0,\tau,L^2(\Gamma))}^2 + \|y\|_{H^{-1/2}(0,\tau,H^1(\Gamma))}^2 \leq k \{ (\|f\|_{L^2(I \times \Omega)} + \|g(t)\|_{L^\infty(I, H^{-1/2}(\Gamma))} + \|\frac{\partial}{\partial t} g(t)\|_{L^1(I, H^{-1/2}(\Gamma))}) \quad \blacksquare$$

(5.18)

$$+ \|f\|_{L^\infty(I, H^{-1}(\Omega))} + \|\frac{\partial}{\partial t} f\|_{L^1(I, H^{-1}(\Omega))} + [E(0) + \|f(0)\|_{H^{-1}(\Omega)} + \|g(0)\|_{H^{-1/2}(\Gamma)}]^{1/2} \quad \blacksquare$$



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