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► **To cite this version:**

Salah-Eddine Rebiai. Exponential Stability of the System of Transmission of the Wave Equation with a Delay Term in the Boundary Feedback. 25th System Modeling and Optimization (CSMO), Sep 2011, Berlin, Germany. pp.276-285, 10.1007/978-3-642-36062-6\_28 . hal-01347547

**HAL Id: hal-01347547**

**<https://inria.hal.science/hal-01347547>**

Submitted on 21 Jul 2016

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# Exponential stability of the system of transmission of the wave equation with a delay term in the boundary feedback

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**Abstract.** We consider a system of transmission of the wave equation with Neumann feedback control that contains a delay term and that acts on the exterior boundary. First, we prove under some assumptions that the closed-loop system generates a  $C_0$ -semigroup of contractions on an appropriate Hilbert space. Then, under further assumptions, we show that the closed-loop system is exponentially stable. To establish this result, we introduce a suitable energy function and use multiplier method together with an estimate taken from [3] (Lemma 7.2) and compactness-uniqueness arguments.

**Keywords:** Wave equation, transmission problem, time delays, boundary stabilization, exponential stability.

## 1 Introduction

It is by now well-known that certain infinite-dimensional second-order systems are not robust with respect to arbitrarily small delays in the damping. This lack of stability robustness was first shown to hold for the one-dimensional wave equation ([2]). Later, further examples illustrating this phenomenon were considered in [1]: the two-dimensional wave equation with damping introduced through Neumann-type boundary conditions on one edge of a square boundary and the Euler-Bernoulli beam equation in one dimension with damping introduced through a specific set of boundary conditions on the right end point.

Recently, Xu et al [9] established sufficient conditions that guarantee the exponential stability of the one-dimensional wave equation with a delay term in the boundary feedback. Nicaise and Pignotti [5] extended this result to the multi-dimensional wave equation with a delay term in the boundary or internal feedbacks. The same type of result was obtained by Nicaise and Rebiai [6] for the Schrödinger equation.

Motivated by the references [9], [5] and [6]; we investigate in this paper the problem of exponential stability for the system of transmission of the wave equation with a delay term in the boundary feedback.

Let  $\Omega$  be an open bounded domain of  $\mathbb{R}^n$  with a boundary  $\Gamma$  of class  $C^2$  which consists of two non-empty parts  $\Gamma_1$  and  $\Gamma_2$  such that  $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$ . Let  $\Gamma_0$

with  $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \overline{\Gamma_0} \cap \overline{\Gamma_2} = \emptyset$  be a regular hypersurface of class  $C^2$  which separates  $\Omega$  into two domains  $\Omega_1$  and  $\Omega_2$  such that  $\Gamma_1 \subset \partial\Omega_1$  and  $\Gamma_2 \subset \partial\Omega_2$ . Furthermore, we assume that there exists a real vector field  $h \in (C^2(\overline{\Omega}))^n$  such that:

(H.1) The Jacobian matrix  $J$  of  $h$  satisfies

$$\int_{\Omega} J(x)\zeta(x) \cdot \zeta(x) d\Omega \geq \alpha \int_{\Omega} |\zeta(x)|^2 d\Omega,$$

for some constant  $\alpha > 0$  and for all  $\zeta \in L^2(\Omega; \mathbb{R}^n)$ ;

(H.2)  $h(x) \cdot \nu(x) \leq 0$  on  $\Gamma_1$ ;

(H.3)  $h(x) \cdot \nu(x) \geq 0$  on  $\Gamma_0$ .

where  $\nu$  is the unit normal on  $\Gamma$  or  $\Gamma_0$  pointing towards the exterior of  $\Omega$  or  $\Omega_1$ .

Let  $a_1, a_2 > 0$  be given. Consider the system of transmission of the wave equation with a delay term in the boundary conditions:

$$y''(x, t) - a(x)\Delta y(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1)$$

$$y(x, 0) = y^0(x), y'(x, 0) = y^1(x, 0) \quad \text{in } \Omega, \quad (2)$$

$$y_1(x, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (3)$$

$$\frac{\partial y_2(x, t)}{\partial \nu} = -\mu_1 y_2'(x, t) - \mu_2 y_2'(x, t - \tau) \quad \text{on } \Gamma_2 \times (0, +\infty), \quad (4)$$

$$y_1(x, t) = y_2(x, t), \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (5)$$

$$a_1 \frac{\partial y_1(x, t)}{\partial \nu} = a_2 \frac{\partial y_2(x, t)}{\partial \nu} \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (6)$$

$$y_2'(x, t - \tau) = f_0(x, t - \tau) \quad \text{on } \Gamma_2 \times (0, \tau). \quad (7)$$

where:

- $a(x) = \begin{cases} a_1, & x \in \Omega_1 \\ a_2, & x \in \Omega_2 \end{cases}$
- $y(x, t) = \begin{cases} y_1(x, t), & (x, t) \in \Omega_1 \times (0, +\infty) \\ y_2(x, t), & (x, t) \in \Omega_2 \times (0, +\infty) \end{cases}$
- $\frac{\partial}{\partial \nu}$  is the normal derivative.
- $\mu_1$  and  $\mu_2$  are positive real numbers.
- $\tau$  is the time delay

-  $y^0, y^1, f_0$  are the initial data which belong to suitable spaces.

In the absence of delay, that is  $\mu_2 = 0$ , Liu and Williams [4] have shown that the solution of (1)-(6) decays exponentially to zero in the energy space  $H_{\Gamma_1}^1(\Omega) \times L^2(\Omega)$  provided that

$$a_1 > a_2 \quad (8)$$

and  $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$  satisfies (H.1), (H.2), (H.3), and

(H.4)  $h(x) \cdot \nu(x) \geq \gamma > 0$ .

The purpose of this paper is to investigate the stability of problem (1) – (7) in the case where both  $\mu_1$  and  $\mu_2$  are different from zero. To this end, assume as in [5] that

$$\mu_1 > \mu_2. \quad (9)$$

and define the energy of a solution of (1) – (7) by

$$E(t) = \frac{1}{2} \int_{\Omega} \left[ |y'(x, t)|^2 + a(x) |\nabla(y(x, t))|^2 \right] dx + \frac{\xi}{2} \int_{\Gamma_2} \int_0^1 |y(x, t - \tau\rho)|^2 d\rho d\sigma(x), \quad (10)$$

where

$$a_2 \tau \mu_2 < \xi < a_2 \tau (2\mu_1 - \mu_2), \quad (11)$$

We show that if  $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$  satisfies (H.1), (H.2) and (H.3), then there is an exponential decay rate for  $E(t)$ . The proof of this result combines multipliers technique and compactness-uniqueness arguments.

The main result of this paper can be stated as follows.

**Theorem 1.** *Assume (H1), (H.2), (H.3), (8) and (9). Then there exist constants  $M \geq 1$  and  $\omega > 0$  such that*

$$E(t) \leq M e^{-\omega t} E(0).$$

Theorem 1 is proved in Section 3. In Section 2, we investigate the well-posedness of system (1) – (7) using semigroup theory.

## 2 Well-posedness of problem (1) – (7)

Inspired from [5] and [6], we introduce the auxilliary variable  $z(x, \rho, t) = y(x, t - \tau\rho)$ . With this new unknown, problem (1) – (7) is equivalent to

$$y''(x, t) - a(x) \Delta y(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (12)$$

$$y(x, 0) = y^0(x), y'(x, 0) = y^1(x) \quad \text{in } \Omega, \quad (13)$$

$$y(x, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (14)$$

$$\frac{\partial z(x, \rho, t)}{\partial t} + \frac{1}{\tau} \frac{\partial z(x, \rho, t)}{\partial \rho} = 0 \quad \text{on } \Gamma_2 \times (0, +\infty) \quad (15)$$

$$\frac{\partial y_2(x, t)}{\partial \nu} = -\mu_1 y_2'(x, t) - \mu_2 z(x, 1, t) \quad \text{on } \Gamma_2 \times (0, +\infty), \quad (16)$$

$$y_1(x, t) = y_2(x, t) \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (17)$$

$$a_1 \frac{\partial y_1(x, t)}{\partial \nu} = a_2 \frac{\partial y_2(x, t)}{\partial \nu} \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (18)$$

$$z(x, 0, t) = y'(x, t) \quad \text{on } \Gamma_2 \times (0, +\infty) \quad (19)$$

$$z(x, \rho, 0) = f_0(x, -\tau\rho) \quad \text{on } \Gamma_2 \times (0, 1) \quad (20)$$

Now, we endow the Hilbert space

$$\mathcal{H} = H_{\Gamma_1}^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2(0, 1))$$

with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}; \begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{z} \end{pmatrix} \right\rangle = \int_{\Omega} (a(x)\nabla u(x)\nabla \bar{u}(x) + v(x)\bar{v}(x)) dx + \xi \int_{\Gamma_2} \int_0^1 z(x, \rho)\bar{z}(x, \rho) d\rho d\sigma(x)$$

and define a linear operator in  $\mathcal{H}$  by

$$D(A) = \{(u, v, z)^T \in H^2(\Omega_1, \Omega_2, \Gamma_1) \times H_{\Gamma_1}^1(\Omega) \times L^2(\Gamma_2; H^1(0, 1)); \frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(\cdot, 1), v = z(\cdot, 0) \text{ on } \Gamma_2\} \quad (21)$$

$$A \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ a(x)\Delta u \\ -\tau^{-1} \frac{\partial z}{\partial \rho} \end{pmatrix} \quad (22)$$

The spaces used for the definition of  $\mathcal{H}$  and  $D(A)$  are

$$H_{\Gamma_1}^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$$

$$H^2(\Omega_1, \Omega_2, \Gamma_1) = \{u_i \in H^2(\Omega_i) : u = 0 \text{ on } \Gamma_1, u_1 = u_2 \text{ and } a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_0\}$$

Then we can rewrite (12) – (20) as an abstract Cauchy problem in  $\mathcal{H}$

$$\begin{cases} \frac{d}{dt} Y(t) = AY(t) \\ Y(0) = Y_0 \end{cases} \quad (23)$$

where

$$Y(t) = (y, y', z)^T \text{ and } Y_0 = (y_0, y_1, f_0(\cdot, -\cdot, \tau))^T$$

**Proposition 1.** *The operator  $A$  defined by (21) and (22) generates a strongly continuous semigroup on  $\mathcal{H}$ . Thus, for every  $Y_0 \in \mathcal{H}$ , problem (23) has a unique solution  $Y$  whose regularity depends on the the initial datum  $Y_0$  as follows:*

$$Y(\cdot) \in C([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in \mathcal{H},$$

$$Y(\cdot) \in C([0, +\infty); D(A)) \cap C^1([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in D(A).$$

*Proof.* Let  $Y = \begin{pmatrix} u \\ v \\ z \end{pmatrix} \in D(A)$ . Then

$$\begin{aligned} \langle AY, Y \rangle &= \int_{\Omega} a(x)\nabla u(x)\nabla v(x) dx + \int_{\Omega} (a(x)\Delta u(x))v(x) dx - \\ &\quad \frac{\xi}{\tau} \int_{\Gamma_2} \int_0^1 z_{\rho}(x, \rho)z(x, \rho) d\rho d\Gamma \end{aligned} \quad (24)$$

Applying Green's first theorem, we obtain

$$\begin{aligned}
 \int_{\Omega} (a(x)\Delta u(x))v(x)dx &= a_1 \int_{\Gamma_1} v(x) \frac{\partial u(x)}{\partial \nu} d\Gamma - a_1 \int_{\Omega_1} \nabla u(x) \cdot \nabla v(x) dx + \\
 & a_2 \int_{\Gamma_2} v(x) \frac{\partial u(x)}{\partial \nu} d\Gamma - a_2 \int_{\Omega_2} \nabla u(x) \cdot \nabla v(x) dx \\
 &= a_2 \int_{\Gamma_2} v(x) \{-\mu_1 v(x) - \mu_2 z(x, 1)\} d\Gamma - \int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) dx \quad (25)
 \end{aligned}$$

Integrating by parts in  $\rho$ , we get

$$\int_{\Gamma_2} \int_0^1 z_{\rho}(x, \rho) z(x, \rho) d\rho d\Gamma = \frac{1}{2} \int_{\Gamma_2} \{z^2(x, 1) - z^2(x, 0)\} d\Gamma \quad (26)$$

Inserting (25) and (26) into (24) results in

$$\begin{aligned}
 \langle AY, Y \rangle &= -a_2 \mu_1 \int_{\Gamma_2} v^2(x) d\Gamma - a_2 \mu_2 \int_{\Gamma_2} v(x) z(x, 1) d\Gamma - \\
 & \frac{\xi}{2\tau} \int_{\Gamma_2} z^2(x, 1) d\Gamma + \frac{\xi}{2\tau} \int_{\Gamma_2} v^2(x) d\Gamma
 \end{aligned}$$

from which follows using the Cauchy-Schwarz inequality

$$\langle AY, Y \rangle \leq -(a_2 \mu_1 - \frac{a_2 \mu_2}{2} + \frac{\xi}{2\tau}) \int_{\Gamma_2} v^2(x) d\Gamma - (\frac{\xi}{2\tau} - \frac{a_2 \mu_2}{2}) \int_{\Gamma_2} z^2(x, 1) d\Gamma \quad (27)$$

(27) implies that

$$\langle AY, Y \rangle \leq 0$$

Thus  $A$  is dissipative.

Now we show that for a fixed  $\lambda > 0$  and  $(g, h, k)^T \in \mathcal{H}$ , there exists  $Y = (u, v, z)^T \in D(A)$  such that

$$(\lambda I - A)Y = (g, h, k)^T$$

or equivalently

$$\lambda u - v = g \quad (28)$$

$$\lambda v - a(x)\Delta u = h \quad (29)$$

$$\lambda z + \frac{1}{\tau} z_{\rho} = k \quad (30)$$

Suppose that we have found  $u$  with the appropriate regularity, then we can determine  $z$ . Indeed, from (21) and (30) we have

$$\begin{cases} z_{\rho}(x, \rho) = -\lambda \tau z(x, \rho) + \tau k(x, \rho) \\ z(x, 0) = v(x) \end{cases}$$

The unique solution of the above initial value problem is

$$z(x, \rho) = e^{-\lambda\tau\rho}v(x) + \tau e^{-\lambda\tau\rho} \int_0^\rho e^{\lambda\tau s}k(x, s)ds$$

and in particular

$$z(x, 1) = \lambda e^{-\lambda\tau}u(x) + z_0(x), \quad x \in \Gamma_2$$

where

$$z_0(x) = -e^{-\lambda\tau}g(x) + \tau e^{-\lambda\tau} \int_0^1 e^{\lambda\tau s}k(x, s)ds$$

By (28) and (29), the function  $u$  satisfies

$$\lambda^2 u - a(x)\Delta u = h + \lambda g \quad (31)$$

Problem (31) can be reformulated as

$$\int_\Omega (\lambda^2 u - a(x)\Delta u)w dx = \int_\Omega (h + \lambda g)w dx, \quad w \in H_{\Gamma_1}^1(\Omega) \quad (32)$$

Using Green's first theorem and recalling (21), we express the right-hand side of (32) as follows

$$\begin{aligned} \int_\Omega (\lambda^2 u - a(x)\Delta u)w dx &= \int_\Omega (\lambda^2 u w + a(x)\nabla u \cdot \nabla w) dx + a_2 \int_{\Gamma_2} \{\mu_1(\lambda u - g)w \\ &+ \mu_2(\lambda e^{-\lambda\tau}u(x) + z_0(x))w\} d\Gamma \end{aligned}$$

Therefore (32), can be rewritten as

$$\begin{aligned} \int_\Omega (\lambda^2 u w + a(x)\nabla u \cdot \nabla w) dx + a_2 \int_{\Gamma_2} (\mu_1 + \mu_2 e^{-\lambda\tau})\lambda u w d\Gamma &= \int_\Omega (h + \lambda g)w d\Gamma \\ + a_2 \mu_1 \int_{\Gamma_2} g w d\Gamma - a_2 \mu_2 \int_{\Gamma_2} z_0 w d\Gamma, \quad \forall w \in H_{\Gamma_1}^1(\Omega). \end{aligned} \quad (33)$$

Since the left-hand side of (33) is coercive on  $H_{\Gamma_1}^1(\Omega)$ , the Lax-Milgram Theorem guarantees the existence and uniqueness of a solution  $y \in H_{\Gamma_1}^1(\Omega)$  of (31). If we consider  $w \in \mathcal{D}(\Omega)$  in (28), then  $y$  is a solution in  $\mathcal{D}'(\Omega)$  of

$$\lambda^2 u - a(x)\Delta u = h + \lambda g \quad (34)$$

and thus  $\Delta u \in L^2(\Omega)$ .

Combining (33) together with (34), we obtain after using Green's first theorem

$$a_2 \int_{\Gamma_2} (\mu_1 + \mu_2 e^{-\lambda\tau})\lambda u w d\Gamma + a_2 \int_\Omega \frac{\partial u}{\partial \nu} w d\Gamma = a_2 \mu_1 \int_{\Gamma_2} g w d\Gamma - a_2 \mu_2 \int_{\Gamma_2} z_0 w d\Gamma$$

which implies that

$$\frac{\partial u(x)}{\partial \nu} = -\mu_1 v(x) - \mu_2 z(x, 1)$$

So, we have found  $(u, v, z)^T \in D(A)$  which satisfies (28) – (30). Thus, by the Lumer-Phillips Theorem (see for instance [8], Theorem 1.4.3), generates a strongly continuous semigroup of contractions on  $\mathcal{H}$ .

### 3 Proof of Theorem 1

We prove Theorem 1 for smooth initial data. The general case follows by a standard density argument.

We proceed in several steps.

**Step 1.**

Since

$$E(t) = \frac{1}{2} \|(y, y', z)\|_{\mathcal{H}}^2$$

Then, we deduce from the proof of Proposition 1 that  $E(t)$  is non-increasing and

$$\frac{d}{dt}E(t) \leq -C \int_{\Gamma_2} \{y^2(x, t) + y'^2(x, t)\} d\Gamma \quad (35)$$

where

$$C = \min\left\{a_2\mu_1 - \frac{a_2\mu_2}{2} + \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{a_2\mu_2}{2}\right\}$$

**Step 2.**

Set

$$E(t) = \mathcal{E}(t) + E_d(t)$$

where

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} \{a(x) |\nabla y(x, t)|^2 + |y'(x, t)|^2\} dx$$

and

$$E_d(t) = \frac{\xi}{2\tau} \int_{\Gamma_2} \int_0^1 |y'(x, t - \tau\rho)|^2 d\rho d\Gamma$$

$E_d(t)$  can be rewritten via a change of variable as

$$E_d(t) = \frac{\xi}{2\tau^2} \int_t^{t+\tau} \int_{\Gamma_2} y'^2(x, s - \tau) d\Gamma ds \quad (36)$$

From (36), we obtain

$$E_d(t) \leq C_1 \int_0^T \int_{\Gamma_2} y'^2(x, s - \tau) d\Gamma ds \quad (37)$$

for  $0 \leq t \leq T$  and  $T$  large enough.

**Step 3.**

By applying energy methods (multiplier  $2h \cdot \nabla y + (\operatorname{div} h - \alpha)y$ ) (see the appendix) to problem (1) – (7), we obtain for all  $T > 0$ ,

$$\begin{aligned} \int_0^T \mathcal{E}(t) dt &\leq C_2(\mathcal{E}(0) + \mathcal{E}(T)) + C_3 \int_0^T \int_{\Gamma_2} \left\{ \left( \frac{\partial y(x, t)}{\partial \nu} \right)^2 + y'^2(x, t) \right\} d\Gamma dt + \\ &C_4 \int_0^T \int_{\Gamma_2} |\nabla_{\sigma} y(x, t)|^2 d\Gamma dt + C_5 \int_0^T \int_{\Omega} |y(x, t)|^2 d\Omega dt \end{aligned} \quad (38)$$



where  $\nabla_\sigma y$  is the tangential gradient of  $y$ .

**Step 4.**

We eliminate the tangential gradient from (38) by using the following estimate due to Lasiecka and Triggiani (Lemma 7.2 in [3])

$$\int_\epsilon^{T-\epsilon} \int_{\Gamma_2} |\nabla_\sigma y(x, t)|^2 d\Gamma dt \leq C_6 \left\{ \int_0^T \int_{\Gamma_2} \left\{ \left( \frac{\partial y(x, t)}{\partial \nu} \right)^2 + y'^2(x, t) \right\} d\Gamma dt + \|y\|_{L^2(0, T; H^{1/2+\delta}(\Omega))}^2 \right\}$$

where  $\epsilon$  and  $\delta$  are arbitrary positive constants. We obtain

$$\int_0^T \mathcal{E}(t) dt \leq C_2(\mathcal{E}(0) + \mathcal{E}(T)) + C_7 \int_0^T \int_{\Gamma_2} \left\{ \left( \frac{\partial y(x, t)}{\partial \nu} \right)^2 + y'^2(x, t) \right\} d\Gamma dt + C_8 \|y\|_{L^2(0, T; H^{1/2+\delta}(\Omega))}^2 \quad (39)$$

**Step 5.**

We differentiate  $\mathcal{E}(t)$  with respect to  $t$  and apply Green's first theorem. We obtain

$$\frac{d}{dt} \mathcal{E}(t) = a_2 \int_{\Gamma_2} y'(x, t) \frac{\partial y(x, t)}{\partial \nu} d\Gamma dt \quad (40)$$

From (40), we get via the Cauchy-Schwarz inequality

$$\mathcal{E}(0) \leq \mathcal{E}(T) + \frac{a_2}{2} \int_0^T \int_{\Gamma_2} \left\{ y'^2(x, t) + \left( \frac{\partial y(x, t)}{\partial \nu} \right)^2 \right\} d\Gamma dt \quad (41)$$

Insertion of (41) into (39) yields

$$\int_0^T \mathcal{E}(t) dt \leq 2C_2 \mathcal{E}(T) + C_9 \int_0^T \int_{\Gamma_2} \left\{ \left( \frac{\partial y(x, t)}{\partial \nu} \right)^2 + y'^2(x, t) \right\} d\Gamma dt + C_8 \|y\|_{L^2(0, T; H^{1/2+\delta}(\Omega))}^2 \quad (42)$$

**Step 6.**

Since  $E(t)$  is non-increasing and  $E(t) = \mathcal{E}(t) + E_d(t)$ , then (42) together with (37) implies that

$$TE(T) \leq 2C_2 \mathcal{E}(T) + C_9 \int_0^T \int_{\Gamma_2} \left\{ \left( \frac{\partial y(x, t)}{\partial \nu} \right)^2 + y'^2(x, t) \right\} d\Gamma dt + C_8 \|y\|_{L^2(0, T; H^{1/2+\delta}(\Omega))}^2 + TC_1 \int_0^T \int_{\Gamma_2} y'^2(x, t - \tau) d\Gamma dt \quad (43)$$

Thus invoking again the identity  $E(t) = \mathcal{E}(t) + E_d(t)$  and recalling the boundary condition (4), we obtain from (43)

$$E(T) \leq C_{10} \int_0^T \int_{\Gamma_2} \left\{ y'^2(x, t) + y'^2(x, t - \tau) \right\} d\Gamma dt + C_{11} \|y\|_{L^2(0, T; H^{1/2+\delta}(\Omega))}^2 \quad (44)$$

for  $T$  large enough.

**Step 7.**

We drop the lower order term on the right-hand side of (44) by a compactness-uniqueness argument to obtain

$$E(T) \leq C_{12} \int_0^T \int_{\Gamma_2} \{y'^2(x, t) + y'^2(x, t - \tau)\} d\Gamma dt \quad (45)$$

**Step 8.**

The estimate (45) together with (35) yields

$$E(T) \leq \frac{C_{13}}{1 + C_{13}} E(0) \quad (46)$$

The desired conclusion follows now from (46) since the system (1) – (7) is invariant by translation.

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**Appendix: Sketch of Proof of (38)**

We multiply both sides of (1) by  $2h \cdot \nabla y + (\operatorname{div} h - \alpha)y$  and integrate over  $(0, T) \times \Omega$ .

We obtain

$$\begin{aligned}
& 2 \int_0^T \int_{\Omega} a(x) J \nabla y \cdot \nabla y d\Omega dt + \alpha \int_0^T \int_{\Omega} \{y'^2 - a(x) |\nabla y|^2\} d\Omega dt = \\
& - \int_{\Omega} \{2y' h \cdot \nabla y + (\operatorname{div} h - \alpha) y' y\}_0^T d\Omega - \int_0^T \int_{\Omega} a(x) y \nabla y \cdot \nabla (\operatorname{div} h - \alpha) d\Omega dt + \\
& a_1 \int_0^T \int_{\Gamma_1} \left| \frac{\partial y_1}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt - (a_1 - a_2) \int_0^T \int_{\Gamma_0} |\nabla y_1|^2 h \cdot \nu d\Gamma dt - \\
& \frac{(a_1 - a_2)^2}{a_2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y_1}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt + \int_0^T \int_{\Gamma_2} |y_2'|^2 h \cdot \nu d\Gamma dt + \\
& 2a_2 \int_0^T \int_{\Gamma_2} \left| \frac{\partial y_2}{\partial \nu} \right|^2 h \cdot \nabla y_2 d\Gamma dt - a_2 \int_0^T \int_{\Gamma_2} |\nabla y_2|^2 h \cdot \nu d\Gamma dt + \\
& a_2 \int_0^T \int_{\Gamma_2} \left| \frac{\partial y_2}{\partial \nu} \right|^2 (\operatorname{div} h - \alpha) d\Gamma dt \tag{47}
\end{aligned}$$

after using the boundary conditions (3) and (5). Identity (47) is used together with (H.1), (H.2), (H.3) and (8) to obtain estimate (38).