



**HAL**  
open science

## Dynamic Contact Problem for Viscoelastic von Kármán-Donnell Shells

Igor Bock, Jiří Jarušek

► **To cite this version:**

Igor Bock, Jiří Jarušek. Dynamic Contact Problem for Viscoelastic von Kármán-Donnell Shells. 25th System Modeling and Optimization (CSMO), Sep 2011, Berlin, Germany. pp.70-75, 10.1007/978-3-642-36062-6\_7. hal-01347524

**HAL Id: hal-01347524**

**<https://inria.hal.science/hal-01347524>**

Submitted on 21 Jul 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Dynamic Contact Problem for Viscoelastic von Kármán-Donnell Shells

Igor Bock and Jiří Jarušek

Institute of Computer Science and Mathematics FEI,  
Slovak University of Technology, 812 19 Bratislava 1, Slovakia,  
`igor.bock@stuba.sk`

Institute of Mathematics, Academy of Sciences of the Czech Republic,  
Žitná 25, 115 67 Praha 1, Czech Republic,  
`jarusek@math.cas.cz`

**Abstract.** We deal with initial-boundary value problems describing vertical vibrations of viscoelastic von Kármán-Donnell shells with a rigid inner obstacle. The short memory (Kelvin-Voigt) material is considered. A weak formulation of the problem is in the form of the hyperbolic variational inequality. We solve the problem using the penalization method.

**Keywords:** Von Kármán-Donnell shell, unilateral dynamic contact, viscoelasticity, solvability, penalty approximation

## 1 Introduction

Contact problems represent an important but complex topic of applied mathematics. Its complexity deepens if the dynamic character of the problem is respected. For elastic problems there is only a very limited amount of results available (cf. [3] and there cited literature). Viscosity makes possible to prove the existence of solutions for a broader set of problems for membranes, bodies as well as for linear models of plates. The presented results extend the research made in [2], where the problem for a viscoelastic short memory von Kármán plate in a dynamic contact with a rigid obstacle was considered. Our results also extend the research made for the quasistatic contact problems for viscoelastic shells (cf. [1]). A thin isotropic shallow shell occupies the domain

$$G = \{(x, z) \in \mathbb{R}^3 : x = (x_1, x_2) \in \Omega, |z - \mathcal{Z}| < h/2\},$$

where  $h > 0$  is the thickness of the shell,  $\Omega \subset \mathbb{R}^2$  is a bounded simply connected domain in  $\mathbb{R}^2$  with a sufficiently smooth boundary  $\Gamma$ . We set  $I \equiv (0, T)$  a bounded time interval,  $Q = I \times \Omega$ ,  $S = I \times \Gamma$ . The unit outer normal vector is denoted by  $\mathbf{n} = (n_1, n_2)$ ,  $\boldsymbol{\tau} = (-n_2, n_1)$  is the unit tangent vector. The displacement is denoted by  $\mathbf{u} \equiv (u_i)$ . The strain tensor is defined as

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - k_{ij} u_3 - x_3 \partial_{ij} u_3, \quad i, j = 1, 2$$

with  $k_{12} = k_{21} = 0$  and the curvatures  $k_{ii} > 0$ ,  $i = 1, 2$ .

Further, we set

$$[u, v] \equiv \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v.$$

In the sequel, we denote by  $W_p^k(M)$ ,  $k \geq 0$ ,  $p \in [1, \infty]$  the Sobolev spaces defined on a domain or an appropriate manifold  $M$ . By  $\dot{W}_p^k(M)$  the spaces with zero traces are denoted. If  $p = 2$  we use the notation  $H^k(M)$ ,  $\dot{H}^k(M)$ . The duals to  $\dot{H}^k(M)$  are denoted by  $H^{-k}(M)$ . For the anisotropic spaces  $W_p^k(M)$ ,  $k = (k_1, k_2) \in \mathbb{R}_+^2$ ,  $k_1$  is related with the time variable while  $k_2$  with the space variables. We shall use also the Bochner-type spaces  $W_p^k(I; X)$  for a time interval  $I$  and a Banach space  $X$ . Let us remark that for  $k \in (0, 1)$  their norm is defined by the relation

$$\|w\|_{W_p^k(I; X)}^p \equiv \int_I \|w(t)\|_X^p dt + \int_I \int_I \frac{\|w(t) - w(s)\|_X^p}{|s - t|^{1+kp}} ds dt.$$

By  $C(M)$  we denote the spaces of continuous functions on a (possibly relatively) compact manifold  $M$ . They are equipped with the max-norm. Analogously the spaces  $C(M; X)$ , are introduced for a Banach space  $X$ . The following generalization of the Aubin's compactness lemma verified in [4] Theorem 3.1 will be essentially used:

**Lemma 1** *Let  $B_0 \hookrightarrow B \hookrightarrow B_1$  be Banach spaces, the first reflexive and separable. Let  $1 < p < \infty$ ,  $1 \leq r < \infty$ . Then*

$$W \equiv \{v; v \in L_p(I; B_0), \dot{v} \in L_r(I, B_1)\} \hookrightarrow L_p(I; B).$$

## 2 Short memory material

### 2.1 Problem formulation

Employing the Einstein summation, the constitutional law has the form

$$\sigma_{ij}(\mathbf{u}) = \frac{E_1}{1 - \mu^2} \partial_t ((1 - \mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{ij}\varepsilon_{kk}(\mathbf{u})) + \frac{E_0}{1 - \mu^2} ((1 - \mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{ij}\varepsilon_{kk}(\mathbf{u})).$$

The constants  $E_0$ ,  $E_1 > 0$  are the Young modulus of elasticity and the modulus of viscosity, respectively. We shall use the abbreviation  $b = h^2/(12\rho(1 - \mu^2))$ , where  $h > 0$  is the shell thickness and  $\rho$  is the density of the material. We involve the rotation inertia expressed by the term  $a\Delta\dot{u}$  in the first equation of the considered system with  $a = \frac{h^2}{12}$ . It will play the crucial role in the deriving a strong convergence of the sequence of velocities  $\{\dot{u}_m\}$  in the appropriate space. We assume the shell clamped on the boundary. We generalize the dynamic elastic model due to the von Kármán-Donnell theory mentioned in [6]. The classical formulation for the deflection  $u_3 \equiv u$  and the Airy stress function  $v$  is then the

initial-value problem

$$\left. \begin{aligned} \ddot{u} + a\Delta\ddot{u} + b(E_1\Delta^2\dot{u} + E_0\Delta^2u) - [u, v] - \Delta_k * v &= f + g, \\ u - \Psi &\geq 0, \quad g \geq 0, \quad (u - \Psi)g = 0, \\ \Delta^2v + E_1\partial_t(\frac{1}{2}[u, u] + k_{11}\partial_{22}u + k_{22}\partial_{11}u) \\ + E_0(\frac{1}{2}[u, u] + \Delta_k u) &= 0 \end{aligned} \right\} \text{ on } Q, \quad (1)$$

$$u = \partial_n u = v = \partial_n v = 0 \text{ on } S, \quad (2)$$

$$u(0, \cdot) = u_0, \quad \dot{u}(0, \cdot) = u_1 \text{ on } \Omega. \quad (3)$$

The obstacle function  $\Psi \in L_\infty(\Omega)$  is fulfilling  $0 < U_0 \leq u_0 - \Psi$  in  $\Omega$  and

$$\Delta_k u \equiv \partial_{11}(k_{22}u) + \partial_{22}(k_{11}u), \quad (4)$$

$$\Delta_k^* v \equiv k_{22}\partial_{11}v + k_{11}\partial_{22}v. \quad (5)$$

We define the operators  $L : H^2(\Omega) \rightarrow \dot{H}^2(\Omega)$ ,  $\Phi : H^2(\Omega) \times H^2(\Omega) \rightarrow \dot{H}^2(\Omega)$  by uniquely solved equations

$$(\Delta Lu, \Delta w) \equiv (\Delta_k u, w) \quad \forall w \in \dot{H}^2(\Omega), \quad (6)$$

$$(\Delta\Phi(u, v), \Delta w) \equiv ([u, v], w) \quad \forall w \in \dot{H}^2(\Omega). \quad (7)$$

with the inner product  $(\cdot, \cdot)$  in the space  $L_2(\Omega)$ . The operator  $L$  is linear and compact. The bilinear operator  $\Phi$  is symmetric and compact. Moreover due to Lemma 1 from [5]  $\Phi : H^2(\Omega)^2 \rightarrow W_p^2(\Omega)$ ,  $2 < p < \infty$  and

$$\|\Phi(u, v)\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)} \quad \forall u \in H^2(\Omega), v \in W_p^1(\Omega). \quad (8)$$

We have also  $L : H^2(\Omega) \mapsto W_p^2(\Omega)$ ,  $2 < p < \infty$  and

$$\|Lu\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)} \quad \forall u \in H^2(\Omega). \quad (9)$$

For  $u, y \in L_2(I; H^2(\Omega))$  we define the bilinear form  $A$  by

$$A(u, y) := b(\partial_{kk}u\partial_{kk}y + \mu(\partial_{11}u\partial_{22}y + \partial_{22}u\partial_{11}y) + 2(1 - \mu)\partial_{12}u\partial_{12}y).$$

We introduce shifted cone  $\mathcal{K}$  by

$$\mathcal{K} := \{y \in H^{1,2}(Q); \dot{y} \in L_2(I, \dot{H}^1(\Omega)); y \geq \Psi\}. \quad (10)$$

Then the variational formulation of the problem (1-3) has the form of

**Problem  $\mathcal{P}$ .** Find  $u \in \mathcal{K}$  such that  $\dot{u} \in L_2(I; \dot{H}^2(\Omega))$  and

$$\begin{aligned} & \int_Q (E_1A(\dot{u}, y - u) + E_0A(u, y - u)) \, dx \, dt \\ & + \int_Q [u, E_1\partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)](y - u) \, dx \, dt \\ & + \int_Q \Delta_k (E_1\partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)) (y - u) \, dx \, dt \\ & - \int_Q (a\nabla\dot{u} \cdot \nabla(\dot{y} - \dot{u}) + \dot{u}(\dot{y} - \dot{u})) \, dx \, dt \\ & + \int_\Omega (a\nabla\dot{u} \cdot \nabla(y - u) + \dot{u}(y - u)) (T, \cdot) \, dx \\ & \geq \int_\Omega (a\nabla u_1 \cdot \nabla(y(0, \cdot) - u_0) + u_1(y(0, \cdot) - u_0)) \, dx \\ & + \int_Q f(y_1 - u) \, dx \, dt \quad \forall y \in \mathcal{K}. \end{aligned} \quad (11)$$

## 2.2 The penalization

For any  $\eta > 0$  we define the *penalized problem*

**Problem  $\mathcal{P}_\eta$ .** Find  $u \in H^{1,2}(Q)$  such that  $\dot{u} \in L_2(I; \dot{H}^2(\Omega))$ ,  $\ddot{u} \in L_2(I; \dot{H}^1(\Omega))$ ,

$$\begin{aligned} & \int_Q (\ddot{u}z + a\nabla\dot{u} \cdot \nabla z + E_1A(\dot{u}, z) + E_0A(u, z)) dx dt \\ & + \int_Q [u, E_1\partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)]z dx dt \\ & + \int_Q \Delta_k (E_1\partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)) z dx dt \\ & = \int_Q (f + \eta^{-1}(u - \Psi)^-)z dx dt \quad \forall z \in L_2(I; H^2(\Omega)) \end{aligned} \quad (12)$$

and the conditions (3) remain valid.

**Lemma 2** Let  $f \in L_2(Q)$ ,  $u_0 \in \dot{H}^2(\Omega)$ , and  $u_1 \in \dot{H}^1(\Omega)$ . Then there exists a solution  $u$  of the problem  $\mathcal{P}_\eta$ .

*Proof.* Let us denote by  $\{w_i \in \dot{H}^2(\Omega); i = 1, 2, \dots\}$  a basis of  $\dot{H}^2(\Omega)$  orthonormal in  $H^1(\Omega)$  with respect to the inner product

$$(u, v)_a = \int_\Omega (uv + a\nabla u \cdot \nabla v) dx, \quad u, v \in H^1(\Omega).$$

We construct the Galerkin approximation  $u_m$  of a solution in a form

$$u_m(t) = \sum_{i=1}^m \alpha_i(t)w_i, \quad \alpha_i(t) \in \mathbb{R}, \quad i = 1, \dots, m, \quad m \in \mathbb{N}, \quad (13)$$

$$\begin{aligned} & (\ddot{u}_m(t), w_i)_a + \int_\Omega (E_1A(\dot{u}_m(t), w_i) + E_0A(u_m(t), w_i)) dx + \\ & \int_\Omega \Delta (E_1\partial_t(\frac{1}{2}\Phi(u_m, u_m) + Lu_m) + E_0(\frac{1}{2}\Phi(u_m, u_m) + Lu_m)) \\ & \times \Delta(\Phi(u_m, w_i) + Lw_i) dx \\ & = \int_\Omega (f(t) + \eta^{-1}(u_m(t) - \Psi)^-)w_i dx, \quad i = 1, \dots, m, \end{aligned} \quad (14)$$

$$u_m(0) = u_{0m}, \quad \dot{u}_m(0) = u_{1m}, \quad u_{0m} \rightarrow u_0 \text{ in } \dot{H}^2(\Omega), \quad u_{1m} \rightarrow u_1 \text{ in } \dot{H}^1(\Omega). \quad (15)$$

After multiplying the equation (14) by  $\dot{\alpha}_i(t)$ , summing up with respect to  $i$ , taking in mind the definitions of the operators  $\Phi, L$  and integrating we obtain the *a priori* estimates not depending on  $m$ :

$$\begin{aligned} & \|\dot{u}_m\|_{L_2(I; \dot{H}^2(\Omega))}^2 + \|\dot{u}_m\|_{L_\infty(I; \dot{H}^1(\Omega))}^2 + \|u_m\|_{L_\infty(I; \dot{H}^2(\Omega))}^2 \\ & + \|\partial_t\Phi(u_m, u_m)\|_{L_2(I; \dot{H}^2(\Omega))}^2 + \|\partial_t Lu_m\|_{L_2(I; \dot{H}^2(\Omega))}^2 \\ & + \eta^{-1}\|(u_m - \Psi)^-\|_{L_\infty(I; L_2(\Omega))} \leq c \equiv c(f, u_0, u_1). \end{aligned} \quad (16)$$

Moreover the estimates (8), (9) imply

$$\|\partial_t\Phi(u_m, u_m)\|_{L_2(I; W_p^2(\Omega))} + \|\partial_t Lu_m\|_{L_2(I; W_p^2(\Omega))} \leq c_p \quad \forall p > 2. \quad (17)$$

After multiplying the equation (14) by  $\ddot{\alpha}_i(t)$ , summing up and integrating we obtain the estimate of  $\ddot{u}_m$

$$\|\ddot{u}_m\|_{L_2(I; H^1(\Omega))} \leq c_\eta, \quad m \in \mathbb{N}. \quad (18)$$

Applying the estimates (16)-(18), the compact imbedding theorem and the interpolation, we obtain for any  $p \in [1, \infty)$ , a subsequence of  $\{u_m\}$  (denoted again by  $\{u_m\}$ ), a function  $u$  and the convergences

$$\begin{aligned}
 \ddot{u}_m &\rightharpoonup \ddot{u} \text{ in } L_2(I; H^1(\Omega)), \\
 \dot{u}_m &\rightharpoonup^* \dot{u} \text{ in } L_\infty(I; \dot{H}^1(\Omega)), \\
 \dot{u}_m &\rightharpoonup \dot{u} \text{ in } L_2(I; \dot{H}^2(\Omega)), \\
 \dot{u}_m &\rightharpoonup \dot{u} \text{ in } L_p(I; \dot{H}^2(\Omega)) \cap L_\infty(I; H^{2-\varepsilon}(\Omega)) \quad \forall \varepsilon > 0, \\
 u_m &\rightharpoonup u \text{ in } C(\bar{I}; W_p^1(\Omega)), \\
 \partial_t(\frac{1}{2}\Phi(u_m, u_m) + Lu_m) &\rightharpoonup \partial_t(\frac{1}{2}\Phi(u, u) + Lu) \text{ in } L_2(I; W_p^2(\Omega))
 \end{aligned} \tag{19}$$

implying that a function  $u$  fulfils the identity (12). The initial conditions (3) follow due to (15) and the proof of the existence of a solution is complete.

### 2.3 Solving the original problem

We verify the existence theorem

**Theorem 1** *Let  $f \in L_2(Q)$ ,  $u_i \in \dot{H}^2(\Omega)$ ,  $i = 0, 1$ ,  $0 < U_0 \leq u_0 - \Psi$ . Then there exists a solution of the Problem  $\mathcal{P}$ .*

*Proof.* We perform the limit process for  $\eta \rightarrow 0$ . We write  $u_\eta$  for the solution of the problem  $\mathcal{P}_{1,\eta}$ . The *a priori* estimates (16) imply the estimates

$$\begin{aligned}
 &\|\dot{u}_\eta\|_{L_2(I; \dot{H}^2(\Omega))}^2 + \|\dot{u}_\eta\|_{L_\infty(I; \dot{H}^1(\Omega))}^2 + \|u_\eta\|_{L_\infty(I; \dot{H}^2(\Omega))}^2 \\
 &+ \|\partial_t \Phi(u_\eta, u_\eta)\|_{L_2(I; W_p^2(\Omega))}^2 + \|\partial_t Lu_\eta\|_{L_2(I; W_p^2(\Omega))}^2 \\
 &+ \eta^{-1} \|(u_\eta - \Psi)^-\|_{L_\infty(I; L_2(\Omega))} \leq c_p, \quad p > 2.
 \end{aligned} \tag{20}$$

To get the crucial estimate for the penalty, we put  $z = u_0 - u_\eta(t, \cdot)$  in (12) and obtain the estimate

$$\begin{aligned}
 0 &\leq U_0 \int_Q \eta^{-1} (u_\eta - \Psi)^- dx dt \leq \int_Q \|\eta^{-1} (u_\eta - \Psi)^-(u_0 - \Psi)\| dx dt \\
 &\leq \int_Q \|\eta^{-1} (u_\eta - \Psi)^-(u_0 - u_\eta)\| dx dt \\
 &= \int_Q (\dot{u}_\eta^2 + a|\nabla \dot{u}_\eta|^2 + A((E_1 \partial_t u_\eta + E_0 u_\eta), u_0 - u_\eta) \\
 &\quad + E_1 \partial_t (\Delta(Lu_\eta + \frac{1}{2}\Phi(u_\eta, u_\eta))) \Delta(L(u_0 - u_\eta) + \Phi(u_\eta, u_0 - u_\eta)) \\
 &\quad + E_0 \Delta(Lu_\eta + \frac{1}{2}\Phi(u_\eta, u_\eta)) \Delta(L(u_0 - u_\eta) + \Phi(u_\eta, u_0 - u_\eta))) dx dt \\
 &\quad - \int_Q f(u_0 - u_\eta) dx dt + \int_\Omega ((\dot{u}_\eta(u_0 - u_\eta) + a\nabla \dot{u}_\eta \cdot \nabla(u_0 - u_\eta))(T, \cdot)) dx.
 \end{aligned}$$

Applying the *a priori* estimates (20) we obtain

$$\|\eta^{-1} u_\eta^-\|_{L_1(Q)} \leq c(f, u_0, u_1, \Psi). \tag{21}$$

With respect to Dirichlet conditions we obtain from (12) and (21) the dual estimate

$$\|-a\Delta \dot{u}_\eta + \ddot{u}_\eta\|_{L_1(I; H^{-2}(\Omega))} \leq c. \tag{22}$$

We take the sequence  $\{u_k\} \equiv \{u_{\eta_k}\}$ ,  $\eta_k \rightarrow 0+$ .

After applying the Lemma 1 with the spaces

$$B_0 = L_2(\Omega), \quad B = H^{-1}(\Omega), \quad B_1 = H^{-2}(\Omega)$$

we obtain the relative compactness of the sequence  $\{-a\Delta\dot{u}_k + \dot{u}_k\}$  in  $L_2(I; H^{-1}(\Omega))$  and with the help of the test function  $\dot{u}_k - \dot{u}$  the crucial strong convergence

$$\dot{u}_k \rightarrow \dot{u} \text{ in } L_2(I; \mathring{H}^1(\Omega)). \quad (23)$$

Simultaneously we have the convergences

$$\begin{aligned} \dot{u}_k &\rightharpoonup \dot{u} \text{ in } L_2(I; \mathring{H}^2(\Omega)), \\ \dot{u}_k &\rightarrow \dot{u} \text{ in } L_2(I; W_p^1(\Omega)), \\ \frac{1}{2}\partial_t\Phi(u_k, u_k) + \partial_t Lu_k &\rightharpoonup \frac{1}{2}\partial_t\Phi(u, u) + \partial_t Lu \text{ in } L_2(I; W_p^2(\Omega)). \end{aligned} \quad (24)$$

It can be verified after inserting the test function  $z = y - u_k$  in (12) for  $y \in \mathcal{K}$ , performing the integration by parts in the terms containing  $\ddot{u}$ , applying the convergences (23), (24), using the definitions of the operators  $L$ ,  $\Phi$  in (6), (7) and the weak lower semicontinuity that the limit function  $u$  is a solution of the original problem  $\mathcal{P}$ .

*Remark 1.* The existence Theorem 1 can be after some modification verified also for another types of boundary conditions.

**Acknowledgments.** The work presented here was supported by the Czech Academy of Sciences under grant P201/12/0671 and under the Institutional research plan RVD 67985840, by the Czech Ministry of Education under grant MEB 0810045, by the APVV Agency of Slovak Republic under grant 0011-09 and by Ministry of Education of the Slovak Republic under VEGA grant 1/0426/12.

## References

1. I. BOCK: On a pseudoparabolic system for a viscoelastic shallow shell. *PAMM Proc. Appl. Math. Mech.* **6** (2006), 621-622.
2. I. BOCK AND J. JARUŠEK: Unilateral dynamic contact of viscoelastic von Kármán plates. *Advances in Math. Sci. and Appl.* **16** (2006), 175-187.
3. C. ECK, J. JARUŠEK AND M. KRBEC: *Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems.* Monographs & Textbooks in Pure & Appl. Math. No. 270 (ISBN 1-57444-629-0). Chapman & Hall/CRC (Taylor & Francis Group), Boca Raton – London – New York – Singapore 2005.
4. J. JARUŠEK, J. MÁLEK, J. NEČAS AND V. ŠVERÁK: Variational inequality for a viscous drum vibrating in the presence of an obstacle. *Rend. Mat., Ser. VII*, **12** (1992), 943-958.
5. H. KOCH AND A. STACHEL: Global existence of classical solutions to the dynamical von Kármán equations. *Math. Methods in Applied Sciences* **16** (1993), 581-586.
6. A.G. VOĽMIR: *Gibkije plastinky i oboločky* (in Russian), Gosizdat, Moskva, 1956.