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# Dynamic Contact Problem for Viscoelastic von Kármán-Donnell Shells

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**Abstract.** We deal with initial-boundary value problems describing vertical vibrations of viscoelastic von Kármán-Donnell shells with a rigid inner obstacle. The short memory (Kelvin-Voigt) material is considered. A weak formulation of the problem is in the form of the hyperbolic variational inequality. We solve the problem using the penalization method.

**Keywords:** Von Kármán-Donnell shell, unilateral dynamic contact, viscoelasticity, solvability, penalty approximation

## 1 Introduction

Contact problems represent an important but complex topic of applied mathematics. Its complexity deepens if the dynamic character of the problem is respected. For elastic problems there is only a very limited amount of results available (cf. [3] and there cited literature). Viscosity makes possible to prove the existence of solutions for a broader set of problems for membranes, bodies as well as for linear models of plates. The presented results extend the research made in [2], where the problem for a viscoelastic short memory von Kármán plate in a dynamic contact with a rigid obstacle was considered. Our results also extend the research made for the quasistatic contact problems for viscoelastic shells (cf. [1]). A thin isotropic shallow shell occupies the domain

$$G = \{(x, z) \in \mathbb{R}^3 : x = (x_1, x_2) \in \Omega, |z - \mathcal{Z}| < h/2\},$$

where  $h > 0$  is the thickness of the shell,  $\Omega \subset \mathbb{R}^2$  is a bounded simply connected domain in  $\mathbb{R}$  with a sufficiently smooth boundary  $\Gamma$ . We set  $I \equiv (0, T)$  a bounded time interval,  $Q = I \times \Omega$ ,  $S = I \times \Gamma$ . The unit outer normal vector is denoted by  $\mathbf{n} = (n_1, n_2)$ ,  $\tau = (-n_2, n_1)$  is the unit tangent vector. The displacement is denoted by  $\mathbf{u} \equiv (u_i)$ . The strain tensor is defined as

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - k_{ij} u_3 - x_3 \partial_{ij} u_3, \quad i, j = 1, 2$$

with  $k_{12} = k_{21} = 0$  and the curvatures  $k_{ii} > 0$ ,  $i = 1, 2$ .

Further, we set

$$[u, v] \equiv \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v.$$

In the sequel, we denote by  $W_p^k(M)$ ,  $k \geq 0$ ,  $p \in [1, \infty]$  the Sobolev spaces defined on a domain or an appropriate manifold  $M$ . By  $\dot{W}_p^k(M)$  the spaces with zero traces are denoted. If  $p = 2$  we use the notation  $H^k(M)$ ,  $\dot{H}^k(M)$ . The duals to  $\dot{H}^k(M)$  are denoted by  $H^{-k}(M)$ . For the anisotropic spaces  $W_p^k(M)$ ,  $k = (k_1, k_2) \in \mathbb{R}_+^2$ ,  $k_1$  is related with the time variable while  $k_2$  with the space variables. We shall use also the Bochner-type spaces  $W_p^k(I; X)$  for a time interval  $I$  and a Banach space  $X$ . Let us remark that for  $k \in (0, 1)$  their norm is defined by the relation

$$\|w\|_{W_p^k(I; X)}^p \equiv \int_I \|w(t)\|_X^p dt + \int_I \int_I \frac{\|w(t) - w(s)\|_X^p}{|s - t|^{1+kp}} ds dt.$$

By  $C(M)$  we denote the spaces of continuous functions on a (possibly relatively) compact manifold  $M$ . They are equipped with the max-norm. Analogously the spaces  $C(M; X)$ , are introduced for a Banach space  $X$ . The following generalization of the Aubin's compactness lemma verified in [4] Theorem 3.1 will be essentially used:

**Lemma 1** *Let  $B_0 \hookrightarrow B \hookrightarrow B_1$  be Banach spaces, the first reflexive and separable. Let  $1 < p < \infty$ ,  $1 \leq r < \infty$ . Then*

$$W \equiv \{v; v \in L_p(I; B_0), \dot{v} \in L_r(I, B_1)\} \hookrightarrow L_p(I; B).$$

## 2 Short memory material

### 2.1 Problem formulation

Employing the Einstein summation, the constitutional law has the form

$$\sigma_{ij}(\mathbf{u}) = \frac{E_1}{1 - \mu^2} \partial_t ((1 - \mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{ij}\varepsilon_{kk}(\mathbf{u})) + \frac{E_0}{1 - \mu^2} ((1 - \mu)\varepsilon_{ij}(\mathbf{u}) + \mu\delta_{ij}\varepsilon_{kk}(\mathbf{u})).$$

The constants  $E_0$ ,  $E_1 > 0$  are the Young modulus of elasticity and the modulus of viscosity, respectively. We shall use the abbreviation  $b = h^2/(12\rho(1 - \mu^2))$ , where  $h > 0$  is the shell thickness and  $\rho$  is the density of the material. We involve the rotation inertia expressed by the term  $a\Delta\dot{u}$  in the first equation of the considered system with  $a = \frac{h^2}{12}$ . It will play the crucial role in the deriving a strong convergence of the sequence of velocities  $\{\dot{u}_m\}$  in the appropriate space. We assume the shell clamped on the boundary. We generalize the dynamic elastic model due to the von Kármán-Donnell theory mentioned in [6]. The classical formulation for the deflection  $u_3 \equiv u$  and the Airy stress function  $v$  is then the

initial-value problem

$$\left. \begin{aligned} \ddot{u} + a\Delta\ddot{u} + b(E_1\Delta^2\dot{u} + E_0\Delta^2u) - [u, v] - \Delta_k * v &= f + g, \\ u - \Psi &\geq 0, \quad g \geq 0, \quad (u - \Psi)g = 0, \\ \Delta^2v + E_1\partial_t(\frac{1}{2}[u, u] + k_{11}\partial_{22}u + k_{22}\partial_{11}u) \\ + E_0(\frac{1}{2}[u, u] + \Delta_k u) &= 0 \end{aligned} \right\} \text{ on } Q, \quad (1)$$

$$u = \partial_n u = v = \partial_n v = 0 \text{ on } S, \quad (2)$$

$$u(0, \cdot) = u_0, \quad \dot{u}(0, \cdot) = u_1 \text{ on } \Omega. \quad (3)$$

The obstacle function  $\Psi \in L_\infty(\Omega)$  is fulfilling  $0 < U_0 \leq u_0 - \Psi$  in  $\Omega$  and

$$\Delta_k u \equiv \partial_{11}(k_{22}u) + \partial_{22}(k_{11}u), \quad (4)$$

$$\Delta_k^* v \equiv k_{22}\partial_{11}v + k_{11}\partial_{22}v. \quad (5)$$

We define the operators  $L : H^2(\Omega) \rightarrow \dot{H}^2(\Omega)$ ,  $\Phi : H^2(\Omega) \times H^2(\Omega) \rightarrow \dot{H}^2(\Omega)$  by uniquely solved equations

$$(\Delta Lu, \Delta w) \equiv (\Delta_k u, w) \quad \forall w \in \dot{H}^2(\Omega), \quad (6)$$

$$(\Delta\Phi(u, v), \Delta w) \equiv ([u, v], w) \quad \forall w \in \dot{H}^2(\Omega). \quad (7)$$

with the inner product  $(\cdot, \cdot)$  in the space  $L_2(\Omega)$ . The operator  $L$  is linear and compact. The bilinear operator  $\Phi$  is symmetric and compact. Moreover due to Lemma 1 from [5]  $\Phi : H^2(\Omega)^2 \rightarrow W_p^2(\Omega)$ ,  $2 < p < \infty$  and

$$\|\Phi(u, v)\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)} \quad \forall u \in H^2(\Omega), v \in W_p^1(\Omega). \quad (8)$$

We have also  $L : H^2(\Omega) \mapsto W_p^2(\Omega)$ ,  $2 < p < \infty$  and

$$\|Lu\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)} \quad \forall u \in H^2(\Omega). \quad (9)$$

For  $u, y \in L_2(I; H^2(\Omega))$  we define the bilinear form  $A$  by

$$A(u, y) := b(\partial_{kk}u\partial_{kk}y + \mu(\partial_{11}u\partial_{22}y + \partial_{22}u\partial_{11}y) + 2(1 - \mu)\partial_{12}u\partial_{12}y).$$

We introduce shifted cone  $\mathcal{K}$  by

$$\mathcal{K} := \{y \in H^{1,2}(Q); \dot{y} \in L_2(I, \dot{H}^1(\Omega)); y \geq \Psi\}. \quad (10)$$

Then the variational formulation of the problem (1-3) has the form of

**Problem  $\mathcal{P}$ .** Find  $u \in \mathcal{K}$  such that  $\dot{u} \in L_2(I; \dot{H}^2(\Omega))$  and

$$\begin{aligned} & \int_Q (E_1 A(\dot{u}, y - u) + E_0 A(u, y - u)) \, dx \, dt \\ & + \int_Q [u, E_1 \partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)](y - u) \, dx \, dt \\ & + \int_Q \Delta_k (E_1 \partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)) (y - u) \, dx \, dt \\ & - \int_Q (a \nabla \dot{u} \cdot \nabla(\dot{y} - \dot{u}) + \dot{u}(\dot{y} - \dot{u})) \, dx \, dt \\ & + \int_\Omega (a \nabla \dot{u} \cdot \nabla(y - u) + \dot{u}(y - u)) (T, \cdot) \, dx \\ & \geq \int_\Omega (a \nabla u_1 \cdot \nabla(y(0, \cdot) - u_0) + u_1(y(0, \cdot) - u_0)) \, dx \\ & + \int_Q f(y_1 - u) \, dx \, dt \quad \forall y \in \mathcal{K}. \end{aligned} \quad (11)$$

## 2.2 The penalization

For any  $\eta > 0$  we define the *penalized problem*

**Problem  $\mathcal{P}_\eta$ .** Find  $u \in H^{1,2}(Q)$  such that  $\dot{u} \in L_2(I; \dot{H}^2(\Omega))$ ,  $\ddot{u} \in L_2(I; \dot{H}^1(\Omega))$ ,

$$\begin{aligned} & \int_Q (\ddot{u}z + a\nabla\dot{u} \cdot \nabla z + E_1 A(\dot{u}, z) + E_0 A(u, z)) dx dt \\ & + \int_Q [u, E_1 \partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)]z dx dt \\ & + \int_Q \Delta_k (E_1 \partial_t(\frac{1}{2}\Phi(u, u) + Lu) + E_0(\frac{1}{2}\Phi(u, u) + Lu)) z dx dt \\ & = \int_Q (f + \eta^{-1}(u - \Psi)^-)z dx dt \quad \forall z \in L_2(I; H^2(\Omega)) \end{aligned} \quad (12)$$

and the conditions (3) remain valid.

**Lemma 2** Let  $f \in L_2(Q)$ ,  $u_0 \in \dot{H}^2(\Omega)$ , and  $u_1 \in \dot{H}^1(\Omega)$ . Then there exists a solution  $u$  of the problem  $\mathcal{P}_\eta$ .

*Proof.* Let us denote by  $\{w_i \in \dot{H}^2(\Omega); i = 1, 2, \dots\}$  a basis of  $\dot{H}^2(\Omega)$  orthonormal in  $H^1(\Omega)$  with respect to the inner product

$$(u, v)_a = \int_\Omega (uv + a\nabla u \cdot \nabla v) dx, \quad u, v \in H^1(\Omega).$$

We construct the Galerkin approximation  $u_m$  of a solution in a form

$$u_m(t) = \sum_{i=1}^m \alpha_i(t) w_i, \quad \alpha_i(t) \in \mathbb{R}, \quad i = 1, \dots, m, \quad m \in \mathbb{N}, \quad (13)$$

$$\begin{aligned} & (\ddot{u}_m(t), w_i)_a + \int_\Omega (E_1 A(\dot{u}_m(t), w_i) + E_0 A(u_m(t), w_i)) dx + \\ & \int_\Omega \Delta (E_1 \partial_t(\frac{1}{2}\Phi(u_m, u_m) + Lu_m) + E_0(\frac{1}{2}\Phi(u_m, u_m) + Lu_m)) \\ & \times \Delta(\Phi(u_m, w_i) + Lw_i) dx \\ & = \int_\Omega (f(t) + \eta^{-1}(u_m(t) - \Psi)^-) w_i dx, \quad i = 1, \dots, m, \end{aligned} \quad (14)$$

$$u_m(0) = u_{0m}, \quad \dot{u}_m(0) = u_{1m}, \quad u_{0m} \rightarrow u_0 \text{ in } \dot{H}^2(\Omega), \quad u_{1m} \rightarrow u_1 \text{ in } \dot{H}^1(\Omega). \quad (15)$$

After multiplying the equation (14) by  $\dot{\alpha}_i(t)$ , summing up with respect to  $i$ , taking in mind the definitions of the operators  $\Phi, L$  and integrating we obtain the *a priori* estimates not depending on  $m$ :

$$\begin{aligned} & \|\dot{u}_m\|_{L_2(I; \dot{H}^2(\Omega))}^2 + \|\dot{u}_m\|_{L_\infty(I; \dot{H}^1(\Omega))}^2 + \|u_m\|_{L_\infty(I; \dot{H}^2(\Omega))}^2 \\ & + \|\partial_t \Phi(u_m, u_m)\|_{L_2(I; \dot{H}^2(\Omega))}^2 + \|\partial_t Lu_m\|_{L_2(I; \dot{H}^2(\Omega))}^2 \\ & + \eta^{-1} \|(u_m - \Psi)^-\|_{L_\infty(I; L_2(\Omega))} \leq c \equiv c(f, u_0, u_1). \end{aligned} \quad (16)$$

Moreover the estimates (8), (9) imply

$$\|\partial_t \Phi(u_m, u_m)\|_{L_2(I; W_p^2(\Omega))} + \|\partial_t Lu_m\|_{L_2(I; W_p^2(\Omega))} \leq c_p \quad \forall p > 2. \quad (17)$$

After multiplying the equation (14) by  $\ddot{\alpha}_i(t)$ , summing up and integrating we obtain the estimate of  $\ddot{u}_m$

$$\|\ddot{u}_m\|_{L_2(I; H^1(\Omega))} \leq c_\eta, \quad m \in \mathbb{N}. \quad (18)$$

Applying the estimates (16)-(18), the compact imbedding theorem and the interpolation, we obtain for any  $p \in [1, \infty)$ , a subsequence of  $\{u_m\}$  (denoted again by  $\{u_m\}$ ), a function  $u$  and the convergences

$$\begin{aligned}
 \ddot{u}_m &\rightharpoonup \ddot{u} \text{ in } L_2(I; H^1(\Omega)), \\
 \dot{u}_m &\rightharpoonup^* \dot{u} \text{ in } L_\infty(I; \dot{H}^1(\Omega)), \\
 \dot{u}_m &\rightharpoonup \dot{u} \text{ in } L_2(I; \dot{H}^2(\Omega)), \\
 \dot{u}_m &\rightharpoonup \dot{u} \text{ in } L_p(I; \dot{H}^2(\Omega)) \cap L_\infty(I; H^{2-\varepsilon}(\Omega)) \quad \forall \varepsilon > 0, \\
 u_m &\rightharpoonup u \text{ in } C(\bar{I}; W_p^1(\Omega)), \\
 \partial_t \left( \frac{1}{2} \Phi(u_m, u_m) + Lu_m \right) &\rightharpoonup \partial_t \left( \frac{1}{2} \Phi(u, u) + Lu \right) \text{ in } L_2(I; W_p^2(\Omega))
 \end{aligned} \tag{19}$$

implying that a function  $u$  fulfils the identity (12). The initial conditions (3) follow due to (15) and the proof of the existence of a solution is complete.

### 2.3 Solving the original problem

We verify the existence theorem

**Theorem 1** *Let  $f \in L_2(Q)$ ,  $u_i \in \dot{H}^2(\Omega)$ ,  $i = 0, 1$ ,  $0 < U_0 \leq u_0 - \Psi$ . Then there exists a solution of the Problem  $\mathcal{P}$ .*

*Proof.* We perform the limit process for  $\eta \rightarrow 0$ . We write  $u_\eta$  for the solution of the problem  $\mathcal{P}_{1,\eta}$ . The *a priori* estimates (16) imply the estimates

$$\begin{aligned}
 &\|\dot{u}_\eta\|_{L_2(I; \dot{H}^2(\Omega))}^2 + \|\dot{u}_\eta\|_{L_\infty(I; \dot{H}^1(\Omega))}^2 + \|u_\eta\|_{L_\infty(I; \dot{H}^2(\Omega))}^2 \\
 &+ \|\partial_t \Phi(u_\eta, u_\eta)\|_{L_2(I; W_p^2(\Omega))}^2 + \|\partial_t Lu_\eta\|_{L_2(I; W_p^2(\Omega))}^2 \\
 &+ \eta^{-1} \|(u_\eta - \Psi)^-\|_{L_\infty(I; L_2(\Omega))} \leq c_p, \quad p > 2.
 \end{aligned} \tag{20}$$

To get the crucial estimate for the penalty, we put  $z = u_0 - u_\eta(t, \cdot)$  in (12) and obtain the estimate

$$\begin{aligned}
 0 &\leq U_0 \int_Q \eta^{-1} (u_\eta - \Psi)^- dx dt \leq \int_Q \|\eta^{-1} (u_\eta - \Psi)^-(u_0 - \Psi)\| dx dt \\
 &\leq \int_Q \|\eta^{-1} (u_\eta - \Psi)^-(u_0 - u_\eta)\| dx dt \\
 &= \int_Q \left( \dot{u}_\eta^2 + a |\nabla \dot{u}_\eta|^2 + A \left( (E_1 \partial_t u_\eta + E_0 u_\eta), u_0 - u_\eta \right) \right. \\
 &\quad \left. + E_1 \partial_t (\Delta(Lu_\eta + \frac{1}{2} \Phi(u_\eta, u_\eta))) \Delta(L(u_0 - u_\eta) + \Phi(u_\eta, u_0 - u_\eta)) \right. \\
 &\quad \left. + E_0 \Delta(Lu_\eta + \frac{1}{2} \Phi(u_\eta, u_\eta)) \Delta(L(u_0 - u_\eta) + \Phi(u_\eta, u_0 - u_\eta)) \right) dx dt \\
 &\quad - \int_Q f(u_0 - u_\eta) dx dt + \int_\Omega \left( (\dot{u}_\eta(u_0 - u_\eta) + a \nabla \dot{u}_\eta \cdot \nabla (u_0 - u_\eta))(T, \cdot) \right) dx.
 \end{aligned}$$

Applying the *a priori* estimates (20) we obtain

$$\|\eta^{-1} u_\eta^-\|_{L_1(Q)} \leq c(f, u_0, u_1, \Psi). \tag{21}$$

With respect to Dirichlet conditions we obtain from (12) and (21) the dual estimate

$$\| -a \Delta \ddot{u}_\eta + \ddot{u}_\eta \|_{L_1(I; H^{-2}(\Omega))} \leq c. \tag{22}$$

We take the sequence  $\{u_k\} \equiv \{u_{\eta_k}\}$ ,  $\eta_k \rightarrow 0+$ .

After applying the Lemma 1 with the spaces

$$B_0 = L_2(\Omega), \quad B = H^{-1}(\Omega), \quad B_1 = H^{-2}(\Omega)$$

we obtain the relative compactness of the sequence  $\{-a\Delta\dot{u}_k + \dot{u}_k\}$  in  $L_2(I; H^{-1}(\Omega))$  and with the help of the test function  $\dot{u}_k - \dot{u}$  the crucial strong convergence

$$\dot{u}_k \rightarrow \dot{u} \text{ in } L_2(I; \mathring{H}^1(\Omega)). \quad (23)$$

Simultaneously we have the convergences

$$\begin{aligned} \dot{u}_k &\rightharpoonup \dot{u} \text{ in } L_2(I; \mathring{H}^2(\Omega)), \\ \dot{u}_k &\rightarrow \dot{u} \text{ in } L_2(I; W_p^1(\Omega)), \\ \frac{1}{2}\partial_t\Phi(u_k, u_k) + \partial_t Lu_k &\rightharpoonup \frac{1}{2}\partial_t\Phi(u, u) + \partial_t Lu \text{ in } L_2(I; W_p^2(\Omega)). \end{aligned} \quad (24)$$

It can be verified after inserting the test function  $z = y - u_k$  in (12) for  $y \in \mathcal{K}$ , performing the integration by parts in the terms containing  $\ddot{u}$ , applying the convergences (23), (24), using the definitions of the operators  $L$ ,  $\Phi$  in (6), (7) and the weak lower semicontinuity that the limit function  $u$  is a solution of the original problem  $\mathcal{P}$ .

*Remark 1.* The existence Theorem 1 can be after some modification verified also for another types of boundary conditions.

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## References

1. I. BOCK: On a pseudoparabolic system for a viscoelastic shallow shell. *PAMM Proc. Appl. Math. Mech.* **6** (2006), 621-622.
2. I. BOCK AND J. JARUŠEK: Unilateral dynamic contact of viscoelastic von Kármán plates. *Advances in Math. Sci. and Appl.* **16** (2006), 175-187.
3. C. ECK, J. JARUŠEK AND M. KRBEC: *Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems.* Monographs & Textbooks in Pure & Appl. Math. No. 270 (ISBN 1-57444-629-0). Chapman & Hall/CRC (Taylor & Francis Group), Boca Raton – London – New York – Singapore 2005.
4. J. JARUŠEK, J. MÁLEK, J. NEČAS AND V. ŠVERÁK: Variational inequality for a viscous drum vibrating in the presence of an obstacle. *Rend. Mat., Ser. VII*, **12** (1992), 943-958.
5. H. KOCH AND A. STACHEL: Global existence of classical solutions to the dynamical von Kármán equations. *Math. Methods in Applied Sciences* **16** (1993), 581-586.
6. A.G. VOĽMIR: *Gibkije plastinky i oboločky* (in Russian), Gosizdat, Moskva, 1956.