



**HAL**  
open science

## Toward an integrated workforce planning framework using structured equations

Marie Doumic, Benoît Perthame, Edouard Ribes, Delphine Salort, Nathan  
Toubiana

► **To cite this version:**

Marie Doumic, Benoît Perthame, Edouard Ribes, Delphine Salort, Nathan Toubiana. Toward an integrated workforce planning framework using structured equations. 2016. hal-01343368v1

**HAL Id: hal-01343368**

**<https://inria.hal.science/hal-01343368v1>**

Preprint submitted on 8 Jul 2016 (v1), last revised 2 Dec 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Toward an integrated workforce planning framework using structured equations

Marie Doumic \*

Benoît Perthame<sup>†</sup>

Edouard Ribes<sup>‡</sup>

Delphine Salort<sup>§</sup>

Nathan Toubiana<sup>‡</sup>

July 8, 2016

## Abstract

Strategic Workforce Planning is a company process providing best in class, economically sound, workforce management policies and goals. Despite the abundance of literature on the subject, this is a notorious challenge in terms of implementation. Reasons span from the youth of the field itself to broader data integration concerns that arise from gathering information from financial, human resource and business excellence systems.

This paper aims at setting the first stones to a simple yet robust quantitative framework for Strategic Workforce Planning exercises. First a method based on structured equations is detailed. It is then used to answer two main workforce related questions: how to optimally hire to keep labor costs flat? How to build an experience constrained workforce at a minimal cost?

---

2010 *Mathematics Subject Classification.* 90B70, 91D35, 92D25, 35B40

*Keywords and phrases.* Human resource planning; Strategic planning; Structured population dynamics; Long time asymptotics

## 1 Introduction

Strategic Workforce Planning (SWP in short) is a company process designed to get the right people at the right place, at the right time, at the right costs. Multiple methodologies exist to sustain it. They all revolve around 5 milestones (see [20, 23]): after a first baselining of the population, demographic forecasts are drafted in order to assess the potential evolution of a company's headcount. Then business needs, both in terms of headcount and competencies, are gathered to perform a gap analysis between a company's desired future state and its natural evolution. Finally solutions to bridge the gaps are proposed, agreed upon and implemented.

If the process in itself seems simple and if many research studies are focused on the topic of manpower /workforce planning (see state of the art), SWP is something most companies struggle to implement

---

\*Inria de Paris, EPC Mamba, UPMC et CNRS, F75005 Paris, France

<sup>†</sup>Sorbonne Universités, UPMC Univ Paris 06, Laboratoire Jacques-Louis Lions UMR CNRS 7598, Inria, F75005 Paris, France

<sup>‡</sup>Strategic Workforce Planning, Global Talent Management, Sanofi, 75008 Paris, France

<sup>§</sup>Sorbonne Universités, UPMC, Laboratoire de Biologie Computationnelle et Quantitative UMR CNRS 7238, F75005 Paris, France

(see [9]). According to the CEB (Corporate Executive Board) latest benchmarks ([1, 10]), only 10% of companies really succeed in aligning their workforce plans to meet strategic objectives. Among the surveyed firms, 70% failed at drafting a workforce plan and 84% of them are not confident in their use of labor market trends. The same study stated that 65% of the respondents felt a disconnection between the business needs and standard Human Resources processes such as recruitment. There is therefore a need to jump from methodological milestones to analytics in order to standardize and industrialize the technical aspects of SWP.

**Goals and motivations.** Companies' Financial Information Systems (IS) and/or Human Resources Information Systems (HRIS) collect both labor costs and demographic data as part of their standard processes. In section 2, the proposal developed in this paper revolves around creating an actionable quantitative framework based upon those data. This enables a workforce evolution forecast and provides a better understanding of the dynamics at stake to manage a company workforce. In section 3, the explanatory power of this framework is stressed by its results on standard workforce management policies. It is shown that moving from a workforce management by operating expenses toward an investment in human capital is economically sound. Empirical evidence is provided.

**State of the art.** Population evolution has been an extensively researched topic, which fields of application are very broad, especially in biology, where partial differential equations (PDE) are frequently used to model real life processes in ecology, immunology, epidemiology (see [6, 16, 21])... What started with Malthusian considerations has now evolved into advanced multidimensional and nonlinear frameworks. Among structured population models, the age-structured also called "renewal" or McKendrick-Von Foerster equation [11, 12], is probably the most famous and most studied equation, under linear or nonlinear forms, and with variants used in many fields, from neurosciences to cancer modeling.

Besides, Manpower planning is a relatively recent area of research (begun in the late 90s). Many frameworks seem to coexist ranging from stochastic formalisms [14] to the PDE framework in place in population evolution models [8]. It appears that most of the manpower based studies are tailored to answer specific questions. For instance, A. C. Georgiou and N. Tsantas chose to divide the population into several classes, and to simulate the evolution with Markov chains [7]. Some studies also propose to determine an optimal hiring policy. With E. G. Anderson, the optimal policy is found by searching the best ratio between apprentices and experienced employees, in a growth context, with a model based on experience and productivity which suggests to strike the happy medium between too many apprentices (that have to be trained by older employees) and too many experienced employees (that are more expensive in the company's point of view) [2]. Other studies also proposed to optimize the required number of staff with a stochastic model [3].

This article is organized as follows. In section 2, we build a preliminary framework with which we determine the workforce evolution and convergence towards a stable age structure. We show that there can be many short term headcount fluctuations, and studying the long term behavior may not be appropriate, due to an exceedingly long time scale. We therefore build another framework in section 3 for which the hire rate structure is driven by an economic constraint: the labor cost. We first determine the workforce evolution, we then optimize the company's expenses with preserved experience, which leads us to an optimal demographic structure and an associated hiring policy. We show that the result is consistent with the idea of human capital investment.

## 2 Understand workforce evolution in a demographic framework

SWP is usually a long term analysis, hence, assessing the stability of the workforce of the firm from a demographic standpoint is of key importance. Workforce usually evolves according to aging, which represents an increasing experience; attrition, which accounts for workers leaving the company, and hiring. Hire is endogenous (depending on firm activity) while attrition is exogenous. Attrition is driven by two factors: market labor demand and company termination policies. In this case, termination is not allowed, because it is not consistent with the idea of human capital investment. This translates into the following age-structured representation

$$\overbrace{\frac{\partial \rho}{\partial t}(t, z) + \frac{\partial \rho}{\partial z}(t, z)}^{\text{Workforce evolution}} = -\overbrace{\mu(z)\rho(t, z)}^{\text{Attrition}} + \overbrace{h(P_t)P_t\gamma(z)}^{\text{Hiring}}, \quad z_{\min} < z < z_{\max},$$

where  $\rho(t, z)$  is the concentration of workers of age  $z$  at time  $t$ ;  $\mu(z)$  the attrition rate;  $\gamma(z)$  the hired population distribution. We assume that  $\mu$  and  $\gamma$  are independent of time because the current framework is built for businesses with long product and research cycles (typically 5 to 10 years), which translates into a relatively stable global labor competition and experience needs.  $P_t$  is the total headcount at time  $t$ :  $P_t = \int_{z_{\min}}^{z_{\max}} \rho(t, z) dz$ , with  $z_{\min}$  the first hiring age and  $z_{\max}$  the retirement age. The coefficient  $h(P_t)$  represents the hiring rate for the population in scope. It is natural to build a model where the number of hired employees is proportional to the total population. However, if  $h(P_t)$  is taken constant, the model is linear, leading to an exponential growth or decay of the population.

### 2.1 Identifying the hiring rate structure

Consequently, we consider here that the hiring profile  $\gamma$  has been defined and propose another hiring rate based only on the total headcount  $P_t$ . We study its ability to stabilize the workforce population towards an age profile  $P_{\text{eq}}$ . Using a standard formulation in population evolution, we choose  $h(P_t) = \frac{1}{1+\alpha P_t^2}$ . Therefore, the temporal evolution of the headcount density is driven by this equation:

$$\begin{cases} \frac{\partial \rho}{\partial t}(t, z) + \frac{\partial \rho}{\partial z}(t, z) = -\mu(z)\rho(t, z) + \frac{P_t}{1+\alpha P_t^2}\gamma(z), & z_{\min} < z < z_{\max}, \\ \rho(t, z_{\min}) = 0, \\ \rho(0, z) = \rho^0(z) \geq 0. \end{cases} \quad (1)$$

The parameter  $\alpha$  is a pressure population constant representing the budget constraint ( $\alpha > 0$ ). Indeed, thanks to  $\alpha$ , the hiring rate increases with the population for small populations, and decreases from a certain population threshold. So workforce cannot grow exponentially, which reflects the fact that companies cannot hire indefinitely.

The hire distribution  $\gamma(z)$  is given equal to its historical value. Under this formalism, stability can be reached. The convergence (see appendix A) is achieved exponentially fast. In order to have a possible non null steady state, We show that the following condition is required:

$$\beta := \int_{z_{\min}}^{z_{\max}} \left( \int_{z_{\min}}^z \gamma(y) e^{-(M(z)-M(y))} dy \right) dz > 1, \quad (2)$$

where  $M$  is an antiderivative of  $\mu$ . This may be interpreted as the fact that the hiring rate must be sufficiently high to counterbalance those leaving the firm.

## 2.2 How to action the framework

In the case of a non null equilibrium, the hiring rate structure and the steady state  $P_{\text{eq}}$  of the workforce are closely connected. Indeed, we show in the appendix (A.1) that:

$$\alpha = \frac{\int_{z_{\min}}^{z_{\max}} \left( \int_{z_{\min}}^z \gamma(y) e^{-(M(z)-M(y))} dy \right) dz - 1}{P_{\text{eq}}^2},$$

which leads us to the condition (2) thanks to  $\alpha > 0$ .

Consider the case of a mature and established business. It can be assumed that its overall workforce is not likely to change over the long term ( $P_0 = P_{\text{eq}}$ ). Indeed, the overall workload can be assumed steady because of the long business cycle hypothesis. According to the previous formalism, the hiring rate is hence fixed. In the next subsection, we analyze the short term workforce evolution according to the current workforce demographic structure.

**Examples: necessity to adjust workforce management practices to reach stability.** We choose to display the workforce analysis for two cases. For both examples, we show the initial workforce structure, the attrition and the hired population distribution, and we then display the associated workforce evolution. We assume  $P_{\text{eq}} = P_0 = 1000$  for both cases. The first example is taken in a fictional business unit A (BU A). In this example, the turnover rate is very low, and employees usually wait until retirement to leave the firm. The second example is taken in another fictional business unit B (BU B). In this example, employees are mainly young, and tend to leave the firm quickly. This is typically the case for sectors in which there are specific labor policies revolving around fixed term contracts and extreme labor demand. The numerical method is described in the appendix (A.3).

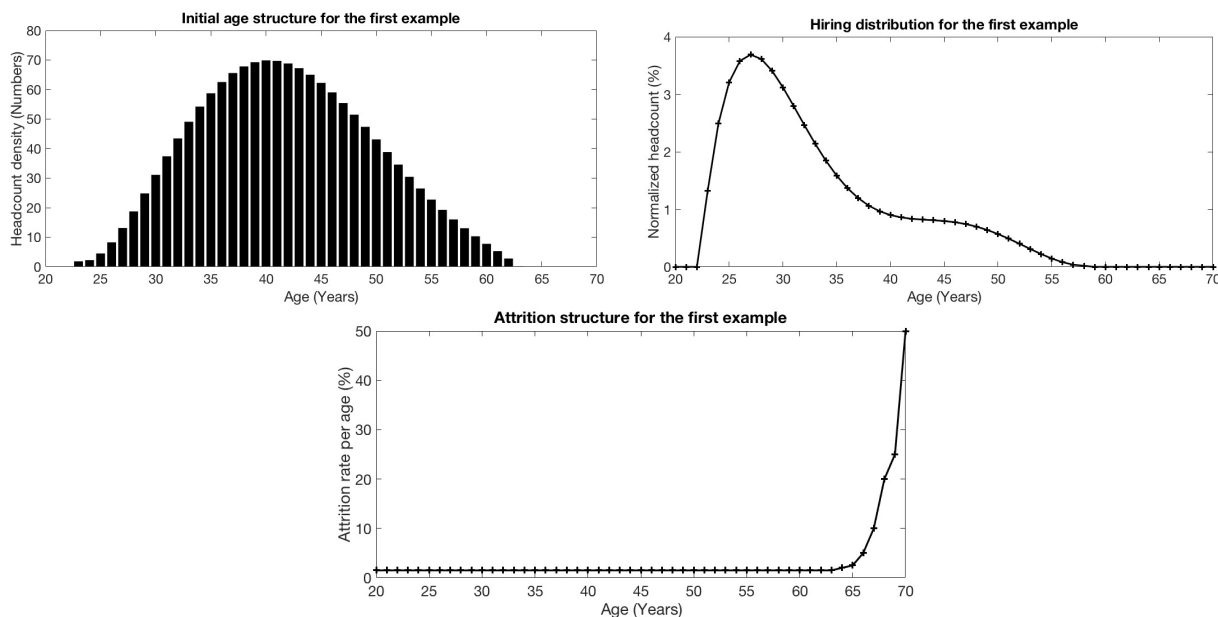


FIG. 1. Initial age structure, historical hired population distribution (normalized), and historical attrition rate (for  $z_{\min} = 20$  years and  $z_{\max} = 70$  years) for the BU A.

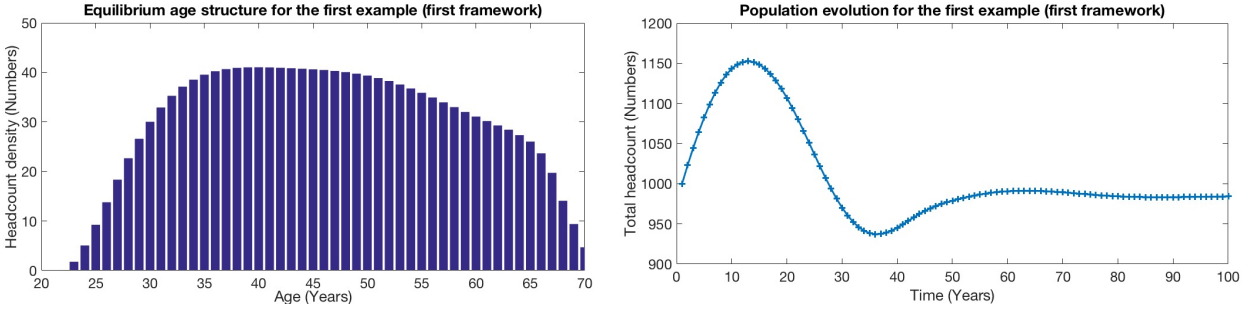


FIG. 2. *Equilibrium age structure and headcount temporal evolution for the BU A, for the discretization  $\delta t = \delta z = 1$  year, and for  $P_{eq} = P_0 = 1000$ .*

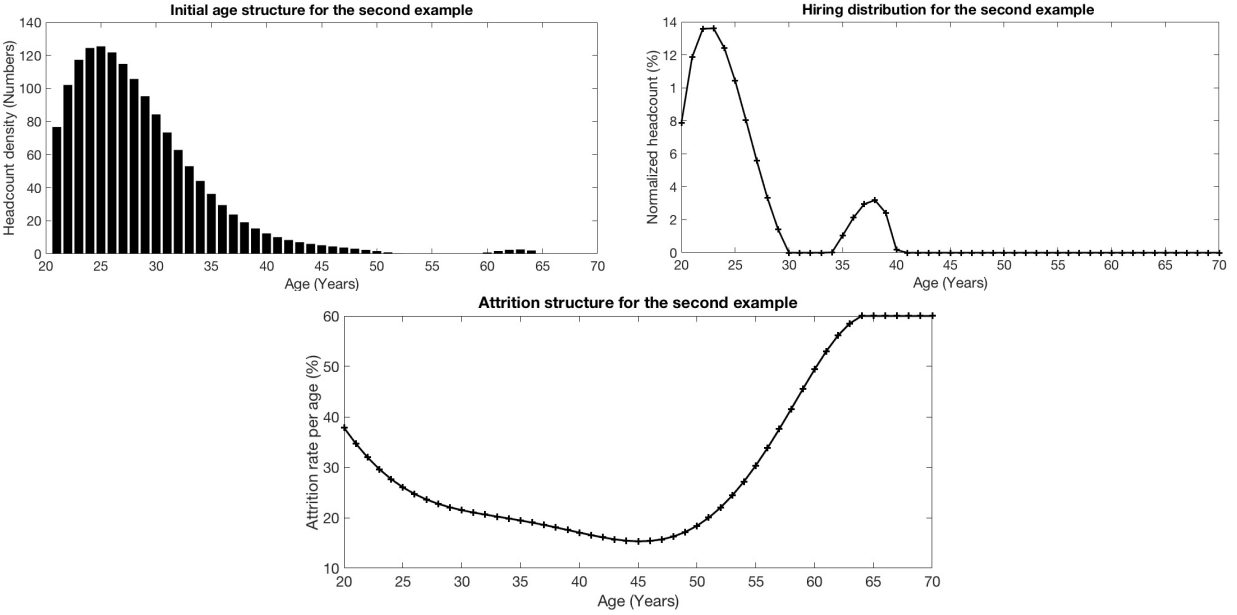


FIG. 3. *Initial age structure, historical hired population distribution (normalized), and historical attrition rate (for  $z_{min} = 20$  years and  $z_{max} = 70$  years) for the BU B.*

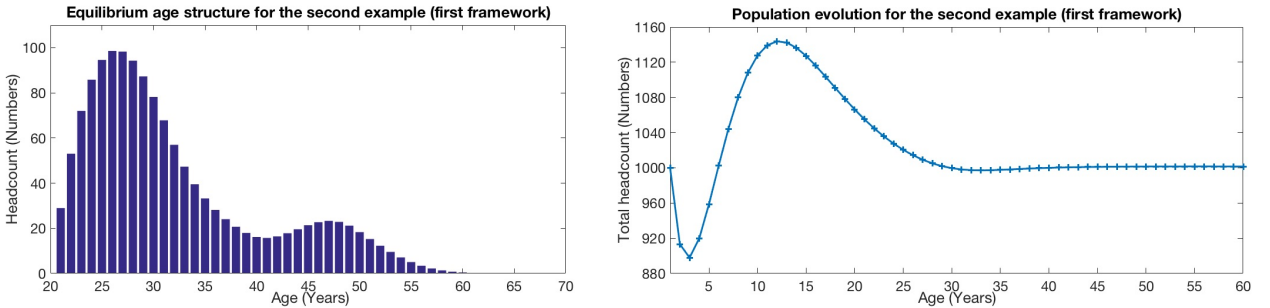


FIG. 4. *Equilibrium age structure and headcount temporal evolution for the BU B, for the discretization  $\delta t = \delta z = 1$  year, and for  $P_{eq} = P_0 = 1000$ .*

For the BU A, we can see that the initial average age is around 45 years. Furthermore, employees are mostly hired when they are young, and the maximum attrition rate is at retirement (see Figure 1). The final average age of the employees is also 45 years, so the overall population did not

age. This is due to the high hiring rate for young employees and the very low attrition rate for all employees until retirement. This also results in a flattening of the age structure. Plus, we note that the equilibrium is reached within around 80 years, and there are substantial headcount fluctuations in-between (see Figure 2).

For the BU B, we can see that the initial average is around 27 years. Furthermore, employees are mostly hired when they are young, and the maximum attrition rate is both for the youngest (fixed term contracts) and oldest (retirement) employees (see Figure 3). The final average age of the employees is around 30 years, 3 years older than the initial average age, which is due to the hiring profile and the attrition rate: young and old employees tend to leave quickly the company, whereas average-aged employees stay (and age) in the company. Plus, we note that the equilibrium is reached within around 30 years, and there are substantial fluctuations in-between (see Figure 4).

As a whole, we find that the equilibrium state is reached very slowly (80 and 30 years), and the fluctuations that we first thought to be short term may not be as short as expected. Indeed, fluctuations can extend up to 60 years, which is higher than an employee's lifetime in the company.

Although determining the steady state seems conceptually appealing, it may not be a relevant option, since the equilibrium will not be reached in a company's activity time scale range. In the next section, we review and modify the hiring rate structure, according to a reasonable economic constraint. The functional  $a(t) = \frac{P_t}{1+\alpha P_t^2}$  has been designed empirically to answer good qualitative properties to the solution, the parameter  $\alpha$  being determined by the target equilibrium  $P_{eq}$ , which happens to be achieved too late to be sound. Plus, each employee does not necessarily have the same impact on the hiring policy of the firm, and this first hiring rate structure does not translate this idea.

### 3 Design of economically sustainable management policies

To complement the use of the total headcount only, we study now another hiring policy based on budget considerations. We assume that workforce needs and expenses are directly tied together, as employees have a certain cost depending on their age. In the first subsection, the workforce evolution will be analyzed with a total budget constraint, in the second subsection, an ideal hiring policy will be searched for, minimizing the cost while keeping a fixed total experience.

#### 3.1 Management policy 1: operational expenditure (opex) adjustments

As a first step, we choose to build the hiring rate with a labor cost constraint: we assume here that the total annual budget (which is assimilated to the sum of the annual salaries) remains constant at all times, and it drives the hiring policy through the modulation of the hiring rate. This translates into the following age-structured representation:

$$\begin{cases} \frac{\partial \rho}{\partial t}(t, z) + \frac{\partial \rho}{\partial z}(t, z) = -\mu(z)\rho(t, z) + h([\rho])\gamma(z), & z_{\min} < z < z_{\max}, \\ \rho(t, z_{\min}) = 0, \\ \rho(0, z) = \rho^0(z) \geq 0, \end{cases} \quad (3)$$

where  $h([\rho])$  depends on the labor cost constraint and does not depend on the age  $z$ .

**Quantitative framework.** To find the hiring rate structure  $h([\rho])$ , we assume here that the hiring profile  $\gamma$  is given (termination is still not allowed) and that the total budget  $\int_{z_{\min}}^{z_{\max}} \rho(z, t)\omega(z)dz$  should

not be time-dependent, where  $w(z)$  is the cost per employee of age  $z$  (given as well). By definition, this makes the equation conservative. We have

$$\omega(z) \frac{\partial \rho}{\partial t}(t, z) + \omega(z) \frac{\partial \rho}{\partial z}(t, z) = -\omega(z) \mu(z) \rho(t, z) + \omega(z) h([\rho]) \gamma(z),$$

and

$$\int_{z_{\min}}^{z_{\max}} \omega(z) \frac{\partial \rho}{\partial t}(t, z) dz = \int_{z_{\min}}^{z_{\max}} \frac{\partial \rho \omega}{\partial t}(t, z) dz = 0,$$

so

$$\underbrace{\int_{z_{\min}}^{z_{\max}} \omega(z) \frac{\partial \rho}{\partial z}(t, z) dz}_{\omega(z_{\max})\rho(t, z_{\max}) - \int_{z_{\min}}^{z_{\max}} \rho(z) \frac{\partial \omega}{\partial z}(t, z) dz} = - \int_{z_{\min}}^{z_{\max}} \omega(z) \mu(z) \rho(t, z) dz + \int_{z_{\min}}^{z_{\max}} \omega(z) h([\rho]) \gamma(z) dz,$$

and thus we obtain the following formula for the hiring rate

$$h([\rho]) = \frac{\overbrace{\int_{z_{\min}}^{z_{\max}} \omega(z) \mu(z) \rho(t, z) dz}^{\text{Attrition}} + \overbrace{\omega(z_{\max}) \rho(t, z_{\max})}^{\text{Retirement}} - \overbrace{\int_{z_{\min}}^{z_{\max}} \rho(t, z) \frac{\partial \omega}{\partial z}(z) dz}^{\text{Cost of aging}}}{\int_{z_{\min}}^{z_{\max}} \omega(z) \gamma(z) dz}.$$

So  $h$  is a linear form, easy to interpret:

- The first term represents the budget available from the attrition of employees of all age bands.
- The second term represents the budget available from the retirement of employees of age  $z_{\max}$ .
- The last term is the cost of aging, which tracks the drift in wages due to seniority and promotions.

Under this formalism, and with the assumption  $\mu\omega \geq \omega'$  (which may be interpreted as a positive balance between the budget earned with the attrition and to the cost of aging), stability can be reached. The convergence is shown in the appendix (B).

**Examples.** Now, we can analyze the workforce evolution for this framework, with the same two examples of the BUs A and B. The historical values (initial age structure, attrition rate and hiring distribution) are the same as before. The numerical method is described in the appendix (B.3).

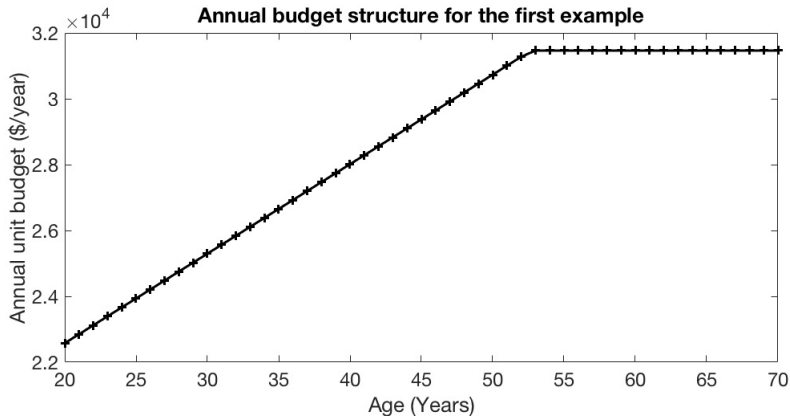


FIG. 5. Budget structure  $\omega(z)$  of the employees of the BU A.



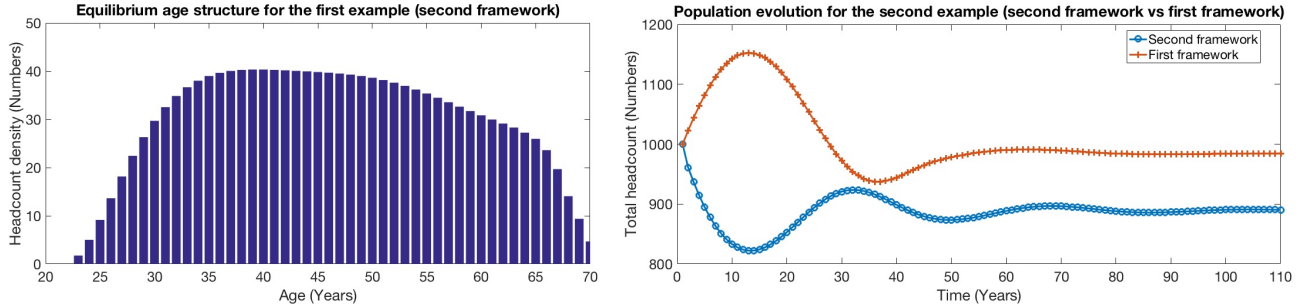


FIG. 6. Equilibrium age structure and headcount temporal evolution for the BU A, for the discretization  $\delta t = 0.5$  year and  $\delta z = 1$  year.

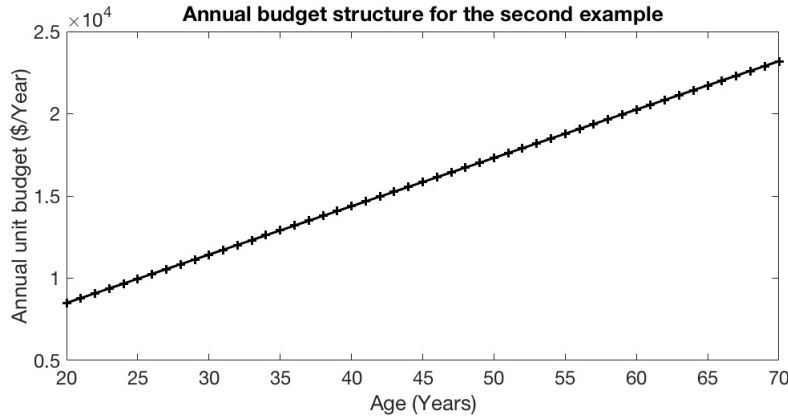


FIG. 7. Budget structure  $\omega(z)$  of the BU B.

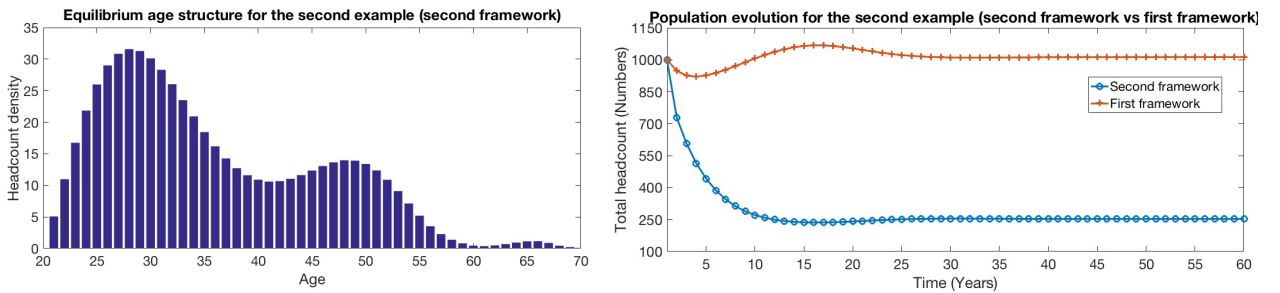


FIG. 8. Equilibrium age structure and headcount temporal evolution for the BU B, for the discretization  $\delta t = 0.5$  year and  $\delta z = 1$  year.

The budget structure of the employees of BU A (see Figure 5) is linear and increases with age, until a certain age (around 55), and then it is constant, reflecting the fact that the maximum experience for an employee is reached at around 55 years. We can see that the final age structure is very similar to the one of the first framework (see Figures 2 and 6). Just as in the previous framework, the high hiring rate for young age and the very low attrition rate for all employees until retirement result in a flattening of the age structure. However, the final headcount is 10% lower (around 900 instead of 1000). The equilibrium is reached around 90 years (see Figure 6).

On the other hand, the budget structure of the employees of BU B is fully linear (see Figure 7). We can see that the final age structure is very similar to the one of the first framework (see Figures 4 and 8). Just as in the previous framework, young and old employees tend to leave quickly the company,

whereas average-aged employees stay in the company. However, the equilibrium headcount is 75% lower (250 instead of 1000), which is due to the flat total budget constraint while having an aging population. The equilibrium is reached within 30 years (see Figure 8).

For both the first and second examples, the equilibrium age structures are very similar for the two frameworks. However, the equilibrium headcount is different (in these cases lower), because we did not fix  $P_{\text{eq}} = P_0$  for the second framework. Plus, the way to reach stable state depends on the framework. By simulating several cases with diverse assumptions, we observe empirically that there seems to be less oscillations for the first one, and the time to reach the equilibrium state is similar for both frameworks.

Even though the two frameworks are similar (in terms of fluctuation and stability), the second one may be more adapted to the SWP analysis. Indeed, this framework makes more economical sense and takes into account labor market trends (through the total budget constraint), contrary to the first one.

### 3.2 Management policy 2: invest in knowledge

Until now, we have kept the hired population distribution constant equal to its historical values. Though this is convenient to analyze the natural workforce evolution, identifying the optimal hiring policies is of key importance regarding the business needs assessment of a given company. This is why the hired population distribution  $\gamma(z)$  is not fixed anymore, and neither is the total budget.

**Identification of the optimal hiring policy.** We now minimize the global labor cost with given total knowledge, and hence find an optimal age structure, and an optimal hiring policy. We consider the case of knowledge workers, in fields for which specific knowledge is required (for instance: experts from the medical field). Knowledge is the sum of aggregated experience and is age dependent. In this case knowledge is assumed to be equal to age.

More precisely, our objective is to minimize the total labor cost defined as  $C = \int_{z_{\min}}^{z_{\max}} \rho^*(z)w(z)dz$  where  $w(z)$  still denotes the cost per employee of age  $z$  and  $\rho^*(z)$  the concentration of workers of age  $z$ ; under the constraint that the total knowledge  $E = \int_{z_{\min}}^{z_{\max}} \rho^*(z)zdz$  is given. This constraint makes especially sense considering the workers population global knowledge. Knowledge (in other words the experience) rather than hourly workload is a better proxy to describe business needs.

Termination is still not allowed. Recalling that  $M$  denotes an antiderivative of the attrition rate  $\mu$ , we show in the appendix (C) that the optimal workforce structure is defined by

$$\rho^*(z) = e^{-M(z)}b\mathbf{1}_{z \geq z_0}, \quad \gamma^*(z) = b\delta_{z_0}e^{-M(z)},$$

with

$$b = \frac{E}{\int_{z_0}^{z_{\max}} ze^{-M(z)}dz}, \quad C = Ed(z_0)$$

and the optimal hiring age  $z_0$  is defined by

$$d(z_0) = \min_z (d(z)),$$

where  $d(z) = \frac{f(z)}{g(z)}$  can be interpreted as follows:

- The numerator  $f(z) = \int_z^{z_{\max}} w(y)e^{-M(y)}dy$  represents the average cost of the tenure of an employee in the firm

- The denominator  $g(z) = \int_z^{z_{\max}} ye^{-M(y)}dy$  represents the average knowledge the employee will own if hired at age  $z$  during its tenure within the firm.

So it makes sense to minimize  $d$ . Hence, people should only be hired at the optimal age ( $z_0$ ). This result is in line with the talent pipeline creation concept (see [4]): the optimal hiring policy is to hire a pool of candidates of the same experience (here linked to their age) qualified to assume newly created or vacated positions. More precisely, according to the cost and the attrition structures, three different cases are possible:

- The "build a talent pipeline" case: if the minimum is reached in  $z_0 \in (z_{\min}, z_{\max})$ , then it is optimal to build internally employees' careers starting from the age  $z_0$ . The firm is here doing long term investments in knowledge workers.
- The "focus on experts" case: if the minimum is reached in  $z_{\max}$ , then it is optimal to hire a pool of experts of maximum experience. However, those experts have to be newly hired each year, and this framework does not take into account the recruiting time and cost. So this solution may not be relevant.
- The "ant colony" case: if the minimum is reached in  $z_{\min}$ , then the firm counts on recruiting a high number of young employees, in order to train and keep them until retirement age.

**Examples.** We choose to apply the previous framework to three examples from three different BUs (BU 1, BU 2, BU 3), in order to cover all three cases considered for the search of the optimal hiring age. We always keep the attrition at  $\mu = 30\%$ . The BU 1 represents a BU of managers, the BU 2 represents a BU of professional workers and the third example represents a support function BU.

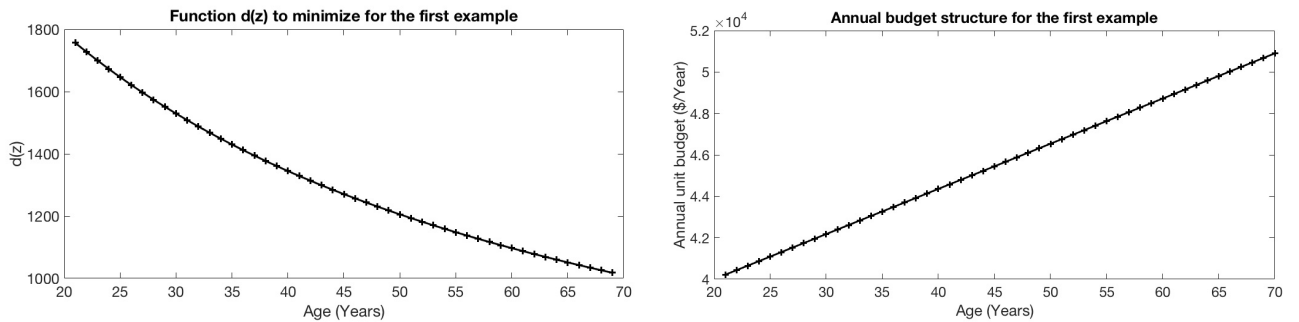


FIG. 9. Function to minimize  $d(z)$  and budget structure  $\omega(z)$  for the BU 1, for which  $E=3500$  years, and, without optimization, average age is 35 years and corresponding labor cost is around 5 million \$/year (for a total headcount of 100). The budget is linear, with positive coefficients.

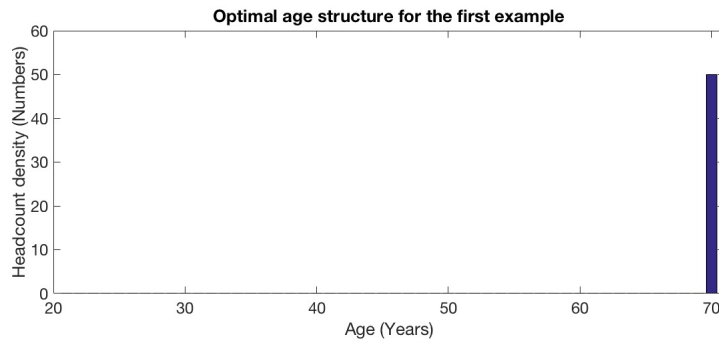


FIG. 10. Optimal age structure for the BU 3.

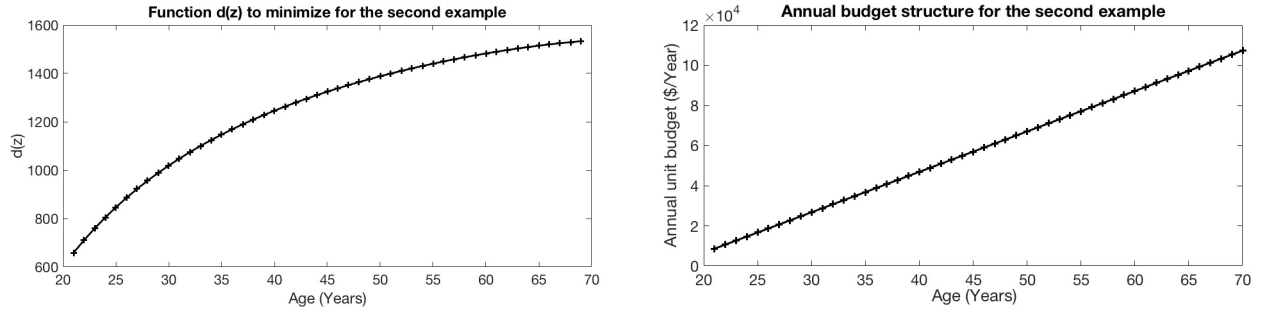


FIG. 11. Function to minimize  $d(z)$  and budget structure  $\omega(z)$  for the BU 2, for which  $E=3000$ , and, without optimization, average age is 30 and corresponding labor cost is around 2 million \$/year (for a total headcount of 100). The budget is linear.

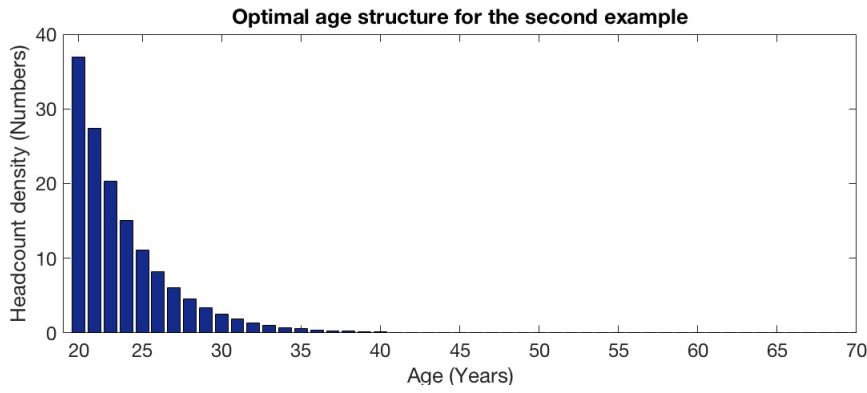


FIG. 12. Optimal age structure for the BU 2.

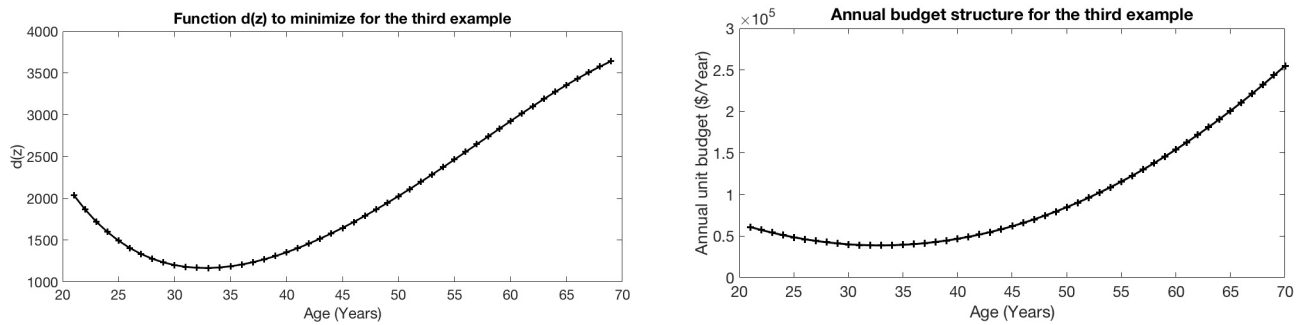


FIG. 13. Function to minimize  $d(z)$  and budget structure  $\omega(z)$  for the BU 3, for which  $E=3700$ , and, without optimization, average age is 37 and corresponding labor cost is around 6 million \$/year (for a total headcount of 100). The unit budget is a polynomial of degree 2.

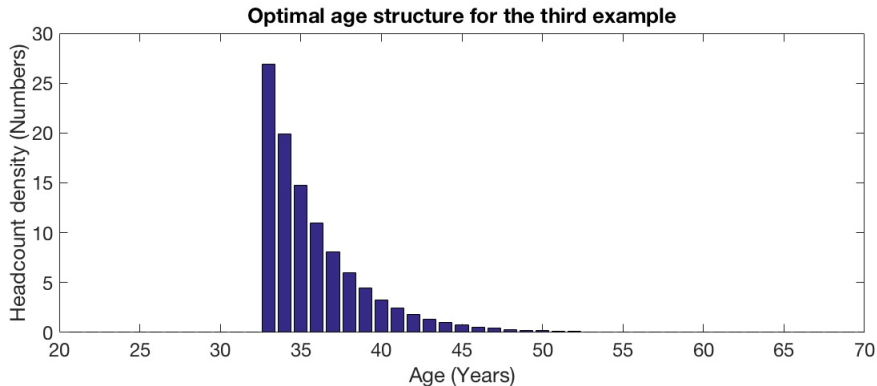


FIG. 14. *Optimal age structure for the BU 3.*

For the BU 1, we can see that the minimum is at the retirement age (see Figure 9), and we can deduce that the ideal age structure of Figure 10 is 50 people of age of retirement (“focus on experts” case). This is a typical scenario for a BU of managers, where many years of experience are usually required. The optimized labor cost is around 3 million \$/year, which represents a 2 million \$/year saving (around 40% of the total labor cost). However, as we said before, this framework does not take into account the recruiting cost and time to fill, which is not realistic. A suboptimal solution should therefore be in order.

For the BU 2, we can see that the minimum is at the minimum age 20 (see Figure 11), and we can deduce the ideal age structure of Figure 12 (“ant colony” case). This is a typical scenario for a BU of professionals, where the salary gap between the young and the old employees overtops the associated experience gap. The optimized labor cost is around 1.8 million \$/year, which represents a 0.2 million \$/year saving (around 10% of the total labor cost). We can see that people are hired at 20 years and they progressively leave the company as they age. The average age is around 25 (instead of 30 for the non optimized situation), and the total headcount is around 120 (instead of 100).

For the BU 3, we can see that the minimum is at the age 33 (see Figure 13), and we can deduce the ideal age structure of Figure 14 (“build a talent pipeline” case). Here, the most experienced employees are expensive and few, whereas the young ones are less expensive and available. Yet, it is risky to hire the youngest employees, so that a happy medium has to be found. The optimized labor cost is around 5 million \$/year, which represents a 1 million \$/year saving (around 15% of the total labor cost). We can see that people are hired at 33 years and they progressively leave the company as they age. The average age is around 37 (just as in the non optimized situation), and the total headcount is also around 100.

This minimization provides generic solutions to workforce design challenges under experience and cost constraints. The three cases scenario that arises from the study described below is in line with the talent pipeline concept, and should provide insight into the business needs assessment of a company.

## 4 Conclusion

The structured equations framework developed in this paper is a suitable first milestone to get preliminary answers to standard long term workforce concerns such as population stability or the mandatory adaptability of a company hiring policies. This framework can also be leveraged to provide generic solutions to workforce design challenges under experience and cost constraints. So far, we have studied two issues. First, assuming the age profile is known, we have considered hiring strategies able to

stabilize the employees population, either based on the total headcount or on a budget constraint. We have also studied the hiring profile in order to reach an optimal age profile at equilibrium under an experience constraint.

Several limits to the current paper arise. First, from a theoretical standpoint, the framework does not allow for time variation in the attrition nor hired population distribution. Second the framework does not account for more than two dimensions (age and time) and one population class, which does not account for the workforce evolution from one job to the other while staying in the same firm. From a practical perspective, the main shortcoming of the study is the lack of productivity function that has been replaced by constraints on experience. Therefore this paper should be considered as a preliminary study case for workforce planning.

A first natural next step would be to optimize the labor costs under population and experience constraints. This type of constraint would be suited to investigate cost-optimal demographic structure for non-knowledge workers. Their overall activity is first determined by workload constraint that is not demographic in nature (for example hours of works spent on a machine) which leads to a population size requirement. Experience would still be important because it represents a knowledge process that cannot be acquired prior to a certain experience threshold. This type of multiple constraints minimization is an extensively researched topic called the linear programming problem. This domain has been pioneered in the 60s [22], and followed by many studies, see e.g. [17, 18, 19]. As another next step, in a continuation of the present analysis, the notion of productivity and a study case on sales representatives could be investigated. The framework could then be expanded to a multi-population framework to better represent layers within a company.

## Appendix

### A Proof of convergence for the non-linear model (1)

We study the convergence for the first workforce evolution model (section 2). We recall the equation (1) under consideration

$$\begin{cases} \frac{\partial \rho}{\partial t}(t, z) + \frac{\partial \rho}{\partial z}(t, z) = -\mu(z)\rho(t, z) + \frac{P_t}{1+\alpha P_t^2}\gamma(z), & z_{\min} < z < z_{\max}, \\ \rho(t, z_{\min}) = 0, \\ \rho(0, z) = \rho^0(z) \geq 0, \end{cases}$$

with  $\alpha > 0$ . In this section, we assume that  $\gamma \in L^\infty((z_{\min}, z_{\max}), \mathbb{R}_+)$ ,  $\mu \in L^\infty((z_{\min}, z_{\max}), \mathbb{R}_+)$ ,  $\rho^0 \in L^1 \cap L^\infty((z_{\min}, z_{\max}), \mathbb{R}_+)$ . Classical arguments (see [16]) allow to prove that (1) admits a unique solution  $\rho \in C_b(\mathbb{R}_+, L^1(z_{\min}, z_{\max}))$ .

The aim of this section is to give the set of possible stationary states of equation (1) and study the asymptotic behavior of the solution. This asymptotic behavior strongly depends on the parameter  $\beta$  defined in (2)

$$\beta := \int_{z_{\min}}^{z_{\max}} \left( \int_{z_{\min}}^z \gamma(y) e^{-(M(z)-M(y))} dy \right) dz$$

which is essentially positively correlated with the mean coefficient of recruitment. We first show that, if  $\beta \leq 1$ , the only stationary state is zero and that, as soon as  $\beta > 1$ , that is, the function of recruitment is big enough compared to the attrition rate, there are two stationary states: zero and a positive non

trivial stationary state. We then show the convergence of the solution to zero when  $\beta < 1$ , and to the positive non trivial equilibrium state when  $\beta > 1$  and small enough, as soon as the initial repartition  $\rho^0(z) \neq 0$ . Let us mention that, the assumption  $\beta$  small enough, which is the case when the non linearity is not too strong, seems to be a technical constraint only. Numerically, we observe that the solution converges to the positive steady state even for a large  $\beta$ .

## A.1 Existence of steady states

**Proposition A.1** *If  $\beta \leq 1$ , the only possible equilibrium of the equation (1) is zero. If  $\beta > 1$ , the system (1) admits two equilibrium states: zero and a positive one.*

**Proof.** The stationary states,  $\rho_{\text{eq}}$ , of equation (1) are solution of the equation

$$\frac{d\rho_{\text{eq}}}{dz}(z) = -\mu(z)\rho_{\text{eq}}(z) + a^*\gamma(z),$$

where  $a^* = \frac{P_{\text{eq}}}{1+\alpha P_{\text{eq}}^2}$  and  $P_{\text{eq}} = \int_{z_{\min}}^{z_{\max}} \rho_{\text{eq}}(z)dz$ . Hence, we have necessarily

$$\left( \frac{d\rho_{\text{eq}}}{dz}(z) + \mu(z)\rho_{\text{eq}}(z) \right) e^{M(z)} = a^*\gamma(z)e^{M(z)},$$

so

$$\frac{d}{dz} \left( \rho_{\text{eq}}(z)e^{M(z)} \right) = a^*\gamma(z)e^{M(z)},$$

and thus

$$\rho_{\text{eq}}(z) = \int_{z_{\min}}^z a^*\gamma(y)e^{-(M(z)-M(y))}dy = \frac{P_{\text{eq}}}{1+\alpha P_{\text{eq}}^2} \int_{z_{\min}}^z \gamma(y)e^{-(M(z)-M(y))}dy.$$

Integrating the above equation, we find that  $P_{\text{eq}}$  must satisfy the equation

$$P_{\text{eq}} = \beta \frac{P_{\text{eq}}}{1+\alpha P_{\text{eq}}^2}. \quad (4)$$

Now, either  $\beta \leq 1$ , and the only possible solution of (4) is  $P_{\text{eq}} = 0$  and hence, the only possible stationary state of the equation (1) is zero. Either,  $\beta > 1$  and there are two solutions of the equation (4) given by

$$P_{\text{eq}} = 0 \text{ and } P_{\text{eq}} = \sqrt{\frac{\beta-1}{\alpha}},$$

and so, there are two different stationary states of the equation (1), which ends the proof of Prop. A.1. □

## A.2 Asymptotic behavior

**Proposition A.2 (case  $\beta < 1$ )** *If  $\beta < 1$ , the total population  $P(t)$  tends exponentially fast to 0 and*

$$\lim_{t \rightarrow \infty} \|\rho(t)\|_{L^\infty(z_{\min}, z_{\max})} = 0.$$

**Proof.** Using the characteristics, we can find a semi-explicit solution in accordance with  $\gamma$ ,  $\mu$  and  $P(s)$ , for  $s \in [t - (z_{\max} - z_{\min}), t]$  and for  $t \geq (z_{\max} - z_{\min})$ :

$$\rho(t, z) = e^{-M(z)} \int_{z_{\min}}^z a(t - z + \tau) \gamma(\tau) e^{M(\tau)} d\tau, \quad z \in [z_{\min}, z_{\max}], \quad a(s) = \frac{P(s)}{1 + \alpha P^2(s)},$$

and by integrating from  $z_{\min}$  to  $z_{\max}$ , we find

$$P(t) = \int_{z_{\min}}^{z_{\max}} e^{-M(z)} \int_{z_{\min}}^z a(t - z + \tau) \gamma(\tau) e^{M(\tau)} d\tau dz \leq \sup_{x \in \mathbb{R}^+} \left( \frac{x}{1 + \alpha x^2} \right) \beta = \frac{1}{2\sqrt{\alpha}} \beta.$$

Yet  $\alpha > 0$ , so  $a(t) = \frac{P(t)}{1 + \alpha P^2(t)} \leq P(t)$ , and

$$P(t) \leq \beta \sup_{s \in [t - (z_{\max} - z_{\min}), t]} P(s).$$

We can now prove that  $\lim_{t \rightarrow +\infty} P(t) = 0$ :  $\sup_{s \in [t - (z_{\max} - z_{\min}), t]} P(s)$  is reached in  $t_0 \in [t - (z_{\max} - z_{\min}), t]$  ( $P$  is continuous because  $\rho$  is uniformly bounded). We obtain that

$$P(t) \leq \beta P(t_0), \quad P(t_0) \leq \beta \sup_{s \in [t_0 - (z_{\max} - z_{\min}), t_0]} P(s).$$

Therefore, we obtain

$$P(t) \leq \beta^2 \sup_{s \in [t_0 - (z_{\max} - z_{\min}), t_0]} P(s),$$

which we write as

$$P(t) \leq \beta^2 \sup_{s \in [t - 2(z_{\max} - z_{\min}), t]} P(s).$$

We may iterate the above argument and deduce that the following estimate holds for  $t \geq n(z_{\max} - z_{\min})$

$$P(t) \leq \beta^n \sup_{s \in [t - n(z_{\max} - z_{\min}), t]} P(s),$$

which ends the proof of exponential convergence of  $P$  to 0 because  $\beta < 1$ . □

**Proposition A.3 (case  $\beta > 1$ )** *Assume that  $\rho^0(z)$  is not identically null and  $1 < \beta < 9$ . Then, the total population  $P(t)$  tends exponentially fast to  $P_{\text{eq}} > 0$  and, denoting  $\rho_{\text{eq}}$  the positive equilibrium state, we have:*

$$\lim_{t \rightarrow \infty} \|\rho(t) - \rho_{\text{eq}}\|_{L^\infty(z_{\min}, z_{\max})} = 0.$$

**Proof.** For  $t \geq z_{\max} - z_{\min}$ , we have

$$P(t) = \int_{z_{\min}}^{z_{\max}} e^{-M(z)} \int_{z_{\min}}^z a(t - z + \tau) \gamma(\tau) e^{M(\tau)} d\tau dz \leq \sup_{x \in \mathbb{R}^+} \left( \frac{x}{1 + \alpha x^2} \right) \beta = \frac{1}{2\sqrt{\alpha}} \beta,$$

and  $\beta = 1 + \alpha P_{\text{eq}}^2$  so, for  $t \geq z_{\max} - z_{\min}$

$$|P(t) - P_{\text{eq}}| \leq \sup_{s \in [t - (z_{\max} - z_{\min}), t]} |a(s) - a^*| (1 + \alpha P_{\text{eq}}^2),$$



which leads us to

$$|P(t) - P_{\text{eq}}| \leq \sup_{s \in [t - (z_{\max} - z_{\min}), t]} \left| \frac{P(s)(1 + \alpha P_{\text{eq}}^2) - P_{\text{eq}}(1 + \alpha P^2(s))}{1 + \alpha P^2(s)} \right|,$$

and

$$|P(t) - P_{\text{eq}}| \leq \sup_{s \in [t - (z_{\max} - z_{\min}), t]} |P(s) - P_{\text{eq}}| \left| \frac{(1 - \alpha P P_{\text{eq}})}{1 + \alpha P^2(s)} \right|.$$

Denoting  $c = \sqrt{\beta - 1}$ , we have

$$P_{\text{eq}} = \frac{c}{\sqrt{\alpha}},$$

which, with  $f_c(x) = \frac{1 - cx}{1 + x^2}$ , leads us to

$$|P(t) - P_{\text{eq}}| \leq \sup_{s \in [t - (z_{\max} - z_{\min}), t]} |P(s) - P_{\text{eq}}| \sup_{s \in [t - (z_{\max} - z_{\min}), t]} |f_c(\sqrt{\alpha}P(s))|.$$

Using Lemma A.4 below, since  $\beta < 9$ , there exists  $C_1 < 1$  such that, for all  $t \geq z_{\max}$ , the following estimate holds

$$\sup_{s \in [t - (z_{\max} - z_{\min}), t]} |f_c(\sqrt{\alpha}P(s))| \leq C_1 < 1,$$

so that we can conclude as for Prop. A.2 that  $P$  converges to  $P_{\text{eq}}$  exponentially fast. Then, we conclude

$$|\rho(t, z) - \rho_{\text{eq}}(z)| \leq \sup_{s \in [t - (z_{\max} - z_{\min}), t]} |a(s) - a^*| e^{-M(z)} \int_{z_0}^z \gamma(\tau) e^{M(\tau)} d\tau,$$

which ends the proof of convergence.  $\square$

**Lemma A.4** *Under assumptions of Prop. A.3, we have*

$$\sup_{s \geq z_{\max} - z_{\min}} |f_c(\sqrt{\alpha}P(s))| < 1.$$

**Proof.** With the assumptions of Prop. A.3, we have  $c < 2\sqrt{2}$ , and so  $f_c(x) < 1$  for all  $x > 0$ . Hence, to prove Lemma A.4, it is enough to show that there exists  $\bar{P} > 0$  such that

$$P(s) \geq \bar{P} \text{ for all } s \geq z_{\max} - z_{\min}. \quad (5)$$

To prove (5), we first show that for any  $t \geq 0$ ,  $P(t) > 0$  and then conclude to estimate (5).

To prove that  $P(t) > 0$  for all  $t \geq 0$ , we first use the semi-explicit formula on  $P$  for  $t \geq z_{\max} - z_{\min}$ , given by

$$P(t) = \int_{z_{\min}}^{z_{\max}} e^{-M(z)} \int_{z_{\min}}^z a(t - z + \tau) \gamma(\tau) e^{M(\tau)} d\tau dz.$$

If we suppose that  $P(t) > 0$  for  $t \leq z_{\max} - z_{\min}$ , then we can deduce that  $P(t) > 0$  for all  $t$ : if it was not the case, we call  $t_0$  the first time for which  $P(t_0) = 0$  (defined because  $P$  is continuous), then

$$P(t_0) = \int_{z_{\min}}^{z_{\max}} e^{-M(z)} \int_{z_{\min}}^z a(t_0 - z + \tau) \gamma(\tau) e^{M(\tau)} d\tau dz.$$

This means that  $a(t_0 - z + \tau) > 0$  everywhere except in  $\tau = z$ , which would bring us to  $P(t_0) > 0$ , in direct contradiction with  $P(t_0) = 0$ . So it remains to show that  $P(t) > 0$  for  $t \leq z_{\max} - z_{\min}$ . For  $t \leq z_{\max} - z_{\min}$ , we have

$$P(t) = \int_{z_{\min}}^{z_{\min}+t} \rho(t, z) dz + \int_{z_{\min}+t}^{z_{\max}} \rho(t, z) dz.$$

Yet

$$\rho(t, z) = e^{-M(t)} \rho(0, z - t) + e^{-M(t)} \int_0^t e^{M(s)} a(s) \gamma(z - t + s) ds \text{ if } t \leq z - z_{\min},$$

and

$$\rho(t, z) = \int_{z_{\min}}^z e^{-M(z)+M(\tau)} \gamma(\tau) a(t - z + \tau) d\tau \text{ if } t \geq z - z_{\min}.$$

By the same reasoning we obtain  $P(t) > 0$  because  $P(0) > 0$ . So we always have  $P(t) > 0$ .

Let us now prove that there exists  $\bar{P} > 0$  such that for  $t \geq z_{\max} - z_{\min}$ ,

$$P(t) \geq \bar{P} > 0.$$

We know that for  $t \geq z_{\max} - z_{\min}$ , and with  $a(x) = \frac{x}{1+\alpha x^2}$

$$P(t) \geq \beta \min_{s \in [t - (z_{\max} - z_{\min}), t]} a(P(s)). \quad (6)$$

Since  $\beta > 1$ , we can find  $\bar{P} > 0$  and  $\varepsilon > 0$  with the following properties:

$$\left\{ \begin{array}{l} \beta a(s) \geq s(1 + \varepsilon), \quad \forall s \leq \bar{P}, \\ a \text{ is strictly increasing on } (0, \bar{P}), \\ \beta \inf_{s \in (\bar{P}, \frac{1}{2\sqrt{\alpha}}\beta)} a(s) \geq \bar{P}(1 + \varepsilon). \end{array} \right. \quad (7)$$

Now, if  $P(t) \geq \bar{P}$  for  $t \geq z_{\max} - z_{\min}$ , our minoration is proved. Else, there exists  $t_0 \geq z_{\max} - z_{\min}$  a time for which  $P(t_0) \leq \bar{P}$ . In this case, we denote

$$\bar{P} \geq P_{\inf}(t_0) := \min_{s \in [t_0 - (z_{\max} - z_{\min}), t_0]} P(s) > 0$$

and let us first prove that

$$P(t_0) \geq P_{\inf}(t_0)(1 + \varepsilon). \quad (8)$$

To this, we write

$$\{s \in [t_0 - (z_{\max} - z_{\min}), t_0]\} = A \cup A^c$$

with

$$A = \{s \in [t_0 - (z_{\max} - z_{\min}), t_0] \text{ such that } P(s) \geq \bar{P}\}$$

and where  ${}^c A$  is the complement of  $A$ . Using (6), we obtain

$$P(t_0) \geq \min(\min_{s \in A} \beta a(P(s)), \min_{s \in {}^c A} \beta a(P(s))).$$

Using the first part of (7), we deduce that

$$\beta \min_{s \in {}^c A} a(P(s)) \geq P_{\inf}(t_0)(1 + \varepsilon),$$

and using the second part of (7), we obtain that

$$\beta \min_{s \in A} a(P(s)) \geq \bar{P}(1 + \varepsilon) \geq P_{\inf}(t_0)(1 + \varepsilon)$$

and so we obtain estimate (8). Let us now prove that, for all  $t \geq t_0$ , the following estimate holds

$$P(t) > P_{\inf}(t_0).$$

Indeed, if it was not the case, since

$$P(t_0) > P_{\inf}(t_0),$$

we would find  $t_1 > t_0$  for which

$$P(t_1) = P_{\inf}(t_0) \text{ and } \min_{s \in [t_1 - (z_{\max} - z_{\min}), t_1]} P(s) \geq P_{\inf}(t_0).$$

Combining again (6) and (7), we would obtain

$$P(t_1) \geq P_{\inf}(t_0)(1 + \varepsilon),$$

which is in contradiction with  $P(t_1) = P_{\inf}(t_0)$ . So our minoration is proved.  $\square$

### A.3 Numerical method

To simulate the workforce evolution  $\rho(t, z)$  according to the framework (1), we discretize time and age. We recall that the initial structure  $\rho^0(z)$ , the attrition  $\mu(z)$  and the hired population distribution  $\gamma(z)$  are given. This numerical method is applied to the examples of subsection 2.2.

We discretize time :  $t = 0 : \delta t : T$  and age :  $z = z_{\min} : \delta z : z_{\max}$ , and we denote by  $z_j = z_{\min} + j\delta z$ ,  $t_k = k\delta t$  and  $\rho(t_k, z_j) = \rho_j^k$ ,  $\mu(z_j) = \mu_j$ ,  $\gamma(z_j) = \gamma_j$  (with  $\rho_0^k = 0$  and  $\rho_j^0$  given). For this simulation, we choose a semi-implicit scheme because there are less restrictions with  $\delta t$  and  $\delta z$ . According to the Courant-Friedrichs-Levi condition (see [5, 13]), we take  $\frac{\delta t}{\delta z} \leq 1$  in order to avoid oscillations. For  $j \geq 1$  and  $k \geq 0$ , for a first order scheme, we have

$$\overbrace{\frac{\rho_j^{k+1} - \rho_j^k}{\delta t} + \frac{\rho_j^k - \rho_{j-1}^k}{\delta z}}^{\text{Workforce evolution}} = - \overbrace{\mu_j \rho_j^{k+1}}^{\text{Attrition}} + \overbrace{\frac{P_k}{1 + \alpha P_k^2} \gamma_j}_{\text{Hiring}},$$

where  $P_k = \sum_j \rho_j^k$ . Then, from time  $t_k$ , we can find the number of employees of age  $z_j$  at time  $t_{k+1}$

$$\rho_j^{k+1} = \frac{1}{1 + \mu_j \delta t} \left( \rho_j^k + \delta t \left( \frac{P_k}{1 + \alpha P_k^2} \gamma_j - \frac{\rho_j^k - \rho_{j-1}^k}{\delta z} \right) \right).$$

## B Proof of convergence for the linear model (3)

We study the convergence for the second workforce evolution model (subsection 3.1). We recall the equation (3) under study

$$\begin{cases} \frac{\partial \rho}{\partial t}(t, z) + \frac{\partial \rho}{\partial z}(t, z) = -\mu(z)\rho(t, z) + h([\rho])\gamma(z), & z_{\min} < z < z_{\max}, \\ \rho(t, z_{\min}) = 0, \\ \rho(0, z) = \rho^0(z) \geq 0, \end{cases}$$

with

$$h([\rho]) = \frac{\int_{z_{\min}}^{z_{\max}} \omega(z)\mu(z)\rho(t, z)dz + \omega(z_{\max})\rho(t, z_{\max}) - \int_{z_{\min}}^{z_{\max}} \rho(t, z)\frac{\partial\omega}{\partial z}(z)dz}{\int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z)dz}.$$

In this section, we assume that  $\gamma \in L^\infty((z_{\min}, z_{\max}), \mathbb{R}_+)$ ,  $\mu \in L^\infty((z_{\min}, z_{\max}), \mathbb{R}_+)$ ,  $\rho^0 \in C((z_{\min}, z_{\max}), \mathbb{R}_+)$ ,  $\omega \in C_1([z_{\min}, z_{\max}], \mathbb{R}_+^*)$  and  $\mu\omega \geq \omega'$ . Classical arguments (see [16]) allow to prove that (3) admits a unique solution  $\rho \in C_b(\mathbb{R}_+, L^1(z_{\min}, z_{\max}))$ .

The aim of this section is to give the set of possible stationary states of the equation (3) and study the asymptotic behavior of the solution.

We first show that the steady states are proportional to the positive state  $\int_{z_{\min}}^z \gamma(y)e^{-(M(z)-M(y))}dy$ . We then show non null convergence under certain assumptions, with the entropy method (see [15, 16]).

## B.1 Existence of steady states

**Proposition B.1** *The steady states are proportional to the positive state  $\int_{z_{\min}}^z \gamma(y)e^{-(M(z)-M(y))}dy$ .*

**Proof.** The equilibrium equation is

$$\frac{d\rho_{\text{eq}}}{dz}(z) = -\mu(z)\rho_{\text{eq}}(z) + h^*\gamma(z), \quad (9)$$

with

$$h^* = \frac{\int_{z_{\min}}^{z_{\max}} \omega(z)\mu(z)\rho_{\text{eq}}(z)dz + \omega(z_{\max})\rho_{\text{eq}}(z_{\max}) - \int_{z_{\min}}^{z_{\max}} \rho_{\text{eq}}(z)\frac{\partial\omega}{\partial z}(z)dz}{\int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z)dz}.$$

So we obtain

$$\rho_{\text{eq}}(z) = \int_{z_{\min}}^z h^*\gamma(y)e^{-(M(z)-M(y))}dy,$$

and injecting this in the first expression of  $h^*$ , we find the condition:

$$\begin{aligned} & h^* \int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z)dz \\ &= h^* \int_{z_{\min}}^{z_{\max}} \left( (\omega(z)\mu(z) - \omega'(z)) \left( \int_{z_{\min}}^z \gamma(y)e^{-(M(z)-M(y))}dy \right) + \omega(z_{\max})\gamma(z)e^{-(M(z_{\max})-M(z))} \right) dz. \end{aligned}$$

Yet, with an appropriate integration by parts, we find

$$\begin{aligned} & \int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z)dz \\ &= \int_{z_{\min}}^{z_{\max}} \left( (\omega(z)\mu(z) - \omega'(z)) \left( \int_{z_{\min}}^z \gamma(y)e^{-(M(z)-M(y))}dy \right) + \omega(z_{\max})\gamma(z)e^{-(M(z_{\max})-M(z))} \right) dz, \end{aligned}$$

so that  $\rho_{\text{eq}}$  is solution of (9). □

## B.2 Asymptotic behavior

In order to prove that the hiring strategy under consideration converges, we introduce some notations. We rewrite the equation as:

$$\frac{\partial\rho}{\partial t}(t, z) + \frac{\partial\rho}{\partial z}(t, z) = -\mu(z)\rho(t, z) + A\gamma(z)\rho(t, z_{\max}) + \gamma(z) \int_{z_{\min}}^{z_{\max}} B(y)\rho(t, y)dy,$$

with

$$A = \frac{\omega(z_{\max})}{\int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z)dz} \geq 0, \quad B(y) = \frac{\mu(y)\omega(y) - \frac{\partial\omega}{\partial y}(y)}{\int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z)dz} \geq 0.$$

Now we denote  $\rho(t, z)$  a solution of system (3), and  $\rho_{\text{eq}}$  a positive solution of the equilibrium equation.

**Proposition B.2 (General Relative Entropy Inequality)** *For all convex function  $H$ , we show that*

$$\frac{d}{dt} \left( \int_{z_{\min}}^{z_{\max}} \omega(z)\rho_{\text{eq}}(z)H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) dz \right) = -D_1^H(t) - D_2^H(t) \leq 0, \quad (10)$$

where  $D_1^H(t) = \int_{z_{\min}}^{z_{\max}} \int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z)B(y)\rho_{\text{eq}}(y) \left\{ H\left(\frac{\rho(t, y)}{\rho_{\text{eq}}(y)}\right) - H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) - H'\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \left(\frac{\rho(t, y)}{\rho_{\text{eq}}(y)} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \right\} dy dz$ ,

and  $D_2^H(t) = A\rho_{\text{eq}}(z_{\max}) \int_{z_{\min}}^{z_{\max}} \omega(z)\gamma(z) \left\{ H\left(\frac{\rho(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})}\right) - H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) - H'\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \left(\frac{\rho(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \right\} dz$ .

**Proof.** An immediate calculation gives

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right) + \frac{\partial}{\partial z} \left( \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right) \\ &= \int_{z_{\min}}^{z_{\max}} \gamma(z)B(y) \frac{\rho_{\text{eq}}(y)}{\rho_{\text{eq}}(z)} \left( \frac{\rho(t, y)}{\rho_{\text{eq}}(y)} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right) dy + A\gamma(z) \frac{\rho_{\text{eq}}(z_{\max})}{\rho_{\text{eq}}(z)} \left( \frac{\rho(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right), \end{aligned}$$

therefore

$$\begin{aligned} & \frac{\partial}{\partial t} \left( H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \right) + \frac{\partial}{\partial z} \left( H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \right) = \\ & H'\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \left( \int_{z_{\min}}^{z_{\max}} \gamma(z)B(y) \frac{\rho_{\text{eq}}(y)}{\rho_{\text{eq}}(z)} \left( \frac{\rho(t, y)}{\rho_{\text{eq}}(y)} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right) dy + A\gamma(z) \frac{\rho_{\text{eq}}(z_{\max})}{\rho_{\text{eq}}(z)} \left( \frac{\rho(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right) \right). \end{aligned}$$

We also have

$$\begin{aligned} & \frac{\partial}{\partial t} (\omega(z)\rho_{\text{eq}}(z)) + \frac{\partial}{\partial z} (\omega(z)\rho_{\text{eq}}(z)) \\ &= A\omega(z)\rho_{\text{eq}}(z_{\max}) + \int_{z_{\min}}^{z_{\max}} (B(y)\rho_{\text{eq}}(y)\gamma(z)\omega(z) - B(z)\rho_{\text{eq}}(z)\gamma(y)\omega(y)) dy, \end{aligned}$$

then

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \omega(z)\rho_{\text{eq}}(z)H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \right) + \frac{\partial}{\partial z} \left( \omega(z)\rho_{\text{eq}}(z)H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \right) \\ &= H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \left( \int_{z_{\min}}^{z_{\max}} (B(y)\rho_{\text{eq}}(y)\gamma(z)\omega(z) - B(z)\rho_{\text{eq}}(z)\gamma(y)\omega(y)) dy \right) \\ &+ \omega(z)\gamma(z)H'\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \left( \int_{z_{\min}}^{z_{\max}} B(y) \frac{\rho_{\text{eq}}(y)}{\rho_{\text{eq}}(z)} \left( \frac{\rho(t, y)}{\rho_{\text{eq}}(y)} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right) dy \right) \\ &+ H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) A\gamma(z)\omega(z)\rho_{\text{eq}}(z_{\max}) + H'\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \gamma(z)\omega(z)A\rho_{\text{eq}}(z_{\max}) \left( \frac{\rho(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right). \end{aligned}$$

We now integrate in  $z$  so we obtain

$$\begin{aligned} & \frac{d}{dt} \left( \int_{z_{\min}}^{z_{\max}} \omega(z)\rho_{\text{eq}}(z)H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) dz \right) + A\rho_{\text{eq}}(z_{\max})H\left(\frac{\rho(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})}\right) \int_{z_{\min}}^{z_{\max}} \gamma(z)\omega(z)dz = -D_1^H(t) + \\ & \int_{z_{\min}}^{z_{\max}} H\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) \gamma(z)\omega(z)A\rho_{\text{eq}}(z_{\max})dz + \int_{z_{\min}}^{z_{\max}} H'\left(\frac{\rho(t, z)}{\rho_{\text{eq}}(z)}\right) A\rho_{\text{eq}}(z_{\max})\omega(z)\gamma(z) \left( \frac{\rho(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})} - \frac{\rho(t, z)}{\rho_{\text{eq}}(z)} \right) dz, \end{aligned}$$

and we obtain the result.  $\square$

**Lemma B.3** *With initial data satisfying  $|\rho^0(z)| \leq C_0\rho_{\text{eq}}(z)$  and  $\frac{\partial}{\partial z}\rho^0(z) \in L^1(\omega(z)dz)$ , we have*

$$\int_{z_{\min}}^{z_{\max}} \left| \frac{\partial}{\partial z} \rho(t, z) \right| \omega(z) dz \leq C(\rho^0), \quad \int_{z_{\min}}^{z_{\max}} \left| \frac{\partial}{\partial t} \rho(t, z) \right| \omega(z) dz \leq C(\rho^0).$$

**Proof.** *First step. Time derivative.* First, from the entropy equation, we have the contraction principle:

$$C_{-\rho_{\text{eq}}}(z) \leq \rho^0(z) \leq C_{+\rho_{\text{eq}}}(z) \Rightarrow C_{-\rho_{\text{eq}}}(z) \leq \rho(t, z) \leq C_{+\rho_{\text{eq}}}(z).$$

Yet,  $q(t, z) = \frac{\partial}{\partial t} \rho(t, z)$  is solution of the first equation of system (3). So, applying the entropy equation to  $q$  with  $H(x) = |x|$ ,

$$\int_{z_{\min}}^{z_{\max}} |q(t, z)| \omega(z) dz \leq \int_{z_{\min}}^{z_{\max}} |q(t=0, z)| \omega(z) dz,$$

but

$$q(t=0, z) = -\frac{\partial}{\partial z} \rho^0(z) - \mu(z)\rho^0(z) + A\gamma(z)\rho^0(z_{\max}) + \gamma(z) \int_{z_{\min}}^{z_{\max}} B(y)\rho^0(y) dy.$$

We may bound  $|\rho^0|$  by  $C_0\rho_{\text{eq}}$ , replace  $A\gamma(z)\rho_{\text{eq}}(z_{\max}) + \gamma(z) \int_{z_{\min}}^{z_{\max}} B(y)\rho_{\text{eq}}(y) dy$  by the other terms of the equation on  $\rho_{\text{eq}}$  and we arrive at

$$\int_{z_{\min}}^{z_{\max}} |q(t=0, z)| \omega(z) dz \leq \int_{z_{\min}}^{z_{\max}} \left( \left| \frac{\partial}{\partial z} \rho^0(z) \right| + C_0 \left| \frac{\partial}{\partial z} \rho_{\text{eq}}(z) \right| \right) dz + 2C_0 \int_{z_{\min}}^{z_{\max}} \rho_{\text{eq}}(z) \mu(z) \omega(z) dz.$$

*Second step. Space derivative.* We have

$$\frac{\partial \rho}{\partial z}(t, z) = -\frac{\partial \rho}{\partial t}(t, z) - \mu(z)\rho(t, z) + A\gamma(z)\rho(t, z_{\max}) + \gamma(z) \int_{z_{\min}}^{z_{\max}} B(y)\rho(t, y) dy.$$

The control of  $\frac{\partial \rho}{\partial t}(t, z)$  in the first step and  $\rho(t, z) \leq C_0\rho_{\text{eq}}(z)$  gives us a control similar to that on the time derivative.  $\square$

**Proposition B.4** *Under assumptions of Lemma. B.3, when  $t \rightarrow +\infty$  we have*

$$\int_{z_{\min}}^{z_{\max}} |\rho(t, z) - m\rho_{\text{eq}}(z)| \omega(z) dz \rightarrow 0, \text{ with } m = \frac{\int_{z_{\min}}^{z_{\max}} \rho^0(z) \omega(z) dz}{\int_{z_{\min}}^{z_{\max}} \rho_{\text{eq}}(z) \omega(z) dz}.$$

**Proof.** First we set

$$n(t, z) = \rho(t, z) - m\rho_{\text{eq}}(z).$$

We notice that  $n$  is solution of (3), so, using (10) with  $H(x) = |x|$ , we obtain

$$\int_{z_{\min}}^{z_{\max}} |\rho(t, z) - m\rho_{\text{eq}}(z)| \omega(z) dz \rightarrow L,$$

and it remains to show that  $L = 0$ .

Yet, we have  $|n| \leq C_0\rho_{\text{eq}}$ ,  $\int_{z_{\min}}^{z_{\max}} \left| \frac{\partial}{\partial t} n(t, z) \right| \omega(z) dz \leq C(\rho^0(z))$  and  $\int_{z_{\min}}^{z_{\max}} \left| \frac{\partial}{\partial z} n(t, z) \right| \omega(z) dz \leq C(\rho^0(z))$ .

We then introduce the sequence of functions  $n_k(t, \cdot) = n(t + t_k, \cdot)$ . After extracting a subsequence,

still denoted  $n_k$ , we have  $n_k \rightarrow g$  strongly in  $L^1([0, T] * [z_{\min}, z_{\max}])$  for all  $T > 0$  because of the global regularity of  $n$ . And we have that  $g$  is solution of (3) and

$$|g(t, z)| \leq C_0 \rho_{\text{eq}}(z).$$

We can now work on the entropy dissipation of  $n$ , using (10) with  $H(x) = x^2$ , we obtain

$$\frac{d}{dt} \left( \int_{z_{\min}}^{z_{\max}} \omega(z) \rho_{\text{eq}}(z) \left( \frac{n(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dz \right) = -D_1^H(t) - D_2^H(t) \leq 0,$$

so  $\int_{z_{\min}}^{z_{\max}} \omega(z) \rho_{\text{eq}}(z) \left( \frac{n(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dz$  is decreasing and yet positive, so has a limit and is bounded, and so is

$$\left| \int_0^\infty \frac{d}{dt} \left( \int_{z_{\min}}^{z_{\max}} \omega(z) \rho_{\text{eq}}(z) \left( \frac{n(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dz \right) dt \right| \leq C,$$

and thus

$$\left| \int_0^\infty (D_1^H(t) + D_2^H(t)) dt \right| \leq C,$$

which brings us to

$$\int_0^\infty \int_{z_{\min}}^{z_{\max}} \int_{z_{\min}}^{z_{\max}} \omega(z) \gamma(z) B(y) \rho_{\text{eq}}(y) \left( \frac{n(t, y)}{\rho_{\text{eq}}(y)} - \frac{n(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dy dz dt \leq C,$$

and

$$A \rho_{\text{eq}}(z_{\max}) \int_0^\infty \int_{z_{\min}}^{z_{\max}} \omega(z) \gamma(z) \left( \frac{n(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})} - \frac{n(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dz dt \leq C.$$

Therefore, as  $k \rightarrow \infty$ :

$$\begin{aligned} & \int_0^\infty \int_{z_{\min}}^{z_{\max}} \int_{z_{\min}}^{z_{\max}} \omega(z) \gamma(z) B(y) \rho_{\text{eq}}(y) \left( \frac{n_k(t, y)}{\rho_{\text{eq}}(y)} - \frac{n_k(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dy dz dt \\ &= \int_k^\infty \int_{z_{\min}}^{z_{\max}} \int_{z_{\min}}^{z_{\max}} \omega(z) \gamma(z) B(y) \rho_{\text{eq}}(y) \left( \frac{n(t, y)}{\rho_{\text{eq}}(y)} - \frac{n(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dy dz dt \rightarrow 0. \end{aligned}$$

By the strong limit of  $n_k$ , we arrive at

$$\int_0^\infty \int_{z_{\min}}^{z_{\max}} \int_{z_{\min}}^{z_{\max}} \omega(z) \gamma(z) B(y) \rho_{\text{eq}}(y) \left( \frac{g(t, y)}{\rho_{\text{eq}}(y)} - \frac{g(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dy dz dt = 0,$$

and by the same reasoning,

$$A \rho_{\text{eq}}(z_{\max}) \int_0^\infty \int_{z_{\min}}^{z_{\max}} \omega(z) \gamma(z) \left( \frac{g(t, z_{\max})}{\rho_{\text{eq}}(z_{\max})} - \frac{g(t, z)}{\rho_{\text{eq}}(z)} \right)^2 dz dt = 0,$$

which brings us to

$$\frac{g(t, z)}{\rho_{\text{eq}}(z)} = c(t).$$

Yet  $\int_{z_{\min}}^{z_{\max}} \omega(z) g(t, z) dz = 0$ , and  $\int_{z_{\min}}^{z_{\max}} \omega(z) \rho_{\text{eq}}(z) dz > 0$ , so  $c(t) = 0$ , and thus  $g = 0$ . We can conclude that  $L = 0$ .  $\square$

### B.3 Numerical method

We take the same notations as in A.3. This numerical method is applied to the examples of subsection 3.1. In order to simplify some calculations, we choose to determine the concentration of workers of age  $z$  at time  $t$ ,  $\rho(t, z)$  with an explicit scheme, because, this way, the equation is kept conservative at a discrete level. Additionally, the expression of the hiring rate  $h([\rho])$  does not depend on the time step  $\delta t$ . However, the Courant-Friedrichs-Levi condition (see [5, 13]) is more restrictive:  $1 - \max_j(\mu_j)\delta t - \frac{\delta t}{\delta z} \geq 0$ . The first order discretization is

$$\overbrace{\frac{\rho_j^{k+1} - \rho_j^k}{\delta t} + \frac{\rho_j^k - \rho_{j-1}^k}{\delta z}}^{\text{Workforce evolution}} = - \overbrace{\mu_j \rho_j^k}^{\text{Attrition}} + \overbrace{h_k \gamma_j}_{\text{Hiring}},$$

or, reorganizing the terms,

$$\rho_j^{k+1} = \rho_j^k (1 - \mu_j \delta t) + \delta t \left( h_k \gamma_j - \frac{\rho_j^k - \rho_{j-1}^k}{\delta z} \right).$$

Since we know that

$$\sum_{j=1}^J \omega_j \rho_j^k = \sum_{j=1}^J \omega_j \rho_j^{k+1},$$

we can write

$$\omega_j \frac{\rho_j^{k+1} - \rho_j^k}{\delta t} + \omega_j \frac{\rho_j^k - \rho_{j-1}^k}{\delta z} = -\omega_j \mu_j \rho_j^k + \omega_j h_k \gamma_j,$$

therefore, by summing from  $j = 1$  to  $J$ , one immediately gets

$$h_k = \frac{\sum_j \left( \frac{\omega_j (\rho_j^k - \rho_{j-1}^k)}{\delta z} + \mu_j \rho_j^k \omega_j \right)}{\sum_j \gamma_j \omega_j}.$$

## C Labor costs minimization

We study here the problem of cost minimization (subsection 3.2). More precisely, we minimize  $C = \int_{z_{\min}}^{z_{\max}} \rho^*(z) w(z) dz$  with a given knowledge  $E = \int_{z_{\min}}^{z_{\max}} \rho^*(z) z dz$ . Here we assume that  $\mu$  and  $\omega$  are smooth enough.

The solution to this minimization problem must also take into account the fact that the workforce structure  $\rho^*$  is driven by this framework:

$$\frac{d\rho^*}{dz}(z) = -\mu(z)\rho^*(z) + \gamma(z).$$

We impose that the hiring profile  $\gamma$  remains nonnegative, that is  $\frac{d(\rho^* e^M)}{dz} = \gamma e^M \geq 0$ . Consequently, this minimization problem has to take into account those two constraints: the knowledge constraint  $E = \int_{z_{\min}}^{z_{\max}} \rho^*(z) z dz$ , and the non-firing structure constraint  $\frac{d(\rho^* e^M)}{dz}(z) \geq 0$ .



**Proposition C.1** *The solution of this minimization problem is*

$$\rho^*(z) = e^{-M(z)} b \mathbf{1}_{z \geq z_0}, \quad C = Ed(z_0),$$

with

$$b = \frac{E}{\int_{z_0}^{z_{\max}} z e^{-M(z)}}, \quad d(z_0) = \min_z (d(z)).$$

**Proof.** We write

$$C = \int_{z_{\min}}^{z_{\max}} w(z) \rho^*(z) dz = \int_{z_{\min}}^{z_{\max}} w(z) e^{-M(z)} \rho^*(z) e^{M(z)} dz.$$

We denote  $Q(z) = \frac{d(\rho^* e^M)}{dz}(z)$ , then, integrating by parts

$$C = \int_{z_{\min}}^{z_{\max}} \left( \int_z^{z_{\max}} w(u) e^{-M(u)} du \right) Q(z) dz + \left[ \left( \int_{z_{\max}}^z w(u) e^{-M(u)} du \right) \rho^*(z) e^{M(z)} \right]_{z=z_{\min}}^{z=z_{\max}},$$

the last term vanishes thanks to the boundary condition  $\rho^*(z_{\min}) = 0$ . Therefore, we obtain

$$C = \int_{z_{\min}}^{z_{\max}} \left( \int_z^{z_{\max}} w(y) e^{-M(y)} dy \right) Q(z) dz = \int_{z_{\min}}^{z_{\max}} f(z) Q(z) dz,$$

and in the same way, we find

$$E = \int_{z_{\min}}^{z_{\max}} \left( \int_z^{z_{\max}} y e^{-M(y)} dy \right) Q(z) dz = \int_{z_{\min}}^{z_{\max}} g(z) Q(z) dz,$$

with  $f(z) = \int_z^{z_{\max}} w(y) e^{-M(y)} dy$  and  $g(z) = \int_z^{z_{\max}} y e^{-M(y)} dy$ . Consequently we obtain:

$$C = \int_{z_{\min}}^{z_{\max}} \frac{f(z)}{g(z)} g(z) Q(z) dz \geq E \min_z (d(z)),$$

where

$$d(z) = \frac{f(z)}{g(z)}.$$

By continuity, this minimum is reached at least on  $z_0$ ; the expression of  $Q$  is then

$$Q(z) = b \delta_{z_0},$$

where  $b \geq 0$  is a positive constant. Finally we obtain

$$Q(z) = b \delta_{z_0} = \frac{d(\rho^* e^M)}{dz}(z), \quad \rho^*(z) = e^{-M(z)} b \mathbf{1}_{z \geq z_0},$$

and the knowledge constraint gives the announced formula for  $b$ .

This gives us the ideal age structure  $\rho^*$  at the equilibrium state, and then we can deduce the hiring rate and profile

$$\gamma^*(z) = \rho^{*'} + \mu \rho^* = b \delta_{z_0} e^{-M(z)}.$$

□

### Acknowledgments.

Marie Doumic was supported by the ERC Starting Grant SKIPPER<sup>AD</sup> (number 306321). Benoit Perthame and Delphine Salort were supported by the french "ANR blanche" project Kibord: ANR-13-BS01-0004 funded by the French Ministry of Research.

## References

- [1] Global talent planning and acquisition solutions. CEB, 2016.
- [2] E G Anderson. Managing the impact of high market growth and learning on knowledge worker productivity and service quality. *European Journal of Operational Research*, 134(3):508–524, 2001.
- [3] D J Bartholomew. Maintaining a grade or age structure in a stochastic environment. *Advances in Applied Probability*, 9:1–17, 1977.
- [4] J Bersin. Predictions for 2014: Building a strong talent pipeline for the global economic recovery. *Bersin by Deloitte*, 2013.
- [5] F Bouchut. *Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources*. Series Frontiers in Mathematics, Birkhäuser, Basel, 2004.
- [6] L Edelstein-Keshet. *Mathematical models in biology*. Random House, New York, 1988.
- [7] A C Georgiou and N Tsantas. Modelling recruitment training in mathematical human resource planning. *Applied Stochastic Models in Business and Industry*, 18(1):53–74, 2002.
- [8] Y Gerchak, M Parlar, and S S Sengupta. On manpower planning in the presence of learning. *Engineering Costs and Production Economics*, 20(3):295–303, 1990.
- [9] M Guthridge, A B Komm, and E Lawson. Making talent a strategic priority. *McKinsey Quarterly*, 1:49–59, 2008.
- [10] J Higgins. Changing the game: how HR can drive strategic planning to impact the bottom line. CEB, 2016.
- [11] W O Kermack and A G McKendrick. A contribution to the mathematical theory of epidemics. 115(772):700–721, 1927.
- [12] B L Keyfitz and N Keyfitz. The mckendrick partial differential equation and its uses in epidemiology and population study. *Mathematical and Computer Modelling*, 26(6):1–9, 1997.
- [13] R J LeVeque. *Numerical methods for conservation laws*. Birkhäuser, Verlag, Boston, 1992.
- [14] S McClean, E Montgomery, and F Ugwuowo. Non-homogeneous continuous-time markov and semi-markov manpower models. *Applied Stochastic Models and Data Analysis*, 13(3-4):191–198, 1997.
- [15] P Michel, S Mischler, and B Perthame. General relative entropy inequality: an illustration on growth models. *Journal de mathématiques pures et appliquées*, 84(9):1235–1260, 2005.
- [16] B Perthame. *Transport equations in biology*. Series Frontiers in Mathematics, Birkhäuser, 2007.
- [17] M C Pullan. An algorithm for a class of continuous linear programs. *SIAM Journal on Control and Optimization*, 31(6):1558–1577, 1993.
- [18] T W Reiland. Optimality conditions and duality in continuous programming ii. the linear problem revisited. *Journal of Mathematical Analysis and Applications*, 77(2):329–343, 1980.
- [19] E Shindin and G Weiss. Symmetric strong duality for a class of continuous linear programs with constant coefficients. *SIAM Journal on Optimization*, 24(3):1102–1121, 2014.
- [20] R Straick, J Baier, and P Zimmermann. Turning the challenge of an older workforce into a managed opportunity. *report, The Boston Consulting Group*, 2011.
- [21] H R Thieme. *Mathematics in population biology*. Princeton University Press, Princeton, 2003.
- [22] W F Tyndall. A duality theorem for a class of continuous linear programming problems. *Journal of the Society for Industrial and Applied Mathematics*, 13(3):644–666, 1965.
- [23] M B Young. Strategic workforce planning: Forecasting human capital needs to execute business strategy. *Conference Board*, (report R-1391-06-WG), 2006.