Faster individual discrete logarithms with the QPA and NFS variants

Aurore Guillevic

FASTER INDIVIDUAL DISCRETE LOGARITHMS WITH THE QPA AND NFS VARIANTS

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Abstract. Computing discrete logarithms in finite fields is a main concern in cryptography. The best algorithms known are the Number Field Sieve and its variants (special, high-degree, tower) in large and medium characteristic fields (e.g. GF(p^2), GF(p^{12})); the Function Field Sieve and the Quasi Polynomial-time Algorithm in small characteristic finite fields (e.g. GF($3^{6\cdot 509}$)). The last step of this family of algorithms is the individual logarithm computation. It computes a smooth decomposition of a given target in two phases: an initial splitting, then a descent tree. While new improvements have been made to reduce the complexity of the dominating relation collection and linear algebra steps, resulting in a smaller factor basis (database of known logarithms of small elements), the last step remains of same difficulty. Indeed, we have to find a smooth decomposition of a typically large element in the finite field. This work improves the initial splitting phase and applies to any non-prime finite field. It is very efficient when the extension degree is composite. It exploits the proper subfields, resulting in a much more smooth decomposition of the target. This leads to a new trade-off between the initial splitting step and the descent step with QPA. Moreover it reduces the width and the height of the subsequent descent tree.

1. Introduction

This work is interested in improving the last step of discrete logarithm (DL) computations in non-prime finite fields. The discrete logarithm instances to be broken arise from Diffie-Hellman (DH) [22] key-exchange instances, or from pairing-based cryptographic instances. In the latter case, the security relies on the hardness of computing discrete logarithms in two groups: the group of points of a particular elliptic curve defined over a finite field, and a small extension of this finite field (in most of the cases of degree 2, 3, 4, 6, or 12).

The finite fields are parted in three categories: small, medium and large characteristic, corresponding to the proportion taken by the characteristic p w.r.t. the total size $Q = p^n$ of the finite field. This is measured with the L notation:

$$L_Q[\alpha, c] = e^{(c+o(1))(\log Q)^{\alpha}(\log\log Q)^{1-\alpha}}, \text{ where } Q = p^n,\ \alpha \in [0,1],\ c \neq 0. \tag{1.1}$$

In large characteristic, that is $p = L_Q[\alpha, c]$ where $\alpha > 2/3$, the Number Field Sieve (NFS) [27, 58, 34] provides the best expected running-time: in $L_Q[1/3, (64/9)^{1/3} \approx 1.923]$ and was used in the latest record computations in a 768-bit prime field [45]. Its special variant in expected running-time $L_Q[1/3, (32/9)^{1/3} \approx 1.526]$ was used to break a 1024-bit trapdoored prime field [26]. In 2015 and 2016, the Tower-NFS construction of Schirokauer was revisited for prime fields [14], then Kim, Barbulescu and Jeong improved it for non-prime finite fields $\mathbb{F}_{p^n}$ where the extension degree n is composite [42, 43], and called it the Extended TNFS algorithm. To avoid a

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confusion due to the profusion of names denoting variants of the same algorithm, we suggest to use TNFS as a generic term to denote the family of all the variants of NFS that use a tower of number fields.

In small characteristic, that is $p = L_Q[\alpha, c]$ where $\alpha < 1/3$, the Function Field Sieve (FFS) [19, 7, 8] applies to any extension degrees [44, 11] and the Quasi-Polynomial-time Algorithm (QPA) [13, 31, 30], a.k.a. Frobenius Representation Algorithm, applies to particular composite extension degrees, typically those corresponding to pairing-based cryptography. It allowed to completely break cryptographic-size instances of discrete logarithms [5, 37, 30].

The NFS and FFS algorithms are made of four phases: polynomial selection (two polynomials are chosen), relation collection where relations between small elements are obtained, linear algebra (computing the kernel of a huge sparse matrix over an auxiliary large prime finite field), and individual discrete logarithm computation. In this work, we improve this last step. All the improvements of NFS, FFS and related variants since the 90's decrease the size of the factor basis, that is the database of known discrete logarithms of *small* elements obtained after the linear algebra step, *small* meaning an element represented by a polynomial of small degree (FFS), resp. an element whose pseudo-norm is small (NFS). The effort required in the individual discrete logarithm step increases: one needs to find decomposition of a given target into *small* elements, to be able to express its discrete logarithm in terms of already known logarithms of elements in the factor basis, while that factor basis has decreased at each major improvement.

The heart of this paper relies on the following two observations. Firstly, to speed-up the individual discrete logarithm phase, we start by speeding-up the initial splitting step, and for that we compute a representation of a preimage of the given target of smaller degree, and/or whose coefficients are smaller. It will improve its smoothness probability. Secondly, to compute this preimage of smaller degree, we exploit the proper subfields of the finite field $\mathrm{GF}(p^n)$, and intensively use this key-ingredient: since we are computing discrete logarithms modulo (a prime divisor of) $\Phi_n(p)$, we can freely multiply or divide the target by any element in a proper subfield without affecting its discrete logarithm modulo $\Phi_n(p)$.

We first present our generic strategy to lower the degree of the polynomial representing a given element in $\mathrm{GF}(p^n)$ (Section 3). We apply it to the small characteristic case, to speed-up the QPA algorithm (Section 4). Then we combine this algorithm with a lattice reduction algorithm (e.g. LLL or BKZ) to reduce the size of the coefficients. We apply it to medium and large characteristic finite fields, that is the NFS case and its tower variant (Section 6). Finally in Section 8 we present a more advanced strategy, to exploit several subfields at a time. We apply this technique to $\mathbb{F}_{p^6}$.

2. Preliminaries

2.1. **Setting.** In all this paper, we are interested in non-prime finite fields $\mathrm{GF}(p^n)$, $n > 1$. To capture all the cases at a glance, we assume that the extension field $\mathbb{F}_{p^n}$ is defined as a first extension of degree n_1, then a second extension of degree n_2 above it, that is $\mathbb{F}_{(p^{n_1})^{n_2}}$. The elements are of the form $T = \sum_{i=0}^{n_2-1} \sum_{j=0}^{n_1-1} a_{ij} y^j x^i$, where the coefficients a_{ij} are in $\mathbb{F}_p$, the coefficients $a_i = \sum_{j=0}^{n_1-1} a_{ij} y^j$ are in $\mathbb{F}_{p^{n_1}} = \mathbb{F}_p[y]/(h(y))$, and $\mathbb{F}_{(p^{n_1})^{n_2}} = \mathbb{F}_{p^{n_1}}[x]/(\psi(x))$, where h has degree n_1 and ψ has degree n_2. In other words, T is represented as a polynomial of degree $n_2 - 1$ in the variable x, and coefficients $a_i \in \mathbb{F}_{p^{n_1}}$. For the FFS and NFS algorithms, $n_1 = 1$ and $n_2 = n$; for the QPA algorithm, $n_2 > 1$ is a strict divisor of n, and for the original version of TNFS, $n_1 = n$ and $n_2 = 1$.

Definition 2.1 (Smoothness)**.** Let B be a positive integer. A polynomial is said to be B-smooth w.r.t. its degree if all its irreducible factors have a degree smaller than B. An integer is said to be B-smooth if all its prime divisors are less than B. An ideal in a number field is said to be B-smooth if it factors into prime ideals whose norms are bounded by B.

Definition 2.2 (Preimage)**.** The preimage of an element $a = \sum_{i=0}^{n_2-1} \sum_{j=0}^{n_1-1} a_{ij} y^j x^i \in \mathbb{F}_{(p^{n_1})^{n_2}}$ will be with NFS the bivariate polynomial $\sum_{i=0}^{n_2-1} \sum_{j=0}^{n_1-1} a'_{ij} y^j x^i \in \mathbb{Z}[x, y]$, where each coefficient a'_{ij} is a lift in $\mathbb{Z}$ of the coefficient a_{ij} in $\mathbb{F}_p$. It is a preimage for the reduction modulo (p, h, ψ) map $\rho : \mathbb{Z}[x, y] \to \mathbb{F}_{(p^{n_1})^{n_2}}$. With QPA and FFS, the preimage of a is a univariate polynomial in $\mathbb{F}_{p^{n_1}}[x]$. It is a preimage for the reduction modulo ψ map $\rho : \mathbb{F}_{p^{n_1}}[x] \to \mathbb{F}_{(p^{n_1})^{n_2}}$.

Definition 2.3 (Pseudo-norm)**.** The integral pseudo-norm w.r.t. a number field $\mathbb{Q}[x]/(f(x))$ (f monic) of a polynomial $T = \sum_{i=0}^{\deg f-1} a_i x^i$ of integer coefficients a_i is computed as $\mathrm{Res}_x(T(x), f(x))$.

Since there is no chance for a preimage of a target T_0 to be B-smooth, the individual discrete logarithm is done in two steps: an *initial splitting* of the target[1], then a *descent* phase[2]. The initial splitting is an iterative process that tries many targets $g^t T_0 \in \mathbb{F}_{p^n}^*$, where t is a known exponent (taken uniformly at random), until a B_1-smooth decomposition of the preimage is found. Here *smooth* stands for a factorization into irreducible polynomials of $\mathbb{F}_{p^{n_1}}[x]$ of degree at most B_1 in the FFS and QPA setting, resp. a pseudo-norm that factors as an integer into a product of primes smaller than B_1 in the NFS (and TNFS) settings.

The second phase starts a recursive process for each element less than B_1 but greater than B_0 obtained after the initial splitting phase. Each of these medium-sized elements are processed until a complete decomposition over the factor basis is found. Each element obtained from the initial splitting is at the root of its descent tree. One finds a relation involving the original one and other ones whose degree, resp. pseudo-norm is strictly smaller than the degree, resp. pseudo-norm of the initial element to be decreased. These smaller elements form the new leaves of the descent tree. For each leaf, the process is repeated, until all the leaves are in the factor basis. The discrete logarithm of an element output by the initial splitting can be computed by a tree traversal. This strategy is considered in [20, §6], [46, §7], [35, §3.5], [18, §4].

In small characteristic, the initial splitting step is known as the Waterloo[3] algorithm [15, 16]. It outputs $T = U(x)/V(x) \mod I(x)$, and U, V are two polynomials of degree $\lfloor (n_2 - 1)/2 \rfloor$. It uses an Extended GCD computation. For prime fields, the continued fraction algorithm was already used with the Quadratic Sieve and Coppersmith-Odlyzko-Schroeppel algorithm. It expresses an integer N modulo p as a fraction $N \equiv u/v \bmod p$, and the numerator and denominator are of size about the square root of p. This technique was generalized to extensions $\mathbb{F}_{p^n}$ in [36]. As for the Waterloo algorithm, this technique provides a very good practical speed-up but does not improve the asymptotic complexity.

In [33] was highlighted this subfield tool that we will intensively use:

[1] also called *boot* or *smoothing step* in large characteristic finite fields

[2] in order to make no confusion with the mathematical *descent*, which is not involved in this process, we mention that in this step, the norm (with NFS) or the degree (with FFS) of the preimage *decreases*.

[3] the name comes from the authors' affiliation: the University of Waterloo, ON, Canada.

Lemma 2.4 ([33, Lemma 1])**.** *Let $T \in \mathbb{F}_{p^n}^*$, and $\deg T < n$. Let ℓ be a non-trivial prime divisor of $\Phi_n(p)$. Let $T' = u \cdot T$ with u in a proper subfield of $\mathbb{F}_{p^n}$. Then*

$$\log T' \equiv \log T \bmod \Phi_n(p) \text{ and in particular } \log T' \equiv \log T \bmod \ell \ . \tag{2.1}$$

3. The heart of our strategy: Representing elements in the cyclotomic subgroup of a non-prime finite field with less coefficients

In the FFS setting, $n_1 = 1$ and usually n_2 is prime and our technique cannot be helpful but if n is not prime, our algorithm applies, and moreover in favorable cases the QPA algorithm can be used and our technique provides a notable speed-up. In the QPA implementations, the factor basis is made of the irreducible polynomials of $\mathbb{F}_{p^{n_1}}[x]$ of very small degree, e.g. of degrees 1, 2, 3 and 4 in [3]. Our aim is to improve the smoothness probability of a preimage $P \in \mathbb{F}_{p^{n_1}}[x]$ of a given target $T \in \mathbb{F}_{(p^{n_1})^{n_2}}$ and for that we want to reduce the degree in x of the preimage P (as a lift of T in $\mathbb{F}_{p^{n_1}}[x]$, P has degree at most $n_2 - 1$ in x), while keeping the property

$$\log(\rho(P)) = \log T \mod \ell \ ,$$

where $\rho : \mathbb{F}_{p^{n_1}}[x] \to \mathbb{F}_{(p^{n_1})^{n_2}}$ is the reduction modulo ψ map.

Let d denotes the largest proper divisor of n, $1 < d < n$ (d might sometimes be equal to n_2 in the QPA setting). We will compute P in $\mathbb{F}_{p^{n_1}}[x]$ of degree at most $n_2 - d/n_1$ in x (and coefficients in $\mathbb{F}_{p^{n_1}}$). There are two strategies: either handle coefficients in $\mathbb{F}_p$, or in $\mathbb{F}_{p^{\gcd(d,n_1)}}$. We will consider the latter case. Let $d' = d/\gcd(d, n_1)$ to simplify the notations, and let $[1, U, \ldots, U^{d'-1}]$ be a polynomial basis of $\mathbb{F}_{p^{d'}}$. Define the $d' \times n_2$ matrix L whose rows are made of the coefficients (in $\mathbb{F}_{p^{n_1}}$) of $U^i T$ for $0 \le i \le d' - 1$:

$$L_{d' \times n_2} = \begin{bmatrix} T \\ UT \\ \vdots \\ U^{d'-1}T \end{bmatrix} \in \mathcal{M}_{d',n}(\mathbb{F}_{p^{n_1}}) \ .$$

Then we compute a row-echelon form of this matrix by performing only $\mathbb{F}_{p^{\gcd(n_1,d)}}$-linear operations over the rows, so that each row of the echeloned matrix is a $\mathbb{F}_{p^{\gcd(n_1,d)}}$-linear combination of the initial rows, that can be expressed as

$$P = \sum_{i=0}^{d'-1} \lambda_i U^i T = uT, \text{ where } \lambda_i \in \mathbb{F}_{p^{\gcd(n_1,d)}}$$

so that $P = uT$ with $u^{p^d - 1} = 1$. Assuming that the matrix is lower-triangular (the other option being an upper-triangular matrix), we take the first row of the matrix, and see it as a polynomial in $\mathbb{F}_{p^{n_1}}[x]$ of degree at most[4] $n_2 - d/n_1$. This is formalized in Algorithm 1.

We obtain the following Theorem 3.1.

Theorem 3.1. *Let $\mathbb{F}_{p^n}$ be a finite field represented as a tower $\mathbb{F}_{(p^{n_1})^{n_2}}$. Let $T \in \mathbb{F}_{p^n}^*$ an element which is not in a proper subfield of $\mathbb{F}_{p^n}$. Let d be the largest proper divisor of n, $1 < d < n$ (n is not prime). Assume that T is represented by a polynomial in $\mathbb{F}_{p^{n_1}}[x]$ of degree larger than $n_2 - \lceil d/n_1 \rceil$. Then there exists a preimage P of T, in $\mathbb{F}_{p^{n_1}}[x]$, of degree $n_2 - \lceil d/n_1 \rceil$ in x and coefficients in $\mathbb{F}_{p^{n_1}}$, and such that*

$$\log(\rho(P)) = \log T \mod \Phi_n(p) \ .$$

[4] $n_2 - d/n_1$ is not necessarily an integer, meaning that the leading coefficient of the polynomial is some element in $\mathbb{F}_{p^{n_1}}$. Its degree in x is actually $n_2 - \lceil d/n_1 \rceil$.

Algorithm 1: Computing a representation with a polynomial of smaller degree

Input: Finite field $\mathbb{F}_{p^n}$ represented as a two-level tower $\mathbb{F}_{(p^{n_1})^{n_2}} = \mathbb{F}_{p^{n_1}}[x]/(\psi(x))$ (one may have $n_1 = 1$), a proper divisor d of n ($d \mid n$, $1 < d < n$), $T \in \mathbb{F}_{p^n}$

Output: $P \in \mathbb{F}_{p^{n_1}}[x]$ a polynomial of degree $\leq n_2 - d/n_1$ satisfying $\rho(P) = uT$, where $u \in \mathbb{F}_{p^d}$

$d' = d/\gcd(d, n_1)$

Compute a polynomial basis $(1, U, U^2, \ldots, U^{d'-1})$ of the subfield $\mathbb{F}_{p^{d'}}$

Define $L = \begin{bmatrix} T \\ UT \\ \vdots \\ U^{d'-1}T \end{bmatrix}$ a $d' \times n_2$ matrix whose coefficients are in $\mathbb{F}_{p^{n_1}}$

$M \leftarrow \text{RowEchelonForm}(L)$ with only $\mathbb{F}_{p^{\gcd(n_1,d)}}$-linear combinations

$P(x) \leftarrow$ the polynomial of lowest degree made of the first row of L

return $P(x)$

Proof. We use Algorithm 1 to compute P. The matrix has full rank since the U^is form a polynomial basis of $\mathbb{F}_{p^{d'}}$. The linear combinations involve T and elements in the proper subfield $\mathbb{F}_{p^d}$ by construction. The first row after Gaussian elimination will have at least $\geq d/n_1 - 1$ coefficients equal to zero at the right, and will represent a polynomial of degree at most $\leq n_2 - d/n_1$. When mapped to $\mathbb{F}_{(p^{n_1})^{n_2}}$, it will satisfy $\log(\rho(P)) = \log T \mod \Phi_n(p)$ since in the process, T was multiplied only by elements whose images in $\mathbb{F}_{(p^{n_1})^{n_2}}$ are in the subfield $\mathbb{F}_{p^d}$: $\rho(P) = uT$, $u \in \mathbb{F}_{p^d}$ and the equality follows by Lemma 2.4. □

We can now directly apply this Algorithm 1 to greatly improve the QPA algorithm in practice.

4. Application to small characteristic finite fields, and cryptographic-size examples

In the QPA setting, n is not prime, for instance $n = 6 \cdot 509$. The notation in [6] was $n = lk$, with the property $p^l \approx k$. With our notations, $n_1 = l$ and $n_2 = k$.

4.1. **Algorithm.**

We directly use Algorithm 1 as a subroutine of Algorithm 2. Then to improve it in practice, we list valuable modifications.

Remark 4.1. As was pointed out to us by F. Rodríguez-Henríquez [53, 3], the elements of the form $x^i R(x)$ where R itself is of degree $\leq n_2 - d/n_1$ are evenly interesting because the discrete logarithm of x^i can be deduced from the discrete logarithm of x, which is known after linear algebra.

So we can increase the number of elements tested for B_1-smoothness for each exponent t by a factor d' almost for free in the following way. We run again a Gaussian elimination algorithm on the matrix M but in the reverse side, for instance from row one to row d' and left to right if it was done from row d' to row one and right to left the first time. The matrix is in row-echelon form on the left-hand side and on the right-hand side (the upper right and lower left corners are filled with

Algorithm 2: Initial splitting in small characteristic with the subfield technique

Input: Finite field $\mathbb{F}_{p^n}$ of small characteristic (e.g. $p = 2, 3$), with a two-level tower representation $\mathbb{F}_{p^n} = \mathbb{F}_{(p^{n_1})^{n_2}} = \mathbb{F}_{p^{n_1}}[x]/(I(x))$ (one may have $n_1 = 1$), generator g (of the order ℓ subgroup of the cyclotomic subgroup of $\mathbb{F}_{p^n}$), target $T_0 \in \mathbb{F}_{(p^{n_1})^{n_2}}$, smoothness bound B_1

Output: t, $P \in \mathbb{F}_{p^{n_1}}[x]$ a polynomial of degree $\leq n_2 - d/n_1$ such that $\mathrm{vlog}_g\, \rho(P) = t + \mathrm{vlog}_g\, T_0 \mod \ell$, and $P(x)$ is B_1-smooth (w.r.t. its degree in x)

$d \leftarrow$ the largest divisor of n, $1 < d < n$

$d' \leftarrow d/\gcd(d, n_1)$

Compute $U(x) \in \mathbb{F}_{(p^{n_1})^{n_2}}$ s.t. $(1, U, U^2, \ldots, U^{d'-1})$ is a polynomial basis of the subfield $\mathbb{F}_{p^{d'}}$

repeat

  take $t \in \{1, \ldots, \ell - 1\}$ at random

  $T \leftarrow g^t T_0$ in $\mathbb{F}_{(p^{n_1})^{n_2}}$

  Define $L = \begin{bmatrix} T \\ UT \\ \vdots \\ U^{d'-1}T \end{bmatrix}$ a $d' \times n_2$ matrix whose coefficients are in $\mathbb{F}_{p^{n_1}}$

  $M \leftarrow \mathrm{RowEchelonForm}(L)$ (with $\mathbb{F}_{p^{\gcd(d,n_1)}}$-linear Gaussian Elimination)

  $P(x) \leftarrow$ the polynomial of lowest degree made of the first row of L

until $P(x)$ *is* B_1*-smooth*

return $t, P(x)$

zeros). We obtain a matrix N of the form

$$N = \begin{bmatrix} * & \ldots & * & * & 0 & \ldots & 0 \\ 0 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \ldots & 0 & * & \ldots & * & * \end{bmatrix}$$

The i-th row represents a polynomial $P_i' = x^{e_i} P_i$, where P_i is of degree at most $n_2 - d/n_1$, and $e_i \approx (i-1)\gcd(n_1, d)/n_1$. Since x is in the factor basis (by construction, like all the degree one polynomials), its logarithm is known at this point (after the relation collection and linear algebra steps), hence the logarithm of any power x^{e_i} is known. It remains to compute the discrete logarithm of P_i.

In practice there are some technicalities: in the second Gaussian Elimination, if the leading coefficient is zero then two rows are swapped, and it cancels the previous Gaussian Elimination (computed at the other end of the matrix) for that row. We end up with a matrix which is in row-echelon form on the right and almost row-echelon form on the left (or vice-versa). Since each set of subsequent $n_1/\gcd(n_1, d)$ rows produces polynomials of same degree, swapping two rows from the same set will not change the degree in x of the polynomial. In average (this is what we observed in our experiments for $\mathbb{F}_{3^{6 \cdot 509}}$ and $\mathbb{F}_{3^{5 \cdot 479}}$), some rare polynomials will have a degree in x increased by one or two. This second Gaussian elimination increases the number of tests by a factor d' at a very cheap cost, since in fact it allows to share the cost of computing the $U^i T$ and the two Gaussian eliminations over d' tests.

Remark 4.2. Other improvements are possible [53, 3], for instance computing $\mathbb{F}_{p^{\gcd(n_1,d)}}-$linear combinations over a small number of rows corresponding to polynomials of almost the same degree. The resulting polynomial will have degree increased by one or two, which does not affect significantly its B_1-smoothness probability in practice for cryptographic sizes. This technique allows to produce many more candidates, at a very cheap cost of linear operations in $\mathbb{F}_{p^{n_1}}[x]$.

4.2. **Complexity analysis.**

4.2.1. *Cost of computing one preimage $P \in \mathbb{F}_{p^{n_1}}[x]$ in the initial splitting step.* We use the notations of Algorithm 2: let d be the largest proper divisor of n ($d \mid n$, $1 < d < n$) and let $d' = d/\gcd(d, n_1)$. Since $d' \mid d \mid n = n_1 n_2$, and $\gcd(d', n_1) = 1$, then $d' \mid n_2$, and $d' \leq n_2$. The computation of all the $U^i T$ of the matrix L costs at most $d' n_2^2$ multiplications in $\mathbb{F}_{p^{n_1}}$, since a schoolbook multiplication in $\mathbb{F}_{(p^{n_1})^{n_2}}$ costs n_2^2 multiplications in $\mathbb{F}_{p^{n_1}}$. There are d' such multiplications. The complexity of a reduced row-echelon form computation of a $(d' \times n_2)$-matrix, $d' \leq n_2$, is less than $O(d'^2 n_2)$ multiplications in $\mathbb{F}_{p^{n_1}}$ [24, §13.4.2]. To simplify, we consider that the computation of the matrix L and of two Gaussian eliminations is done in time at most $O(d' n_2^2)$. This cost is shared over d' polynomials P_i to be tested for B_1-smoothness. In this way, the complexity of computing a preimage P with our initial splitting algorithm is the same as in the Waterloo algorithm: $O(n_2^2)$, and moreover the smoothness probabilities are much higher for the cryptographic cases coming from supersingular pairing-friendly curves, for which QPA is very efficient. We also replace two B_1-smoothness tests by only one and that might save some time in practice (this saving disappears in the O notation). We present the theoretical costs in Table 1 from [25]. XGCD stands for extended Euclidean algorithm, SQF stands for SQuare-free Factorization, DDF stands for Distinct Degree Factorization and EDF stands for Equal Degree Factorization. All the polynomials to be factored are of degree smaller than n_2, we take n_2 as an upper bound.

algorithm	XGCD(T, I)	matrix $[U^i T]_{0 \leq i \leq d'-1}$ and Echelon Form	SQF	DDF	EDF
cost	$O(n_2^2)$	$O(d' n_2^2)$	$O(d_P^2)$	$O(d_P^3 \log p^{n_1})$	$O(d_P^2 \log p^{n_1})$
Waterloo, $d_P = \lfloor n_2/2 \rfloor$	1 time	–	2 times	1 time/SQF factor	1 time/DDF factor
Alg. 2, $d_P = \lfloor n_2(1 - d/n) \rfloor$	–	1 time	1 time	1 time/SQF factor	1 time/DDF factor
Alg. 2+Rem. 4.1	–	1 time/d' tests	1 time	1 time/SQF factor	1 time/DDF factor

TABLE 1. Cost of one iteration of the initial splitting step. The preimage obtained with Alg. 2 has degree $d_P \leq n_2 - d/n_1$. The Waterloo algorithm [15, 16] output two polynomials of degree $d_P = \lfloor n_2/2 \rfloor$.

algorithm	Waterloo	this work	this work, improved
av. cost to compute one preimage P, $\mathbb{F}_{p_1^n}$ op.	$O(n_2^2)$	$O(d' n_2^2)$	$O(n_2^2)$

TABLE 2. Average cost to compute one preimage to be tested for smoothness, in the initial splitting step, in $\mathbb{F}_{p^{n_1}}$ operations.

4.2.2. *Running-time of the initial splitting step.* To start, we recall some results on the smoothness probability of a polynomial of given degree.

Definition 4.3. Let $N_q(b;d)$ denote the number of monic polynomials over $\mathbb{F}_q$ of degree d which are b-smooth. Let $N_q(b;d_1,d_2)$ denote the number of coprime pairs of monic polynomials over $\mathbb{F}_q$ of degrees d_1 and d_2, respectively, which are b-smooth.

Let $\Pr_q(b;d)$ denote the probability of a monic polynomial over $\mathbb{F}_q$ of degree d to be b-smooth. Let $\Pr_q(b;d_1,d_2)$ denote the probability of two coprime monic polynomials over $\mathbb{F}_q$ of degrees d_1 and d_2 to be both b-smooth.

Odlyzko gave the following estimation for $\Pr_q(b;d)$ in [51, eq. (4.5) p. 14].

$$\Pr_q(b,d)^{-1} = \exp\left((1+o(1))\frac{d}{b}\log_e\frac{d}{b}\right) \text{ for } d^{1/100} \leq b \leq d^{99/100} \tag{4.1}$$

Writing the smoothness bound degree $b = \log L_Q[\alpha_b, c_b]/\log p^{n_1}$ to match Odlyzko's convention $b = c_b n_2^{\alpha_b}(\log n_2)^{1-\alpha_b}$, and the degree of the polynomial to be tested for smoothness $d = an_2$, where $a \in]0,1[$ and $n_2 = \log Q/\log p^{n_1}$, one obtains

$$\Pr_{p^{n_1}}(b,d) = L_Q\left[1-\alpha_b, -(1-\alpha_b)a/\gamma\right] \text{ , where } Q = p^{n_1 n_2}.$$

Theorem 4.4 ([23, Theorem 1])**.** *Let $\delta > 0$ be given. Then we have, uniformly for $b, d_1, d_2 \to \infty$ with $d_1^\delta \leq b \leq d_1^{1-\delta}$ and $d_2^\delta \leq b \leq d_2^{1-\delta}$,*

$$N_q(b;d_1,d_2) \sim \left(1-\frac{1}{q}\right) N_q(b;d_1)N_q(b;d_2) \ .$$

Corollary 4.5 ([23, Theorem 1])**.** *Let $\delta > 0$ be given. Then we have, uniformly for $b, d_1, d_2 \to \infty$ with $d_1^\delta \leq b \leq d_1^{1-\delta}$ and $d_2^\delta \leq b \leq d_2^{1-\delta}$,*

$$\Pr_q(b;d_1,d_2) \sim \left(1-\frac{1}{q}\right) \Pr_q(b;d_1)\Pr_q(b;d_2) \ .$$

We can now compare the Waterloo algorithm with this work. Assuming that $B_1 = \log_{p^{n_1}} L_{p^n}[2/3,\gamma]$ for a certain γ, then the probability of a polynomial of degree an_2, $0 < a < n_2$, to be B_1-smooth is $L_{p^n}[1/3, -a/(3\gamma)]$. In the Waterloo algorithm, two polynomials of degree $n_2/2$ should be B_1-smooth at the same time, and the expected running-time to find such a pair is $L_{p^n}[1/3, 1/(3\gamma)]$ (the square of $L_{p^n}[1/3, 1(6\gamma)]$). In our algorithm, a polynomial of degree $\lfloor n_2 - d/n_1 \rfloor = \lfloor n_2(1-d/n) \rfloor$ is tested for B_1-smoothness, so finding a good one requires

$$L_{p^n}[1/3, a/(3\gamma)] \text{ tests, where } a \approx 1 - d/n \ , \tag{4.2}$$

which is always faster than the Waterloo algorithm, for which $a = 1$. When n is even (this is always the case for finite fields of supersingular pairing-friendly curves), one can choose $d = n/2$, hence $a = 1/2$ and our algorithm has running-time the square-root of the running-time of the Waterloo algorithm.

4.3. **Improving the record computation in GF$(3^{6\cdot 509})$.** Adj, Menezes, Oliveira and Rodríguez-Henríquez estimated in [6] the cost to compute discrete logarithms in the 4841-bit finite field GF$(3^{6\cdot 509})$ and announced their record computation in July 2016 [4]. The details of the computations are available in Adj's PhD thesis [2] and details for initial splitting and descent can be found in [17]. The elements are represented by polynomials of degree at most 508 whose coefficients are in $\mathbb{F}_{3^6}$. The initial splitting made with the Waterloo algorithm outputs two polynomials of degree 254. The probability that two independent and relatively prime polynomials of degree 254 over $\mathbb{F}_{3^6}$ are simultaneously b-smooth is $(1-1/3^6)\Pr_{3^6}^2(254,b)$ [23].The term $(1-1/3^6)$ is negligible in practice for the values we are looking at.

4.3.1. *Improvements.* Our Algorithm 2 outputs *one* polynomial of degree 254, whose probability to be b-smooth is $\Pr_{3^6}(n, b)$, i.e. the square root of the previous one. So we can take a much smaller b while reaching the same probability as before with the Waterloo algorithm. We list in Tab. 2a p. 10 the values of b to obtain a probability between 2^{-40} and 2^{-20}. For instance, if we allow 2^{30} trials, then we can set $b = 28$ with our algorithm, instead of $b = 43$ previously: we have $\Pr^2_{3^6}(254, 43) = 2^{-30.1}$ and we only need to take $b = 28$ to get the same probability with this work: $\Pr_{3^6}(254, 28) = 2^{-29.6}$. This will provide a good practical speed-up of the descent phase: much less elements need to be "reduced": this reduces the initial width of the tree, and they are of much smaller degree: this reduces the the depth of the descent tree.

4.3.2. *A 30-smooth initial splitting.* The finite field is represented as a first extension $\mathbb{F}_{3^6} = \mathbb{F}_{p^{n_1}} = \mathbb{F}_3[y]/(y^6 + 2y^4 + y^2 + 2y + 2)$, then a second extension $\mathbb{F}_{3^{6\cdot 509}} = \mathbb{F}_{3^6}[x]/(I(x))$, where $I(x)$ is the degree 509 irreducible factor of $h_1 x^{q_1} - h_0$, where $q_1 = p^{n_1}$, $h_1 = x^2 + y^{424}x$ and $h_0 = y^{316}x + y^{135}$. The generator is $g = x + y^2$. As a proof of concept, we computed a 30-smooth initial splitting of the target $T_0 = \sum_{i=0}^{508}(y^{\lfloor \pi(3^6)^{i+1} \rfloor} \bmod 3^6)x^i$, with the parameters $d = 3 \times 509$, $d' = d/\gcd(d, n_1) = 509$. Each trial $g^t T_0$ produces $d' = 509$ polynomials to test for smoothness. We found that $g^{47233}T_0 = uvx^{230}P$ where $u = 1 \in \mathbb{F}_{3^6}$, $v \in \mathbb{F}_{3^{3\cdot 509}}$ and P is of degree 255 and 30-smooth. The equality $(g^{47233}T_0)^{\frac{p^n-1}{\ell}} = (uvx^{230}P)^{\frac{p^n-1}{\ell}}$ is satisfied. The explicit value of P is available at `https://members.loria.fr/AGuillevic/files/F3_6_509_30smooth.mag.txt`.

The whole computation took less than 6 days (real time) on 48 cores Intel Xeon E5-2609 2.40GHz (274 core days, i.e. 0.75 core-years). This is obviously overshot compared to the estimate of $2^{26.6}$, but this was done with a non-optimized Magma implementation.

As a comparison, with classical Waterloo algorithm, Adj et al. computed a 40-smooth initial splitting in 51.71 CPU (2.87GHz) years [2, Tab. 5.2 p. 87] and [4]. They obtained irreducible polynomials of degree 40, 40, 39, 38, 37, and seven polynomials of degree between 22 and 35. They needed another 9.99 CPU (2.66 GHz) years to compute a classical descent from 40-smooth to 21-smooth polynomials. A complete comparison can be found in [3] and [1]. In [3], Adj *et al.* estimated that with our Algorithm 2 enriched as in Remarks 4.1 and 4.2, it is possible to compute discrete logarithms in $\mathbb{F}_{3^{6\cdot 709}}$ at the same cost as in $\mathbb{F}_{3^{6\cdot 509}}$ with the former Waterloo algorithm.

4.4. **Computing discrete logarithms in $\mathbb{F}_{2^{512}}$ and $\mathbb{F}_{2^{1024}}$.** In [28, 29, §3.6] discrete logarithms in $\mathbb{F}_{2^{512}}$ and $\mathbb{F}_{2^{1024}}$ need to be computed modulo the full multiplicative group order $2^n - 1$. As pointed to us by R. Granger, our technique can be used to compute discrete logarithms in $\mathbb{F}_{2^{1024}}$. Our algorithm provides a decomposition of the target as the product uR where u is an element in the largest proper subfield $\mathbb{F}_{2^{512}}$, and P is an element of $\mathbb{F}_{2^{1024}}$ of degree 512 instead of 1023. The discrete logarithm of the subfield cofactor u can be obtained by a discrete logarithm computation in $\mathbb{F}_{2^{512}}$. More generally, our technique is useful when discrete logarithms in nested finite fields such as $\mathbb{F}_{2^{2^i}}$ are computed recursively.

4.5. **Improving the record computation in GF$(3^{5\cdot 479})$.** Joux and Pierrot announced a discrete logarithm record computation in GF$(3^{5\cdot 479})$ in [39] (then published in [37]). They defined a first degree 5 extension GF(3^5) = GF$(3)[y]/(y^5 - y + 1)$ then a degree 479 extension above it. The irreducible degree 479 polynomial $I(x)$ was a divisor of $xh_1(x^{q_1}) - h_0(x^{q_1})$, where $q_1 = p^{n_1} = 3^5$, $h_0 = x^2 + y^{111}x$ and $h_1 = yx + 1$. Given a target $T \in \mathbb{F}_{3^{5\cdot 479}}$, the Waterloo initial splitting outputs two polynomials $u(x), v(x) \in \mathbb{F}_{3^5}[x]$ of degree $\lfloor 478/2 \rfloor = 239$. Our Algorithm 1

(A) Probabilities for $\mathrm{GF}(3^{6\cdot 509})$

Waterloo alg.		Algorithm 2	
b	$\Pr^2_{3^6}(254, b)$	b	$\Pr_{3^6}(254, b)$
36	$2^{-40.1}$	22	$2^{-42.3}$
37	$2^{-38.4}$	23	$2^{-39.6}$
38	$2^{-36.8}$	24	$2^{-37.2}$
39	$2^{-35.3}$	25	$2^{-35.1}$
40	$2^{-33.9}$		
41	$2^{-32.5}$	26	$2^{-33.1}$
42	$2^{-31.3}$	27	$2^{-31.3}$
43	$2^{-30.1}$	28	$2^{-29.6}$
44	$2^{-28.9}$		
45	$2^{-27.9}$	29	$2^{-28.1}$
46	$2^{-26.9}$		
47	$2^{-25.9}$	30	$2^{-26.6}$
48	$2^{-25.0}$	31	$2^{-25.3}$
49	$2^{-24.1}$	32	$2^{-24.1}$
50	$2^{-23.3}$	33	$2^{-23.0}$
51	$2^{-22.5}$		
52	$2^{-21.8}$	34	$2^{-21.9}$
53	$2^{-21.1}$	35	$2^{-21.0}$
54	$2^{-20.4}$	36	$2^{-20.1}$
55	$2^{-19.7}$		
56	$2^{-19.1}$	37	$2^{-19.2}$
57	$2^{-18.5}$		
58	$2^{-18.0}$	38	$2^{-18.4}$
59	$2^{-17.4}$	39	$2^{-17.6}$
60	$2^{-16.9}$	40	$2^{-16.9}$
61	$2^{-16.4}$	41	$2^{-16.3}$
62	$2^{-15.9}$	42	$2^{-15.6}$

(B) For $\mathrm{GF}(3^{5\cdot 479})$

	Waterloo alg.	Algorithm 2
b	$\Pr^2_{3^5}(239, b)$	$\Pr_{3^5}(383, b)$
24	$2^{-67.96}$	$2^{-67.59}$
25	$2^{-63.95}$	$2^{-63.86}$
26	$2^{-60.30}$	$2^{-60.45}$
27	$2^{-56.95}$	$2^{-57.32}$
28	$2^{-53.89}$	$2^{-54.44}$
29	$2^{-51.07}$	$2^{-51.79}$
30	$2^{-48.46}$	$2^{-49.34}$
31	$2^{-46.06}$	$2^{-47.06}$
32	$2^{-43.83}$	$2^{-44.95}$
33	$2^{-41.76}$	$2^{-42.99}$
34	$2^{-39.83}$	$2^{-41.16}$
35	$2^{-38.03}$	$2^{-39.44}$
36	$2^{-36.35}$	$2^{-37.84}$
37	$2^{-34.77}$	$2^{-36.34}$
38	$2^{-33.30}$	$2^{-34.92}$
39	$2^{-31.91}$	$2^{-33.60}$
40	$2^{-30.61}$	$2^{-32.34}$
41	$2^{-29.39}$	$2^{-31.16}$
42	$2^{-28.23}$	$2^{-30.05}$
43	$2^{-27.14}$	$2^{-28.99}$
44	$2^{-26.11}$	$2^{-27.99}$
45	$2^{-25.13}$	$2^{-27.04}$
46	$2^{-24.21}$	$2^{-26.14}$
47	$2^{-23.33}$	$2^{-25.29}$
48	$2^{-22.50}$	$2^{-24.47}$
49	$2^{-21.71}$	$2^{-23.70}$
50	$2^{-20.96}$	$2^{-22.96}$

TABLE 3. Smoothness probabilities of polynomials over finite fields, comparison of the Waterloo algorithm and Algorithm 2. The values were computed with Odlyzko's induction formula [51] and Drmota and Panario's Theorem 4.4, as in [17].

outputs one polynomial of degree $\lfloor \frac{4}{5}479 \rfloor = 383$. This example is interesting because the smoothness probabilities are very close. We computed the exact values with Drmota–Panario's formulas, and give them in table 2b p. 10. We obtain $\Pr^2_{3^5}(239, 50) = 2^{-20.96}$ (Waterloo) and $\Pr_{3^5}(383, 50) = 2^{-22.96}$, i.e. our Algorithm 2 would be four times slower compared to Joux–Pierrot record; $\Pr^2_{3^5}(239, 40) = 2^{-30.61}$ and $\Pr_{3^5}(383, 40) = 2^{-32.34}$; $\Pr^2_{3^5}(239, 30) = 2^{-48.46}$ and $\Pr_{3^5}(383, 30) = 2^{-49.34}$; and the cross-over point is for $b = 24$: in this case, we have $\Pr^2_{3^5}(239, 24) = 2^{-67.96}$ and $\Pr_{3^5}(383, 24) = 2^{-67.59}$ which is slightly larger.

The probabilities would advise to use the classical Initial Splitting with the Waterloo (extended GCD) algorithm. We remark that this algorithm would output two B_1-smooth polynomials of degree $(n_2 - 1)/2$. Each would factor into at least $(n_2 - 1)/(2B_1)$ irreducible polynomials of degree at most B_1. Each such factor is sent as an input to the second step (descent step), that is roughly $n_2 B_1$ factors. If we use Algorithm 2, the initial splitting will outputs one polynomial of degree $4/5n_2 = 383$ that factors into at least $4/5n_2/B_1$ polynomials of degree at most B_1,

each of them sent as input to the second step, that is, the descent step is called 20% time less, and that would reduce the total width of the descent tree in the same proportion. Since the descent is the most costly part, and in particular, the memory size required is huge, this remark would need to be taken into consideration for a practical implementation.

As a proof of concept of our algorithm, we implemented in Magma our algorithm, took the same parameters, generator and target as in [39] and found a 50-smooth decomposition for the target given by the 471-th row of the matrix computed for $g^{23940}T_0$ in 1239 core-hours (22.12 hours over 56 cores) on an Intel Xeon E5-2609 2.40GHz (compared to 5000 core-hours announced in [39]).

The value can be found at `https://members.loria.fr/AGuillevic/files/F3_5_479_50smooth.mag.txt`. In our technique, we compute $g^t T_0 = uvR$ where $u \in \mathbb{F}_{3^5}$ (this is the leading term of the polynomial), $v \in \mathbb{F}_{3^{479}}$ and R which is 50-smooth. The discrete logarithm of u can be tabulated however it remains quite hard to compute the discrete logarithm of v. Our technique is useful if it is easy (or not required) to compute discrete logarithms in the subfields.

5. Preliminaries before medium and large characteristic cases

In the first part of this paper, we were considering polynomials, and wanted them to have degree as small as possible. Now we turn to the medium and large characteristic cases, where we do not deal with polynomials but with ideals in number fields, and want them of small norm. It requires to test for smoothness large integers as fast as possible. We recall the results of Pomerance and Barbulescu on the *early abort strategy*.

5.1. Pomerance's Early Abort Strategy.

Pomerance in [52] introduced the *Early Abort Strategy* (EAS) to speed-up the factorization of large integers, within Dixon's algorithm, Morrison-Brillhart (continued fraction) algorithm, Schroeppel (linear sieve) and Quadratic Sieve, with several variations in the factorization subroutine (trial-division, Pollard-Strassen method). The Early Abort Strategy provides an asymptotic improvement in the expected running-time. Two versions are studied in [52]: one early-abort test, then many tests. In the relation collection step of the NFS algorithm, the partial factorization of the pseudo-norms is done with ECM in time $L_Q[1/6]$ ($Q = p^n$), which is negligible compared to the total cost in $L_Q[1/3]$. So Pomerance's EAS does not provide an asymptotic speed-up, but a practical one. However, in the individual discrete logarithm computation, the initial splitting requires to find smooth integers (pseudo-norms) of larger size: $L_Q[1]$. This time the ECM cost is not negligible, and Pomerance's EAS matters. The speed-up was analyzed by Barbulescu in [9].

Remark 5.1. Instead of the ECM test, one can use the Hyperelliptic curve method test of H. Lenstra, Pila and Pomerance [48, 49].

Pomerance's analysis is presented in the general framework of testing integers for smoothness. This is named *smoothing problem* in [9, Chapter 4]. In the individual discrete logarithm context, the numbers we want to test for smoothness are not integers in an interval, but pseudo-norms, and their chances of being smooth do not exactly match the chances of random integers of the same size. However, we will make the usual heuristic assumption that for our asymptotic computations, the pseudo-norms considered behave as integers of the same size. We give Pomerance's Early Abort Strategy with one test in Algorithm 3 and with k tests in Algorithm 4.

Writing the complexities as in Pomerance's paper, in terms of k early-abort tests, one obtains

Algorithm 3: Pomerance's early abort strategy (EAS)

Input: Integer m, smoothness bound B_1, real numbers $\theta, b \in]0,1[$
Output: B_1-smooth decomposition of m, or $\perp$

$(m_0, m_1) \leftarrow \texttt{ECM}\ (m, B_1^\theta)$ `// cost:` $L_{B_1^\theta}[1/2, \sqrt{2}]$
`//` m_0 `is a` B_1^θ`-smooth part of` m
`//` m_1 `is the non-factorized part of` m
if $m_1 \leq m^{1-b}$ **then**
  $(m_2, m_3) \leftarrow \texttt{ECM}(m_1, B_1)$ `// cost:` $L_{B_1}[1/2, \sqrt{2}]$
  if $m_3 = 1$ **then**
    return B_1-*smooth decomposition* m_1, m_2 *of* m
return $\perp$

Algorithm 4: Pomerance's early abort strategy with k tests (k-EAS)

Input: Integer m, smoothness bound B_1, number of tests $k \geq 0$,
array of positive real numbers $\mathbf{b} = [b_0, b_1, \ldots, b_k]$ where $0 < b_i \leq 1$, and $\sum_{i=0}^{k} b_i = 1$
array of positive real numbers $\boldsymbol{\theta} = [\theta_0, \ldots, \theta_k = 1]$ where $\theta_i < \theta_{i+1}$
Output: B_1-smooth decomposition of m, or $\perp$

$m_i \leftarrow m$
$i \leftarrow 0$
$S \leftarrow \emptyset$
repeat
  $(s_i, m_{i+1}) \leftarrow \texttt{ECM}\ (m_i, B_1^{\theta_i})$ `// cost:` $L_{B_1^{\theta_i}}[1/2, \sqrt{2}]$
    `//` s_i `is a` $B_1^{\theta_i}$`-smooth part of` m_i`,` m_{i+1} `is not factorized`
  $S \leftarrow S \cup s_i$
  $m_i \leftarrow m_{i+1}$
  $i \leftarrow i + 1$
until $(i > k)$ *OR* $(m_i = 1)$ *OR* $(m_i > m^{1-\sum_{j=0}^{i-1} b_j})$
if $m_i == 1$ **then**
  return B_1-*smooth decomposition* S *of* m
return $\perp$

Theorem 5.2 ([9, § 4.3])**.** *The expected running time of the smoothing problem of an integer N with Pomerance's EAS and the ECM smoothness test is $L_N[1/3, c]$ where $c = (23/3)^{2/3}/3$, the smoothness bound is $B = L_N[2/3, \gamma]$, where $\gamma = 1/c$, $\theta = 4/9$, and $b = 8/23$.*

Theorem 5.3 ([9, § 4.5 Th. 4.5.1])**.** *The expected running-time of the smoothing problem of an integer N with k tests of Pomerance's EAS and the ECM smoothness test is $L_N[1/3, c]$ where*

$$c = 3^{1/3}((15 + 4(2/3)^{3k})/19)^{2/3} ,$$

the smoothness bound is $B = L_N[2/3, \gamma]$, where

$$\gamma = 1/c ,$$

the bound b_i for $0 \leq i \leq k-1$ on the remaining part m_i in Algorithm 4 is

$$b_i = (2/3)^{3(k-i)}19/(15 + 4(2/3)^{3k})$$

and the exponent θ_i for $0 \leq i \leq k$ is

$$\theta_i = (4/9)^{k-i} \ .$$

In Section 6.3, we will consider that pseudo-norms behave in terms of smoothness like integers bounded by N^e (instead of N). We will need

Lemma 5.4 ([18, 33, Lemma 1] Running-time of B-smooth decomposition of integers with ECM)**.** *Let N_i be integers taken uniformly at random and bounded by N^e, for a fixed real number $e > 0$. Write $B = L_N[\alpha_B, \gamma]$ the smoothness bound. Then the expected running-time to obtain a B-smooth N_i, using ECM for B-smooth tests, is $L_N[1/3, (3e)^{1/3}]$, obtained with $B = L_N[2/3, e/c = (e^2/3)^{1/3}]$.*

Lemma 5.5 ([52, 9] Running-time of B-smooth decomposition of integers with ECM and k-EAS)**.** *Let N_i be integers taken uniformly at random and bounded by N^e, for a fixed real number $e > 0$. Write $B = L_N[\alpha_B, \gamma]$ the smoothness bound. Then the expected running-time to obtain a B-smooth N_i, using ECM for B-smooth tests and Pomerance's Early Abort Strategy with one test, is $L_N[1/3, c = (3e)^{1/3}(23/27)^{2/3}]$, obtained with $B = L_N[2/3, e/c]$. The expected running-time with k-EAS is $L_N[1/3, c = (3e)^{1/3}((15 + 4(2/3)^{3k})/19)^{2/3}]$, with $B = L_N[2/3, e/c]$.*

We will mix Pomerance' strategy with our new initial splitting step to improve its running-time.

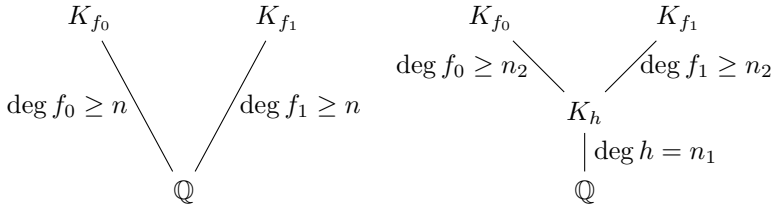
5.2. **LLL algorithm.** We recall an important property of the LLL algorithm [47] that we will widely use in this paper. Given a lattice $\mathcal{L}$ of $\mathbb{Z}^n$ defined by a basis given by an $n \times n$ matrix L, and parameters $\frac{1}{4} < \delta < 1$, $\frac{1}{2} < \eta < \sqrt{\delta}$, the LLL algorithm outputs a (η, δ)-*reduced basis* of the lattice. the coefficients of the first (shortest) vector are bounded by

$$(\delta - \eta^2)^{\frac{n-1}{4}} \det(L)^{1/n} \ .$$

In the remaining of this paper, we will simply denote by C this LLL approximation factor.

5.3. **NFS and Tower variants.**

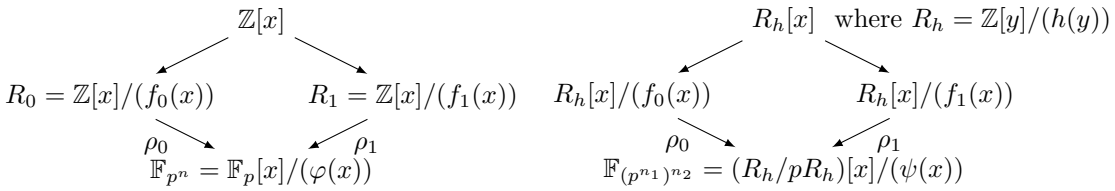
5.3.1. *Settings.* There exists many polynomial selection methods to initialize the NFS algorithm for large and medium characteristic finite fields. We give in Table 4 the properties of the polynomials that we need (degree and coefficient size) to deduce an upper bound of the pseudo-norm, as in (5.3), (5.4).



(A) NFS number fields (B) Tower-NFS number fields

FIGURE 1. Extensions of number fields for NFS and Tower variants

Three polynomials define the NFS setting: ψ, f_0, f_1, where f_0, f_1 are two polynomials of integer coefficients, irreducible over $\mathbb{Q}$, of degree $\geq n$, defining two non-isomorphic number fields, and whose GCD modulo p is an irreducible polynomial ψ of degree n, used to define the extension $\mathbb{F}_{p^n} = \mathbb{F}_p[x]/(\psi(x))$.



(A) NFS diagram for $\mathbb{F}_{p^n}$ (B) Tower-NFS diagram

FIGURE 2. NFS and Tower variant diagrams for $\mathbb{F}_{p^n}$

In a tower-NFS setting, one has $n = n_1 n_2$, $n_1, n_2 \neq 1$ and four polynomials are defined: h, ψ, f_0, f_1, where $\deg h = n_1$ and h is irreducible modulo p, $\deg \psi = n_2$ and ψ is irreducible modulo p, and $\gcd(f_0 \bmod (p,h), f_1 \bmod (p,h)) = \psi$. It can be seen as a generalization of the NFS setting as follows: writing $n = n_1 n_2$, one starts by defining a field extension $\mathbb{F}_{p^{n_1}} = \mathbb{F}_p[y]/(h(y))$, and then adapts any previously available polynomial selection designed for NFS in $\mathrm{GF}(p^{n_2})$, using $\mathbb{F}_{p^{n_1}}$ as the base field instead of $\mathbb{F}_p$. When $\gcd(n_1, n_2) > 1$, the polynomials f_0, f_1, resp. ψ will have coefficients in $\mathbb{Q}[y]/(h(y))$, resp. $\mathbb{F}_{p^{n_1}}$ instead of $\mathbb{Q}$, resp. $\mathbb{F}_p$. Then one defines the second extension $\mathbb{F}_{p_1^n}[x]/(\psi(x))$ of degree $n_2 = \deg_x \psi$.

Again, to cover all the cases, we consider $\mathbb{F}_{p^n} = \mathbb{F}_{(p^{n_1})^{n_2}}$. The NFS case will correspond to $n_1 = 1$, $n_2 = n$ and the original TNFS case to $n_1 = n$, $n_2 = 1$.

5.3.2. *Pseudo-Norm and Upper Bound.* Let f be a monic irreducible polynomial over $\mathbb{Q}$ and let $K = \mathbb{Q}[x]/(f(x))$ be a number field. Write $T \in K$ as a polynomial in x: $T = \sum_{i=0}^{\deg f - 1} a_i x^i$. The norm is defined by a resultant computation:

$$\mathrm{Norm}_{K/\mathbb{Q}}(T) = \mathrm{Res}(f, T) \ . \tag{5.1}$$

In the NFS case, we will consider elements expressed as polynomials in x whose coefficients are integers. We define the pseudo-norm as the resultant of the element with the given polynomial f:

$$T = \sum_{i=0}^{\deg f - 1} a_i x^i, \ \text{pseudo-norm}(T(x)) = \mathrm{Res}(T(x), f(x)) \ .$$

We use Kalkbrener's bound [40, Corollary 2] for an upper bound:

$$|\mathrm{Res}(f, T)| \leq \kappa(\deg f, \deg T) \|f\|_\infty^{\deg T} \|T\|_\infty^{\deg f}, \tag{5.2}$$

where $\kappa(n,m) = \binom{n+m}{n}\binom{n+m-1}{n}$ and $\|f\|_\infty = \max_{0 \leq j \leq \deg f} |f_j|$ is the absolute value of the largest coefficient. An upper bound for $\kappa(n,m)$ is $(n+m)!$. We will use the following bound in Section 6:

$$\mathrm{Norm}_{K_f/\mathbb{Q}}(T) \leq (\deg f + \deg T)! \|f\|_\infty^{\deg T} \|T\|_\infty^{\deg f} \ . \tag{5.3}$$

In a Tower-NFS case, we nest two resultants:

$$T = \sum_{i=0}^{\deg f - 1} \sum_{j=0}^{\deg h - 1} a_{ij} y^j x^i, \ \text{pseudo-norm}(T(x,y)) = \mathrm{Res}_y(\mathrm{Res}_x(T(x), f(x)), h(y)) \ .$$

A bound is [42, §A Lemma 2]

(5.4)

$$|N_{K_f/\mathbb{Q}} \sum_{i=0}^{\deg_x P} \sum_{j=0}^{\deg h - 1} a_{ij} \alpha_h^j \alpha_f^i| < \|a_{ij}\|_\infty^{\deg h \deg f} \|f\|_\infty^{\deg_x P \deg h} \|h\|_\infty^{(\deg_x P + \deg f)(\deg h - 1)} D(\deg h, \deg f) \ ,$$

where $\|a_{ij}\|_\infty = max_{i,j}|a_{ij}|$ and $D(d_1, d_2)$ is a combinatorial term, $D(d_1, d_2) = ((2d_2 - 1)(d_1 - 1) + 1)^{d_1/2}(d_1 + 1)^{(2d_2-1)(d_1-1)/2}((2d_2 - 1)!d_1^{2d_2})^{d_1}$.

6. Faster Initial Splitting with NFS and Tower variants for medium and large characteristic finite fields

We apply Algorithm 1 to the medium and large characteristic cases. For a general exposition, we assume that we are in a tower setting, where $Q = p^n = (p^{n_1})^{n_2}$, The elements of $\mathbb{F}_{p^n}$ are represented as $T = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} a_{i,j}y^j x^i$. the NFS setting corresponds to $n_1 = 1$, $n_2 = n$. When n is prime, the tower setting is $n_1 = n$, $n_2 = 1$ but our algorithm does not apply. Here we are not interested (only) in computing a preimage of degree as small as possible, but more generally whose size of pseudo-norm is as small as possible. According to the bounds (5.3), (5.4), we need to combine small coefficients $a_{i,j}$ (to reduce the contribution of $\|a_{ij}\|_\infty^{\deg h \deg f}$) with a small degree in x (to reduce the contribution of $\|f\|_\infty^{\deg_x P \deg h}$), and adjust the two, to find a pseudo-norm of smaller size.

6.1. The algorithm.

We start again with Algorithm 1, but we consider the whole row-echeloned matrix M, modify it, then use a lattice reduction algorithm (*e.g.* LLL or BKZ) to reduce the coefficients of the preimage we are computing.

To use a lattice reduction algorithm, we first need to compute a matrix representing the lattice, with integer coefficients. We modify Algorithm 1 as follows. Let d be the largest proper divisor of n, $1 < d < n$ ($d = \deg(h) = n_1$ is the case studied independently in the preprint [61]). We compute a polynomial basis $\{1, U, U^2, \ldots, U^{d-1}\}$ of the (proper) subfield $\mathbb{F}_{p^d}$. This time we will handle coefficients in $\mathbb{F}_p$ instead of $\mathbb{F}_{p^{n_1}}$ in the small characteristic case. Defines the $d \times n$ matrix L whose rows are made of the coefficients of $U^i T$ (the coefficients in $\mathbb{F}_{p^{n_1}}$ are exploded as n_1 coefficients in $\mathbb{F}_p$, which makes n coefficients) for $0 \le i \le d - 1$:

$$L_{d\times n} = \begin{bmatrix} T \\ UT \\ \vdots \\ U^{d-1}T \end{bmatrix} \in \mathcal{M}_{d,n}(\mathbb{F}_p) \ .$$

Then we compute a row-echelon form M of this matrix L by performing only $\mathbb{F}_p$-linear operations over the rows, so that each new row M_k is a $\mathbb{F}_p$-linear combination of the initial rows, that is

$$M_k = \sum_{i=0}^{d-1} \lambda_i U^i T = uT, \text{ where } \lambda_i \in \mathbb{F}_p$$

so that $M_k = uT$ with $u^{p^d-1} = 1$. To continue, we take a representative in $\{0, \ldots, p - 1\}$ for each coefficient of M to obtain a matrix $N \in \mathcal{M}_{d\times n}(\mathbb{Z})$. We join $n - d$ rows above N, made of the horizontal join of two matrices: on the left side, the square matrix $pI_{n-d,n-d}$, where $I_{n-d,n-d}$ is the $(n-d) \times (n-d)$ identity matrix, and on the right side, the $(n - d) \times d$ zero matrix.

The second modification is the following: when the desired degree $\deg_x P$ of the preimage P is $\ge n_2$, one has to enlarge below the row-echeloned matrix N with rows representing multiples of the polynomial ψ. In the classical NFS setting, the multiples are $x^i\psi$ and this trick is well-known for polynomial selection algorithms. Here we can use all the multiples of the form $x^i(y^j\psi \bmod h(y))$.

Remark 6.1. The important point to be noted is that the coefficients of $y^j\psi(x)$ should be reduced modulo the polynomial $h(y)$ (in particular if ψ has coefficients in K_h instead of integer coefficients).

Algorithm 5: Initial splitting,Tower-NFS setting

Input: Finite field $\mathbb{F}_{p^n}$, $n = n_1 n_2$, monic irreducible polynomials h, ψ s.t. $\mathbb{F}_{p^{n_1}} = \mathbb{F}_p[y]/(h(y))$, $\mathbb{F}_{(p^{n_1})^{n_2}} = \mathbb{F}_{p^{n_1}}[x]/(\psi(x))$, prime order subgroup $\ell \mid \Phi_n(p)$, generator g (of the order ℓ subgroup), target $T_0 \in \mathbb{F}_{p^n}$, degree of the preimage $\deg P$, polynomial f_i, smoothness bound B_1

Output: $t \in \{1, \ldots, \ell - 1\}$, $P \in \mathbb{Z}[x]$ s.t. $\log_g \rho(P) \equiv t + \log_g T_0$, and the pseudo-norm $\mathrm{Res}_y(\mathrm{Res}_x(P, f_i), h)$ is B_1-smooth

$d \leftarrow$ the largest divisor of n, $1 \le d < n$

Compute a polynomial basis $(1, U, U^2, \ldots, U^{d-1})$ of the subfield $\mathbb{F}_{p^d}$, where U satisfies $U^{p^d - 1} = 1 \in \mathbb{F}_{p^n}$

repeat

  take $t \in \{1, \ldots, \ell - 1\}$ uniformly at random

  $T \leftarrow g^t T_0 \in \mathbb{F}_{p^n}$

  $L \leftarrow \begin{bmatrix} T \\ UT \\ \vdots \\ U^{d-1}T \end{bmatrix} \in \mathcal{M}_{d \times n}(\mathbb{F}_p)$

  $M \leftarrow \mathbb{F}_p$– `ReducedRowEchelonForm`$(L)$

  $N \leftarrow$ lift M in $\mathcal{M}_{d \times n}(\mathbb{Z})$

  Define

  a $(\deg P + 1)n_2 \times (\deg P + 1)n_2$ lattice by the lower triangular matrix $N =$

$$\left[\begin{array}{ccccccccc}
p & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & p & & & & & & \\
 & & & N & & & & & \\
\psi(x) & & & & 1 & & & & \\
 & \ddots & & & & \ddots & & & \\
y^{\deg h - 1}\psi(x) \bmod h(y) & & & & & & 1 & & \\
 & & x\psi(x) & & & & & 1 & \\
 & & & \ddots & & & & & \ddots \\
x^{\deg_x P - n_2}(y^{\deg h - 1}\psi(x) \bmod h(y)) & & & & & & & & 1
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l} pI_{n-d}|O_{d,d},\ n-d \text{ rows} \\ N,\ d \text{ rows} \end{array}\right\} n \text{ rows} \\
\left.\begin{array}{l} y^j\psi(x) \bmod h(y), \\ 0 \le j \le n_1 - 1, \\ n_1 = \deg h \text{ rows} \end{array}\right\} \\
\left.\begin{array}{l} (y^j\psi(x) \bmod h(y))x^i, \\ 0 \le j \le n_1 - 1, \\ 1 \le i \le \deg_x P, \\ (\deg_x P - n_2)n_1 \text{ rows} \end{array}\right\} \begin{array}{l} (\deg_x P + 1 - n_2)n_1 \\ = (\deg_x P + 1)n_1 - n \\ \text{rows} \end{array}
\end{array}$$

  $N \leftarrow$ `LatticeReduction`(N)

  $P \leftarrow$ polynomial in $\mathbb{Z}[y, x]$ made of the shortest vector output by the `LatticeReduction` algorithm

until $\mathrm{Res}_y(\mathrm{Res}_x(P, f_i), h)$ *is* B_1*-smooth* `// ECM, ECM+EAS, or ECM+`k`-EAS`

return t, P, *factorization of* $\mathrm{Res}_y(\mathrm{Res}_x(P, f_i), h)$

6.2. **Properties and pseudo-norm size bound.**

Proposition 6.2. *The preimage P output by Algorithm 5 satisfies* $\log_g \rho(P) \equiv \log_g g^t T_0 = \log_g T_0 + t \bmod \Phi_n(p)$, *where* $\rho : \mathbb{Z}[x, y] \to \mathbb{F}_{(p^{n_1})^{n_2}}$ *was defined in Fig. 2.*

Proof of Proposition 6.2. Each row of the row-echelon matrix M represents a $\mathbb{F}_p$-linear combination of the d elements $U^i T$, $0 \le i \le d-1$, i.e. an element $\sum_{i=0}^{d-1} \lambda_i U^i T$, where $\lambda_i \in \mathbb{F}_p$. We can factor T in the expression. Each element $u_j = \sum_{i=0}^{d-1} \lambda_i U^i$

satisfies $u_j^{p^d-1} = 1$, *i.e.* is implicitly in $\mathbb{F}_{p^d}$ by construction. So each row represents an element $T_j = u_j T$, where $u_j^{p^d-1} = 1$ ($u_j \in \mathbb{F}_{p^d}$), so that $\log T_j \equiv \log T \bmod \Phi_n(p)$ by Lemma 2.4.

The second part of the proof uses the same argument: the short vector output by the LLL algorithm is a linear combination of the rows of the matrix N. Each row represents either 0 or a $\mathbb{F}_{p^d}$-multiple T_j of T, hence the short vector is also a $\mathbb{F}_{p^d}$-multiple of T. We conclude thanks to Lemma 2.4 that $\log \rho(P) \equiv \log T \bmod \Phi_n(p)$. □

Proposition 6.3. *The pseudo-norm of P in Algorithm 5 has size*

$$|\operatorname{Res}_y(\operatorname{Res}_x(P, f_i), h)| = O\left(Q^{(1-\frac{d}{n})\frac{\deg f_i}{\deg_x P+1}} \|f_i\|_\infty^{n_1 \deg_x P}\right) \tag{6.1}$$

assuming that $\|h\|_\infty = O(1)$.

Proof of Proposition 6.3. The matrix N computed in Algorithm 5 is a square matrix of $(\deg_x P + 1)n_1$ rows and columns, whose coefficients are in $\mathbb{F}_p$. Its determinant is $\det N = p^{n-d} = Q^{1-d/n}$. Using the LLL algorithm for the lattice reduction, the coefficients of the shortest vector P are bounded by $CQ^{(1-d/n)/((\deg_x P+1)n_1)}$, where C is the LLL factor. We obtain the bound (6.1) according to the bound formula (5.4), and neglecting the combinatorial factor $D(n_1, \deg f_i)$. Moreover in the Tower-NFS setting, the polynomial selection is designed such that $\|h\|_\infty = O(1)$. □

We finally obtain

Theorem 6.4. *Let $\mathrm{GF}(p^n)$ a finite field and d be the largest divisor of n, $d < n$ and $d = 1$ if n is prime. Let $n = n_1 n_2$ and h, ψ, f_i be given by a polynomial selection method. Let $T \in \mathbb{F}_{(p^{n_1})^{n_2}}$ an element which is not in a proper subfield of $\mathbb{F}_{p^n}$. Then there exists a preimage $P \in \mathbb{Z}[x, y]$ of T, of any degree (in x) between $\lfloor n_2 - d/n_1 \rfloor$ and $\deg f_i - 1$, of coefficients bounded by $O(Q^{(1-\frac{d}{n})\frac{1}{(\deg P+1)n_1}})$, and such that when P is mapped in $\mathbb{F}_{(p^{n_1})^{n_2}}$ as $\rho(P)$, its discrete logarithm is equal to the discrete logarithm of T modulo $\Phi_n(p)$ (and in particular modulo any prime divisor ℓ of $\Phi_n(p)$), that is*

$$\log \rho(P) \equiv \log T \bmod \Phi_n(p) \ .$$

The degree of P in x and the polynomial f_i can be chosen to minimize the resultant (pseudo-norm):

$$\min_i \min_{\lfloor n_2 - d/n_1 \rfloor \le \deg_x P \le \deg f_i - 1} \|f_i\|_\infty^{n_1 \deg_x P} Q^{(1-\frac{d}{n})\frac{\deg f_i}{\deg_x P+1}} \ .$$

We recall in Table 4 the degree and coefficient sizes of the polynomial selections published as July 2017.

Corollary 6.5. *With the notations of Table 4, and the NFS setting corresponding to $n_2 = n$ and $n_1 = 1$,*

1. *For the polynomial selection methods where there is a side i such that $\|f_i\|_\infty = O(1)$ (GJL, Conjugation, Joux–Pierrot and Sarkar–Singh up to now), we do the initial splitting on this side and choose $\deg_x P = \deg_x f_i - 1$ to obtain the smallest norm: $|\operatorname{Res}_y(\operatorname{Res}_x(P, f_i), h)| = O\left(Q^{1-\frac{d}{n}}\right)$. We obtain the same bound for NFS and its tower variants.*
2. *When $\|f_i\|_\infty = Q^{1/(2n)}$ as for the JLSV$_1$ method, the bound is $Q^{(1-\frac{d}{n})\frac{n_2}{\deg_x P+1} + \frac{\deg_x P}{2n_2}}$. When $\deg_x P = \deg_x f_i - 1 = n_2 - 1$, one obtains $Q^{\frac{3}{2} - \frac{d}{n} - \frac{1}{2n_2}}$. In the NFS setting, $n_2 = n$ while in the tower setting, $n_2 < n$ and the pseudonorm is slightly smaller.*

(3) *When* $\|f_i\|_\infty = Q^{1/(n_1(D+1))}$ *as for the JLSV*$_2$ *method, the lower bound is* $Q^{\frac{\deg_x P}{D+1}+(1-\frac{d}{n})\frac{n_2}{\deg_x P+1}}$ *on the* f_0*-side where* $\deg f_0 = n_2$*, and* $Q^{\frac{\deg_x P}{D+1}+(1-\frac{d}{n})\frac{D}{\deg_x P+1}}$ *on the* f_1*-side, where* $\deg f_1 = D \geq n_2$*. According the a given value of* n*, one can decide which value of* $\deg_x P$ *will produce a smaller norm.*

TABLE 4. Properties: degree and coefficient size of the main polynomial selection methods for NFS-DL in $\mathbb{F}_Q$, where $Q = p^n$

The coefficient sizes are in $O(x)$. To lighten the notations, we simply write the x term. In the Joux–Pierrot method, the prime p can be written $p = p_x(x_0)$, where p_x is a polynomial of tiny coefficients and degree at least 2. This table takes into account the methods published until July 2017.

method	$\deg h$	$\deg f_0$	$\deg f_1$	$\|f_0\|_\infty$	$\|f_1\|_\infty$
NFS					
JLSV$_1$ [36]		n	n	$Q^{1/2n}$	$Q^{1/2n}$
JLSV$_2$ [36]		n	$D > n$	$Q^{1/(D+1)}$	$Q^{1/(D+1)}$
GJL [50, 9, 12]		$D+1$	$D \geq n$	$\log p$	$Q^{1/(D+1)}$
Conjugation [12]		$2n$	n	$\log p$	$Q^{1/2n}$
Joux-Pierrot [38], $p = p_x(x_0)$		$n(\deg p_x)$	n	$\log p$	$Q^{1/(n \deg p_x)}$
Sarkar-Singh [56], $n = n_1 n_2,\ D \geq n_2$		$(D+1)n_1$	Dn_1	$\log p$	$Q^{1/(n_1(D+1))}$
Tower-NFS					
TNFS + base $-\ m$ [14]	n	D	1	$p^{1/D}$	$p^{1/D}$
Tower-JLSV1 $n = n_1 n_2$	n_1	n_2	n_2	$Q^{1/(2n)}$	$Q^{1/(2n)}$
Tower-JLSV2, $n = n_1 n_2$ [41, 42]	n_1	n_2	$D \geq n_2$	$Q^{1/(n_1(D+1))}$	$Q^{1/(n_1(D+1))}$
Tower-GJL $n = n_1 n_2$ [42]	n_1	$D+1$	$D \geq n_2$	$\log p$	$Q^{1/(n_1(D+1))}$
Tower-Conjugation $n = n_1 n_2$ [10, 42, 43]	n_1	$2n_2$	n_2	$\log p$	$Q^{1/(2n)}$
Tower-Joux–Pierrot $n = n_1 n_2, p = p_x(x_0)$ [42, 43]	n_1	$n_2(\deg p_x)$	n_2	$\log p$	$Q^{1/(n \deg p_x)}$
Tower-Sarkar–Singh $n = n_1 n_2 n_3,\ D \geq n_3$ [54, 57, 55]	n_1	$(D+1)n_2$	Dn_2	$\log p$	$Q^{1/(n_1 n_2(D+1))}$

6.3. **Running-time.** To apply Lemma 5.4 to the initial splitting case, we make the usual heuristic assumption that the pseudo-norms of the elements $g^t T_0$ behave asymptotically like random integers of the same size. Their size is $O(Q^e)$, so we replace N^e by Q^e. The basis $\{1, U, \ldots, U^{d-1}\}$ can be precomputed. The cost of computing the $U^i T$ for $0 \leq i \leq d-1$ is at most dn^2 multiplications in $\mathbb{F}_p$ with a schoolbook multiplication algorithm. We can roughly upper-bound it by $O(n^3)$. The time needed to compute the reduced row-echelon form of a $d \times n$ matrix is in $O(n^3)$ which is polynomial in n [24]. These two complexities are asymptotically negligible compared to any $L_Q[\alpha > 0]$. We obtain

Corollary 6.6. *The running-time of the initial splitting step with Algorithm 5 to find a* B*-smooth pseudo-norm, where the pseudo-norm has size* $O(Q^e)$ *for a fixed real number* $e > 0$ *determined by the polynomial selection (Table 4, two right-most colums), is*

(1) $L_Q[1/3, c = (3e)^{1/3}]$ *with ECM to perform the smoothness tests;*

(2) $L_Q[1/3, c = (3e)^{1/3}(23/27)^{2/3}]$ *with ECM and EAS;*
(3) $L_Q[1/3, c = (3e)^{1/3}((15 + 4(2/3)^{3k})/19)^{2/3}]$ *with ECM and k-EAS.*

For each case, the lower bound was obtained for $B = L_Q[2/3, e/c]$.

Corollary 6.5 gives a bound on the size of the pseudo-norms, from which we can deduce e to apply Corollary 6.6, and get the expected running-time.

7. Examples

Example 7.1. Let $p = \lfloor 10^{25}\pi \rfloor + 7926 = 31415926535897932384634359$ a 85-bit prime made of the first 26 decimals of π, so that $\mathbb{F}_{p^6}$ is a 509-bit finite field. Moreover $\Phi_6(p) = p^2-p+1$ is the 170-bit prime $\ell = 986960440108935861883947021513080740536833738706523$. We want to compute discrete logarithms in the order-ℓ cyclotomic subgroup of $\mathbb{F}_{p^6}$. The JLSV$_1$ method computes two polynomials f_0, f_1 where $\deg f_0 = \deg f_1 = 6$, and $\|f_i\|_\infty \approx p^{1/2}$. In our example, we have $\log_2 \|f_0\|_\infty = 44.67$ and $\log_2 \|f_1\|_\infty = 46.67$ (and $\frac{1}{2}\log_2 p = 42.35$).

$$\begin{aligned} f_0 &= x^6 - 11209975711932\, x^5 - 28024939279845\, x^4 - 20\, x^3 + 28024939279830\, x^2 \\ &\quad +11209975711938\, x + 1 \\ f_1 &= 5604994576830\, x^6 + 20986447533158\, x^5 - 31608799819555\, x^4 - 112099891536600\, x^3 \\ &\quad -52466118832895\, x^2 + 12643519927822\, x + 5604994576830 \end{aligned}$$

Since f_0 is already of degree 6 and monic, it can define the extension $\mathbb{F}_{p^6} = \mathbb{F}_p[x]/(f_0(x))$. Let T_0 be our target in $\mathbb{F}_{p^6}$ whose coefficients are made of the decimals of π (starting at the 26-th decimal, since the first 25 ones were already used for p).

$$\begin{aligned} T_0 &= 6427704988581508162162455\, x^5 + 16240052432693899613177738\, x^4 \\ &\quad +45093902837809499090201 39\, x^3 + 3868374359445757647591444\, x^2 \\ &\quad +82097559136021129208081 22\, x + 3279502884197169399375105 \end{aligned}$$

Let $g = x + 3$ be a generator of $\mathbb{F}_{p^6}$. Let $(1, U, U^2)$ be a polynomial basis of $\mathbb{F}_{p^3}$ considered as an implicit subfield of $\mathbb{F}_{p^6}$, where $U = g^{1+p^3} = \mathrm{Norm}_{\mathbb{F}_{p^6}/\mathbb{F}_{p^3}}(g)$. We run Algorithm 5 and found that the fourth preimage of $T = g^{812630}T_0$ gives a 61-smooth pseudo-norm. We compute the reduced row-echelon form M of the matrix $\begin{bmatrix} T \\ UT \\ U^2T \end{bmatrix}$ to be $M =$

$$\begin{bmatrix} 309307783589872533731980 53 & 161722767329614778864718 65 & 251875570676859576731124 & 1 & 0 & 0 \\ 898107170664718087063300 8 & 262971212330086624765059 21 & 499954586742598970758992 7 & 438055394047024712492645 1 & 1 & 0 \\ 478750294182786678769808 5 & 188554197294627445369875 06 & 154503476287753387686732 52 & 310921634924444115970112 43 & 982438275618110988698846 1 & 1 \end{bmatrix}$$

Then we reduce with LLL the following lattice defined by the (6×6)-matrix, where $m_{i,j}$ stands for the coefficient of the i-th row and j-th column of the above matrix M, and $m_{i,3+i} = 1$:

$$N = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ m_{1,1} & \dots & & m_{1,4} & 0 & 0 \\ m_{2,1} & & \dots & & m_{2,5} & 0 \\ m_{3,1} & & & \dots & & m_{3,6} \end{bmatrix}$$

Each row of LLL(N) gives us a preimage $P \in \mathbb{Z}[x]$ of short coefficients, such that $\log_2 \|P\|_\infty \approx \frac{1}{2}\log_2 p = 42.34$ bits and $\log \rho(P) \equiv \log T \bmod \ell$ (in other words,

$(T/\rho(P))^{\frac{p^6-1}{\ell}} = 1$). The fourth row has coefficients of at most 41.82 bits and gives

$$\begin{aligned} P &= 4821654023653x^5 + 3892831179802x^4 + 2694050932529x^3 + 2325450478817x^2 \\ &\quad +1117470283668x + 3688595236671 \ . \end{aligned}$$

The pseudo-norm of P w.r.t. f_0 is

$$\begin{aligned} \mathrm{Res}(P, f_0) &= 3260155118418797860288782022278028036855679121340635278795985947888200 9\backslash \\ &\quad 8941171005210581276328537987769936351535827542939231218958274136018656 1 \end{aligned}$$

of 471 bits, which is very close to $\log_2 Q^{11/12} = 466$ bits. Its factorization in prime ideals of K_{f_0} is
$\langle 3, x+2\rangle^3\langle 11, x+5\rangle\langle 17, x+4\rangle\langle 67, x+44\rangle\langle 2011, x+463\rangle\langle 501997, x+18312\rangle$
$\langle 340575947, x+27999767\rangle\langle 506032577, x+177467846\rangle\langle 604579099, x+309800481\rangle$
$\langle 1402910243559283, x+1034551157262971\rangle\langle 1587503571970639, x+524543605465730\rangle$
$\langle 36834399852305717, x+2491650720793075 2\rangle\langle 242270403627311729, x+170018299727614229\rangle$
$\langle 1070632553963863603, x+408232161861505290\rangle\langle 4305864084909925127, x+3252872861595329896\rangle$.
A common choice for the factor basis would be to set its smoothness bound to 30 or 32 bits. There are six prime ideals whose norm is larger than 30 bits, and that should be retreated to reach the factor basis. This initial splitting: testing all speudo-norms obtained for $g^i T_0$, i from 0 to 930000, that is $5.58 \cdot 10^6$ speudo-norms, with our Magma implementation, took 0.95 day on one node of 16 physical cores (32 virtual cores thanks to hyperthreading) Intel Xeon E5-2650 @ 2.0GHz, that is 15.2 core-days.

Example 7.2 (A more general example with NFS)**.** Assume that n is even and let $T \in \mathbb{F}_{p^n}$. Compute a polynomial basis $(1, U, U^2, \ldots, U^{n/2-1})$ of the subfield $\mathbb{F}_{p^{n/2}}$. Let

$$L = \begin{bmatrix} T \\ UT \\ \vdots \\ U^{n/2-1}T \end{bmatrix} \text{ and compute } M = \begin{bmatrix} m_{1,1} & \ldots & m_{1,\frac{n}{2}-1} & 1 & 0 & \ldots & 0 \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & 0 \\ m_{\frac{n}{2}} & & & \ldots & & m_{\frac{n}{2},\frac{n}{2}-1} & 1 \end{bmatrix}$$

to be the reduced echelon form of L. Then we define the lower triangular matrix made of the $n/2 \times n/2$ identity matrix with p on the diagonal in the upper left quarter, the $n/2 \times n/2$ zero matrix in the upper right quarter, and the $n/2 \times n$ matrix M in reduced echelon form in the lower half. Moreover if $\deg(f) > n$ then we add $(\deg f - n - 1)$ rows made of the coefficients of $x^i\psi$ where $\mathbb{F}_{p^n} = \mathbb{F}_p[x]/(\psi(x))$, for $0 \le i < \deg f - n - 1$. Finally we apply the LLL algorithm to this matrix. The short vector gives us a preimage P whose pseudo-norm is bounded by $Q^{1/2}$, with a polynomial selection such that $\|f\|_\infty = O(1)$ (such as Conj and GJL). Applying Lemma 5.4, we set the bound B_1 to be $B_1 = L_Q[2/3, ((1/2)^2/3)^{1/3} \approx 0.436]$. The running-time of Algorithm 5 will be $L_q[1/3, (3/2)^{1/3} \approx 1.144]$. We obtain preimages P whose pseudo-norm is bounded by $Q^{1-\frac{1}{2n}}$ with the JLSV_1 polynomial selection method as shown in Example 7.1. Applying Lemma 5.4, we set the bound B_1 to be $B_1 = L_Q[2/3, ((1-\frac{1}{2n})^2/3)^{1/3}]$. The running-time of Algorithm 5 will be $L_Q[1/3, (3(1-\frac{1}{2n}))^{1/3}]$.

8. Optimal representation: monic polynomial of degree $\varphi(n)$

In Section 3, we exploited the largest proper subfield $\mathbb{F}_{p^d}$ of $\mathbb{F}_{p^n}$ to find an alternative representation of a given element $T \in \mathbb{F}_{p^n}$, with $n-d$ non-zero coefficients, and $d-1$ coefficients (in $\mathbb{F}_p$) set to zero. The key-ingredient was to compute an expression of the form $P = uT$, where P has $d-1$ coefficients set to zero, and $u \in \mathbb{F}_{p^d}$, so that we have the equality $(P/T)^{(p^n-1)/\Phi_n(p)} = 1$. We can generalize this

strategy: given an element T in the cyclotomic subgroup of $\mathbb{F}_{p^n}$, of order $\Phi_n(p)$, we would like to compute an element $P \in \mathbb{F}_{p^n}$ such that $(P/T)^{(p^n-1)/\Phi_n(p)} = 1$ and P has only $\varphi_n(p) = \deg \Phi_n(x)$ non-zero coefficients in $\mathbb{F}_p$. To achieve that, we would like to compute an expression

$$T = u_1 u_2 \dots u_i P, \quad \text{where each } u_i \text{ is in a proper subfield } \mathbb{F}_{p^{d_i}} \text{ of } \mathbb{F}_{p^n} .$$

Given an element $T \in \mathbb{F}_{p^n}$ such that $T^{(p^n-1)/\Phi_n(p)} \neq 1$ (in other words, its order in the cyclotomic subgroup of $\mathbb{F}_{p^n}$ is not zero), we can sometimes compute an element P with $\varphi(n)$ non-zero coefficients, where $\varphi(n)$ is the Euler totient function, plus a monic leading term. Since in Algorithm 5 we do not need a one-to-one correspondence between the given elements of the cyclotomic subgroup on one hand, and their representation with only $\varphi(n)$ non-zero non-one coefficients on the other hand, we can just solve a system of equations even if we do not expect a solution at all times. If no such compact representation is found, one picks a new t and tests for the next $g^t T_0$. To define the system to be solved, we list all the distinct subfields $\mathbb{F}_{p^d}$ of $\mathbb{F}_{p^n}$ that are not contained themselves in another proper subfield, compute a polynomial basis for each of them, and allow a degree of freedom for the coefficients to be $\varphi(d)$ for each subfield $\mathbb{F}_{p^d}$. If we consider the system as Gröbner basis computation, it becomes very costly even for $\mathbb{F}_{p^{30}}$, where we need to handle $n - \varphi(n) - 1 = 21$ variables. We give a numerical example for $\mathbb{F}_{p^6}$.

What we do is different than what is done in XTR and CEILIDH compact representations: in [60, 59] for instance, the aim is to define a one-to-one correspondance between the elements in the torus of $\mathbb{F}_{p^n}$, and the set of coefficients $(\mathbb{F}_p)^{\varphi(n)}$. This optimal compression was achieved for $n = 6$ but not for $n = 30$.

8.1. **Compressed representation of elements in the cyclotomic subgroup of $\mathbb{F}_{p^6}$ by a monic polynomial of degree 2.** We consider the finite field $\mathbb{F}_{p^6}$. We will use the two subfields $\mathbb{F}_{p^2}$ and $\mathbb{F}_{p^3}$ to cancel three coefficients. Let $U \in \mathbb{F}_{p^6}$ such that $(1, U, U^2)$ is a basis of $\mathbb{F}_{p^3} \subset \mathbb{F}_{p^6}$. Let $V \in \mathbb{F}_{p^6}$ such that $(1, V)$ is a basis of $\mathbb{F}_{p^2} \subset \mathbb{F}_{p^6}$. We want to solve

$$uvwT = (u_0 + u_1 U + u_2 U^2)(v_0 + v_1 V)wT = P$$

where $u \in \mathbb{F}_{p^3}$, $v \in \mathbb{F}_{p^2}$, $w \in \mathbb{F}_p$ and $P \in \mathbb{F}_{p^6}$ is represented by a monic polynomial in x of degree 2. To simplify, we set $u_2 = v_1 = 1$ so that we obtain equations where we can eliminate recursively the variables by computing resultants. We compute u, v, w such that $uvwT = P$, where $P = a_0 + a_1 x + x^2$ is monic of degree 2. We define the lattice

$$L = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ a_0 & a_1 & 1 \end{bmatrix}$$

The determinant of L is p^2 hence $\mathrm{LLL}(L)$ computes a short vector P of coefficient size bounded by $Cp^{2/3}$, with C the LLL approximation factor (we can take $C \approx 1$ in this practical case). The pseudo-norm of P will be in the JLSV_1 case $|\operatorname{Res}(P, f)| \approx \|P\|_\infty^6 \|f\|_\infty^2 = p^5 = Q^{5/6}$. This is better than the bound $Q^{11/12}$ obtained with the cubic subfield cofactor method. This specific method can be generalized to specific cases of finite fields where reducing as most as possible the degree of the target is the best strategy, like in Example 8.1. This technique was implemented in [32] for computing a new discrete logarithm record in $\mathbb{F}_{p^6}$ of 422 bits.

Example 8.1. We take the same finite field parameters as in Example 7.1, where $\mathbb{F}_{p^6} = \mathbb{F}_p[x]/(f(x))$. $g = x + 3$ is a generator of $\mathbb{F}_{p^6}$. $(1, U, U^2)$ where $U = g^{1+p^3}$ is a basis of $\mathbb{F}_{p^3}$ and $(1, V)$ where $V = g^{1+p^2+p^4}$ is a basis of $\mathbb{F}_{p^2}$. We solve the system $(u_0 + u_1 U + U^2)(v_0 + V)T = P$ where $u_i, v_i \in \mathbb{F}_p$ and P is monic and represented by

a polynomial of degree 2 instead of 5. We ran Algorithm 5 with this modification on the same machine (Intel Xeon E5-2650 @ 2.0GHz with hyperthreading turned on), from g^0T_0 to $g^{90000}T_0$. In average, the set of I candidates g^iT_0 led to six times more monic degree two polynomials P_i. We found that the third polynomial output for $T = g^{60928}T_0$ has a 64-bit-smooth pseudonorm. Testing the 90000 g^iT_0 (that is, $2.7 \cdot 10^5$ pseudo-norms) took 1.2 core-day.

$$\begin{aligned} u &= 1230723276504067753226029 3 + 1811688736376198892741749 7U + U^2 \\ v &= 3042251478862957549502540 1 + V \\ w &= 2147088856371930500490085 1 \\ P &= uvwT) = x^2 + 4791904874308502360876 13x + 6943966382910680737931850 \end{aligned}$$

We checked that $(P/T)^{\frac{p^6-1}{\ell}} = 1$, meaning that $\log_g P = \log_g T = 60928 + \log_g T_0$. Then we reduce the lattice defined by the matrix

$$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 6943966382910680737931850 & 4791904874308502360876 13 & 1 \end{bmatrix}$$

to get three polynomials of smaller coefficients, the third one being

$$R = 1073014026134419 38x^2 + -3201464245272711 1x + 60125316588415598$$

whose pseudo-norm is

$$\begin{aligned} \mathrm{Res}(R, f) &= 1247420065593933976264772085368689393082237317268524580013893532 0\backslash \\ &\quad 2251491895904106662360530142149762187886749730229487340028599492 1 \end{aligned}$$

of 429 bits, which corresponds to the estimate $\log_2 Q^{5/6} = 423$ bits. We still have $\log_g \rho(P) \equiv \log_g T_0 + 60928 \bmod \ell$. The pseudo-norm is 64-bit-smooth, its factorization into prime ideals is
$\langle 11, x + 8\rangle\langle 23, x + 15\rangle\langle 12239, x + 482\rangle$
$\langle 1144616018827, x + 218590032699\rangle\langle 2682498999539, x + 1582479651452\rangle$
$\langle 42175797334421, x + 14828919302862\rangle\langle 1195156519724071, x + 966160984838340\rangle$
$\langle 13533793331200309, x + 1222425903090272\rangle\langle 92644276473186311, x + 5754482791048201\rangle$
$\langle 101186915694167857, x + 4282643286676490 5\rangle\langle 20516170632026633467, x + 14633926248916275064\rangle$
The first three ideals being small enough to be in the factor basis, eight ideals on side 0 remain to be descended.

Conclusion

The algorithms presented in this paper were implemented in Magma and used for cryptographic-size record computations. It was shown in [3] that combined with a practical variant of QPA, our Algorithm 2 allows to compute a discrete logarithm in the finite field $\mathbb{F}_{3^{6\cdot 709}}$ at the same cost as in $\mathbb{F}_{3^{6\cdot 509}}$ with the previous state of the art. The large characteristic variant (Algorithm 5) was used in [32] for a 422-bit record computation in $\mathbb{F}_{p^6}$. It would be interesting to be able to generalize it further, to be able to exploit at the same time several subfields, and provide a practical implementation of it for cryptographic sizes.

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Inria Nancy–Grand Est, Équipe Caramba, 615 rue du jardin botanique, CS 20101, 54603 Villers-lès-Nancy Cedex, France

E-mail address: `aurore.guillevic@inria.fr`

URL: `https://members.loria.fr/AGuillevic`