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Julio Araujo, Frédéric Havet, Claudia Linhares Sales, Ana Silva. Proper orientation of cacti. Theoretical Computer Science, 2016, 639, pp.14-25. 10.1016/j.tcs.2016.05.016 . hal-01338646

**HAL Id: hal-01338646**

**<https://inria.hal.science/hal-01338646>**

Submitted on 29 Jun 2016

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# Proper orientation of cacti <sup>☆</sup>

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## Abstract

An orientation of a graph  $G$  is *proper* if two adjacent vertices have different in-degrees. The *proper-orientation number*  $\vec{\chi}(G)$  of a graph  $G$  is the minimum maximum in-degree of a proper orientation of  $G$ .

In [1], the authors ask whether the proper orientation number of a planar graph is bounded.

We prove that every cactus admits a proper orientation with maximum in-degree at most 7. We also prove that the bound 7 is tight by showing a cactus having no proper orientation with maximum in-degree less than 7. We also prove that any planar claw-free graph has a proper orientation with maximum in-degree at most 6 and that this bound can also be attained.

*Keywords:* proper orientation, graph coloring, cactus graph, claw-free graph, planar graph, block graph.

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## 1. Introduction

For basic notions and notations on Graph Theory and Computational Complexity, the reader is referred to [2, 3]. All graphs in this work are considered to be simple.

An *orientation*  $D$  of a graph  $G = (V, E)$  is a digraph obtained from  $G$  by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each  $v \in V(G)$ , the *in-degree* of  $v$  in  $D$ , denoted by  $d_D^-(v)$ , is the number of arcs with root  $v$  in  $D$ . We use the notation  $d^-(v)$  when the orientation  $D$  is clear from the context. An orientation  $D$  of  $G$  is *proper* if  $d^-(u) \neq d^-(v)$ , for all  $uv \in E(G)$ . An orientation with maximum in-degree at most  $k$  is called a *k-orientation*. The *proper-orientation number* of a graph

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<sup>☆</sup>This work was partially supported by the FUNCAP/CNRS project GAIATO INC-0083-00047.01.00/13 and by the CNPq-Brazil Universal project 459466/2014-3.

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<sup>2</sup>Partially supported by ANR under contract STINT ANR-13-BS02-0007.

<sup>3</sup>Partially supported by CAPES 99999.000458/2015-05 and CNPq 307252/2013-2.

<sup>4</sup>Partially supported by CNPq PDE 232891/2014-1.

$G$ , denoted by  $\vec{\chi}(G)$ , is the minimum integer  $k$  such that  $G$  admits a proper  $k$ -orientation.

This graph parameter was introduced by Ahadi and Dehghan [4]. They observed that this parameter is well-defined for any graph  $G$  since one can always obtain a proper  $\Delta(G)$ -orientation. Note that every proper orientation of a graph  $G$  induces a proper vertex coloring of  $G$ . Hence, we have the following sequence of inequalities:

$$\omega(G) - 1 \leq \chi(G) - 1 \leq \vec{\chi}(G) \leq \Delta(G).$$

These inequalities are best possible since, for a complete graph  $K_n$ :

$$\omega(K_n) - 1 = \chi(K_n) - 1 = \vec{\chi}(K_n) = \Delta(K_n) = n - 1.$$

In [4], the authors characterize the proper-orientation number of regular bipartite graphs, study other particular subclasses of regular graphs and prove the NP-hardness of the problem even when restricted to planar graphs.

Recently, it has been shown that the problem remains NP-hard for subclasses of planar graphs that are also bipartite and of bounded degree [1]. In the same paper, it is proved that the proper-orientation number of a tree is at most 4.

**Theorem 1 ([1]).** *Every tree has proper-orientation number at most 4.*

A natural question is to ask how this theorem can be generalized.

**Problem 2.** *Which graph classes containing the trees have bounded proper-orientation number?*

In [1], several generalizations are suggested: on the one hand, the authors ask whether the proper-orientation number of planar graphs is bounded; on the other hand, they asked whether the proper-orientation number can be bounded by a function of the treewidth. We pose a similar, but simpler, question.

**Problem 3.** *Is there a constant  $c$  such that  $\vec{\chi}(G) \leq c$ , for every outerplanar graph  $G$ ?*

Already this question seems highly non-trivial. One of the reasons is that, contrary to many other parameters like the chromatic number, the proper-orientation number is not monotonic. Recall that a graph parameter  $\gamma$  is *monotonic* if  $\gamma(H) \leq \gamma(G)$  for every (induced) subgraph  $H$  of  $G$ . For example, the tree  $T^*$ , depicted in Figure 1, satisfies  $\vec{\chi}(T^*) = 2$ , while  $\vec{\chi}(T^* \setminus \{x\}) = 3$  as  $T^* \setminus \{x\}$  is exactly the tree  $T_3$  mentioned in [1]. Its non-monotonicity makes it difficult to handle the proper-orientation number.

In this paper, we consider a standard graph class containing the trees, namely the cacti. A graph  $G$  is a *cactus* if every 2-connected component of  $G$  is either an edge or a cycle. Clearly, every cactus is an outerplanar graph. We prove that the proper orientation of such graphs is bounded by 7.

**Theorem 4.** *If  $G$  is a cactus, then  $\vec{\chi}(G) \leq 7$ .*

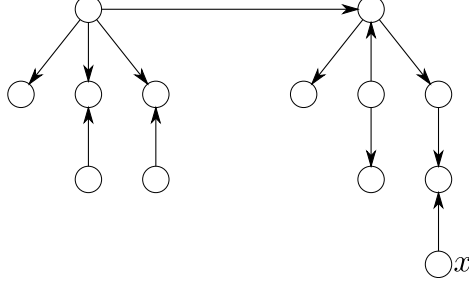


Figure 1: Tree  $T^*$  and a proper 2-orientation of it.

Furthermore, we show in Corollary 20 that this upper bound 7 is attained.

We conclude this section by introducing some definitions and previous results that we need in different sections of this work.

Let  $S \subseteq V(G)$  be a subset of vertices of  $G$  and  $F \subseteq E(G)$  be a subset of its edges. We denote by  $G[S]$  the subgraph of  $G$  induced by  $S$ , by  $G \setminus F$  the graph obtained from  $G$  by removing the edges in  $F$  from its edge set  $E(G)$ , and by  $G - S$  the graph  $G[V(G) \setminus S]$ .

For any two adjacent vertices  $u$  and  $v$  of  $G$ , the edge  $(u, v)$  is denoted by  $uv$ . Given an orientation  $D$  of  $G$ , we denote the orientation of  $uv$  towards  $v$  by  $(u, v)$ .

Let  $T$  be a tree. A *leaf* of  $T$  is a vertex with degree 1. A *twig* of  $T$  is a vertex which is not a leaf and whose neighbors are all leaves except possibly one. A *bough* of  $T$  is a vertex which is neither a leaf nor a twig and whose neighbors are all leaves or twigs except possibly one. A *branch* of  $T$  is a vertex which is neither a leaf nor a twig nor a bough and whose neighbors are all leaves or twigs or boughs except possibly one.

The definitions above are the same as the ones used in [1] and we borrow from them. Let  $G$  be a graph. The *block tree* associated to  $G$  is the tree  $T(G)$  with vertex set the set of blocks of  $G$  such that two vertices are adjacent in  $T(G)$  if and only if the blocks intersect. A block of order  $i$  is said to be an  *$i$ -block*. A *leaf block* (resp. *twig block*, *bough block*, *branch block*) is a block which is a leaf (resp. twig, bough, branch) in  $T(G)$ . By the definitions in the previous paragraph, observe that if  $B$  is of one of these types of blocks, then  $B$  may have a neighbor in  $T(G)$  that is an exception in its neighborhood. If such a neighbor  $B'$  exists and  $u \in B$  separates  $B$  from  $B'$ , then we call  $u$  the *root* of  $B$ . Otherwise, we pick any vertex of  $B$  to be the root of  $B$ . If  $B$  is a twig block with root  $r$ , then the *twig subgraph of  $G$  with root  $r$*  is the union of  $B$  and all leaf blocks with root in  $V(B) \setminus \{r\}$ . If  $B$  is a bough block with root  $r$ , then the *bough subgraph of  $G$  with root  $r$*  is the union of  $B$  and all twig subgraphs with root in  $V(B) \setminus \{r\}$ . Observe that twig and bough subgraphs are connected.

Let  $B$  be a block in  $G$ . For any vertex  $v \in B$  we denote by  $G_B \langle v \rangle$  the connected component of  $G \setminus E(B)$  containing  $v$ . If the block  $B$  is clear from the context, we often drop the subscript  $B$ .

## 2. Proper 7-orientation of cacti

In this section, we prove Theorem 4 by considering a minimum counter-example. Such a counter-example is a cactus  $G$  that admits no proper 7-orientation, and such that every cactus  $H$  with fewer vertices than  $G$  has a proper 7-orientation. Observe that such a counter-example  $G$  is clearly a connected graph.

The idea of the proof is to analyse the structure of the leaf, twig and bough subgraphs of  $G$  and observe that there is always one such subgraph in  $G$  with root  $r$  such that any proper 7-orientation of  $G\langle r \rangle$  (which exists by the minimality of  $G$ ) can be extended in a proper 7-orientation of  $G$ , which is a contradiction.

If  $B$  is a block of  $G$  with vertex set  $\{v_1, \dots, v_p\}$  appearing in this order on the cycle (or edge), then we write  $B$  as  $\langle v_1, \dots, v_p \rangle$ .

**Lemma 5.** *Let  $P = (v_1, \dots, v_n)$  be a path on  $n$  vertices,  $n \neq 2$ . Then, there exists a proper 2-orientation of  $P$  such that  $v_1$  and  $v_n$  have in-degree 0.*

*Proof.* If  $n$  is odd, it suffices to orient the arcs of  $P$  from vertices with odd indices towards vertices with even indices. This yields an alternating in-degree sequence of 0's and 2's that starts and ends with 0. If  $n$  is even, orient  $(v_1, \dots, v_{n-1})$  as above and  $v_{n-1}v_n$  towards  $v_{n-1}$  in order to obtain the desired orientation.  $\square$

Now we show that, in  $G$ , every vertex of small degree has a neighbor of higher degree.

**Proposition 6.** *Let  $u$  be a vertex of  $G$ . If  $d(u) \leq 7$ , then there exists  $v \in N(u)$  such that  $d(v) > d(u)$ .*

*Proof.* Suppose for a contradiction that  $d(u) \leq 7$  and all vertices in  $N(u)$  have degree at most  $d(u)$ . Let  $D$  be a proper 7-orientation of  $G - u$ . For each  $v \in N_G(u)$ , since  $d_{G-u}(v) = d_G(v) - 1 \leq d_G(u) - 1$ , we know that  $d_D^-(v) < d_G(u)$ . Therefore, because  $d_G(u) \leq 7$ , one can extend  $D$  by orienting every edge incident to  $u$  in  $G$  towards  $u$  to obtain a proper 7-orientation of  $G$ , a contradiction.  $\square$

**Proposition 7.** *Every leaf block of  $G$  is either a 2-block or a 3-block.*

*Proof.* Observe that, for any leaf block with at least four vertices, there must be at least one vertex of degree 2 whose neighbors also have degree 2, contradicting Proposition 6.  $\square$

Proposition 7 implies that a leaf block is either a 1-*path* (i.e. a path of length 1) or a *triangle* (i.e. a cycle of length 3). In Figure 2, we present every possible proper orientation of a leaf block.

**Proposition 8.** *Every vertex of  $G$  is contained in at most one leaf 2-block.*

*Proof.* By contradiction, suppose that it is not the case and let  $\langle u, v \rangle, \langle u, w \rangle$  be two leaf 2-blocks containing  $u$ . Let  $D$  be a proper 7-orientation of  $G - u$ . If  $d_D^-(u) \neq 1$ , orienting  $uw$  towards  $w$  extends  $D$  into a proper 7-orientation of  $G$ , a contradiction. Hence  $d_D^-(u) = 1$ . Since  $D$  is proper, the edge  $uv \in E(G)$  must

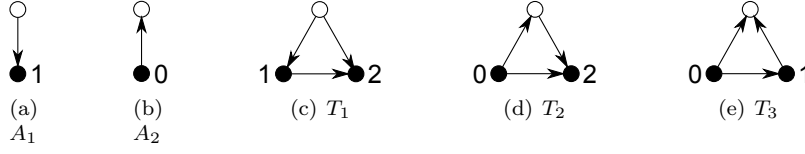


Figure 2: Leaf blocks and their possible proper orientations.

be the only one oriented towards  $u$  in  $D$ . Therefore all neighbors of  $u$  distinct from  $v$  and  $w$  have in-degree greater than 1 in  $D$ . Reverting the orientation of  $uv$  in  $D$  and orienting  $uw$  towards  $w$ , we obtain a 7-orientation of  $G$ , which is proper because the in-degree of  $u$  is 0, hence different from the in-degree of all of its neighbors. This is a contradiction.  $\square$

**Proposition 9.** *Every twig block is a 2-block or a 3-block.*

*Proof.* Let  $B$  be a twig block of order  $q$  at least 4, say  $B = \langle u_1, \dots, u_q \rangle$  with  $u_1$  the root of  $B$ .

**Claim 9.1.**  $d(u_i) \neq 3$ , for every  $i \in \{2, \dots, q\}$ .

*Subproof.* By contradiction, suppose that there exists a vertex  $u_i \in \{u_2, \dots, u_q\}$  of degree 3 in  $G$ . Note that  $u_i$  is contained in the block  $B$  and in a leaf 2-block, say  $\langle u_i, v \rangle$ .

First suppose that  $i \notin \{2, q\}$  and let  $G' = G - \{u_i, v\}$ . By the minimality of  $G$ , there exists a proper 7-orientation  $D$  of  $G'$ . If  $\{d_D^-(u_{i-1}), d_D^-(u_{i+1})\} \neq \{2, 3\}$ , then one could extend  $D$  to a proper 7-orientation of  $G$  by orienting  $u_i u_{i-1}$  and  $u_i u_{i+1}$  towards  $u_i$  and choosing the orientation of  $u_i v$  according to the in-degrees of  $u_{i-1}$  and  $u_{i+1}$  in  $D$ , a contradiction. Hence, without loss of generality, consider that  $d_D^-(u_{i-1}) = 2$  and  $d_D^-(u_{i+1}) = 3$ .

Let us extend  $D$  by orienting all the arcs incident to  $u_i$  away from this vertex. The resulting orientation  $D'$  is not yet proper but we shall prove how to change it into a proper 7-orientation of  $G$ . Problems could only appear in edges incident to  $u_{i-1}$  or  $u_{i+1}$  which had in-degree 3 and 4 respectively in  $D$ . Observe that these two vertices have degree more than 2 and thus belong to some other blocks which must be leaf blocks since  $u_1$  is the root of  $B$ . One can reorient the leaf blocks containing  $u_{i+1}$  using the orientations of Figure 2 so that the in-degree of  $u_{i+1}$  becomes 3 again. Similarly, if  $d(u_{i-1}) = 4$ , one can reorient the leaf blocks containing  $u_{i-1}$  so that the in-degree of  $u_{i-1}$  is in  $\{3, 4\} \setminus \{d_D^-(u_{i-2})\}$ , and if  $d(u_{i-1}) = 3$  (that is  $u_{i-1}$  is in a unique leaf 2-block), one can reorient the leaf block containing  $u_{i-1}$  so that the in-degree of  $u_{i-1}$  becomes 2 again. The resulting orientation is then a proper 7-orientation of  $G$ , a contradiction.

Suppose now that  $i \in \{2, q\}$ . Without loss of generality, we may assume that  $i = 2$ . Let  $G'$  be the connected component of  $G - u_3$  containing  $u_2$ . Let  $D'$  be a proper 7-orientation of  $G'$ . Clearly,  $d_{D'}^-(u_2) \leq 2$ . By the previous paragraphs and because  $q \geq 4$ , we know that  $d_G(u_3) \neq 3$ . If  $d_G(u_3) > 3$ , we

can obtain a proper orientation of  $G$  by orienting edges  $u_2u_3$  and  $u_3u_4$  towards  $u_3$  and orienting the leaf blocks containing  $u_3$  in such a way that  $d^-(u_3) \in \{3, 4\} \setminus d_D^-(u_4)$ ; this is a contradiction. Consequently,  $d(u_3) = 2$ , and we can suppose that  $2 \in \{d_D^-(u_2), d_D^-(u_4)\}$ , as otherwise we get a contradiction by adding  $(u_2, u_3)$  and  $(u_4, u_3)$ .

First, suppose that  $d_D^-(u_2) \neq 2$ , in which case one can verify that we can suppose that  $d_D^-(u_2) = 0$ . If  $d(u_4) > 3$ , we reorient the leaf blocks and  $u_3u_4$  so that  $u_4$  has in-degree in  $\{3, 4\} \setminus \{d_D^-(u_5)\}$ , then we let  $u_3$  have in-degree 1 or 2, depending on the orientation of  $u_3u_4$ . This gives us a contradiction, and, because  $d_D^-(u_4) = 2$ , we get that  $d(u_4) = 3$  and, by the previous paragraphs, that  $q = 4$ . Let  $v' \in N(u_4) \setminus B$ . We get a contradiction by reversing  $(v', u_4)$  and adding  $(u_2, u_3)$  and  $(u_3, u_4)$ .

Finally, suppose that  $d_D^-(u_2) = 2$ . Then we can also suppose that  $d_D^-(u_4) = 1$  as otherwise we can reverse  $(v, u_2)$ , orient  $u_2u_3$  towards  $u_2$ , and orient  $u_3u_4$  towards  $u_3$  to obtain a proper 7-orientation of  $G$ . By similar arguments, if  $d(u_4) > 3$ , then we can change its in-degree to some  $c \in \{3, 4\} \setminus \{d_D^-(u_5)\}$ ; hence  $d(u_4) \in \{2, 3\}$  and we analyse the cases below:

- $d(u_4) = 3$ : let  $v' \in N(u_4) \setminus B$ . Because  $d^-(u_4) = 1$ , we know that  $(v', u_4), (u_4, u_1) \in D$ . Reverse  $(v, u_2)$  and  $(v', u_4)$ , and add  $(u_3, u_2)$  and  $(u_4, u_3)$  to obtain a contradiction;
- $d(u_4) = 2$ : if  $q = 4$ , because  $d_D^-(u_2) = 2$  we know that  $d_D^-(u_1) \neq 2$ . Reverse  $(v, u_2)$  and add  $(u_3, u_2)$  and  $(u_3, u_4)$  to obtain a contradiction. Otherwise, by similar arguments we can suppose that  $d_D^-(u_5) = 2$ . Suppose that  $d(u_5) > 3$  and reorient the leaf blocks containing  $u_5$  and  $u_4u_5$  so that  $u_5$  has in-degree in  $\{3, 4\} \setminus \{d_D^-(u_6)\}$ . After this,  $u_4$  has in-degree either 0 or 1, in which case we reverse  $(v, u_2)$ , add  $(u_3, u_2)$  and either  $(u_4, u_3)$  or  $(u_3, u_4)$ , depending on  $u_4$ . Finally, we can suppose that  $d(u_5) = 3$  and  $q = 5$ . Let  $v' \in N(u_5) \setminus B$ . Reverse  $(v, u_2)$  and  $(v', u_5)$ , and add  $(u_3, u_2)$ ,  $(u_4, u_5)$  and  $(u_3, u_4)$  to get a contradiction.

◇

Now we return to the proof of the proposition. By the minimality of  $G$ , there is a proper 7-orientation  $D$  of  $G\langle u_1 \rangle$ .

We shall extend  $D$  into a proper 7-orientation of  $G$ , which gives us the desired contradiction. We first add  $(u_1, u_2), (u_1, u_q)$ . We then distinguish some cases according to  $d_D^-(u_1)$ .

Assume first  $d_D^-(u_1) \notin \{2, 4\}$ . Add  $(u_3, u_2), (u_{q-1}, u_q)$  and orient the path  $(u_3, \dots, u_{q-1})$  according to Lemma 5. So far the vertices  $u_2, \dots, u_q$  have in-degree 0, 1, or 2 in  $B$ . For each  $i \in \{2, \dots, q\}$ , if  $u_i$  is contained in some leaf block, then by Claim 9.1  $d(u_i) \geq 4$ . Thus, by Proposition 8,  $u_i$  is in at least one leaf 3-block. If  $u_i$  has in-degree 0 in  $B$ , then we orient all the leaf blocks containing  $u_i$  with  $A_1$  or  $T_1$ , so that  $u_i$  still has in-degree 0. If  $u_i$  has in-degree 1 (resp. 2) in  $B$ , we orient one leaf 3-block according to

$T_3$  and all other blocks according to  $A_1$  and  $T_1$ , so that its in-degree is 3 (resp. 4). It is now a simple matter to check that the obtained orientation is a proper 7-orientation of  $G$ .

Assume now  $d_D^-(u_1) \in \{2, 4\}$ . If  $q = 4$ , add  $(u_2, u_3)$  and  $(u_4, u_3)$ , and one can verify that we can get a contradiction again by orienting the leaf blocks containing vertices in  $B$  in the same way as above. So, suppose that  $q \geq 6$ . Add  $(u_2, u_3)$ ,  $(u_4, u_3)$ ,  $(u_q, u_{q-1})$ , and  $(u_{q-2}, u_{q-1})$ . Furthermore, if  $q = 7$  then add  $(u_4, u_5)$ , and if  $q > 7$  apply Lemma 5 to orient the path  $(u_4, \dots, u_{q-2})$ . We then orient the leaf blocks containing vertices in  $B$  in the same way as above to get a contradiction.

Therefore, we can consider  $q = 5$ . Add the arcs  $(u_1, u_2)$ ,  $(u_1, u_5)$ ,  $(u_3, u_4)$ , and  $(u_5, u_4)$  to  $D$ .

If  $d(u_2) > 2$ , then  $u_2$  is in a leaf 3-block. Add  $(u_3, u_2)$ , and orient one leaf 3-block containing  $u_2$  with  $T_2$  and the other leaf blocks with  $A_1$  or  $T_1$  so that  $u_2$  has in-degree 3. For  $j \in \{3, 4, 5\}$ , if  $u_j$  is contained in some leaf block, orient its leaf blocks so that the in-degree of  $u_j$  increases by 2 (using one  $T_3$  and possibly some  $A_1$  and  $T_1$ ). It is simple matter to check that it gives a proper 7-orientation of  $G$ . By symmetry, we get the result if  $d(u_5) = 2$ .

Finally, consider  $d(u_2) = d(u_5) = 2$ , and since  $B$  is not a leaf block, we can suppose, without loss of generality, that  $d(u_3) > 2$ . In this case, add  $(u_2, u_3)$ ,  $(u_3, u_4)$  and  $(u_5, u_4)$ , orient the leaf block(s) containing  $u_3$  so that its in-degree is 3 and, if  $d(u_4) > 2$ , orient the leaf block(s) containing  $u_4$  so that its in-degree is 4.

□

**Proposition 10.** *Let  $B$  be a twig block with root  $u_1$ .*

- (a) *If  $B = \langle u_1, u_2 \rangle$ , then either  $d(u_2) = 2$  or  $u_2$  belongs exactly to  $B$  and to a leaf 3-block.*
- (b) *If  $B = \langle u_1, u_2, u_3 \rangle$ , then, for each  $j \in \{2, 3\}$ ,  $u_j$  belongs exactly to  $B$  and either a leaf 2-block or a leaf 3-block.*

*Proof.* (a) Assume that  $d(u_2) > 2$ . Let  $D$  be a proper 7-orientation of  $G\langle u_1 \rangle$ . We can suppose that  $d(u_2) = 3$ , as otherwise we extend  $D$  to a proper 7-orientation of  $G$  by orienting  $u_1 u_2$  towards  $u_2$  and orienting the leaf blocks with root  $u_2$  in such a way that its in-degree belongs to  $\{3, 4\} \setminus \{d_D^-(u_1)\}$ . Consequently, by Proposition 7 and Proposition 8, we obtain that  $u_2$  is contained exactly in  $B$  and in a leaf 3-block.

(b) Suppose first that one vertex of  $\{u_2, u_3\}$ , say  $u_3$ , is in no leaf block, since  $B$  is a twig block, so  $d(u_2) \geq 3$ .

Suppose  $d(u_2) > 3$  and let  $D$  be a proper 7-orientation of  $G\langle u_1 \rangle$ . One can orient the edges  $u_1 u_2$  and  $u_1 u_3$  from  $u_1$  to its neighbors and then orient the



leaf block(s) containing  $u_2$  and the edge  $u_2u_3$  in such a way that the in-degree of the pair  $(u_2, u_3)$  is  $(3, 2)$ , in case  $d_D^-(u_1) \notin \{2, 3\}$ , or  $(4, 1)$ , otherwise. This results in a proper 7-orientation of  $G$ , a contradiction.

If  $d(u_2) = 3$ , then let  $D$  be a proper 7-orientation of  $G - v$ , where  $v$  is the neighbor of  $u_2$  not in  $B$ . Since  $u_2$  and  $u_3$  are symmetric in  $G - v$ , we can suppose that  $d_D^-(u_2) \neq 1$ , in which case we can extend  $D$  into a proper 7-orientation of  $G$  by orienting  $u_2v$  towards  $v$ . This is a contradiction.

Suppose now that  $d(u_2) > 2$ , and  $d(u_3) > 2$ . If  $d(u_2) \geq 5$ , let  $G'$  be the component of  $G - u_2$  containing  $u_1$ . Let  $D$  be a proper 7-orientation of  $G'$ . One could then extend  $D$  to a proper 7-orientation of  $G$  by orienting the edges  $u_1u_2$  and  $u_2u_3$  towards  $u_2$  and orienting the leaf blocks containing  $u_2$  in such a way that its in-degree belongs to  $\{3, 4, 5\} \setminus \{d_D^-(u_1), d_D^-(u_3)\}$ . By symmetry, we get a contradiction in the same way if  $d(u_3) \geq 5$ . Therefore  $d(u_2) \leq 4$  and  $d(u_3) \leq 4$ . Then, the proposition follows by Proposition 7 and by Proposition 8.  $\square$

The *2-path*, the *kite*, the *bull*, the *elk*, and the *moose* are the rooted graphs depicted in Figure 3 where the root is the white vertex.

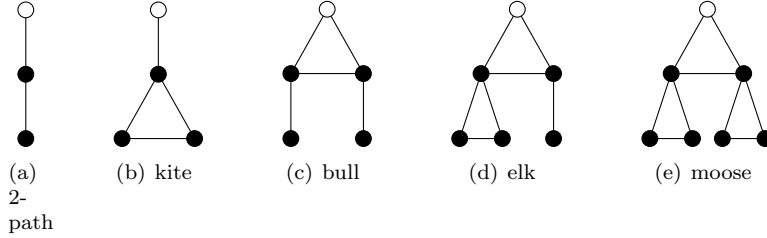


Figure 3: The five possible twig subgraphs.

Propositions 8, 9, and 10 imply directly the following.

**Corollary 11.** *Every twig subgraph in  $G$  is either a 2-path, or a kite, or a bull, or an elk, or a moose.*

In the following we will very often use this corollary without referring explicitly to it.

All the possible (partial) proper orientations of the twig subgraphs are depicted in Figures 4 to 8. In these figures, the notation  $i - j$  means that the corresponding vertex can have any in-degree in this range, depending on the orientation given to the non-oriented edges.

**Proposition 12.** *Let  $B$  be a bough block with root  $u$ . Every vertex  $v$  in  $V(B) \setminus \{u\}$  with degree at least 3 is the root of a twig subgraph or a leaf block that is neither a kite nor a moose.*

*Proof.* Let  $v$  be a vertex in  $V(B) \setminus \{u\}$  with degree at least 3. It must be the root of at least one twig subgraph or leaf block. Suppose the contrary that  $v$  is only root of kites and moose. Let  $S$  be the set of vertices that belong to such

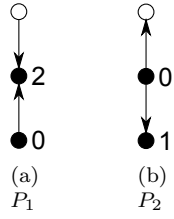


Figure 4: Proper orientations of the 2-path.

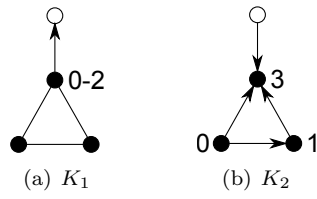


Figure 5: Proper orientations of the kite.

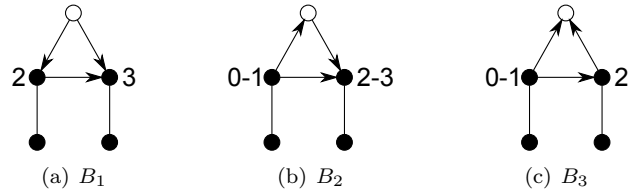


Figure 6: Proper orientations of the bull.

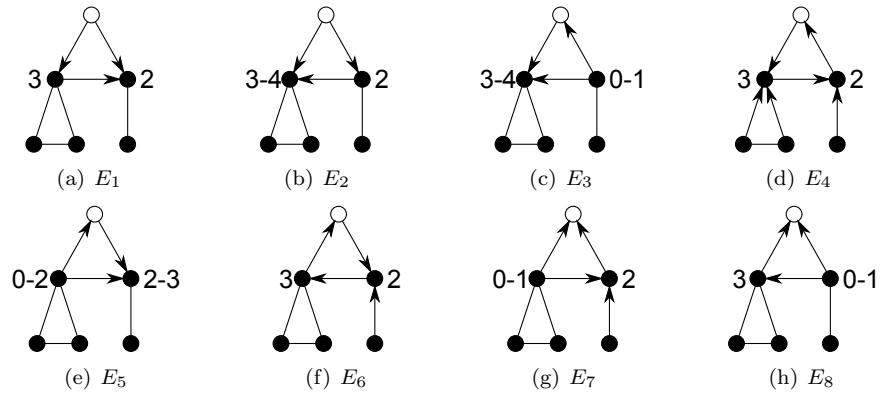


Figure 7: Proper orientations of the elk.

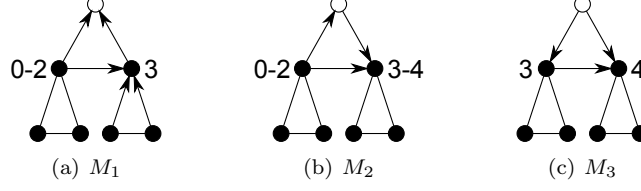


Figure 8: Proper orientations of the moose.

kites and moose rooted at  $v$ . Let  $D$  be a proper 7-orientation of the subgraph of  $G$  induced by  $(V(G) \setminus S) \cup \{v\}$ . Observe that  $d_D^-(v) \leq 2$ . Thus, one could extend  $D$  to  $G$  by orienting the kites and moose rooted at  $v$  according to  $K_2$  or  $M_3$ .  $\square$

**Proposition 13.** *Let  $B = \langle u_1, \dots, u_q \rangle$  be a bough block with root  $u_1$ . For all  $i \in \{2, \dots, q\}$ ,  $d(u_i) \leq 4$ .*

*Proof.* Let  $i \in \{2, \dots, q\}$ . Let  $G'$  be the connected component containing  $u_1$  in  $G - u_i$ . By the minimality of  $G$ ,  $G'$  admits a proper 7-orientation  $D$ . Set  $F = \{d_D^-(u_{i-1}), d_D^-(u_{i+1})\}$ . Add the arcs  $(u_{i-1}, u_i)$  and  $(u_{i+1}, u_i)$ .

If  $d(u_i) \geq 7$ , then one can properly orient the twig subgraphs and leaf blocks with root  $u_i$  in such a way that  $u_i$  has in-degree in  $\{5, 6, 7\} \setminus F$ . Observe that all other vertices of those graphs have in-degree at most 4, so we obtain a proper 7-orientation of  $G$ , a contradiction.

If  $d(u_i) = 6$ , by Proposition 12, it is contained in at most one moose. Therefore, one can orient the twig and leaf subgraphs containing  $u_i$  so that  $u_i$  has in-degree in  $\{4, 5, 6\} \setminus F$ , taking care to use  $M_1$  for the possible moose. Observe that every other possible twig can avoid a 4 from appearing in  $N(u_i)$ . Hence, we have a proper 7-orientation of  $G$ , a contradiction.

Thus, we can suppose that  $d(u_i) \leq 5$ , for all  $i \in \{2, \dots, q\}$ .

Assume now for a contradiction that there is some  $i \in \{2, \dots, q\}$  such that  $d(u_i) = 5$ .

If  $q = 2$ , then  $|F| = 1$  and one can extend  $D$  to a proper 7-orientation of  $G$  by orienting the twig and leaf blocks containing  $u_i$  so that the in-degree of  $u_i$  belongs to  $\{4, 5\} \setminus F$ . This is a contradiction so  $q \geq 3$ .

Observe that if  $\{4, 5\} \neq F$ , then one can extend  $D$  to  $G$  by orienting the twig and leaf blocks containing  $u_i$  in such a way that its in-degree belong to  $\{4, 5\} \setminus F$ . Consequently, we can assume that  $F = \{4, 5\}$ . But  $d(u_j) \geq d_D^-(u_j) + 1$ . So one vertex in  $\{u_{i-1}, u_{i+1}\}$  is  $u_1$ . Free to relabel the vertices in the other sense around  $B$ , we may assume that  $i = 2$ . Hence  $d_D^-(u_1) = 5$  and  $d_D^-(u_3) = 4$ . So  $d(u_3) = 5$ . Applying the same reasoning to  $u_3$ , we obtain that  $q = 3$ .

**Claim 13.1.** *There is a proper 7-orientation  $D'$  of  $G'$  such that  $d_{D'}^-(u_3) \in \{2, 3\}$ .*

*Subproof.* The idea is to start from  $D$  and to reorient the edges of the leaf blocks and twig subgraphs with root  $u_3$ . Observe that in  $D$  all the edges incident to  $u_3$  are directed towards  $u_3$ . In particular  $(u_1, u_3)$  is an arc of  $D$ .

By Propositions 7, 9 and 10,  $u_3$  is the root of:

1. two subgraphs,  $H_1$  and  $H_2$ , with  $H_1$  being a triangle, a bull, an elk or a moose, and  $H_2$  being a 1-path, a 2-path, or a kite; or
2. three subgraphs,  $H_1$ ,  $H_2$  and  $H_3$ , each of them being a 1-path, a 2-path, or a kite.

If Case 1 occurs, then we are in one of the following subcases.

- 1.1.  $H_1$  is a moose. Orient it using  $M_3$  and  $H_2$  using  $A_2$ ,  $P_2$  or  $K_1$  (with the in degree of its neighbor 0). This yields the desired proper orientation  $D'$  with  $d_{D'}^-(u_3) = 2$ .
- 1.2.  $H_1$  is an elk or a bull. If  $H_2$  is a 1-path or 2-path, then orient  $H_1$  with  $E_7$  or  $B_3$  and  $H_2$  with  $A_1$  or  $P_1$  to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 3$ . If not, then  $H_2$  is a kite. Orient  $H_1$  with  $E_3$  or  $B_2$  (with the neighbor of  $u_3$  having in degree different from 2) and  $H_2$  with  $K_2$  to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 2$ .
- 1.3.  $H_1$  is a triangle. Orient  $H_1$  with  $T_2$  and  $H_2$  with  $A_2$ ,  $P_2$  or  $K_1$  to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 3$ .

If Case 2 occurs, without loss of generality, we are in one of the following subcases.

- 2.1.  $H_1$  and  $H_2$  are kites. Orient  $H_1$  and  $H_2$  using  $K_2$  and  $H_3$  using  $A_2$ ,  $P_2$  or  $K_1$ , to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 2$ .
- 2.2.  $H_1$  is a kite or a 1-path or a 2-path, and  $H_2$  and  $H_3$  are 1-path or a 2-path. Orient  $H_1$  using  $K_1$  or  $A_2$  or  $P_2$ ,  $H_2$  using  $A_2$  or  $P_2$ , and  $H_3$  using  $A_1$  or  $P_1$ , to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 3$ .

◇

Now apply the above reasoning with the orientation  $D'$  given by Claim 13.1: we have  $F \neq \{4, 5\}$  because  $d_{D'}^-(u_3) \in \{2, 3\}$ . Therefore, we obtain a proper 7-orientation of  $G$ , a contradiction. □

Proposition 6 implies the following.

**Proposition 14.** *Let  $u$  be a vertex in  $G$ .*

- (a) *if  $u$  is the root of a kite or a bull, then  $d(u) \geq 4$ ;*
- (b) *if  $u$  is the root of an elk or a moose, then  $d(u) \geq 5$ .*

**Proposition 15.** *Every bough block is a 3-block.*

*Proof.* Let  $B = \langle u_1, \dots, u_q \rangle$  be a block with root  $u_1$ .

Assume first that  $q = 2$ . Let  $D$  be a proper 7-orientation of  $G\langle u_1 \rangle$ . By Proposition 13, we know that  $d(u_2) \leq 4$ . If  $d(u_2) = 4$ , we can orient the remaining edges in such a way that  $u_2$  has in-degree in  $\{3, 4\} \setminus \{d_D^-(u_1)\}$  taking care that all kites are oriented using  $K_1$ . This is possible because  $u_2$  is the root of at most two kites thanks to Proposition 12. This yields a proper 7-orientation of  $G$ , a contradiction.

Henceforth, since  $B$  is a bough block,  $u_2$  is the root of a twig subgraph  $H_1$ . In particular,  $d(u_2) = 3$ , and by Proposition 14,  $H_1$  is a 2-path, say  $(u_2, x, x')$ . Vertex  $u_2$  must also be the root of another subgraph  $H_2$  that is either a 2-path  $(u_2, y, y')$  or a 1-path  $(u, y)$ . Add the arc  $(u_1, u_2)$ . If  $d_D^-(u_1) \neq 3$ , one can orient  $H_1$  and  $H_2$  using  $P_2$  and  $A_2$  so that  $u_2$  get in-degree 3. This yields a proper 7-orientation of  $G$ , a contradiction. Assume  $d_D^-(u_1) = 3$ . If  $H_2$  is a 2-path, then orient  $H_1$  and  $H_2$  using  $P_1$  so that  $u_2$  get in-degree 1. If  $H_2$  is a 1-path, then orient  $H_1$  using  $P_2$  and  $H_2$  using  $A_1$  so that  $u_2$  get in-degree 2. In both cases, it results in a proper 7-orientation of  $G$ , a contradiction.

Now, suppose that  $q \geq 4$ . Note that Propositions 6 and 13 imply  $d(u_3) \leq 3$ , and that Proposition 14 implies that  $u_3$  is not root of a kite. So, either  $d(u_3) = 2$  or  $u_3$  is the root of a 1-path or a 2-path.

Suppose first that  $d(u_2) = 4$ . Let  $D$  be a proper orientation of  $G\langle u_2 \rangle - u_2$ . Because  $d(u_3) \leq 3$ , we get that  $d_D^-(u_3) \leq 2$ . Add the arcs  $(u_1, u_2)$  and  $(u_3, u_2)$ . By Proposition 12,  $u_2$  is neither the root of a moose nor of two kites. Therefore, one can orient the twig subgraphs and leaf blocks with root  $u_2$  so that its in-degree belongs to  $\{3, 4\} \setminus \{d_D^-(u_1)\}$ . This results in a proper 7-orientation of  $G$ , a contradiction.

Similarly, we get a contradiction if  $d(u_{q-1}) = 4$ , so we can assume that:  $(\star)$   $d(u_i) \leq 3$ , for all  $i \in \{2, \dots, q\}$ .

Now by Proposition 14-(a), if  $d(u_i) = 3$  for some  $i \in \{2, \dots, q\}$ , it is the root of a 1-path or a 2-path. Consequently, by  $(\star)$ , for all  $i \in \{3, \dots, q-1\}$ , it has no neighbor of degree more than 3. Thus, by Proposition 6, we get  $d(u_i) = 2$ , for every  $i \in \{3, \dots, q-1\}$ , and  $q \leq 5$ , for otherwise  $u_4$  has degree 2 and no neighbor of degree more than 2.

Since  $B$  is a bough block and not a twig block, one of its vertices distinct from the root  $u_1$  must be the root of a twig subgraph. Necessarily, it must be  $u_2$  or  $u_q$  as all other vertices have degree 2. By symmetry, we may assume that it is  $u_2$ . Furthermore, since  $d(u_2) = 3$ , by Proposition 14-(a),  $u_2$  is necessarily the root of a 2-path, say  $(u_2, x, x')$ .

Let  $D$  be a proper 7-orientation of  $G\langle u_1 \rangle$ . Orient the edges  $u_1 u_2$ ,  $u_1 u_q$  and  $u_2 u_3$  towards  $u_2$ ,  $u_q$  and  $u_3$ , respectively. We now describe how to extend this orientation in a proper 7-orientation of  $G$ , yielding the contradiction. We distinguish two cases depending on whether  $q = 4$  or  $q = 5$ .

- $q = 4$ . Assume first  $d(u_4) = 2$ . If  $d_D^-(u_1) \neq 2$ , add  $(u_3, u_4)$ ,  $(x, u_2)$  and  $(x, x')$ ; otherwise, add their reverses. So suppose that  $d(u_4) = 3$ . Then  $u_4$  is the root of either a 1-path  $(u_4, y)$  or a 2-path  $(u_4, y, y')$  by Proposition 14. If  $d_D^-(u_1) \neq 3$ , then  $D$  can be extended to  $G$  by reversing

$u_2u_3$  and orienting the remaining edges so that the in-degrees of  $u_2$  and  $u_4$  will be 3. If  $d_D^-(u_1) = 3$ . Add  $(u_4, y)$ . If  $u_4$  is the root of a 1-path, add  $(u_3, u_4)$ ,  $(x, u_2)$  and  $(x, x')$ . Otherwise,  $u_4$  is the root of a 2-path : add  $(u_4, u_3)$ ,  $(u_2, x)$ ,  $(x', x)$ , and  $(y', y)$ .

- $q = 5$ . By Proposition 6, we have  $d(u_5) = 3$ . So  $u_5$  is the root of either a 1-path  $(u_5, y)$  or a 2-path  $(u_5, y, y')$  by Proposition 14. If  $d_D^-(u_1) \neq 3$ , reverse  $u_2u_3$  and orient properly the remaining edges in a way that the in-degrees of  $u_2$  and  $u_5$  is 3. If  $d_D^-(u_1) = 3$ , first add  $(u_2, x)$ ,  $(x', x)$  and  $(u_4, u_3)$  to  $D$ . If  $u_5$  is the root of a 1-path, then add  $(u_5, y)$  and  $(u_4, u_5)$ ; otherwise,  $u_5$  is the root of a 2-path : add  $(y, u_5)$ ,  $(u_5, u_4)$  and  $(y, y')$ .

□

A *reindeer* is the graph depicted in Figure 9, where the root is the white vertex. It also depicts all possible orientations of the reindeer.

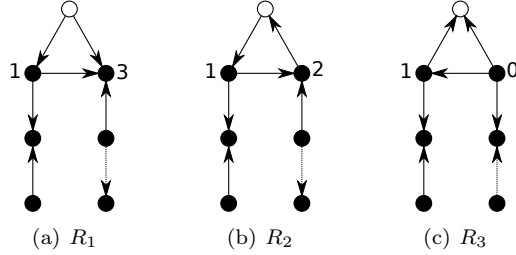


Figure 9: The reindeer and its possible orientations. The dashed edge may or may not exist.

**Proposition 16.** *Every bough subgraph is a reindeer.*

*Proof.* Let  $H$  be a bough subgraph rooted at  $u_1$ . It contains a bough block  $B$ . By Proposition 15,  $B$  is a 3-block, say  $B = \langle u_1, u_2, u_3 \rangle$ . By Proposition 13,  $d(u_2) \leq 4$  and  $d(u_3) \leq 4$ .

Let  $G'$  be the connected component of  $G - u_2$  containing  $u_1$ . Let  $D$  be a proper 7-orientation of  $G'$ .

Assume  $d(u_2) = 4$ . By Proposition 14,  $u$  is the root of no moose nor elk, and by Proposition 12, it is the root of at most one kite. If  $\{d_D^-(u_1), d_D^-(u_3)\} \neq \{3, 4\}$ , then adding  $(u_1, u_2)$  and  $(u_3, u_2)$  and using appropriate orientations of the twig subgraphs and leaf blocks with root  $u_2$ , one can get an orientation of  $D$  such that  $d^-(u_2) \in \{3, 4\} \setminus \{d_D^-(u_1), d_D^-(u_3)\}$ . This is a proper 7-orientation of  $D$ , a contradiction. Consequently,  $\{d_D^-(u_1), d_D^-(u_3)\} = \{3, 4\}$ , and so  $d_D^-(u_3) = d_G(u_3) - 1 = 3$ . Let  $x$  be a neighbor of  $u_3$  not in  $B$  and let  $H$  be the twig subgraph or leaf block with root  $u_3$  containing  $x$ . By Proposition 12, one can choose  $x$  so that  $H$  is not in a kite. Add  $(u_2, u_3)$  and use  $A_1$ ,  $T_2$ ,  $P_1$ , or  $B_2$  to reverse  $(x, u_3)$ . If  $u_2$  is not the root of two 2-paths, we can orient the twig subgraphs and leaf blocks with root  $u_2$  so that its in-degree becomes 2 by using orientations  $A$ ,  $T_2$ ,  $P_2$ ,  $K$  or  $B_2$ . If  $u_2$  is the root of two 2-paths, we can orient

these 2-paths using  $P_2$  so that  $u_2$  gets in-degree 1. In both cases, we obtain a proper 7-orientation of  $D$ , a contradiction.

Similarly, we get a contradiction if  $d(u_3) = 4$ . Therefore  $d(u_2) \leq 3$  and  $d(u_3) \leq 3$ . Since  $B$  is a bough block,  $u_2$  or  $u_3$  must be the root of a twig subgraph. Without loss of generality, we may assume that  $u_2$  is. By Proposition 14,  $u_2$  must be the root of a 2-path, say  $(u_2, x, x')$ .

Assume  $d(u_3) = 2$ . If  $d^-(u_1) \notin \{1, 2\}$ , add  $(u_2, x), (u_2, u_3), (x', x)$ , and if  $(u_3, u_1) \in D$ , reverse it and add  $(u_2, u_1)$ ; otherwise, add  $(u_1, u_2)$ . And if  $d^-(u_1) \in \{1, 2\}$ , add  $(u_1, u_2), (u_3, u_2), (x, u_2)$  and  $(x, x')$ . In both cases, it results in a proper 7-orientation of  $D$ , a contradiction.

Hence  $d(u_3) = 3$ , which by Proposition 12 implies that  $u_3$  is the root of either a 2-path or a 1-path. Therefore  $H$  is a reindeer.  $\square$

We can finally prove the main result of this paper.

*Proof of Theorem 4.* If  $G$  has no branch blocks, then there exists a vertex  $u$  such that  $G$  is the union of bough subgraphs, twig subgraphs and leaf blocks with root  $u$ . In this case, one may obtain a proper 4-orientation of  $G$  by orienting all bough subgraphs, twig subgraphs and leaf blocks so that the in-degree of  $u$  is 0.

Thus,  $G$  contains a branch block  $B$ . It must contain a vertex  $u$  which is the root of a bough subgraph  $R$ . By Proposition 16,  $R$  is a reindeer, and by Proposition 6, we have  $d(u) \geq 4$ . Denote by  $Q$  the subgraph rooted at  $u$  containing exactly all the bough, twig and leaf blocks rooted at  $u$ .

Let  $H$  be the component of  $G - u$  that contains  $B - u$ ; then  $u$  has at most 2 neighbors in  $H$ . By minimality of  $G$ ,  $H$  has a proper 7-orientation  $D$ . Let  $F$  be the set of in-degrees of neighbors of  $u$  in  $H$ . Orient the edges of  $H$  incident to  $u$  towards  $u$ .

If  $d(u) \geq 7$ , we can orient  $G \setminus \{u\}$  in such a way that  $u$  has in-degree in  $\{5, 6, 7\} \setminus F$  and no vertex in  $Q$  has in-degree more than 4. This gives a proper 7-orientation of  $G$ , a contradiction.

Assume  $d(u) = 6$ . Let  $\alpha$  be an integer in  $\{4, 5, 6\} \setminus F$ . We can orient  $Q$  in such a way that  $u$  has in-degree  $\alpha$  and no vertex of  $Q - u$  has in-degree  $\alpha$ . This is possible because no vertex has in-degree 5 in the orientations depicted in Figures 2, 4–8 and 9 and  $u$  is in at most two moose, so if  $\alpha = 4$ , we can orient the moose first using  $M_1$  or  $M_2$ . This gives a proper 7-orientation of  $G$ , a contradiction.

Assume  $d(u) = 4$ . If  $u$  has two neighbors in  $H$ , then  $Q = R$ . Let  $\alpha$  be an integer in  $\{2, 3, 4\} \setminus F$ . If  $\alpha = 2$ , then orient  $R$  with  $R_1$ ; if  $\alpha = 3$ , then orient  $R$  with  $R_2$ ; if  $\alpha = 4$ , then orient  $R$  with  $R_3$ . In each case, this yields a proper 7-orientation of  $G$ , a contradiction. If  $u$  has a unique neighbor in  $H$ , then  $Q$  is the union of  $R$  and either a 1-path, or a 2-path, or a kite. Orient that subgraph using  $A_2$ ,  $P_2$  or  $K_1$ . Now, since  $|F| = 1$ , we can orient  $R$  using  $R_2$  or  $R_3$  so that the in-degree of  $u$  in  $\{3, 4\} \setminus F$ . This yields a proper 7-orientation of  $G$ , a contradiction.

Finally assume  $d(u) = 5$ . If  $F \neq \{4, 5\}$ , we can orient the edges of  $Q$  so that the in-degree of  $u$  is some  $\alpha \in \{4, 5\} \setminus F$ , and no vertex of  $Q - u$  has in-degree

$\alpha$ . If  $\alpha = 4$ , this is possible because  $u$  is in at most one moose, and we can start orienting the moose with  $M_2$ . This yields a proper 7-orientation of  $G$ , a contradiction. If  $F = \{4, 5\}$ , then  $Q$  is the union of  $R$  and either a 1-path or a 2-path or a kite. In the first two cases, orient the 1-path or 2-path by using  $A_1$  or  $P_1$ , and  $R$  with  $R_2$ , so that vertex  $u$  has in-degree 3. In the latter case, orient the kite with  $K_2$  and  $R$  with  $R_1$ , so that vertex  $u$  has in-degree 2. In both cases, we obtain a proper 7-orientation of  $G$ , a contradiction.  $\square$

### 3. A tight example

Recall that a *block* graph is a graph such that each block is a clique. In the sequel, we find a tight example for Theorem 4. As a drawback, we obtain another tight example for Theorem 1 different to the one the authors in [1] propose and an example of a planar graph whose proper orientation must be at least 10.

**Theorem 17.** *Let  $k$  be a positive integer. There exists a block graph  $G(k)$  such that  $\omega(G) = k$  and  $\vec{\chi}(G) \geq 3k - 2$ .*

Let  $G$  be a connected graph, and  $K$  be a clique in  $G$ . We say that  $K$  is a *pending clique* of  $G$  if there exists  $u \in K$  such that there are no edges between  $K - u$  and  $V(G) - u$ . We say that  $u$  is the *root* of  $K$ .

**Lemma 18.** *Let  $G$  be a connected graph,  $K$  be a pending clique of  $G$  with size  $k$  and root  $u$ , and  $D$  be a proper orientation of  $G$ . If  $u$  has an in-neighbor in  $V(G) \setminus K$ , then  $d_D^-(u) \geq k$ .*

*Proof.* By contradiction, suppose that  $u$  has an in-neighbor in  $V(G) \setminus K$  and that  $d_D^-(u) = d \in \{1, \dots, k-1\}$ . Because  $d \in \{1, \dots, k-1\}$ , and  $d(v) = k-1$  for every  $v \in K \setminus \{u\}$ , we necessarily have that  $\{d_D^-(v) \mid v \in K\} = \{0, \dots, k-1\}$ . Consequently, there exist  $d$  vertices  $k_{i_0}, \dots, k_{i_{d-1}}$  in  $K$  such that  $d_D^-(k_{i_j}) = j$ , for every  $j \in \{0, \dots, d-1\}$ . Define  $k_{i_d} = u$ , similarly. Observe that all edges  $k_{i_j}u$  must be oriented towards  $u$ , since  $k_{i_j}k_{i_\ell}$  must be oriented towards  $k_{i_\ell}$ , whenever  $0 \leq j < \ell \leq d$ . This is a contradiction, because  $u$  has another in-neighbor that does not belong to  $K$  and thus  $d_D^-(u) \geq d+1$ .  $\square$

A *k-chandelier* is the graph obtained from a  $k$ -clique  $K = \{v_0, \dots, v_{k-1}\}$  by adding  $k-1$  pending  $k$ -cliques in the vertices  $v_1, \dots, v_{k-1}$ . We say  $v_0$  is the *root* of the  $k$ -chandelier and  $K$  is its *base*.

**Lemma 19.** *Let  $G$  be a  $k$ -chandelier with root  $v_0$  and base  $K = \{v_0, \dots, v_{k-1}\}$ . If  $D$  is a proper orientation of  $G$  such that  $(v_0, v_i) \in A(D)$  for every  $i \in \{1, \dots, k-1\}$ , then  $d_D^-(v_0) \notin \{k, \dots, 2k-2\}$ .*

*Proof.* Consider any  $i \in \{1, \dots, k-1\}$ . Since  $(v_0, v_i) \in A(D)$ , Lemma 18 yields  $d_D^-(v_i) \geq k$ . In addition  $d_D^-(v_i) \leq 2k-2$ , because  $d(v_i) = 2k-2$ . Therefore, we must have  $\{d_D^-(v_i) \mid v_i \in K - v_0\} = \{k, \dots, 2k-2\}$  and the lemma follows, because  $D$  is a proper orientation.  $\square$



*Proof of Theorem 17.* Let  $G(k)$  be the graph obtained as follows: we start with a  $k$ -clique  $K = \{v_1, \dots, v_k\}$  and then we add  $2k - 1$  pending  $k$ -cliques  $C_{i,j}$  on each  $v_i$ , for every  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, 2k - 1\}$ . Define

$$B = \bigcup_{i=1}^k \bigcup_{j=1}^{2k-1} C_{i,j}.$$

Note that at this point  $B$  contains all vertices we have added to  $G(k)$  so far. Then, for each  $u \in B$ , we add  $3k - 2$  copies of a  $k$ -chandelier and 2 pending  $k$ -cliques, all of them with root  $u$ . This finishes the construction of  $G(k)$ .

Suppose for a contradiction that there exists a proper orientation  $D$  of  $G(k)$  such that  $\Delta^-(D) \leq 3k - 3$ .

We claim that, for every  $u \in B$ , we have that  $d_D^-(u) \notin \{1, \dots, 2k - 2\}$ . Indeed, suppose that  $d_D^-(u) \neq 0$  and thus that  $u$  has an in-neighbor  $v$ . One of the two  $k$ -cliques pending in  $u$  does not contain  $v$ , so by Lemma 18,  $d_D^-(u) \geq k$ . Now recall that  $d_D^-(u) \leq \Delta^-(D) \leq 3k - 3$ . Therefore,  $u$  has no in-neighbors in at least one of the  $3k - 2$   $k$ -chandeliers with root  $u$ . Hence, by Lemma 19,  $d_D^-(u) \notin \{k, \dots, 2k - 2\}$ . This proves our claim.

Therefore the in-degrees of the vertices of  $B$  are in  $\{0, 2k - 1, \dots, 3k - 3\}$ . There are exactly  $k$  values in this set, so each  $k$ -clique in  $B$  must have exactly one vertex of each in-degree in this set. In particular, each of these cliques of  $B$  must contain a vertex of in-degree 0. Consider the vertex  $v_i \in K$  such that  $d_D^-(v_i) = 2k - 1$ . Let  $u_0 \in K$  be such that  $d_D^-(u_0) = 0$ , and, for each  $j \in \{1, \dots, 2k - 1\}$ , let  $u_j \in C_{i,j}$  be such that  $d_D^-(u_j) = 0$ . Since all edges  $u_j v_i$  are oriented towards  $v_i$ , we have that  $d_D^-(v_i) \geq 2k$ , a contradiction.  $\square$

One may see that Theorem 17 provides a tight example for Theorem 1 when  $k = 2$  and a tight example for Theorem 4 for  $k = 3$ .

**Corollary 20.** *There exist cacti  $G$  such that  $\overrightarrow{\chi}(G) \geq 7$ .*

Since every block graph  $G$  with  $\omega(G) = 4$  is planar, we also have the following corollary:

**Corollary 21.** *There exist planar graphs  $G$  such that  $\overrightarrow{\chi}(G) \geq 10$ .*

## 4. Further Research

### 4.1. Proper-orientation number of planar graphs

We believe that Problem 3 must be answered in the affirmative: outerplanar graphs have proper-orientation number bounded by a constant  $c$ . If such a  $c$  exists, then  $c \geq 7$ , since cacti (and in particular, the one described in Section 3) are outerplanar. A first step would be to establish the result for 2-connected outerplanar graphs. We actually believe that in this case this constant should be smaller than 7 and that it should not be much greater than 3. One can easily attain 3 as a lower bound using the following lemma.

**Lemma 22** [1]). *Let  $k$  be a positive integer, and let  $G$  be a graph containing a clique  $K$  of size  $k+1$ . In any proper  $k$ -orientation of  $G$ , all edges between  $V(K)$  and  $V(G) \setminus V(K)$  are oriented from  $V(K)$  to  $V(G) \setminus V(K)$ .*

**Proposition 23.** *There exists a 2-connected outerplanar graph  $G$  such that  $\vec{\chi}(G) = 3$ .*

*Proof.* Let  $G$  be the graph on six vertices defined by  $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  and  $E(G) = \{v_1v_2, v_2v_3, v_1v_3, v_4v_5, v_5v_6, v_4v_6, v_1v_4, v_2v_5\}$ . Suppose by way of contradiction that  $G$  has a proper 2-orientation  $D$ . Observe that the sets  $\{v_1, v_2, v_3\}$  and  $\{v_4, v_5, v_6\}$  are cliques in  $G$ . Thus Lemma 22 implies that the edges  $v_1v_4$  and  $v_2v_5$  must be oriented in both ways, a contradiction.  $\square$

To the more general case of planar graphs, similarly, it would be interesting to find a constant  $c'$ , if it exists, satisfying  $\vec{\chi}(G) \leq c'$ , for every planar graph  $G$ . We provided in Section 3 a planar graph whose proper orientation number is 10 and thus  $c' \geq 10$ .

#### 4.2. $\vec{\chi}$ -bounded families of graphs

Gyárfás [5] introduced the concept of  $\chi$ -bounded graph classes. A class of graph  $\mathcal{G}$  is said to be  $\chi$ -bounded if there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $G \in \mathcal{G}$ . Similarly, one can define  $\vec{\chi}$ -bounded graph classes. A class of graph  $\mathcal{G}$  is said to be  $\vec{\chi}$ -bounded if there is a function  $f$  such that  $\vec{\chi}(G) \leq f(\omega(G))$  for every  $G \in \mathcal{G}$ . Because  $\chi \leq \vec{\chi}$ , a  $\vec{\chi}$ -bounded graph class is also  $\chi$ -bounded. Conversely, one might wonder which  $\chi$ -bounded graph classes are also  $\vec{\chi}$ -bounded.

The  $\chi$ -boundedness of graph classes defined by forbidden induced subgraphs have been particularly investigated. For a fixed graph  $F$ , let us denote by  $\text{Forb}(F)$  the class of graphs that do not contain  $F$  as an induced subgraph. Erdős [6] showed that there are graphs with arbitrarily high girth and chromatic number. This implies that if  $F$  contains a cycle, then  $\text{Forb}(F)$  is not  $\chi$ -bounded. Conversely, Gyárfás [7] and Sumner [8] independently made the following beautiful and difficult conjecture

**Conjecture 24** ([7] and [8]). *For every tree  $T$ , the class  $\text{Forb}(T)$  is  $\chi$ -bounded.*

It is natural to ask whether this conjecture generalizes to proper orientations.

**Problem 25.** *Is the class  $\text{Forb}(T)$   $\vec{\chi}$ -bounded for all tree  $T$ ?*

Gyárfás [5] establishes Conjecture 24 for stars by showing that a graph in  $\text{Forb}(K_{1,n})$  has maximum degree  $R(n, \omega(G))$ , where  $R(p, q)$  denotes the Ramsey number  $(p, q)$ . In particular, this shows that  $\text{Forb}(K_{1,n})$  is also  $\vec{\chi}$ -bounded.

In particular, if  $G$  is a planar claw-free graph (recall that the claw is the graph  $K_{1,3}$ ), Gyárfás result gives us that  $\vec{\chi}(G) \leq \Delta(G) \leq R(3, 4) = 9$ . This is also a partial answer to whether planar graphs have bounded proper orientation number. However, this bound is not tight, as we show next. In [9], Plummer showed that any claw-free 3-connected planar graph has maximum degree at most 6. His result can be extended to any claw-free planar graph.

**Theorem 26.** *If  $G$  is a claw-free planar graph, then  $\Delta(G) \leq 6$ .*

*Proof.* The proof is by induction on the number of vertices of  $G$ . If  $G$  is disconnected, then, by the induction hypothesis, each connected component of  $G$  has maximum degree at most 6 and so  $\Delta(G) \leq 6$ .

Assume that  $G$  has a cut-vertex  $u$ . As  $G$  is claw-free,  $G - u$  has exactly two components  $C_i$ ,  $i = 1, 2$ , and the neighborhood of  $u$  in each  $C_i$  is a clique  $N_i$ . Observe that  $N_i \cup \{u\}$  is a clique, which has size at most 4 because  $G$  is planar, so  $|N_i| \leq 3$ . Hence  $d(u) = |N_1| + |N_2| \leq 6$ . Now by the induction hypothesis applied to  $G[V(C_1) \cup \{u\}]$  and  $G[V(C_2) \cup \{u\}]$ , we obtain that every vertex distinct from  $u$  has degree at most 6. Therefore  $\Delta(G) \leq 6$ . Henceforth we may assume that  $G$  is 2-connected.

Assume that  $G$  has a 2-cut  $\{u, v\}$  (that is  $G - \{u, v\}$  is disconnected). The graph  $G' = G - v$  is connected with cut-vertex  $u$ . As above,  $G' - u$  has exactly two components,  $C_1$  and  $C_2$ , and  $N_i = N(u) \cap C_i$  is a clique, for  $i = 1, 2$  of size at most 3. We claim that  $d(u) \leq 6$ . If  $uv \notin E(G)$ , then  $d(u) = |N_1| + |N_2|$ , so  $d(u) \leq 6$ . If  $uv \in E(G)$ , then  $d(u) = |N_1| + |N_2| + 1$ . But  $|N_1| + |N_2| \leq 5$  for otherwise there exist  $u_1 \in N_1$  and  $u_2 \in N_2$  non-adjacent to  $v$  (because  $G$  has no clique of size 5), so  $G[\{u, v, u_1, u_2\}]$  is a claw, a contradiction. Therefore  $d(u) \leq 6$ . Similarly, one proves  $d(v) \leq 6$ . Now by the induction hypothesis applied to  $G[V(C) \cup \{u, v\}]$  for each connected component of  $G - \{u, v\}$ , we obtain that every vertex distinct from  $u$  and  $v$  has degree at most 6; hence  $\Delta(G) \leq 6$ .

Henceforth, we may assume that  $G$  is 3-connected and the result follows by Plummer [9].  $\square$

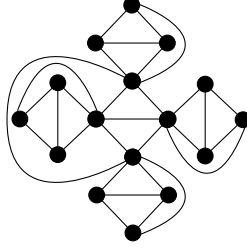


Figure 10: A planar claw-free graph  $G^*$  with maximum degree 6 and proper orientation number 6.

Theorem 26 is tight as shown by the graph  $G$  depicted in Figure 10 which is claw-free, planar and has maximum degree 6. Moreover, Theorem 26 implies that every planar claw-free graph has proper-orientation number at most 6. This is tight as shown by the following proposition.

**Proposition 27.** *The graph  $G^*$ , depicted in Figure 10, has proper orientation number equal to 6.*

*Proof.* The graph  $G^*$  is made of 5 blocks isomorphic to  $K_4$ . One of them (in the center of the figure), denoted by  $C$  intersects the four others. For every vertex

$v$  of  $C$ , let  $B(v)$  be the block intersecting  $C$  in  $v$ . Assume for a contradiction that  $G$  has a proper 5-orientation  $D$ . There are two vertices  $v_1$  and  $v_2$  in  $C$ , such that  $d_D^-(v_i) \in \{0, 1, 2, 3\}$ . Now the set of in-degrees of the other vertices of  $B(v_i)$  is exactly  $\{0, 1, 2, 3\} \setminus \{d_D^-(v_i)\}$ . Thus inside  $B(v)$  there are exactly  $6 - (0 + 1 + 2 + 3 - d_D^-(v_i)) = d_D^-(v_i)$  arcs towards  $v$ . Hence all the edges such that  $v_i$  is an endpoint are oriented from  $v_i$  to its neighbors in  $C$ . This is a contradiction, because the edge  $v_1v_2$  cannot be oriented both ways.  $\square$

## 5. Acknowledgements

The authors would like to thank an anonymous referee for pointing out interesting questions that helped us to improve Section 3.

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