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# Deriving Extrusion on Constraint Systems from Concurrent Constraint Programming Calculi

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**Abstract.** Spatial constraint systems (scs) are algebraic structures from concurrent constraint programming to specify spatial and epistemic behavior in multi-agent systems. In this paper we shall use spatial constraint systems to give an abstract characterization of the notion of normality in modal logic and to derive right inverses/reverse modalities. In particular, we shall identify minimal conditions for the existence of right inverses and show that the abstract notion of normality corresponds to the preservation of finite suprema. We shall apply and discuss our results in the context of some specific modal languages.

## 1 Introduction

*Constraint systems* provide the basic domains and operations for the semantic foundations of the family of *formal declarative models* from concurrency theory known as concurrent constraint programming (ccp) process calculi [24,3,13,8,22,16]. In these calculi, processes can be thought of as both concurrent computational entities and logic specifications (e.g., process composition can be seen as parallel execution and conjunction). All ccp process calculi are parametric in a constraint system (cs) that specifies partial information upon which programs (processes) may act.

A cs is often represented as a complete algebraic lattice  $(Con, \sqsubseteq)$ . The elements of  $Con$ , the *constraints*, represent partial information and we shall think of them as being *assertions*. The order  $\sqsubseteq$ , the join  $\sqcup$ , the bottom *true* and the top *false* of the lattice correspond respectively to entailment, conjunction, the empty information and the join of all (possibly inconsistent) information. The ccp operations and their logical counterparts typically have a corresponding elementary construct or operation on the elements of the constraint system. In particular, parallel composition and conjunction correspond to the *join* operation, and existential quantification and local variables correspond to a cylindrification operation on the set of constraints [24].

Similarly, the notion of computational space and the epistemic notion of belief in the sccp process calculi [13] correspond to a family of functions  $[\cdot]_i : Con \rightarrow Con$  on the elements of the constraint system  $Con$  that preserve finite suprema. These functions are called *space functions*. A cs equipped with space functions is called a *spatial constraint system* (scs). From a computational point of view the assertion (constraint)  $[c]_i$  specifies that  $c$  resides within the space of agent  $i$ . From an epistemic point of view, the assertion

$[c]_i$  specifies that agent  $i$  considers  $c$  to be true (i.e. that in the world of agent  $i$  assertion  $c$  is true). Both intuitions convey the idea of  $c$  being local (subjective) to agent  $i$ .

**The Extrusion Problem.** Given a spatial function  $[\cdot]_i$ , the *extrusion problem* consists in finding/constructing a *right inverse* of  $[\cdot]_i$ , called *extrusion function*, satisfying some basic requirements (e.g., preservation of finite suprema). By right inverse of  $[\cdot]_i$  we mean a function  $\uparrow_i : Con \rightarrow Con$  such that  $[\uparrow_i c]_i = c$ . Computationally, since  $[\uparrow_i(c)]_i = c$ , we can think of  $\uparrow_i$  as being able to extrude any  $c$  from the space  $[\cdot]_i$ . Epistemically, we can use  $\uparrow_i$  to express *utterances* by agent  $i$ , e.g., to let others see agent  $i$  believes  $c$ . One can then think of extrusion/utterance as the *right inverse* of space/belief.

Modal logics [20] extend classical logic to include operators expressing modalities. Depending on the intended meaning of the modalities, a particular modal logic can be used to reason about space, knowledge, belief or time, among others. Although the notion of spatial constraint system is intended to give an algebraic account of spatial and epistemic assertions, we shall show in this paper that it is sufficiently robust to give an algebraic account of more general modal assertions.

The main focus of this paper is the study of the above-mentioned *extrusion problem* for a meaningful family of scs's that can be used as semantic structures for modal logics. They are called *Kripke spatial constraint systems* because its elements are *Kripke Structures* (KS's). KS's can be seen as transition systems with some additional structure on their states.

**Contributions.** We shall show that the extrusion functions (i.e., right inverses of space) of the Kripke scs's correspond to (right) inverse modalities in modal logic. (Inverses modalities arise in the form of past operators in temporal logic [23], utterances in epistemic logic [11], backward moves in modal logic for concurrency [18], among others.) We shall derive a complete characterization for the existence of right inverses of space functions: The weakest restriction on the elements of the constraint systems (i.e., KS's) that guarantees the existence of right inverses. We shall also give an algebraic characterization of the notion of normality in modal logic by building upon work on Geometric Logic [25] and show that the abstract notion corresponds exactly to preservation of finite suprema. We then give a characterization and derivations of extrusion functions that are normal (and thus corresponding to normal inverse modalities). Finally we discuss the application of our results in the context of specific modal systems and related concepts such as the minimal modal logic  $K_n$  [9], a modal logic of linear-time [19], and bisimilarity.

## 2 Background: Spatial Constraint Systems

In this section we recall the notion of basic constraint system [3] and the more recent notion of spatial constraint system [13]. We presuppose basic knowledge of order theory and modal logic [1,20,9,2].

The concurrent constraint programming model of computation [24] is parametric in a *constraint system* (cs) specifying the structure and interdependencies of the partial information that computational agents can ask of and post in a *shared store*. This information is represented as *assertions* traditionally referred to as *constraints*.

Constraint systems can be formalized as *complete algebraic lattices* [3]<sup>4</sup>. The elements of the lattice, the *constraints*, represent (partial) information. A constraint  $c$  can be viewed as an *assertion* (or a *proposition*). The lattice order  $\sqsubseteq$  is meant to capture entailment of information:  $c \sqsubseteq d$ , alternatively written  $d \sqsupseteq c$ , means that the assertion  $d$  represents as much information as  $c$ . Thus we may think of  $c \sqsubseteq d$  as saying that  $d$  *entails*  $c$  or that  $c$  can be *derived* from  $d$ . The *least upper bound (lub)* operator  $\sqcup$  represents join of information;  $c \sqcup d$ , the least element in the underlying lattice above  $c$  and  $d$ . Thus  $c \sqcup d$  can be seen as an assertion stating that both  $c$  and  $d$  hold. The top element represents the lub of all, possibly inconsistent, information, hence it is referred to as *false*. The bottom element *true* represents the empty information.

**Definition 1 (Constraint Systems [3]).** *A constraint system (cs)  $\mathbf{C}$  is a complete algebraic lattice  $(Con, \sqsubseteq)$ . The elements of  $Con$  are called constraints. The symbols  $\sqcup$ , *true* and *false* will be used to denote the least upper bound (lub) operation, the bottom, and the top element of  $\mathbf{C}$ , respectively.*

We shall use the following notions and notations from order theory.

**Notation 1 (Lattices and Limit Preservation)** *Let  $\mathbf{C}$  be a partially ordered set (poset)  $(Con, \sqsubseteq)$ . We shall use  $\sqcup S$  to denote the least upper bound (lub) (or supremum or join) of the elements in  $S$ , and  $\sqcap S$  is the greatest lower bound (glb) (infimum or meet) of the elements in  $S$ . We say that  $\mathbf{C}$  is a complete lattice iff each subset of  $Con$  has a supremum and a infimum in  $Con$ . A non-empty set  $S \subseteq Con$  is directed iff every finite subset of  $S$  has an upper bound bound in  $S$ . Also  $c \in Con$  is compact iff for any directed subset  $D$  of  $Con$ ,  $c \sqsubseteq \sqcup D$  implies  $c \sqsubseteq d$  for some  $d \in D$ . A complete lattice  $\mathbf{C}$  is said to be algebraic iff for each  $c \in Con$ , the set of compact elements below it forms a directed set and the lub of this directed set is  $c$ . A self-map on  $Con$  is a function  $f : Con \rightarrow Con$ . Let  $(Con, \sqsubseteq)$  be a complete lattice. The self-map  $f$  on  $Con$  preserves the supremum of a set  $S \subseteq Con$  iff  $f(\sqcup S) = \sqcup\{f(c) \mid c \in S\}$ . The preservation of the infimum of a set is defined analogously. We say  $f$  preserves finite/infinite suprema iff it preserves the supremum of arbitrary finite/infinite sets. Preservation of finite/infinite infima is defined similarly.*

**Spatial Constraint Systems.** The authors of [13] extended the notion of cs to account for distributed and multi-agent scenarios where agents have their own space for their local information and performing their computations.

Intuitively, each agent  $i$  has a *space* function  $[\cdot]_i$  from constraints to constraints. Recall that constraints can be viewed as assertions. We can then think of  $[c]_i$  as an assertion

<sup>4</sup> An alternative syntactic characterization of cs, akin to Scott information systems, is given in [24].

stating that  $c$  is a piece of information residing *within a space attributed to agent  $i$* . An alternative *epistemic logic* interpretation of  $[c]_i$  is an assertion stating that agent  $i$  *believes  $c$*  or that  $c$  holds within the space of agent  $i$  (but it may not hold elsewhere). Both interpretations convey the idea that  $c$  is local to agent  $i$ . Similarly,  $[[c]_j]_i$  is a hierarchical spatial specification stating that  $c$  holds within the local space the agent  $i$  attributes to agent  $j$ . Nesting of spaces can be of any depth. We can think of a constraint of the form  $[c]_i \sqcup [d]_j$  as an assertion specifying that  $c$  and  $d$  hold within two *parallel/neighbor* spaces that belong to agents  $i$  and  $j$ , respectively. From a computational/ concurrency point of view, we think of  $\sqcup$  as parallel composition. As mentioned before, from a logic point of view the join of information  $\sqcup$  corresponds to conjunction.

**Definition 2 (Spatial Constraint System [13]).** An  $n$ -agent spatial constraint system ( $n$ -scs)  $\mathbf{C}$  is a cs  $(Con, \sqsubseteq)$  equipped with  $n$  self-maps  $[\cdot]_1, \dots, [\cdot]_n$  over its set of constraints  $Con$  such that: (S.1)  $[true]_i = true$ , and (S.2)  $[c \sqcup d]_i = [c]_i \sqcup [d]_i$  for each  $c, d \in Con$ .

Axiom S.1 requires space functions to be strict maps (i.e bottom preserving). Intuitively, it states that having an empty local space amounts to nothing. Axiom S.2 states that the information in a given space can be distributed. Notice that requiring S.1 and S.2 is equivalent to requiring that each  $[\cdot]_i$  preserves *finite suprema*. Also S.2 implies that each  $[\cdot]_i$  is monotonic: I.e., if  $c \sqsubseteq d$  then  $[c]_i \sqsubseteq [d]_i$ .

**Extrusion and utterance.** We can also equip each agent  $i$  with an *extrusion* function  $\uparrow_i : Con \rightarrow Con$ . Intuitively, within a space context  $[\cdot]_i$ , the assertion  $\uparrow_i c$  specifies that  $c$  must be posted outside of (or extruded from) agent  $i$ 's space. This is captured by requiring the *extrusion axiom*  $[\uparrow_i c]_i = c$ . In other words, we view *extrusion/utterance* as the right inverse of *space/belief* (and thus space/belief as the left inverse of extrusion/utterance).

**Definition 3 (Extrusion).** Given an  $n$ -scs  $(Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$ , we say that  $\uparrow_i$  is an *extrusion function for the space  $[\cdot]_i$*  iff  $\uparrow_i$  is a right inverse of  $[\cdot]_i$ , i.e., iff  $[\uparrow_i c]_i = c$ .

From the above definitions it follows that  $[c \sqcup \uparrow_i d]_i = [c]_i \sqcup d$ . From a spatial point of view, agent  $i$  *extrudes  $d$*  from its local space. From an epistemic view this can be seen as an agent  $i$  that believes  $c$  and *utters  $d$*  to the outside world. If  $d$  is inconsistent with  $c$ , i.e.,  $c \sqcup d = false$ , we can see the utterance as an intentional *lie* by agent  $i$ : The agent  $i$  utters an assertion inconsistent with their own beliefs.

**The Extrusion/Right Inverse Problem.** A legitimate question is: Given space  $[\cdot]_i$  can we derive an extrusion function  $\uparrow_i$  for it? From set theory we know that there is an extrusion function (i.e., a right inverse)  $\uparrow_i$  for  $[\cdot]_i$  iff  $[\cdot]_i$  is *surjective*. Recall that the *pre-image* of  $y \in Y$  under  $f : X \rightarrow Y$  is the set  $f^{-1}(y) = \{x \in X \mid y = f(x)\}$ . Thus the extrusion  $\uparrow_i$  can be defined as a function, called *choice* function, that maps each element  $c$  to some element from the pre-image of  $c$  under  $[\cdot]_i$ .

The existence of the above-mentioned choice function assumes the Axiom of Choice. The next proposition from [11] gives some constructive extrusion functions. It also identifies a distinctive property of space functions for which a right inverse exists.

**Proposition 1.** *Let  $[\cdot]_i$  be a space function of scs. Then*

1. *If  $[\text{false}]_i \neq \text{false}$  then  $[\cdot]_i$  does not have any right inverse.*
2. *If  $[\cdot]_i$  is surjective and preserves arbitrary suprema then  $\uparrow_i : c \mapsto \bigsqcup [c]_i^{-1}$  is a right inverse of  $[\cdot]_i$  and preserve arbitrary infima.*
3. *If  $[\cdot]_i$  is surjective and preserves arbitrary infima then  $\uparrow_i : c \mapsto \bigsqcap [c]_i^{-1}$  is a right inverse of  $[\cdot]_i$  and preserve arbitrary suprema.*

We have presented spatial constraint systems as algebraic structures for spatial and epistemic behaviour as that was their intended meaning. Nevertheless, we shall see that they can also provide an algebraic structure to reason about Kripke models with applications to modal logics.

In Section 4 we shall study the existence, constructions and properties of right inverses for a meaningful family of scs's; the Kripke scs's. The importance of such a study is the connections we shall establish between right inverses and reverse modalities which are present in temporal, epistemic and other modal logics. Property (1) in Proposition 1 can be used as a test for the existence of a right-inverse. The space functions of Kripke scs's preserve arbitrary suprema, thus Property (2) will be useful. They do not preserve in general arbitrary (or even finite) infima so we will not apply Property (3).

It is worth to point out that the derived extrusion  $\uparrow_i$  in Property (3), preserves arbitrary suprema, this implies  $\uparrow_i$  is *normal* in a sense we shall make precise next. Normal self-maps give an abstract characterization of normal modal operators, a fundamental concept in modal logic. We will be therefore interested in deriving normal inverses.

### 3 Constraint Frames and Normal Self Maps

Spatial constraint systems are algebraic structures for spatial and mobile behavior. By building upon ideas from Geometric Logic and Heyting Algebras [25] we can also make them suitable as semantics structures for modal logic. In this section we give an algebraic characterization of the concept of normal modality as those maps that preserve finite suprema.

We can define a general form of implication by adapting the corresponding notion from Heyting Algebras to constraint systems. Intuitively, a *Heyting implication*  $c \rightarrow d$  in our setting corresponds to the *weakest constraint* one needs to join  $c$  with to derive  $d$ : The greatest lower bound  $\bigsqcap \{e \mid e \sqcup c \sqsupseteq d\}$ . Similarly, the negation of a constraint  $c$ , written  $\sim c$ , can be seen as the *weakest constraint inconsistent* with  $c$ , i.e., the greatest lower bound  $\bigsqcap \{e \mid e \sqcup c \sqsupseteq \text{false}\} = c \rightarrow \text{false}$ .

**Definition 4 (Constraint Frames).** A constraint system  $(Con, \sqsubseteq)$  is said to be a constraint frame iff its joins distribute over arbitrary meets: More precisely,  $c \sqcup \prod S = \prod \{c \sqcup e \mid e \in S\}$  for every  $c \in Con$  and  $S \subseteq Con$ . Given a constraint frame  $(Con, \sqsubseteq)$  and  $c, d \in Con$ , define Heyting implication  $c \rightarrow d$  as  $\prod \{e \in Con \mid c \sqcup e \sqsubseteq d\}$  and Heyting negation as  $\sim c \stackrel{\text{def}}{=} c \rightarrow \text{false}$ .

The following basic properties of Heyting implication are immediate consequences of the above definitions.

**Proposition 2.** Let  $(Con, \sqsubseteq)$  be a constraint frame. For every  $c, d, e \in Con$ :

1.  $c \sqcup (c \rightarrow d) = c \sqcup d$
2.  $c \sqsubseteq (d \rightarrow e)$  iff  $c \sqcup d \sqsubseteq e$
3.  $c \rightarrow d = \text{true}$  iff  $c \sqsubseteq d$

*Proof.* The proof proceeds as follows:

1.  $c \sqcup (c \rightarrow d) = c \sqcup d$

This is proved in Lemma 1 (Modus Ponens) of [11].

2.  $c \sqsubseteq (d \rightarrow e)$  iff  $c \sqcup d \sqsubseteq e$

– Assume  $c \sqsubseteq (d \rightarrow e)$ . Then we have  $c \sqcup d \sqsubseteq (d \rightarrow e) \sqcup d$ . Therefore, from Proposition 2 (1) (modus ponens) we obtain  $c \sqcup d \sqsubseteq d \sqcup e$  and  $c \sqcup d \sqsubseteq e$  as wanted.

– Suppose  $c \sqcup d \sqsubseteq e$ . Because  $d \rightarrow e = \prod S$  where  $S = \{a \mid a \sqcup d \sqsubseteq e\}$ , and  $c \in S$ , then  $c \sqsubseteq (d \rightarrow e)$ .

3.  $c \rightarrow d = \text{true}$  iff  $c \sqsubseteq d$ . By Proposition 2 (2), assume  $e \sqsubseteq (c \rightarrow d)$  iff  $e \sqcup c \sqsubseteq d$ . Let us take  $e = \text{true}$ . Then,  $\text{true} \sqsubseteq (c \rightarrow d)$  iff  $\text{true} \sqcup c \sqsubseteq d$ . Therefore,  $\text{true} = (c \rightarrow d)$  iff  $c \sqsubseteq d$  as wanted.

In modal logics one is often interested in *normal modal* operators. The formulae of a modal logic are those of propositional logic extended with modal operators. Roughly speaking, a modal logic operator  $m$  is normal iff (1) the formula  $m(\phi)$  is a theorem (i.e., true in all models for the underlying modal language) whenever the formula  $\phi$  is a theorem, and (2) the implication formula  $m(\phi \Rightarrow \psi) \Rightarrow (m(\phi) \Rightarrow m(\psi))$  is a theorem. Since constraints can be viewed as logic assertions, we can think of modal operators as self-maps on constraints. Thus, using Heyting implication, we can express the normality condition in constraint frames as follows.

**Definition 5 (Normal Maps).** Let  $(Con, \sqsubseteq)$  be a constraint frame. A self-map  $m$  on  $Con$  is said to be normal if (1)  $m(\text{true}) = \text{true}$  and (2)  $m(c \rightarrow d) \rightarrow (m(c) \rightarrow m(d)) = \text{true}$  for each  $c, d \in Con$ .

We now prove that the normality requirement is equivalent to the requirement of preserving finite suprema. The next theorem basically states that Condition (2) in Definition 5 is equivalent to the seemingly simpler condition:  $m(c \sqcup d) = m(c) \sqcup m(d)$ .

**Theorem 1 (Normality & Finite Suprema).** *Let  $\mathbf{C}$  be a constraint frame  $(Con, \sqsubseteq)$  and let  $f$  be a self-map on  $Con$ . Then  $f$  is normal if and only if  $f$  preserves finite suprema.*

*Proof.* It suffices to show that for any bottom preserving self-map  $f$ ,  $\forall c, d \in Con : f(c \rightarrow d) \rightarrow (f(c) \rightarrow f(d)) = true$  iff  $\forall c, d \in Con : f(c \sqcup d) = f(c) \sqcup f(d)$ . (Both conditions require  $f$  to be bottom preserving, i.e.,  $f(true) = true$ , and preservation of non-empty finite suprema is equivalent to the preservation of binary suprema.)

- Assume that  $\forall c, d \in Con : f(c \rightarrow d) \rightarrow (f(c) \rightarrow f(d)) = true$ . Take two arbitrary  $c, d \in Con$ . We first prove  $f(c \sqcup d) \sqsupseteq f(c) \sqcup f(d)$ . From the assumption and Proposition 2(3) we obtain

$$f((c \sqcup d) \rightarrow d) \sqsupseteq f(c \sqcup d) \rightarrow f(d). \quad (1)$$

From Proposition 2(3)  $(c \sqcup d) \rightarrow d = true$ . Since  $f(true) = true$  we have  $f((c \sqcup d) \rightarrow d) = true$ . We must then have, from Equation 1,  $f(c \sqcup d) \rightarrow f(d) = true$  as well. Using Proposition 2(3) we obtain  $f(c \sqcup d) \sqsupseteq f(d)$ . In a similar fashion, by exchanging  $c$  and  $d$  in Equation 1, we can obtain  $f(d \sqcup c) \sqsupseteq f(c)$ . We can then conclude  $f(c \sqcup d) \sqsupseteq f(c) \sqcup f(d)$  as wanted.

We now prove  $f(c) \sqcup f(d) \sqsupseteq f(c \sqcup d)$ . From the assumption and Proposition 2(3) we have

$$f(c \rightarrow (d \rightarrow c \sqcup d)) \sqsupseteq f(c) \rightarrow f(d \rightarrow c \sqcup d). \quad (2)$$

Using Proposition 2 one can verify that  $c \rightarrow (d \rightarrow c \sqcup d) = true$ . Since  $f(true) = true$  then  $f(c \rightarrow (d \rightarrow c \sqcup d)) = true$ . From Equation 2, we must then have  $f(c) \rightarrow f(d \rightarrow c \sqcup d) = true$  and by using Proposition 2(3) we conclude  $f(c) \sqsupseteq f(d \rightarrow c \sqcup d)$ . From the assumption and Proposition 2(3)  $f(d \rightarrow c \sqcup d) \sqsupseteq f(d) \rightarrow f(c \sqcup d)$ . We then have  $f(c) \sqsupseteq f(d \rightarrow c \sqcup d) \sqsupseteq f(d) \rightarrow f(c \sqcup d)$ . Thus  $f(c) \sqsupseteq f(d) \rightarrow f(c \sqcup d)$  and then using Proposition 2(2) we obtain  $f(c) \sqcup f(d) \sqsupseteq f(c \sqcup d)$  as wanted.

- Assume that  $\forall c, d \in Con : f(c \sqcup d) = f(c) \sqcup f(d)$ . Take two arbitrary  $c, d \in Con$ . We shall prove  $f(c \rightarrow d) \rightarrow (f(c) \rightarrow f(d)) = true$ . From Proposition 2(2-3) it suffices to prove  $f(c \rightarrow d) \sqcup f(c) \sqsupseteq f(d)$ . Using the the assumption and Proposition 2(1) we obtain  $f(c \rightarrow d) \sqcup f(c) = f(c \sqcup (c \rightarrow d)) = f(c \sqcup d) = f(c) \sqcup f(d) \sqsupseteq f(d)$  as wanted.

□

By applying the above theorem, we can conclude that space functions from constraint frames are indeed normal self-maps, since they preserve finite suprema.



Now we introduce an important corollary of the above theorem stating that every normal self-map is also monotone.

**Corollary 1.** *Let  $\mathbf{C}$  be a constraint frame  $(Con, \sqsubseteq)$  and let  $f$  be a normal self-map on  $Con$ . Then,*

$$\text{If } c \sqsubseteq d \text{ then } f(c) \sqsubseteq f(d).$$

*Proof.* Assume  $c \sqsubseteq d$ . Then  $d = c \sqcup d$  so  $f(d) = f(c \sqcup d)$ . Then by Theorem 1 we have  $f(c \sqcup d) = f(c) \sqcup f(d)$ . Since  $f(c) \sqcup f(d) = f(d)$  therefore  $f(c) \sqsubseteq f(d)$ .

## 4 Extrusion Problem for Kripke Constraint Systems

This is the main and more technical part of the paper. Here we will study the extrusion/right inverse problem for a meaningful family of spatial constraint systems (scs's); the Kripke scs. In particular we shall derive and give a *complete* characterization of normal extrusion functions as well as identify the *weakest* condition on the elements of the scs under which extrusion function may exist. To illustrate the importance of this study is convenient to give some intuition first.

Kripke structures (KS) are a fundamental mathematical tool in logic and computer science. They can be seen as transition systems and they are used to give semantics to modal logics. A KS  $M$  provides a relational structure with a set of states and one or more accessibility relations  $\xrightarrow{i}_M$  between them:  $s \xrightarrow{i}_M t$  can be seen as a transition, labelled with  $i$ , from  $s$  to  $t$  in  $M$ . Broadly speaking, a model-based Kripke semantics equates each modal formula  $\phi$  to a certain set  $\llbracket \phi \rrbracket$  of pairs  $(M, s)$ , called pointed KS's, where  $s$  is a state of the KS  $M$ . In particular, in modal logics with one or more modal (box) operators  $\Box_i$ , the formula  $\Box_i \phi$  is equated to  $\llbracket \Box_i \phi \rrbracket = \{(M, s) \mid \forall t : s \xrightarrow{i}_M t, (M, t) \in \llbracket \phi \rrbracket\}$ .

Analogously, in a Kripke scs each constraint  $c$  is equated to a set of pairs  $(M, s)$  of pointed KS. Furthermore, for each space  $[\cdot]_i$  we have  $[c]_i = \{(M, s) \mid \forall t : s \xrightarrow{i}_M t, (M, t) \in c\}$ . This means that formulae can be interpreted as constraints and in particular  $\Box_i$  can be interpreted by  $[\cdot]_i$  as  $\llbracket \Box_i \phi \rrbracket = [\llbracket \phi \rrbracket]_i$ .

Inverse modalities  $\Box_i^{-1}$ , also known as a reverse modalities, are used in many modal logics. In tense logics they represent past operators [21] and in epistemic logic they represent utterances [11]. The basic property of a (right) inverse modality is given by the axiom  $\Box_i(\Box_i^{-1}\phi) \Leftrightarrow \phi$ . In fact, given a modal logic one may wish to see if it can be extended it with reverse modalities (e.g., is there a reverse modality for the always operator of temporal logic?).

Notice that if we have an extrusion function  $\uparrow_i$  for  $[\cdot]_i$  we can provide the semantics for inverse modalities  $\Box_i^{-1}$  by letting  $\llbracket \Box_i^{-1}\phi \rrbracket = \uparrow_i(\llbracket \phi \rrbracket)$  (thus  $\llbracket \Box_i(\Box_i^{-1}\phi) \rrbracket = \llbracket \phi \rrbracket$ ). Therefore deriving extrusion functions and establishing the weakest conditions under which they exists is a relevant issue. Furthermore, the algebraic structure of Kripke scs

may help us stating derived properties of the reverse modality such that of being normal (Definition 5).

#### 4.1 KS and Kripke SCS

We begin by recalling some notions and notations related to Kripke models.

**Definition 6 (Kripke Structures)** . An  $n$ -agent Kripke Structure (KS)  $M$  over a set of atomic propositions  $\Phi$  is a tuple  $(S, \pi, \mathcal{R}_1, \dots, \mathcal{R}_n)$  where  $S$  is a nonempty set of states,  $\pi : S \rightarrow (\Phi \rightarrow \{0, 1\})$  is an interpretation associating with each state a truth assignment to the primitive propositions in  $\Phi$ , and  $\mathcal{R}_i$  is a binary relation on  $S$ . A pointed KS is a pair  $(M, s)$  where  $M$  is a KS and  $s$ , called the actual world, is a state of  $M$ .  $\square$

We shall use the following notation in the rest of the paper.

**Notation 2** Each  $\mathcal{R}_i$  is referred to as the accessibility relation for agent  $i$ . We shall use  $\xrightarrow{i}_M$  to refer to the accessibility relation of agent  $i$  in  $M$ . We write  $s \xrightarrow{i}_M t$  to denote  $(s, t) \in \mathcal{R}_i$ . We use  $\triangleright_i(M, s) = \{(M, t) \mid s \xrightarrow{i}_M t\}$  to denote the pointed KS reachable from the pointed KS  $(M, s)$ . The interpretation function  $\pi$  tells us what primitive propositions are true at a given world:  $p$  holds at state  $s$  iff  $\pi(s)(p) = 1$ . We shall use  $S_M$  and  $\pi_M$  to denote the set of states and interpretation function of  $M$ .

We now define the Kripke scs wrt a set  $\mathcal{S}_n(\Phi)$  of pointed KS.

**Definition 7 (Kripke Spatial Constraint Systems [13]).** Let  $\mathcal{S}_n(\Phi)$  be a non-empty set of  $n$ -agent Kripke structures over a set of primitive propositions  $\Phi$ . We define the Kripke  $n$ -scs for  $\mathcal{S}_n(\Phi)$  as  $\mathbf{K}(\mathcal{S}_n(\Phi)) = (Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  where  $Con = \mathcal{P}(\Delta)$ ,  $\sqsubseteq = \supseteq$ , and

$$[c]_i \stackrel{\text{def}}{=} \{(M, s) \in \Delta \mid \triangleright_i(M, s) \subseteq c\} \quad (3)$$

where  $\Delta$  is the set of all pointed Kripke structures  $(M, s)$  such that  $M \in \mathcal{S}_n(\Phi)$

The structure  $\mathbf{K}(\mathcal{S}_n(\Phi)) = (Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  is a complete algebraic lattice given by a powerset ordered by reversed inclusion  $\supseteq$ . The join  $\sqcup$  is set intersection, the meet  $\sqcap$  is set union, the top element *false* is the empty set  $\emptyset$ , and bottom *true* is the set  $\Delta$  of all pointed Kripke structures  $(M, s)$  with  $M \in \mathcal{S}_n(\Phi)$ . Notice that  $\mathbf{K}(\mathcal{S}_n(\Phi))$  is a frame since meets are unions and joins are intersections so the distributive requirement is satisfied. Furthermore, each  $[\cdot]_i$  preserves arbitrary suprema (intersection) and thus, from Theorem 1 it is a normal self-map.

**Proposition 3.** Let  $\mathbf{K}(\mathcal{S}_n(\Phi)) = (Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  as in Definition 7.

1.  $\mathbf{K}(\mathcal{S}_n(\Phi))$  is an  $n$ -agent spatial constraint frame.
2. Each  $[\cdot]_i$  preserves arbitrary suprema .

## 4.2 Existence of Right Inverses

We shall now address the question of whether a given Kripke constraint system can be extended with extrusion functions. We shall identify a sufficient and necessary condition on accessibility relations for the existence of an extrusion function  $\uparrow_i$  given the space  $[\cdot]_i$ . We shall also give explicit right inverse constructions.

**Notation 3** For notational convenience, we take the set  $\Phi$  of primitive propositions and  $n$  to be fixed from now on and omit them from the notation. E.g., we write  $\mathcal{M}$  instead of  $\mathcal{M}_n(\Phi)$ .

The following notions play a key role in our complete characterization, in terms of classes of KS, of the existence of right inverses for Kripke space functions.

**Definition 8 (Determinacy and Unique-Determinacy).** Let  $S$  and  $\mathcal{R}$  be the set of states and an accessibility relation of a KS  $M$ , respectively. Given  $s, t \in S$ , we say that  $s$  determines  $t$  wrt  $\mathcal{R}$  if  $(s, t) \in \mathcal{R}$ . We say that  $s$  uniquely determines  $t$  wrt  $\mathcal{R}$  if  $s$  is the only state in  $S$  that determines  $t$  wrt  $\mathcal{R}$ . A state  $s \in S$  is said to be determinant wrt  $\mathcal{R}$  if it uniquely determines some state in  $S$  wrt  $\mathcal{R}$ . Furthermore,  $\mathcal{R}$  is determinant-complete if every state in  $S$  is determinant wrt  $\mathcal{R}$ .  $\square$

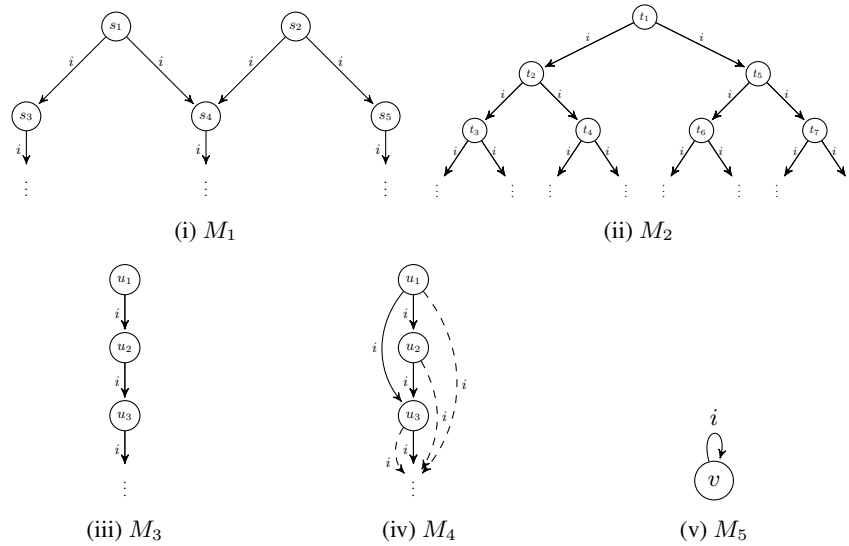


Fig. 1: Accessibility relations for an agent  $i$ . In each sub-figure we omit the corresponding KS  $M_k$  from the edges and draw  $s \xrightarrow{i} t$  whenever  $s \xrightarrow{i}_{M_k} t$ .

*Example 1.* Figure 1 illustrates some typical determinant-complete accessibility relations for agent  $i$ . Notice that any determinant-complete relation  $\xrightarrow{i}_M$  is necessarily

*serial* (or left-total): i.e., For every  $s \in S_M$ , there exists  $t \in S_M$  such that  $s \xrightarrow{i}_M t$ . Tree-like accessibility relations where all paths are infinite are determinant-complete (Figure 1.(ii) and Figure 1.(iii)). Also some non-tree like structures such as Figure 1.(i) and Figure 1.(v). Figure 1.(iv) shows a non determinate-complete accessibility relation by taking the transitive closure of Figure 1.(iii).

We need to introduce some notation.

**Notation 4** Recall that  $\triangleright_i(M, s) = \{(M, t) \mid s \xrightarrow{i}_M t\}$  where  $\xrightarrow{i}_M$  denotes the accessibility relation of agent  $i$  in the KS  $M$ . We extend this definition to sets of states as follows  $\triangleright_i(M, S) = \bigcup_{s \in S} \triangleright_i(M, s)$ . Furthermore, we shall write  $s \xrightarrow{i}_M t$  to mean that  $s$  uniquely determines  $t$  wrt  $\xrightarrow{i}_M$ .

The following proposition gives an alternative definition of determinant states.

**Proposition 4.** Let  $s \in S_M$ . The state  $s$  is determinant wrt  $\xrightarrow{i}_M$  if and only if for every  $S' \subseteq S_M$ : If  $\triangleright_i(M, s) \subseteq \triangleright_i(M, S')$  then  $s \in S'$ .

*Proof.* The proof proceeds as follows :

- Assume that  $s \in S_M$  is determinant wrt  $\xrightarrow{i}_M$ . Let us proceed by contradiction and assume that there exists a  $S' \subseteq S_M$  such that  $\triangleright_i(M, s) \subseteq \triangleright_i(M, S')$  and  $s \notin S'$ . Since  $s$  is determinant wrt  $\xrightarrow{i}_M$  there must exist a  $t$  such that  $s \xrightarrow{i}_M t$  and  $s' \not\xrightarrow{i}_M t$  for any  $s' \in S'$ . By definition of  $\triangleright_i$ ,  $(M, t) \in \triangleright_i(M, s)$  and  $(M, t) \notin \triangleright_i(M, S')$  which is a contradiction since  $\triangleright_i(M, s) \subseteq \triangleright_i(M, S')$ .
- Assume for every  $S' \subseteq S_M$ , if  $\triangleright_i(M, s) \subseteq \triangleright_i(M, S')$  then  $s \in S'$ . To reach a contradiction, let us suppose that  $s$  is not determinant wrt  $\xrightarrow{i}_M$ . By our assumption that  $s$  is not determinant wrt  $\xrightarrow{i}_M$  it follows that for any  $t_i$  such that  $s \xrightarrow{i}_M t_i$  there must be  $s_i \neq s$  such that  $s_i \xrightarrow{i}_M t_i$ . Now, take  $S'' = \{s_1, s_2, \dots\}$ . Notice that  $\triangleright_i(M, s) \subseteq \triangleright_i(M, S'')$ . Applying the assumption, we conclude  $s \in S''$ , a contradiction.

The following theorem provides a complete characterization, in terms of classes of KS, of the existence of right inverses for space functions.

**Theorem 2 (Completeness).** Let  $[\cdot]_i$  be a spatial function of a Kripke scs  $\mathbf{K}(S)$ . Then  $[\cdot]_i$  has a right inverse if and only if for every  $M \in \mathcal{S}$  the accessibility relation  $\xrightarrow{i}_M$  is determinant-complete.

*Proof.* – Suppose that for every  $M \in \mathcal{S}$ ,  $\xrightarrow{i}_M$  is determinant-complete. By the Axiom of Choice,  $[\cdot]_i$  has a right inverse if  $[\cdot]_i$  is surjective. Thus, it suffices to show that for every set of pointed KS  $d$ , there exists a set of pointed KS  $c$  such that  $[c]_i = d$ . Take an arbitrary  $d$  and let  $c = \triangleright_i(M', S')$  where  $S' = \{s \mid (M, s) \in d\}$ . From Definition 7 we conclude  $d \subseteq [c]_i$ . It remains to prove  $d \supseteq [c]$ . Suppose

$d \not\subseteq [c]$ . Since  $d \subseteq [c]$  we have  $d \subset [c]$ . Then there must be a  $(M', s')$ , with  $M' \in \mathcal{S}$ , such that  $(M', s') \notin d$  and  $(M', s') \in [c]$ . But if  $(M', s') \in [c]_i$  then from Definition 7 we conclude that  $\triangleright_i(M', s') \subseteq c = \triangleright_i(M', S')$ . Furthermore  $(M', s') \notin d$  implies  $s' \notin S'$ . It then follows from Proposition 4 that  $s'$  is not determinant wrt  $\xrightarrow{i}_{M'}$ . This leads us to a contradiction since  $\xrightarrow{i}_{M'}$  is supposed to be determinant-complete.

- Suppose  $[\cdot]_i$  has a right inverse. By the Axiom of Choice,  $[\cdot]_i$  is surjective. We claim that  $\xrightarrow{i}_M$  is determinant-complete for every  $M \in \mathcal{S}$ . To show this claim let us assume that there is  $M' \in \mathcal{S}$  such that  $\xrightarrow{i}_{M'}$  is not determinant-complete. From Proposition 4 we should have  $s \in S$  and  $S' \subseteq S$  such that  $\triangleright_i(M', s) \subseteq \triangleright_i(M', S')$  and  $s \notin S'$ . Since  $[c']_i$  is surjective there must be a set of pointed KS  $c'$  such that  $\{(M', s') \mid s' \in S'\} = [c']_i$ . We can then verify, using Definition 7, that  $\triangleright_i(M, S') \subseteq c'$ . Since  $\triangleright_i(M', s) \subseteq \triangleright_i(M', S')$  then  $\triangleright_i(M', s) \subseteq c'$ . It then follows from Definition 7 that  $(M', s) \in [c']_i$ . But  $[c']_i = \{(M', s') \mid s' \in S'\}$  then  $s \in S'$ , a contradiction.

□

Henceforth we use  $\mathcal{M}^D$  to denote the class of KS's whose accessibility relations are determinant-complete. It follows from Theorem 2 that  $\mathcal{S} = \mathcal{M}^D$  is the largest class for which space functions of a Kripke  $\mathbf{K}(\mathcal{S})$  have right inverses.

### 4.3 Right Inverse Constructions

Let  $\mathbf{K}(\mathcal{S}) = (Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  be the Kripke scs. The Axiom of Choice and Theorem 2 tell us that each  $[\cdot]_i$  has a right inverse (extrusion function) if and only if  $\mathcal{S} \subseteq \mathcal{M}^D$ . We are interested, however, in explicit constructions of the right inverses.

*Remark 1.* Recall that any Kripke scs  $\mathbf{K}(\mathcal{S}) = (Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  is ordered by reversed inclusion (i.e.,  $c \sqsubseteq d$  iff  $d \subseteq c$ ). Thus, for example, saying that some  $f$  is the least function wrt  $\subseteq$  satisfying certain conditions is equivalent to saying that  $f$  is the greatest function wrt  $\sqsubseteq$  satisfying the same conditions. As usual given two self-maps  $f$  and  $g$  over  $Con$  we define  $f \sqsubseteq g$  iff  $f(c) \sqsubseteq g(c)$  for every  $c \in Con$ .

Since any Kripke scs space function preserve arbitrary suprema (Proposition 3), we can apply Proposition 1.2 to obtain the following canonical greatest right-inverse construction. Recall that the pre-image of  $c$  under  $[\cdot]_i$  is given by the set  $[c]_i^{-1} = \{d \mid c = [d]_i\}$ .

**Definition 9 (Max Right Inverse).** Let  $\mathbf{K}(\mathcal{S}) = (Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  be a Kripke scs over  $\mathcal{S} \subseteq \mathcal{M}^D$ . We define  $\uparrow_i^M$  as the following self-map on  $Con$ :  $\uparrow_i^M : c \mapsto \bigsqcup [c]_i^{-1}$ .

It follows from Proposition 1.2 that  $\uparrow_i^M$  is a right inverse for  $[\cdot]_i$ , and furthermore, from its definition it is clear that  $\uparrow_i^M$  is the greatest right inverse of  $[\cdot]_i$  wrt  $\sqsubseteq$ .

Nevertheless, as stated in the following proposition,  $\uparrow_i^M$  is not necessarily *normal* in the sense of Definition 5. To state this more precisely, let us first extend the terminology in Definition 8.

**Definition 10 (Indeterminacy and Multiple Determinacy).** *Let  $S$  and  $\mathcal{R}$  be the set of states and an accessibility relation of a KS  $M$ , respectively. Given  $t \in S$ , we say that  $t$  is determined wrt  $\mathcal{R}$  if there is  $s \in S$  such that  $s$  determines  $t$  wrt  $\mathcal{R}$ , else we say that  $t$  is indetermined (or initial) wrt  $\mathcal{R}$ . Similarly, we say that  $t$  is multiply, or ambiguously, determined if it is determined by at least two different states in  $S$  wrt  $\mathcal{R}$ .*

The following statement and Theorem 1 lead us to conclude that the presence of indetermined/initial states or multiple-determined states causes  $\uparrow_i^M$  not to be normal.

**Proposition 5.** *Let  $\mathbf{K}(\mathcal{S}) = (Con, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  and  $\uparrow_i^M$  as in Definition 9. Let  $nd(\mathcal{S}) = \{(M, t) \mid M \in \mathcal{S} \ \& \ t \text{ is indetermined wrt } \xrightarrow{i}_M\}$  and  $md(\mathcal{S}) = \{(M, t) \mid M \in \mathcal{S} \ \& \ t \text{ is multiply determined wrt } \xrightarrow{i}_M\}$ :*

- *If  $nd(\mathcal{S}) \neq \emptyset$  then  $\uparrow_i^M(true) \neq true$ .*
- *If  $md(\mathcal{S}) \neq \emptyset$  then  $\uparrow_i^M(c \sqcup d) \neq \uparrow_i^M(c) \sqcup \uparrow_i^M(d)$  for some  $c, d \in Con$ .*

*Proof.* - If  $nd(\mathcal{S}) \neq \emptyset$  then  $\uparrow_i^M(true) \neq true$ .

Assume  $nd(\mathcal{S}) \neq \emptyset$ . Take  $c$  to be the complement of  $nd(\mathcal{S})$  in  $Con$ . Using Equation 3 we have  $[c]_i = true$  and from the assumption  $c \sqsupset true$ . Therefore  $\uparrow_i^M(true) = \bigsqcup [c]_i^{-1} \neq true$ .

- If  $md(\mathcal{S}) \neq \emptyset$  then  $\uparrow_i^M(c \sqcup d) \neq \uparrow_i^M(c) \sqcup \uparrow_i^M(d)$  for some  $c, d \in Con$ .

Suppose  $md(\mathcal{S}) \neq \emptyset$ . Take  $t \in md(\mathcal{S})$ , then there exist  $s, s'$  s.t.  $s \xrightarrow{i}_M t$  and  $s' \xrightarrow{i}_M t$ . Let  $c = \{(M, s)\}$  and  $d = \{(M, s')\}$ . Recall that  $\uparrow_i^M$  is a right inverse of  $[\cdot]_i$ . Because of this any constraint  $e = [\uparrow_i^M(e)]_i = \{(M, s'') \mid \triangleright_i(M, s'') \subseteq \uparrow_i^M(e)\}$ , therefore  $(M, t) \in \uparrow_i^M(c) \cap \uparrow_i^M(d)$ . Now,  $\uparrow_i^M(c \cap d) = \uparrow_i^M(false) = false$  because Proposition 1 and  $\uparrow_i^M(false) = \bigsqcup [false]_i^{-1}$ . We therefore conclude  $\uparrow_i^M(c \sqcup d) = \uparrow_i^M(c \cap d) \neq \uparrow_i^M(c) \cap \uparrow_i^M(d) = \uparrow_i^M(c) \sqcup \uparrow_i^M(d)$

In what follows we shall identify right inverse constructions that are normal. The notion of indeterminacy and multiply determinacy we just introduced in Definition 10 will play a central role.

#### 4.4 Normal Right Inverses

The following central lemma provides distinctive properties of any normal right inverse.

**Lemma 1.** Let  $\mathbf{K}(\mathcal{S}) = (\text{Con}, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  be the Kripke scs over  $\mathcal{S} \subseteq \mathcal{M}^D$ . Suppose that  $f$  is a normal right-inverse of  $[\cdot]_i$ . Then for every  $M \in \mathcal{S}$ ,  $c \in \text{Con}$ :

1.  $\triangleright_i(M, s) \subseteq f(c)$  if  $(M, s) \in c$ ,
2.  $\{(M, t)\} \subseteq f(c)$  if  $t$  is multiply determined wrt  $\xrightarrow{i}_M$ , and
3.  $\text{true} \subseteq f(\text{true})$ .

*Proof.* The proof proceeds as follows :

1.  $\triangleright_i(M, s) \subseteq f(c)$  if  $(M, s) \in c$ . Assume  $(M, s) \in c$ . By definition of  $\triangleright_i$  we have  $\triangleright_i(M, s) = \{(M, t) \mid s \xrightarrow{i}_M t\}$ . Then for every  $(M, t)$  belonging to  $\triangleright_i(M, s)$   $t$  is determined wrt  $\xrightarrow{i}_M$ . Given that  $f$  is a right-inverse of  $[\cdot]_i$ , it must satisfy  $[f(c)]_i = c$ . By Definition 7 we have  $[c]_i = \{(M, s) \mid \triangleright_i(M, s) \subseteq c\}$ , then  $[f(c)]_i = \{(M, s) \mid \triangleright_i(M, s) \subseteq f(c)\}$ . Therefore, since  $[f(c)]_i = c = \{(M, s) \mid \triangleright_i(M, s) \subseteq f(c)\}$  then  $\triangleright_i(M, s) \subseteq f(c)$  as wanted.
2.  $\{(M, t)\} \subseteq f(c)$  if  $t$  is multiply determined wrt  $\xrightarrow{i}_M$ . Assume  $t$  is multiply determined wrt  $\xrightarrow{i}_M$ . Therefore we know  $\exists s, s'$  with  $s \neq s'$  such that  $s \xrightarrow{i}_M t$  and  $s' \xrightarrow{i}_M t$ . From Corollary 1 we know if  $c \subseteq d$  then  $f(c) \subseteq f(d)$ . Then it follows that  $\text{false} \subseteq d$  and  $f(\text{false}) \subseteq f(d)$ . Thus, it suffices to prove that  $(M, t) \in f(\text{false})$ . From the assumption that  $f$  is a normal right-inverse then by Theorem 1 it preserves suprema  $f(\{(M, s)\} \cap \{(M, s')\}) = f(\{(M, s)\}) \cap f(\{(M, s')\})$ . Since  $s \neq s'$  then  $\{(M, s)\} \cap \{(M, s')\} = \emptyset$ . From Lemma 1 (1)  $s \xrightarrow{i}_M t$  and  $s' \xrightarrow{i}_M t$  we know  $\{(M, t)\} \subseteq f(\{(M, s)\}) \cap f(\{(M, s')\})$ . Therefore by Corollary 1,  $\{(M, t)\} \subseteq f(\{(M, s)\} \cap \{(M, s')\})$  which is the same as  $\{(M, t)\} \subseteq f(\text{false})$ . Then, we know  $f(\text{false}) \subseteq f(c)$  therefore  $\{(M, t)\} \subseteq f(c)$  as wanted.
3.  $\text{true} \subseteq f(\text{true})$ . Trivial from the assumption that  $f$  is a normal-right inverse.

The above property tell us what sets should necessarily be included in every  $f(c)$  if  $f$  is to be both normal and a right inverse of  $[\cdot]_i$ . It turns out that it is sufficient to include exactly those sets to obtain a normal right inverse of  $[\cdot]_i$ . In other words the above lemma gives us a *complete* set of conditions for normal right inverses. In fact, the least self-map  $f$  wrt  $\subseteq$ , i.e., the greatest one wrt the lattice order  $\sqsubseteq$ , satisfying Conditions 1,2 and 3 in Lemma 1 is indeed a normal right-inverse. We call such a function the *max normal right inverse*  $\uparrow_i^{\text{MN}}$  and is given below.

**Definition 11 (Max Normal-Right Inverse).** Let  $\mathbf{K}(\mathcal{S}) = (\text{Con}, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  be a Kripke scs over  $\mathcal{S} \subseteq \mathcal{M}^D$ . We define the max normal right inverse for agent  $i$ ,  $\uparrow_i^{\text{MN}}$  as the following self-map on  $\text{Con}$ :

$$\uparrow_i^{\text{MN}}(c) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } c = \text{true} \\ \{(M, t) \mid t \text{ is determined wrt } \xrightarrow{i}_M \text{ \& } \forall s : s \xrightarrow{i}_M t, (M, s) \in c\} & \end{cases} \quad (4)$$

(Recall that  $s \xrightarrow{i}_M t$  means that  $s$  uniquely determines  $t$  wrt  $\xrightarrow{i}_M$ .)

The following theorem states that  $\uparrow_i^{\text{MN}}(c)$  is the greatest normal right inverse of  $[\cdot]_i$  wrt  $\sqsubseteq$ .

**Theorem 3.** Let  $\mathbf{K}(S) = (\text{Con}, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  and  $\uparrow_i^{\text{MN}}$  as in Definition 11.

- The self-map  $\uparrow_i^{\text{MN}}$  is a normal right inverse of  $[\cdot]_i$ ,
- For every normal right-inverse  $f$  of  $[\cdot]_i$ , we have  $f \sqsubseteq \uparrow_i^{\text{MN}}$ .

*Proof.* – The self-map  $\uparrow_i^{\text{MN}}$  is a normal right inverse of  $[\cdot]_i$ .

To prove it is a right inverse we prove  $[\uparrow_i^{\text{MN}}(c)]_i = c$ . To prove that is a normal map, by Theorem 1, it suffices to prove  $\uparrow_i^{\text{MN}}(\text{true}) = \text{true}$  and  $\uparrow_i^{\text{MN}}(c \sqcup d) = \uparrow_i^{\text{MN}}(c) \sqcup \uparrow_i^{\text{MN}}(d)$  (equivalent to proving  $\uparrow_i^{\text{MN}}(c \cap d) = \uparrow_i^{\text{MN}}(c) \cap \uparrow_i^{\text{MN}}(d)$ ).

- $[\uparrow_i^{\text{MN}}(c)]_i = c$ . Suppose  $(M, s) \in [\uparrow_i^{\text{MN}}(c)]_i$ , this means  $\triangleright_i(M, s) \subseteq \uparrow_i^{\text{MN}}(c)$ . As  $s$  is determinant, there exists  $t$  s.t.  $s \xrightarrow{i}_M t$  and  $(M, t) \in \uparrow_i^{\text{MN}}(c)$ . Therefore by definition of  $\uparrow_i^{\text{MN}}$ ,  $s \in c$ . Now suppose  $(M, s) \in c$ , then for all states in  $\triangleright_i(M, s)$ , they are either uniquely determined by  $s$  or multiply determined, therefore  $\triangleright_i(M, s) \subseteq \uparrow_i^{\text{MN}}(c)$ , and consequently  $(M, s) \in [\uparrow_i^{\text{MN}}(c)]_i$ .
- $\uparrow_i^{\text{MN}}(\text{true}) = \text{true}$ . By definition of  $\uparrow_i^{\text{MN}}$ .
- $\uparrow_i^{\text{MN}}(c \cap d) = \uparrow_i^{\text{MN}}(c) \cap \uparrow_i^{\text{MN}}(d)$ . We first prove monotonicity (i.e. if  $c \subseteq d$  then  $\uparrow_i^{\text{MN}}(c) \subseteq \uparrow_i^{\text{MN}}(d)$ ). For this, suppose  $c \subseteq d$ . If  $(M, t) \in \uparrow_i^{\text{MN}}(c)$  then by definition of  $\uparrow_i^{\text{MN}}$   $t$  is determined wrt  $\xrightarrow{i}_M$ . If there exists a state  $(M, s) \in c$  s.t.  $s \xrightarrow{i}_M t$  then  $(M, s) \in d$ , thus  $(M, t) \in \uparrow_i^{\text{MN}}(d)$ . If there does not exist such state, then  $(M, t)$  is multiply determined and consequently  $(M, t) \in \uparrow_i^{\text{MN}}(d)$ . We now proceed with the proof.
  - \*  $\uparrow_i^{\text{MN}}(c \cap d) \subseteq \uparrow_i^{\text{MN}}(c) \cap \uparrow_i^{\text{MN}}(d)$ . Because  $c \cap d \subseteq c$  and  $c \cap d \subseteq d$ . Then  $\uparrow_i^{\text{MN}}(c \cap d) \subseteq \uparrow_i^{\text{MN}}(c)$  and  $\uparrow_i^{\text{MN}}(c \cap d) \subseteq \uparrow_i^{\text{MN}}(d)$  by monotonicity of  $\uparrow_i^{\text{MN}}$ . Thus  $\uparrow_i^{\text{MN}}(c \cap d) \subseteq \uparrow_i^{\text{MN}}(c) \cap \uparrow_i^{\text{MN}}(d)$ .
  - \*  $\uparrow_i^{\text{MN}}(c) \cap \uparrow_i^{\text{MN}}(d) \subseteq \uparrow_i^{\text{MN}}(c \cap d)$ . Suppose  $(M, t) \in \uparrow_i^{\text{MN}}(c) \cap \uparrow_i^{\text{MN}}(d)$ , this means that, by definition of  $\uparrow_i^{\text{MN}}$ ,  $t$  is determined wrt  $\xrightarrow{i}_M$  of  $\uparrow_i^{\text{MN}}$ . If there exists  $s$  s.t.  $s \xrightarrow{i}_M t$ , then  $s \in c$  and  $s \in d$ . Now suppose there exists such  $s$ , therefore  $(M, s) \in c \cap d$  and  $(M, t) \in \uparrow_i^{\text{MN}}(c \cap d)$ . If not, then  $(M, t)$  is multiply determined hence  $(M, t) \in \uparrow_i^{\text{MN}}(c \cap d)$  by definition of  $\uparrow_i^{\text{MN}}$ .
- For every normal right-inverse  $f$  of  $[\cdot]_i$ , we have  $f \sqsubseteq \uparrow_i^{\text{MN}}$ .

Suppose  $f$  is a normal right inverse. We then need to prove  $\uparrow_i^{\text{MN}}(c) \subseteq f(c)$  for every  $c$ . Keep in mind that  $\sqsubseteq$  is  $\supseteq$  in  $\mathbf{K}(S)$ . It suffices to prove that if  $(M, t) \in \uparrow_i^{\text{MN}}(c)$  then  $(M, t) \in f(c)$ .

For this, take any  $(M, t) \in \uparrow_i^{\text{MN}}(c)$ . This mean that:

1.  $(M, t)$  is determined wrt  $\xrightarrow{i}_M$ , and



2. if  $(M, t)$  is uniquely determined, then  $s \xrightarrow{i}_M t$  for some  $s \in c$ .

Suppose  $(M, t)$  is uniquely determined, therefore by point 1 of Lemma 1 (i.e.  $\triangleright_i(M, s) \subseteq f$  if  $s \in c$ ) we have that  $(M, t) \in f(c)$ . Now suppose it is not uniquely determined, then  $(M, t)$  is multiply determined, therefore by condition 2 (i.e.  $\{(M, t)\} \subseteq f(c)$  if  $(M, t)$  is multiply determined) then  $(M, t) \in f(c)$ .

Notice that  $\uparrow_i^{\text{MN}}(c)$  excludes undetermined states if  $c \neq \text{true}$ . It turns out that we can add them and obtain a more succinct normal right inverse:

**Definition 12 (Normal-Right Inverse).** Let  $\mathbf{K}(\mathcal{S}) = (\text{Con}, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  be a Kripke scs over  $\mathcal{S} \subseteq \mathcal{M}^D$ . Define  $\uparrow_i^{\text{N}} : \text{Con} \rightarrow \text{Con}$  as  $\uparrow_i^{\text{N}}(c) \stackrel{\text{def}}{=} \{(M, t) \mid \forall s : s \xrightarrow{i}_M t, (M, s) \in c\}$ .

Clearly  $\uparrow_i^{\text{N}}(c)$  includes every  $(M, t)$  such that  $t$  is indetermined wrt  $\xrightarrow{i}_M$ .

**Theorem 4.** Let  $\mathbf{K}(\mathcal{S}) = (\text{Con}, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  and  $\uparrow_i^{\text{N}}$  as in Definition 12. The self-map  $\uparrow_i^{\text{N}}$  is a normal right inverse of  $[\cdot]_i$ .

*Proof.* The proof is very similar to the proof of the first point of Theorem 3 (i.e. The self-map  $\uparrow_i^{\text{MN}}$  is a normal right inverse of  $[\cdot]_i$ ). To prove  $\uparrow_i^{\text{N}}$  is a right inverse we prove  $[\uparrow_i^{\text{N}}(c)]_i = c$ . To prove that is a normal map, by Theorem 1, it suffices to prove  $\uparrow_i^{\text{N}}(\text{true}) = \text{true}$  and  $\uparrow_i^{\text{N}}(c \sqcup d) = \uparrow_i^{\text{N}}(c) \sqcup \uparrow_i^{\text{N}}(d)$  (equivalent to proving  $\uparrow_i^{\text{N}}(c \cap d) = \uparrow_i^{\text{N}}(c) \cap \uparrow_i^{\text{N}}(d)$ ).

- $[\uparrow_i^{\text{N}}(c)]_i = c$ . Suppose  $(M, s) \in [\uparrow_i^{\text{N}}(c)]_i$ , this means  $\triangleright_i(M, s) \subseteq \uparrow_i^{\text{N}}(c)$ . As  $s$  is determinant, there exists  $t$  s.t.  $s \xrightarrow{i}_M t$  and  $(M, t) \in \uparrow_i^{\text{N}}(c)$ . Therefore by definition of  $\uparrow_i^{\text{N}}$ ,  $s \in c$ . Now suppose  $(M, s) \in c$ , then for all states in  $\triangleright_i(M, s)$ , they are either uniquely determined by  $s$  or multiply determined, therefore  $\triangleright_i(M, s) \subseteq \uparrow_i^{\text{N}}(c)$ , and consequently  $(M, s) \in [\uparrow_i^{\text{N}}(c)]_i$ .
- $\uparrow_i^{\text{N}}(\text{true}) = \text{true}$ . We have  $\uparrow_i^{\text{N}}(\text{true}) = \{(M, t) \mid \forall s : s \xrightarrow{i}_M t, (M, s) \in \text{true}\}$ . Since  $(M, s) \in \text{true}$  always holds, all  $(M, t) \in \uparrow_i^{\text{N}}(\text{true})$ , thus  $\uparrow_i^{\text{N}}(\text{true}) = \text{true}$ .
- $\uparrow_i^{\text{N}}(c \cap d) = \uparrow_i^{\text{N}}(c) \cap \uparrow_i^{\text{N}}(d)$ . We first prove monotonicity (i.e. if  $c \subseteq d$  then  $\uparrow_i^{\text{N}}(c) \subseteq \uparrow_i^{\text{N}}(d)$ ). For this, suppose  $c \subseteq d$  and  $(M, t) \in \uparrow_i^{\text{N}}(c)$ . If there exists a state  $(M, s) \in c$  s.t.  $s \xrightarrow{i}_M t$  then  $(M, s) \in d$ , thus  $(M, t) \in \uparrow_i^{\text{N}}(d)$ . If there does not exists such state, then  $(M, t) \in \uparrow_i^{\text{N}}(d)$ . We now proceed with the proof.
  - $\uparrow_i^{\text{N}}(c \cap d) \subseteq \uparrow_i^{\text{N}}(c) \cap \uparrow_i^{\text{N}}(d)$ . Because  $c \cap d \subseteq c$  and  $c \cap d \subseteq d$ . Then  $\uparrow_i^{\text{N}}(c \cap d) \subseteq \uparrow_i^{\text{N}}(c)$  and  $\uparrow_i^{\text{N}}(c \cap d) \subseteq \uparrow_i^{\text{N}}(d)$  by monotonicity of  $\uparrow_i^{\text{N}}$ . Thus  $\uparrow_i^{\text{N}}(c \cap d) \subseteq \uparrow_i^{\text{N}}(c) \cap \uparrow_i^{\text{N}}(d)$ .
  - $\uparrow_i^{\text{N}}(c) \cap \uparrow_i^{\text{N}}(d) \subseteq \uparrow_i^{\text{N}}(c \cap d)$ . Suppose  $(M, t) \in \uparrow_i^{\text{N}}(c) \cap \uparrow_i^{\text{N}}(d)$ . If there exists  $s$  s.t.  $s \xrightarrow{i}_M t$ , then  $s \in c$  and  $s \in d$ . Now suppose there exists such  $s$ , therefore  $(M, s) \in c \cap d$  and  $(M, t) \in \uparrow_i^{\text{N}}(c \cap d)$ . If not, then  $(M, t)$  is not determine by any other state, hence  $(M, t) \in \uparrow_i^{\text{N}}(c \cap d)$  by definition of  $\uparrow_i^{\text{N}}$ .

We conclude this section with the order of the right-inverses we identified.

**Corollary 2.** *Let  $\mathbf{K}(\mathcal{S}) = (\text{Con}, \sqsubseteq, [\cdot]_1, \dots, [\cdot]_n)$  be a Kripke scs over  $\mathcal{S} \subseteq \mathcal{M}^D$ . Let  $\uparrow_i^M, \uparrow_i^{MN}$ , and  $\uparrow_i^N$  as in Definitions 9, 11 and 12, respectively. Then  $\uparrow_i^N \sqsubseteq \uparrow_i^{MN} \sqsubseteq \uparrow_i^M$ .*

## 5 Applications

In this section we will illustrate and briefly discuss the results obtained in the previous section in the context of modal logic.

Modal formulae can be interpreted as constraints in the scs  $\mathbf{K}(\mathcal{S}_n(\Phi))$ . First we recall the notion of modal language.

**Definition 13 (Modal Language).** *Let  $\Phi$  be a set of primitive propositions. The modal language  $\mathcal{L}_n(\Phi)$  is given by the following grammar:  $\phi, \psi, \dots := p \mid \phi \wedge \psi \mid \neg\phi \mid \Box_i\phi$  where  $p \in \Phi$  and  $i \in \{1, \dots, n\}$ . We shall use the abbreviations  $\phi \vee \psi$  for  $\neg(\neg\phi \wedge \neg\psi)$ ,  $\phi \Rightarrow \psi$  for  $\neg\phi \vee \psi$ ,  $\phi \Leftrightarrow \psi$  for  $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$ , the constant false  $\mathbb{F}$  for  $p \wedge \neg p$ , and the constant  $\mathbb{T}$  for  $\neg\mathbb{F}$ .*

We say that a pointed KS  $(M, s)$  satisfies  $\phi$  iff  $(M, s) \models \phi$  where  $\models$  is defined inductively as follows:  $(M, s) \models p$  iff  $\pi_M(s)(p) = 1$ ,  $(M, s) \models \phi \wedge \psi$  iff  $(M, s) \models \phi$  and  $(M, s) \models \psi$ ,  $(M, s) \models \neg\phi$  iff  $(M, s) \not\models \phi$ , and  $(M, s) \models \Box_i\phi$  iff  $(M, t) \models \phi$  for every  $t$  such that  $s \xrightarrow{i}_M t$ . This notion of satisfiability is invariant under a standard equivalence on Kripke structures: *Bisimilarity*, itself a central equivalence in concurrency theory [15].

**Definition 14 (Bisimilarity).** *Let  $\mathcal{B}$  a symmetric relation on pointed KS's. The relation is said to be a bisimulation iff for every  $((M, s), (N, t)) \in \mathcal{B}$ : (1)  $\pi_M(s) = \pi_N(t)$  and (2) if  $s \xrightarrow{i}_M s'$  then there exists  $t'$  s.t.  $t \xrightarrow{i}_N t'$  and  $((M, s'), (N, t')) \in \mathcal{B}$ . We say that  $(M, s)$  and  $(N, t)$  are bisimilar, written  $(M, s) \sim (N, t)$  if there exists a bisimulation  $\mathcal{B}$  such that  $((M, s), (N, t)) \in \mathcal{B}$ .*

The well-known result of *bisimilarity-invariance* of modal satisfiability [12] implies that if  $(M, s) \sim (N, t)$  then  $(M, s)$  and  $(M, t)$  satisfy exactly the same formulae in  $\mathcal{L}_n(\Phi)$ .

Modal logics are typically interpreted over different classes of KS's obtained by imposing conditions on their accessibility relations. Let  $\mathcal{S}_n(\Phi)$  be a non-empty set of  $n$ -agent Kripke structures over a set of primitive propositions  $\Phi$ . A modal formula  $\phi$  is said to be *valid* in  $\mathcal{S}_n(\Phi)$  iff  $(M, s) \models \phi$  for each  $(M, s)$  such that  $M \in \mathcal{S}_n(\Phi)$ .

We can interpret modal formulae as constraints in a given Kripke scs  $\mathbf{C} = \mathbf{K}(\mathcal{S}_n(\Phi))$  as follows.

**Definition 15 (Kripke Constraint Interpretation).** Let  $\mathbf{C}$  be a Kripke scs  $\mathbf{K}(\mathcal{S}_n(\Phi))$ . Given a modal formula  $\phi$  in the modal language  $\mathcal{L}_n(\Phi)$ , its interpretation in the Kripke scs  $\mathbf{C}$  is the constraint  $\mathbf{C}[\![\phi]\!]$  inductively defined as follows:

$$\begin{aligned}\mathbf{C}[\![p]\!] &= \{(M, s) \mid \pi_M(s)(p) = 1\} \\ \mathbf{C}[\![\phi \wedge \psi]\!] &= \mathbf{C}[\![\phi]\!] \sqcup \mathbf{C}[\![\psi]\!] \\ \mathbf{C}[\![\neg\phi]\!] &= \sim \mathbf{C}[\![\phi]\!] \\ \mathbf{C}[\![\Box_i\phi]\!] &= [\mathbf{C}[\![\phi]\!]]_i\end{aligned}$$

*Remark 2.* One can verify that for any Kripke scs  $\mathbf{K}(\mathcal{S}_n(\Phi))$ , the Heyting negation  $\sim c$  (Def. 4) is  $\Delta \setminus c$  where  $\Delta$  is the set of all pointed Kripke structures  $(M, s)$  such that  $M \in \mathcal{S}_n(\Phi)$  (i.e., boolean negation). Similarly, Heyting implication  $c \rightarrow d$  is equivalent to  $(\sim c) \cup d$  (i.e., boolean implication).

It is easy to verify that the constraint  $\mathbf{C}[\![\phi]\!]$  includes those pointed KS  $(M, s)$ , where  $M \in \mathcal{S}_n(\Phi)$ , such that  $(M, s) \models \phi$ . Thus,  $\phi$  is valid in  $\mathcal{S}_n(\Phi)$  if and only if  $\mathbf{C}[\![\phi]\!] = \text{true}$ .

Notice that from Proposition 3 and Theorem 1, each space function  $[\cdot]_i$  of  $\mathbf{K}(\mathcal{S}_n(\Phi))$  is a normal self-map. From Definitions 5 and 15 we can derive the following standard property stating that  $\Box_i$  is a normal modal operator: (*Necessitation*) If  $\phi$  is valid in  $\mathcal{S}_n(\Phi)$  then  $\Box_i\phi$  is valid in  $\mathcal{S}_n(\Phi)$ , and (*Distribution*)  $\Box_i(\phi \Rightarrow \psi) \Rightarrow (\Box_i\phi \Rightarrow \Box_i\psi)$  is valid in  $\mathcal{S}_n(\Phi)$ .

**Right-Inverse Modalities.** Reverse modalities, also known as inverse modalities, arise naturally in many modal logics. For example in temporal logics they are past operators [19], in modal logics for concurrency they represent backward moves [18], in epistemic logic they correspond to utterances [11].

To illustrate our results in the previous sections, let us fix a modal language  $\mathcal{L}_n(\Phi)$  (whose formulae are) interpreted in an arbitrary Kripke scs  $\mathbf{C} = \mathbf{K}(\mathcal{S}_n(\Phi))$ . Suppose we wish to extend it with modalities  $\Box_i^{-1}$ , called reverse modalities also interpreted over the same set of KS's  $\mathcal{S}_n(\Phi)$  and satisfying some minimal requirement. The new language is given by the following grammar.

**Definition 16 (Modal Language with Reverse Modalities).** Let  $\Phi$  be a set of primitive propositions. The modal language  $\mathcal{L}_n^{+r}(\Phi)$  is given by the following grammar:  $\phi, \psi, \dots := p \mid \phi \wedge \psi \mid \neg\phi \mid \Box_i\phi \mid \Box_i^{-1}\phi$  where  $p \in \Phi$  and  $i \in \{1, \dots, n\}$ .

The minimal semantic requirement for each  $\Box_i^{-1}$  is that, regardless of the interpretation we give to  $\Box_i^{-1}\phi$ , we should have:

$$\Box_i\Box_i^{-1}\phi \Leftrightarrow \phi \quad \text{valid in } \mathcal{S}_n(\Phi). \quad (5)$$

We then say that  $\Box_i^{-1}$  is a *right-inverse* modality for  $\Box_i$  (by analogy to the notion of right-inverse of a function).

Since  $\mathbf{C}[\Box_i \phi] = [\mathbf{C}[\phi]]_i$ , we can use the results in the previous sections to derive semantic interpretations for  $\Box_i^{-1} \phi$  by using a right inverse  $\uparrow_i$  for the space function  $[\cdot]_i$  in Definition 15. Assuming that such a right inverse exists, we can then interpret the reverse modality in  $\mathbf{C}$  as

$$\mathbf{C}[\Box_i^{-1} \phi] = \uparrow_i(\mathbf{C}[\phi]). \quad (6)$$

Since each  $\uparrow_i$  is a right inverse of  $[\cdot]_i$ , it is easy to verify that the interpretation satisfies the requirement in Equation 5. Furthermore, from Theorem 2 we can conclude that for each  $M \in \mathcal{S}_n(\Phi)$ ,  $\xrightarrow{i}_M$  must necessarily be determinant-complete.

**Normal Inverses Modalities.** We can choose  $\uparrow_i$  in Equation 6 from the set  $\{\uparrow_i^N, \uparrow_i^{MN}, \uparrow_i^M\}$  of right-inverse constructions in Section 4.3. Assuming that  $\uparrow_i$  is a normal self-map (e.g., either  $\uparrow_i^N$  or  $\uparrow_i^M$ ), we can show from Definition 5 and Equation 6 that  $\Box_i^{-1}$  is itself a normal modal operator in the following sense: (1) If  $\phi$  is valid in  $\mathcal{S}_n(\Phi)$  then  $\Box_i^{-1} \phi$  is valid in  $\mathcal{S}_n(\Phi)$ , and (2)  $\Box_i^{-1}(\phi \Rightarrow \psi) \Rightarrow (\Box_i^{-1} \phi \Rightarrow \Box_i^{-1} \psi)$  is valid in  $\mathcal{S}_n(\Phi)$ .

**Inconsistency Invariance.** We can conclude from Proposition 1(1) that since we assumed a right inverse for  $[\cdot]_i$ , we should have

$$\neg \Box_i F \text{ valid in } \mathcal{S}_n(\Phi). \quad (7)$$

Indeed using the fact that  $[\cdot]_i$  is a normal self-map with an inverse  $\uparrow_i$  and Theorem 1, we can verify the following:

$$\mathbf{C}[\Box_i F] = \mathbf{C}[\Box_i(F \wedge \Box_i^{-1} F)] = \mathbf{C}[\Box_i(F) \wedge \Box_i(\Box_i^{-1} F)] = \mathbf{C}[\Box_i(F) \wedge F] = \mathbf{C}[F].$$

This implies  $\Box_i F \Leftrightarrow F$  is valid in  $\mathcal{S}_n(\Phi)$  and this amounts to say that  $\neg \Box_i F$  is valid in  $\mathcal{S}_n(\Phi)$ .

Modal systems such  $K_n$  or HM [12] where  $\neg \Box_i F$  is not an axiom cannot be extended with a reverse modality satisfying Equation 5 (without restricting their models). The issue is that the axiom  $\neg \Box_i F$ , typically needed in epistemic, doxastic and temporal logics, would require the accessibility relations of agent  $i$  to be serial (recall that determinant-complete relations are necessarily serial). In fact  $\Box_i F$  is used in HM logic to express deadlocks wrt to  $i$ ;  $(M, s) \models \Box_i F$  iff there is no  $s'$  such that  $s \xrightarrow{i}_M s'$ . Clearly there cannot be state deadlocks wrt  $i$  if  $\xrightarrow{i}_M$  is required to be serial for each  $M$ .

**Bisimilarity Invariance.** Recall that bisimilarity invariance states that bisimilar pointed KS's satisfy the same formulae in  $\mathcal{L}_n(\Phi)$ . The addition of a reverse modality  $\Box_i^{-1}$  may violate this invariance, in the sense that bisimilar pointed KS's may not longer satisfy the same formulae in  $\mathcal{L}_n^{+r}(\Phi)$ . This can be viewed as saying that the addition of inverse modalities increases the distinguishing power of the original modal language. We prove this next.

Let us suppose that the chosen right inverse  $\uparrow_i$  in Equation 6 is any normal self-map whatsoever. It follows from Lemma 1(2) and Equation 6 that if  $t$  is *multiply-determined*

wrt  $\xrightarrow{i}_M$  then  $(M, t) \models \Box_i^{-1}\text{F}$ . We can use Lemma 1(1) and Equation 6 to show that if  $t$  is *uniquely determined* wrt  $\xrightarrow{i}_M$  then  $(M, t) \not\models \Box_i^{-1}\text{F}$ .

Now take  $v$  and  $s_4$  as in Figure 1. Suppose that  $\pi_{M_5}(v) = \pi_{M_1}(s_i)$  for every  $s_i$  in the states of  $M_1$ . Clearly  $(M_1, s_4) \sim (M_5, v)$ . Since  $s_4$  is multiply determined and  $v$  is uniquely determined, we conclude that  $(M_1, s_4) \models \Box_i^{-1}\text{F}$  but  $(M_1, v) \not\models \Box_i^{-1}\text{F}$ . Thus  $\Box_i^{-1}\text{F}$  can tell uniquely determined states from multiply determined ones but bisimilarity cannot.

**Temporal Operators.** We conclude this section with a brief discussion on some right-inverse linear-time modalities. Let us suppose that  $n = 2$  in our modal language  $\mathcal{L}_n(\Phi)$  under consideration (thus interpreted in Kripke scs  $\mathbf{C} = \mathbf{K}(\mathcal{S}_2(\Phi))$ ). Assume further that the intended meaning of the two modalities  $\Box_1$  and  $\Box_2$  are the *next* operator ( $\circ$ ) and the *henceforth/always* operator ( $\square$ ), respectively, in a *linear-time* temporal logic. To obtain the intended meaning we take  $\mathcal{S}_2(\Phi)$  to be the largest set such that: If  $M \in \mathcal{S}_2(\Phi)$ ,  $M$  is a 2-agent KS where  $\xrightarrow{1}_M$  is isomorphic to the successor relation on the natural numbers and  $\xrightarrow{2}_M$  is the reflexive and transitive closure of  $\xrightarrow{1}_M$ . The relation  $\xrightarrow{1}_M$  is intended to capture the linear flow of time. Intuitively,  $s \xrightarrow{1}_M t$  means  $t$  is the only next state for  $s$ . Similarly,  $s \xrightarrow{2}_M t$  for  $s \neq t$  is intended to capture the fact that  $t$  is one of the infinitely many future states for  $s$ .

Let us first consider the next operator  $\Box_1 = \circ$ . Notice that  $\xrightarrow{1}_M$  is determinant-complete. If we apply Equation 6 with  $\uparrow_1 = \uparrow_1^M$ , i.e., the greatest right inverse of  $[\cdot]_1$ , we obtain  $\Box_1^{-1} = \ominus$ , a past modality known in the literature as *strong* previous operator [19]. The operator  $\ominus$  is given by  $(M, t) \models \ominus \phi$  iff there exists  $s$  such that  $s \xrightarrow{M}_1 t$  and  $(M, s) \models \phi$ . If we take  $\uparrow_i$  to be the normal right inverse  $\uparrow_i^N$ , we obtain  $\Box_1^{-1} = \tilde{\ominus}$  the past modality known as *weak* previous operator [19]. The operator  $\tilde{\ominus}$  is given by  $(M, t) \models \tilde{\ominus} \phi$  iff for every  $s$  if  $s \xrightarrow{M}_1 t$  then  $(M, s) \models \phi$ . Notice that the only difference between the two operators is the following: If  $s$  is an indetermined/initial state wrt  $\xrightarrow{1}_M$  then  $(M, s) \not\models \ominus \phi$  and  $(M, s) \models \tilde{\ominus} \phi$  for any  $\phi$ . It follows that  $\ominus$  is not a normal operator, since  $\ominus\top$  is not valid in  $\mathcal{S}_2(\Phi)$  but  $\top$  is.

Let us now consider the always operator  $\Box_2 = \square$ . Notice that  $\xrightarrow{2}_M$  is not determinant-complete: Take any increasing chain  $s_0 \xrightarrow{1}_M s_1 \xrightarrow{1}_M \dots$ . The state  $s_1$  is not determinant because for every  $s_j$  such that  $s_1 \xrightarrow{2}_M s_j$  we also have  $s_0 \xrightarrow{2}_M s_j$ . Theorem 2 tells us that there is no right-inverse  $\uparrow_2$  of  $[\cdot]_i$  that can give us an operator  $\Box_2^{-1}$  satisfying Equation 5.

By analogy to the above-mentioned past operators, one may think that the past operator *it-has-always-been*  $\boxminus$  [23] may provide a reverse modality for  $\square$  in the sense of Equation 5. The operator is given by  $(M, t) \models \boxminus \phi$  iff  $(M, s) \models \phi$  for every  $s$  such that  $s \xrightarrow{2}_M t$ . Clearly  $\square \boxminus \phi \Rightarrow \phi$  is valid in  $\mathcal{S}_2(\Phi)$  but  $\phi \Rightarrow \square \boxminus \phi$  is not.

## 6 Concluding Remarks and Related Work

We studied the existence and derivation of right inverses (extrusion) of space functions for the Kripke spatial constraint systems. We showed that being *determinant-complete* is the weakest condition on KS's that guarantees the existence of such right inverses. We identified the greatest normal right inverse of any given space function. We applied these results to modal logic by using space functions and their right inverses as the semantic counterparts of box modalities and their right inverse modalities. We discussed our results in the context of modal concepts such as bisimilarity invariance, inconsistency invariance and temporal modalities.

Most of the related work was discussed in the previous sections. In previous work [11] the authors derived an inverse modality but only for the specific case of a logic of belief. The work was neither concerned with giving a *complete* characterization of the existence of right inverse nor deriving *normal* inverses. The constraint systems in this paper can be seen as modal extension of geometric logic [25]. Modal logic have also been studied from an algebraic perspective by using modal extensions of boolean and Heyting algebras in [14,2,4]. These works, however, do not address issues related to inverse modalities. Inverse modalities have been used in temporal, epistemic and logic for concurrency. In [23] discusses inverse temporal and epistemic modalities from a proof theory perspective. The works [18,5,10] use modal logic with reverse modalities for specifying true concurrency and [6,7] use backward modalities for characterizing branching bisimulation. None of these works is concerned with an algebraic approach or with deriving inverse modalities for modal languages.

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