A practical guide and new trends to price European options under Exponential Lévy models

Khaled Salhi

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Abstract

In this paper we develop a thorough survey of the European option pricing under exponential Lévy models. We sweep all steps from equivalent martingale measures construction to numerical valuation of the option price under these measures. We apply the Esscher transform technique to provide two examples of equivalent martingale measures: the Esscher martingale measure and the minimal entropy martingale measure. We numerically compute the option price using the fast Fourier transform. The results are detailed with an example of each exponential Lévy class. The main contribution of this paper is to build a comprehensive study from the theoretical point of view to practical numerical illustration and to give a complete characterization of the studied equivalent martingale measures by discussing their similarity and their applicability in practice.

Keywords: Lévy process, incomplete market, Esscher martingale measure, minimal entropy martingale measure, fast Fourier transform, Merton model, variance gamma model.

1 Introduction

Stochastic processes are intensively used for modeling financial markets. The Black & Scholes model is one of the most known models. It describes the stock price as a geometric Brownian motion. In this context, the option pricing problem is solved using the risk neutral approach [7]. The key tool is the uniqueness of the equivalent martingale measure (EMM) and the derivative price is therefore the unique arbitrage-free contingent claim value.

It has become clear, however, that this option pricing model is inconsistent with options data. In the real world, we observe that asset price processes have jumps or spikes and risk managers have to take them into consideration. Moreover, the empirical distributions of asset returns exhibit fat tails and skewness behaviors that deviate from

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normality [4]. Hence, models that accurately fit return distributions are essential to estimate profit ans loss (P&L) distributions. Similarly, in the risk-neutral world, we observe that implied volatilities are constant neither across strikes nor across maturities as stipulated by the Black & Scholes model [35, 36]. Therefore, traders need models that can capture the behavior of the implied volatility smiles more accurately, in order to handle the risk of trades. Lévy processes provide the appropriate tools to adequately and consistently describe all these observations, both in the real world and in the risk-neutral world [3, 14, 15, 31].

By allowing the stock price process to jump, problems become more complicated. As soon as the security can have more than a single jump size, the market will be incomplete. Thus, under the assumption of no arbitrage, there are infinitely many equivalent martingale measures. This induces an interval of arbitrage-free prices. In order to construct an option pricing model, we have to select a suitable martingale measure. Once an equivalent martingale measure $\mathbb{P}^*$ is selected, the price $\pi(H)$ of an option H is given by

$$\pi(H) = \mathbb{E}^*[e^{-rT}H], \tag{1}$$

where r is the risk-free interest rate and T is a maturity. This is the idea of the equivalent martingale measure method.

This survey is a practical guide to option pricing when the log of the stock price is modeled with a Lévy process. This work aims at explaining in a single document all stages of the option valuation process, as a global understanding of this process is necessary and useful for a practical purposes. Considering a price process model under the historical probability measure, we explicit this model under an equivalent martingale measure. Then, we numerically compute the option price using the fast Fourier transform technique (FFT) developed in [10].

Outline. This paper is organized as follows. In Section 2, we give an overview of the exponential Lévy model and the option pricing in this context. For background information on exponential Lévy models, the reader may refer to textbooks [1, 12]. In Section 3, we explain how to define an equivalent martingale measure $\mathbb{P}^*$ using the Esscher transform technique. We detail two examples: the Esscher martingale measure and the minimal entropy martingale measure. For these measures, the logarithm stock price process is still a Lévy process under $\mathbb{P}^*$ and its characteristic triplet is known. The similarity between the Esscher martingale measure and the minimal entropy martingale measure is studied. Otherwise, we show that while the former is still a good tool for pricing applications, the latter cannot be applied in a practical context. In Section 4, we give the Madan-Carr method and develop an expression of the option price, based on the characteristic function of the log price process. The application of the FFT here is possible and will be the subject of Section 5. Finally, in Section 6, we detail our approach on three examples of exponential Lévy models: the standard Black & Scholes model, the Merton model and the variance gamma model.

2 Exponential Lévy model

In this section, we introduce the exponential Lévy model and give some of its properties.

Definition 2.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. The exponential Lévy model is defined by an asset price process $(S_t)_{t\in[0,T]}$ of the form:*

$$S_t = S_0\, e^{X_t}, \tag{2}$$

where $S_0 > 0$ is a constant and $(X_t)_{t\in[0,T]}$ is a one-dimensional Lévy process [1, 6, 38] with a characteristic triplet (b, σ^2, ν). The discounted price is given by $\tilde{S}_t = e^{-rt}S_t$ where r is the risk-free interest rate.

Notation. *In the sequel, the notations used in Definition 2.1 will always be valid.*

The measure ν on $\mathbb{R}$, called the Lévy measure, determines the intensity of jumps of different sizes: $\nu([a_1, a_2])$ is the expected number of jumps on the time interval $[0, 1]$, whose sizes fall in $[a_1, a_2]$. The Lévy measure satisfies the integrability condition

$$\int_{\mathbb{R}} 1 \wedge |x|^2 \nu(\mathrm{d}x) < \infty.$$

Note that $\nu([a_1, a_2])$ is still finite for any compact set $[a_1, a_2]$ such that $0 \notin [a_1, a_2]$. If not, the process $(X_t)_{t\in[0,T]}$ would have an infinite number of jumps of finite size on every time interval $[0, t]$, which contradicts the càdlàg property of $(X_t)_{t\in[0,T]}$. Thus ν defines a Radon measure on $\mathbb{R}\backslash\{0\}$. However, ν is not necessarily a finite measure. The above restriction still allows it to blow up at zero and $(X_t)_{t\in[0,T]}$ may have an infinite number of small jumps on $[0, t]$. In this case, the sum of the jumps becomes an infinite series and its convergence imposes some additional conditions on the measure ν.

The law of X_t at any time t is determined by the triplet (b, σ^2, ν). In particular, the Lévy-Khintchine representation gives the characteristic function of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}$

$$\Phi_t(u) := \mathbb{E}[e^{iuX_t}] = e^{t\Psi(u)}, \quad u \in \mathbb{R} \tag{3}$$

where Ψ, called the characteristic exponent, is given by

$$\Psi(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|\leqslant 1}\right) \nu(\mathrm{d}x). \tag{4}$$

Furthermore, if the Lévy measure also satisfies the condition $\int_{|x|\leqslant 1} |x|\nu(\mathrm{d}x) < \infty$, the jump part process, defined by

$$X_t^J = \sum_{\substack{s\in(0,t] \\ \Delta X_s \neq 0}} \Delta X_s,$$

becomes a finite variation process. In this case, the process $(X_t)_{t\in[0,T]}$ can be expressed as the sum of a linear drift, a Brownian motion and a jump part process:

$$X_t = \gamma t + \sigma B_t + X_t^J,$$

where $\gamma = b - \int_{|x|\leqslant 1} x\nu(\mathrm{d}x)$. The characteristic exponent can be expressed by

$$\Psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1\right) \nu(\mathrm{d}x).$$

Note that the Lévy triplet of $(X_t)_{t\in[0,T]}$ is not given by (γ, σ^2, ν), but by (b, σ^2, ν). In fact, b is not an intrinsic quantity and depends on the truncation function used in the Lévy-Khintchine representation while γ has an intrinsic interpretation as the expectation slope of the continuous part process of $(X_t)_{t\in[0,T]}$. The expectation $\mathbb{E}[X_t]$ is given by the sum of the linear drift and the expectation of jump part equal to $(\gamma + \int_{\mathbb{R}} x\nu(\mathrm{d}x))t$.

Now, by using Itô's formula, we can observe that $(S_t)_{t\in[0,T]}$ is the solution to the following SDE

$$S_t = S_0 + \int_{(0,t]} S_{s-} d\hat{X}_s, \tag{5}$$

where

$$\hat{X}_t := X_t + \frac{1}{2}\langle X^c\rangle_t + \sum_{s\in(0,t]} \{e^{\Delta X_s} - 1 - \Delta X_s\} \tag{6}$$

and $(X^c_t)_{t\in[0,T]}$ is the continuous part of $(X_t)_{t\in[0,T]}$. Hence, $(S_t)_{t\in[0,T]}$ can be rewritten as:

$$S_t = S_0\ \mathcal{E}(\hat{X})_t, \tag{7}$$

where $(\mathcal{E}(\hat{X})_t)_{t\in[0,T]}$ stands for the Doléans-Dade exponential of $(\hat{X}_t)_{t\in[0,T]}$, [27]. Furthermore, $(\hat{X}_t)_{t\in[0,T]}$ is still a Lévy process under $\mathbb{P}$. By expressing its Lévy-Itô decomposition, we obtain that the characteristic triplet of $(\hat{X}_t)_{t\in[0,T]}$ is given by $(\hat{b}, \sigma^2, \hat{\nu})$ (see [12, 19] for details) where

$$\hat{b} = b + \frac{1}{2}\sigma^2 + \int_{|x|\leqslant 1} x\hat{\nu}(\mathrm{d}x) - \int_{|x|\leqslant 1} x\nu(\mathrm{d}x) \tag{8}$$

and

$$\hat{\nu}(\mathrm{d}x) = \nu \circ J^{-1}(\mathrm{d}x) \text{ where } J(x) := e^x - 1 \text{ for } x \in \mathbb{R}. \tag{9}$$

Remark 2.1. *(i) It holds that*

$$\mathrm{supp}\{\hat{\nu}\} \subset (-1, \infty).$$

(ii) If ν has a density $\nu(x)$, then $\hat{\nu}$ has a density $\hat{\nu}(x)$ given by

$$\hat{\nu}(x) = \frac{1}{1+x}\ \nu(\log(1+x)).$$

(iii) From the economical point of view, $(X_t)_{t\in[0,T]}$ represents the logarithmic return process of $(S_t)_{t\in[0,T]}$, while $(\hat{X}_t)_{t\in[0,T]}$ represents the simple return process of $(S_t)_{t\in[0,T]}$.

We consider a call with maturity T and strike K. The payoff of this option is given by the random variable $H = (S_T - K)_+$. Let $\mathcal{P}$ denote the set of all equivalent martingale measures (also called risk-neutral measures)

$$\mathcal{P} = \left\{\mathbb{P}^* \sim \mathbb{P},\quad (\tilde{S}_t)_{t\in[0,T]} \text{ is a martingale under } \mathbb{P}^*\right\}.$$

In a complete market, there is only one equivalent martingale measure $\mathbb{P}^*$. Then, the risk-neutral price of the option at $t = 0$ is given by

$$C(K) = e^{-rT}\mathbb{E}^*[(S_T - K)_+], \tag{10}$$

where $\mathbb{E}^*$ is the expectation under $\mathbb{P}^*$.

With exponential Lévy models, we are mostly in the incomplete market case. Therefore, several equivalent martingale measures can be used to price the option. The range of option prices is given by

$$\left[\inf_{\mathbb{P}^*\in\mathcal{P}} e^{-rT}\mathbb{E}^*[(S_T - K)_+],\ \sup_{\mathbb{P}^*\in\mathcal{P}} e^{-rT}\mathbb{E}^*[(S_T - K)_+]\right].$$

So, one can always choose a measure $\mathbb{P}^* \in \mathcal{P}$ according to some criteria and price the option using the formula (10).

Another difficulty with exponential Lévy models is that closed-form expressions exist for their characteristic function while their density function is usually unknown. It is thus difficult to find a closed-form formula of $C(K)$, and even not possible for some pricing measures and Lévy processes. Nevertheless, the analytic expression of the characteristic function Φ_t^* under the pricing measure $\mathbb{P}^*$ is known, one can use the fast Fourier transform (FFT) method developed by Carr & Madan [10] to numerically compute the option price.

Assume now that the characteristic function Φ_t of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}$ is analytically known. The pricing procedure has two steps:

- Choose an equivalent martingale measure $\mathbb{P}^* \in \mathcal{P}$ under which we have an analytic expression of the characteristic function, called Φ_t^*.
- Apply the FFT in Φ_T^* to compute the option price.

3 Equivalent martingale measure

The equivalent martingale measure method is one of the most powerful methods of option pricing. The no-arbitrage assumption can be expressed by the existence of at least one equivalent martingale measure. If the market is arbitrage-free and incomplete, there are several equivalent martingale measures and we have to select, with respect to some criteria, the most suitable one in order to price options.

Several candidates for an equivalent martingale measure are proposed in the literature. To construct them, two different approaches are employed:

- Esscher transform method: The Esscher transform method is widely used in risk theory. It consists in applying an Esscher transform with respect to some risk process. This risk process can be the logarithmic return $(X_t)_{t\in[0,T]}$ in the case of the *Esscher martingale measure* [8, 21], the simple return $(\hat{X}_t)_{t\in[0,T]}$ in the case of the *Minimal entropy martingale measure* [18, 19, 33] or the continuous martingale part $(X_t^c)_{t\in[0,T]}$ of the Lévy process $(X_t)_{t\in[0,T]}$ in the case of the *mean correcting martingale measure* [42].
- Minimal distance method: This method is more related to the maximization of expected utility and hedging problem. This includes the *utility-based martingale measure* [26], the *minimal martingale measure* [17] and the *variance optimal martingale measure* [39].

3.1 Esscher martingale measure

The Esscher martingale measure is constructed by applying an Esscher transform with respect to the process $(X_t)_{t\in[0,T]}$. One of the greatest advantages is that $(X_t)_{t\in[0,T]}$ is still a Lévy process under this equivalent measure. Let us give the definition of the Esscher transform and the condition under which we obtain an equivalent martingale measure.

Definition 3.1. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. We call Esscher transform with respect to $(X_t)_{t\in[0,T]}$ any change of $\mathbb{P}$ to an equivalent measure $\mathbb{P}^*$ by a density process $Z_t = \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t}$ of the form:*

$$Z_t = \frac{e^{\theta X_t}}{\mathbb{E}\left[e^{\theta X_t}\right]}, \tag{11}$$

where $\theta \in \mathbb{R}$.

The Esscher density process $Z_t = \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t}$, which formally looks like the density of a one-dimensional Esscher transform, leads to one-dimensional Esscher transforms of the marginal distributions, with the same parameter θ:

$$\mathbb{P}^*(X_t \in B) = \int \mathbf{1}_B(X_t)\frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}]}\mathrm{d}\mathbb{P} = \int \mathbf{1}_B(x)\frac{e^{\theta x}}{\mathbb{E}[e^{\theta X_t}]}\mathrm{d}\mathbb{P}^{X_t}(\mathrm{d}x),$$

for any set $B \in \mathcal{B}(\mathbb{R})$.

One advantage of using $(X_t)_{t\in[0,T]}$ as a risk process is that the density process only depends on the current stock price. In what follows, we give the condition for the existence of the density process $Z_t = \frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t}$ and the characteristic triplet of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}^*$ in such case.

Proposition 3.1. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process on $\mathbb{R}$ with characteristic triplet (b, σ^2, ν) and let $\theta \in \mathbb{R}$. The exponential moment $\mathbb{E}[e^{\theta X_t}]$ is finite for some t or, equivalently, for all $t > 0$ if and only if $\int_{|x|\geqslant 1} e^{\theta x}\nu(\mathrm{d}x) < \infty$. In this case,*

$$\mathbb{E}[e^{\theta X_t}] = e^{t\Psi(-i\theta)},$$

where Ψ is the characteristic exponent of the Lévy process defined by (4).

For a proof, see [38, Theorem 25.17].

Proposition 3.2. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process on $\mathbb{R}$ with characteristic triplet (b, σ^2, ν) under $\mathbb{P}$. For all $\theta \in \mathbb{R}$ such that $\mathbb{E}\left[e^{\theta X_1}\right] < \infty$,*

(i) The process $(Z_t)_{t\in[0,T]}$ given by (11) defines a density process.

(ii) The process $(X_t)_{t\in[0,T]}$ is a Lévy process with triplet (b^*, σ^2, ν^*) under $\mathbb{P}^*$ where

$$\nu^*(\mathrm{d}x) = e^{\theta x}\nu(\mathrm{d}x), \quad \textit{for } x \in \mathbb{R},$$

and

$$b^* = b + \sigma^2\theta + \int_{|x|\leqslant 1} x\nu^*(\mathrm{d}x) - \int_{|x|\leqslant 1} x\nu(\mathrm{d}x).$$

Proof. (i) Recall that $\mathbb{E}[e^{\theta X_t}] = e^{t\Psi(-i\theta)} = \left(\mathbb{E}[e^{\theta X_1}]\right)^t$. Then, Z_t is integrable for all t. Using the independence and stationary properties of the Lévy process $(X_t)_{t\in[0,T]}$, we have for $s < t$,

$$\begin{aligned}
\mathbb{E}[Z_t|\mathcal{F}_s] &= \frac{1}{\mathbb{E}\left[e^{\theta X_t}\right]}\mathbb{E}[e^{\theta(X_t - X_s + X_s)}|\mathcal{F}_s] = \frac{1}{\mathbb{E}\left[e^{\theta X_t}\right]}\ \mathbb{E}[e^{\theta(X_t - X_s)}|\mathcal{F}_s]\ \mathbb{E}[e^{\theta X_s}|\mathcal{F}_s] \\
&= \frac{1}{\mathbb{E}\left[e^{\theta X_t}\right]}\ \mathbb{E}[e^{\theta(X_t - X_s)}]\ e^{\theta X_s} = \frac{e^{\theta X_s}}{\mathbb{E}\left[e^{\theta X_s}\right]} = Z_s.
\end{aligned}$$

Thus, $(Z_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale.

(ii) We prove that $(X_t)_{t\in[0,T]}$ is a Lévy process under the probability measure $\mathbb{P}^*$ by computing its characteristic function under $\mathbb{P}^*$:

$$\begin{aligned}
\Phi^*(u) = \mathbb{E}^*[e^{iuX_t}] &= \int e^{iuX_t}\mathrm{d}\mathbb{P}^* = \int e^{iuX_t}\frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}]}\mathrm{d}\mathbb{P} \\
&= \frac{\mathbb{E}[e^{(\theta+iu)X_t}]}{\mathbb{E}[e^{\theta X_t}]} = \exp\left(t\left(\Psi(-i(\theta + iu)) - \Psi(-i\theta)\right)\right).
\end{aligned}$$

Define $\Psi^*(u) \equiv \Psi(-i(\theta+iu)) - \Psi(-i\theta)$ for $u \in \mathbb{R}$. Then,

$$\begin{aligned}
\Psi^*(u) &= \left(b(\theta+iu) + \frac{\sigma^2}{2}(\theta+iu)^2 + \int_{\mathbb{R}} \left(e^{(\theta+iu)x} - 1 - (\theta+iu)x\mathbf{1}_{|x|\leqslant 1}\right) \nu(\mathrm{d}x) \right) \\
&\quad - \left(b\theta + \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} \left(e^{\theta x} - 1 - \theta x\mathbf{1}_{|x|\leqslant 1}\right) \nu(\mathrm{d}x) \right) \\
&= i(b+\sigma^2\theta)u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{\theta x}(e^{iux}-1) - iux\mathbf{1}_{|x|\leqslant 1}\right) \nu(\mathrm{d}x) \\
&= i\left(b + \sigma^2\theta + \int_{|x|\leqslant 1} (e^{\theta x}-1)x\nu(\mathrm{d}x) \right) u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{\theta x}(e^{iux} - 1 - iux\mathbf{1}_{|x|\leqslant 1})\right) \nu(\mathrm{d}x).
\end{aligned}$$

By defining

$$\nu^*(x) = e^{\theta x}\nu(x), \quad \text{for } x \in \mathbb{R}$$

and

$$b^* = b + \sigma^2\theta + \int_{|x|\leqslant 1} x\nu^*(\mathrm{d}x) - \int_{|x|\leqslant 1} x\nu(\mathrm{d}x),$$

we obtain a Lévy-Khintchine representation for Φ^*. Thus, $(X_t)_{t\in[0,T]}$ is a Lévy process under $\mathbb{P}^*$ with triplet (b^*, σ^2, ν^*). □

To interpret the expression of b^*, let us remember that the jump measure ν can have a singularity at zero. Thus, there can be infinitely many small jumps and the characteristic function of their sum $\int_{|x|\leqslant 1}(e^{iux}-1)\nu(\mathrm{d}x)$ does not necessarily converge. To obtain convergence, this jump integral was centered and replaced by its compensated version in the Lévy-Khintchine representation. We integrate this compensator $\int_{|x|\leqslant 1} x\nu(\mathrm{d}x)$ in the drift. When we change the measure $\mathbb{P}$ to $\mathbb{P}^*$, we must naturally truncate the compensator of ν from the drift and add the one of ν^*. We thus obtain b^*.

Theorem 3.1. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process with triplet (b, σ^2, ν) under $\mathbb{P}$. Suppose that X_1 is non-degenerate and has a moment generating function $u \mapsto \mathbb{E}[\exp(uX_1)]$ on some open interval (a_1, a_2) with $a_2 - a_1 > 1$. Assume that there exists a real number $\theta \in (a_1, a_2 - 1)$ such that*

$$b + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{\theta x}(e^x - 1) - x\mathbf{1}_{|x|\leqslant 1}\right) \nu(\mathrm{d}x) = r, \tag{12}$$

or equivalently

$$\mathbb{E}[e^{(\theta+1)X_t}] = e^{rt}\mathbb{E}[e^{\theta X_t}], \tag{13}$$

where r is the risk-free interest rate. Then the real θ is unique and the equivalent measure $\mathbb{P}^*$ given by the Esscher transform with respect to $(X_t)_{t\in[0,T]}$

$$\left.\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{\mathbb{E}\left[e^{\theta X_t}\right]}$$

is an equivalent martingale measure.

Proof. Proposition 3.2 guarantees that $(X_t)_{t\in[0,T]}$ is a Lévy process under all measures $\mathbb{P}^*$ given by an Esscher transform. By the independence and stationarity of increments of

$(X_t)_{t\in[0,T]}$, the martingale property of $(S_t)_{t\in[0,T]}$ under $\mathbb{P}^*$ is implied by $\mathbb{E}^*[\tilde{S}_t] = S_0$ for $t > 0$. From the definition of the Esscher measure transform,

$$\mathbb{E}^*[\tilde{S}_t] = \mathbb{E}^*[S_0\ e^{X_t - rt}] = \mathbb{E}\left[S_0\ e^{X_t - rt}\frac{e^{\theta X_t}}{\mathbb{E}[e^{\theta X_t}]}\right] = S_0\ e^{-rt}\ \frac{\mathbb{E}[e^{(\theta+1)X_t}]}{\mathbb{E}[e^{\theta X_t}]}. \tag{14}$$

Thus, $\mathbb{E}^*[\tilde{S}_t] = S_0$ if and only if there exists a real θ such that (13) holds. This real θ must be in $(a_1, a_2 - 1)$ to ensure the existence of the moment generating function in θ and $\theta + 1$.

Using Proposition 3.1, we rewrite (13) in terms of the characteristic exponent under $\mathbb{P}$

$$\Psi(-i(\theta+1)) - \Psi(-i\theta) = r.$$

We then develop the expression of Ψ given by (4) and we obtain the condition (12). The discounted price $(\tilde{S}_t)_{t\in[0,T]}$ is a martingale under the equivalent measure given by the Esscher transform with θ (if it exists) solution to this equation (12). $\square$

3.2 Minimal Entropy martingale measure (MEMM)

The MEMM has been investigated in various settings by several authors [16, 17, 18, 33, 39]. In particular, the MEMM for Exponential Lévy process has been discussed in [11, 19, 23, 34]. It turns out that this measure can be obtained by applying an Esscher transform with respect to the simple return process $(\hat{X}_t)_{t\in[0,T]}$. Furthermore, $(X_t)_{t\in[0,T]}$ is still a Lévy process under this measure. In this section, we recall the definition of the relative entropy and give the condition on the Esscher parameter for the existence of the MEMM, as well as the characteristic triplet of $(X_t)_{t\in[0,T]}$ under this measure.

Definition 3.2. *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ and $\mathbb{Q}$ a probability measure on $\mathcal{G}$. The relative entropy on $\mathcal{G}$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined by*

$$\mathbb{H}_{\mathcal{G}}(\mathbb{Q}|\mathbb{P}) := \begin{cases} \displaystyle\int \log\left(\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}}\right) d\mathbb{Q}, & \text{if } \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{G}, \\ +\infty, & \text{otherwise,} \end{cases} \tag{15}$$

where $\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}}$ stands for the Radon-Nikodym derivative of $\mathbb{Q}|_{\mathcal{G}}$ with respect to $\mathbb{P}|_{\mathcal{G}}$.

Theorem 3.2. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process with triplet (b, σ^2, ν) under $\mathbb{P}$. Suppose that there exists a real number $\beta \in \mathbb{R}$ such that*

$$\int_{x>1} e^x e^{\beta(e^x - 1)}\nu(\mathrm{d}x) < \infty \tag{16}$$

and

$$b + \sigma^2\beta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left((e^x - 1)e^{\beta(e^x-1)} - x\boldsymbol{1}_{|x|\leqslant 1}\right)\nu(\mathrm{d}x) = r, \tag{17}$$

where r is the risk-free interest rate. Then,

1. *The real β is unique and the equivalent measure $\mathbb{Q}^*$ given by the Esscher transform with respect to $(\hat{X}_t)_{t\in[0,T]}$,*

$$\left.\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{e^{\beta\hat{X}_t}}{\mathbb{E}\left[e^{\beta\hat{X}_t}\right]}$$

   is an equivalent martingale measure, where $(\hat{X}_t)_{t\in[0,T]}$ is given by (6).

2. *The stochastic process* $(X_t)_{t\in[0,T]}$ *is still a Lévy process under* $\mathbb{Q}^*$ *with the following characteristic triplet*

$$\left(b + \beta\sigma^2 + \int_{|x|\leqslant 1} x\nu^*(\mathrm{d}x) - \int_{|x|\leqslant 1} x\nu(\mathrm{d}x), \sigma^2, \nu^{\mathbb{Q}^*}\right),$$

where

$$\nu^{\mathbb{Q}^*}(\mathrm{d}x) = e^{\beta(e^x-1)}\nu(\mathrm{d}x).$$

3. *The probability measure* $\mathbb{Q}^*$ *attains the minimal entropy in* $\mathcal{P}$

$$\mathbb{H}_{\mathcal{F}_T}(\mathbb{Q}^*|\mathbb{P}) = \min_{\mathbb{Q}\in\mathcal{P}} \mathbb{H}_{\mathcal{F}_T}(\mathbb{Q}|\mathbb{P}).$$

For a proof, see [19].

Thus, the minimal entropy martingale measure can be simply expressed as an Esscher transform with respect to the simple return process $(\hat{X}_t)_{t\in[0,T]}$. Note that although we have the characteristic triplet of the process $(X_t)_{t\in[0,T]}$ under $\mathbb{Q}^*$, the analytic expression of the characteristic function under this equivalent martingale measure is often difficult to express and the pricing with the characteristic function is not possible in this case.

4 Pricing with characteristic function

We consider here the problem of European call valuation of maturity T. Various techniques have been applied to answer this question. For example, one can resort to Monte Carlo techniques to simulate sample paths for the asset. Averaging a sufficiently large number of realized payoffs then yields the required price, see for example [5, 22]. One can also attempt to derive a partial differential equation for pricing which can be solved using numerical methods [41]. Yet another method is based on the Fourier analysis, which is the subject of the current section.

Two methods based on the Fourier analysis exist in the literature. Both of them rely on the availability of the characteristic function of the stock price logarithm. Indeed, for a wide class of stock models characteristic functions have been obtained in a closed-form formula even if the risk-neutral densities (or probability mass function) themselves are not explicitly available. Examples of Lévy process characteristic functions have been derived in [24, 30, 43].

The first of these Fourier methods is actually the application of the Gil-Palaez inversion formula in finance. This idea originates from [24]. However, singularities in the integrand prevent it to be an accurate method. The second, called the Carr-Madan method, was first proposed by [10]. It ensures that the Fourier transform of the call price exists thanks to the inclusion of a damping factor. Moreover, the Fourier inversion can be accomplished by the fast Fourier transform (FFT) in this case. The tremendous speed of the FFT allows option pricing for a huge number of strikes to be evaluated very rapidly. In this section, we illustrate the Carr-Madan method.

Let $S_T = S_0 \exp(X_T)$ be the terminal price of the underlying asset of a European call with strike K, where $(X_t)_{t\in[0,T]}$ is a Lévy process with triplet (b, σ^2, ν). Denote by $\mathbb{P}^*$ the selected equivalent martingale measure and by f_T^* (its analytic expression is unknown) the risk-neutral density of X_T. The characteristic function of X_T under $\mathbb{P}^*$ can be written as

$$\Phi_T^*(u) = \int_{\mathbb{R}} e^{iux} f_T^*(x)\mathrm{d}x. \tag{18}$$

Let $k = \log(K/S_0)$ be the logarithm of the normalized strike. The risk-neutral valuation under $\mathbb{P}^*$ yields

$$\begin{aligned} C(K) &= e^{-rT}\mathbb{E}^*[(S_T - K)_+] \\ &= S_0 e^{-rT}\mathbb{E}^*[(e^{X_T} - e^k)_+] \\ &= S_0 e^{-rT}\int_k^\infty (e^x - e^k) f_T^*(x)\mathrm{d}x. \end{aligned}$$

Define the function c_0 by $c_0(k) = C(S_0 e^k)$. The Fourier inversion technique consists in the following assertion: $C(K) = c_0(k)$ and $c_0 = \mathrm{FT}^{-1} \circ \mathrm{FT}(c_0)$ where FT is the Fourier transform operator. Since

$$\lim_{k \to -\infty} c_0(k) = \lim_{K \to 0} C(K) = S_0,$$

we see that c_0 is not in L^1, the space of integrable functions, as the limit of $c_0(k)$ as k goes to minus infinity is different from zero. For that, we cannot directly apply the Fourier inversion technique as the Fourier transform of $c_0(k)$ does not converge. To get around this problem of integrability, we consider the modified call price

$$c_\alpha(k) = e^{\alpha k} c_0(k)$$

where $\alpha > 0$.

In the next two propositions, we develop a closed-form formula for the Fourier transform of $c_\alpha(k)$ and obtain the option price $C(K)$ by applying the inverse Fourier transform to the developed formula.

Proposition 4.1. *Let $\alpha > 0$ such that $\mathbb{E}^*[e^{(\alpha+1)X_T}] < \infty$. The Fourier transform of $c_\alpha(k)$ is well defined and given by:*

$$\hat{c}_\alpha(v) = \frac{S_0 e^{-rT}\Phi_T^*(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}, \quad \forall v \in \mathbb{R}, \tag{19}$$

where Φ_T^* is the characteristic function of X_T under $\mathbb{P}^*$.

Proof. Assume for the moment that $\hat{c}_\alpha(v)$ is well defined. We have

$$\begin{aligned} \hat{c}_\alpha(v) &= \int_{-\infty}^\infty e^{ivk} c_\alpha(k)\mathrm{d}k \\ &= \int_{-\infty}^\infty e^{ivk} e^{\alpha k} C(S_0 e^k)\mathrm{d}k \\ &= \int_{-\infty}^\infty e^{ivk} e^{\alpha k}\left(S_0 e^{-rT}\int_k^\infty (e^x - e^k) f_T^*(x)\mathrm{d}x\right)\mathrm{d}k \\ &= S_0 e^{-rT}\int_{-\infty}^\infty f_T^*(x)\left(\int_{-\infty}^x e^{(\alpha+iv)k}(e^x - e^k)\mathrm{d}k\right)\mathrm{d}x \\ &= S_0 e^{-rT}\int_{-\infty}^\infty f_T^*(x)\left(e^x\int_{-\infty}^x e^{(\alpha+iv)k}\mathrm{d}k - \int_{-\infty}^x e^{(\alpha+1+iv)k}\mathrm{d}k\right)\mathrm{d}x \\ &= S_0 e^{-rT}\int_{-\infty}^\infty f_T^*(x)\left(\frac{e^{(\alpha+1+iv)x}}{\alpha+iv} - \frac{e^{(\alpha+1+iv)x}}{\alpha+1+iv}\right)\mathrm{d}x. \end{aligned}$$

By substituting (18) here in, we obtain the expression (19).

We now prove the existence of $\hat{c}_\alpha(v)$. First note that $\mathbb{E}^*[e^{(\alpha+1)X_T}] < \infty$ implies

$$\hat{c}_\alpha(0) < \infty, \tag{20}$$

since

$$\hat{c}_\alpha(0) = \frac{S_0 e^{-rT}\Phi_T^*(-(\alpha+1)i)}{\alpha^2+\alpha} = \frac{S_0 e^{-rT}\mathbb{E}^*[e^{(\alpha+1)X_T}]}{\alpha^2+\alpha}.$$

On the other hand, as $c_\alpha(k)$ is positive, we have

$$|\hat{c}_\alpha(v)| = \left|\int_{-\infty}^{\infty} e^{ivk} c_\alpha(k)\mathrm{d}k\right| \leqslant \int_{-\infty}^{\infty} c_\alpha(k)\mathrm{d}k = \hat{c}_\alpha(0).$$

Combining this with (20) completes the proof. □

Proposition 4.2. *Let $(X_t)_{t\in[0,T]}$ be a Lévy process with characteristic function Φ^* under an equivalent martingale measure $\mathbb{P}^*$. The option price is given by*

$$C(K) = \frac{e^{-\alpha\log(K/S_0)}}{\pi}\,\mathrm{Re}\left\{\int_0^{\infty} e^{-iv\log(K/S_0)}\hat{c}_\alpha(v)\mathrm{d}v\right\}, \tag{21}$$

where $\hat{c}_\alpha$ is given by (19).

Proof. The inverse Fourier transform gives us

$$c_\alpha(k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ivk}\hat{c}_\alpha(v)\mathrm{d}v. \tag{22}$$

Then,

$$C(K) = \frac{e^{-\alpha\log(K/S_0)}}{2\pi}\int_{\mathbb{R}} e^{-iv\log(K/S_0)}\hat{c}_\alpha(v)\mathrm{d}v = \frac{e^{-\alpha\log(K/S_0)}}{\pi}\,\mathrm{Re}\left\{\int_0^{\infty} e^{-iv\log(K/S_0)}\hat{c}_\alpha(v)\mathrm{d}v\right\}, \tag{23}$$

where the last equality follows from the observation that

$$\int_{\mathbb{R}} e^{-iv\log(K/S_0)}\hat{c}_\alpha(v)\mathrm{d}v = \int_0^{\infty} e^{-iv\log(K/S_0)}\hat{c}_\alpha(v)\mathrm{d}v + \int_{-\infty}^{0} e^{-iv\log(K/S_0)}\hat{c}_\alpha(v)\mathrm{d}v,$$

and where the second term on the right-hand side can be written as

$$\begin{aligned}\int_{-\infty}^{0} e^{-iv\log(K/S_0)}\hat{c}_\alpha(v)\mathrm{d}v &= \int_0^{\infty} e^{iu\log(K/S_0)}\hat{c}_\alpha(-u)\mathrm{d}u = \int_0^{\infty} \overline{e^{-iu\log(K/S_0)}\hat{c}_\alpha(u)}\mathrm{d}u \\ &= \overline{\int_0^{\infty} e^{-iu\log(K/S_0)}\hat{c}_\alpha(u)\mathrm{d}u}.\end{aligned}$$

This concludes the proof. □

We only have considered the pricing of vanilla calls. Obviously, one can obtain prices of vanilla puts by using the put-call parity. The price $P_T(K)$ of a vanilla put can alternatively be obtained with the Carr-Madan inversion by choosing a negative value for α, see [29].

5 Discretization and FFT

Computing the price of a call option $C(K) = e^{-rT}\mathbb{E}^*[(S_T - K)_+]$ under the pricing rule $\mathbb{P}^*$ requires the inversion of the Fourier transform in (22). In general, this will not be analytically tractable. A numerical approach is necessary. In doing so we give a formulation to which we can apply the fast Fourier transform (FFT) [13, 40]. Here we define the discrete Fourier transform (DFT) as

$$F_u = \sum_{n=1}^{N} f_n \omega_N^{(n-1)(u-1)}, \quad u = 1, \dots, N \tag{24}$$

where $\omega_N = e^{-\frac{2\pi i}{N}}$. The software R provides an efficient FFT-algorithm for this formulation.

We are interested in computing the integral

$$\int_0^\infty e^{-ivk}\hat{c}_\alpha(v)\mathrm{d}v,$$

where $\hat{c}_\alpha(v)$ is given by (19).

For $g_k(v) \equiv e^{-ivk}\hat{c}_\alpha(v)$, the trapezoidal rule yields

$$\int_0^A g_k(v)\mathrm{d}v \approx \frac{\Delta v}{2}\left[g_k(v_1) + 2\sum_{n=2}^{N-1} g_k(v_n) + g_k(v_N)\right] \tag{25}$$

$$= \Delta v \left[\sum_{n=1}^{N} g_k(v_n) - \frac{1}{2}[g_k(v_1) + g_k(v_N)]\right], \tag{26}$$

where $A = (N-1)\Delta v$. As we truncated the interval of integration, a truncation error will result and we refer to [10] for discussions on this topic. Let

$$v_n = (n-1)\Delta v \tag{27}$$

where $n = 1, \dots, N$. Furthermore, let

$$k_u = k_1 + (u-1)\Delta k, \tag{28}$$

where $u = 1, \dots, N$, be the grid in the k-domain. The constant $k_1 \in \mathbb{R}$ can be tuned such that the grid is laid around aimed strikes. If we are interested in options with particular strikes around a value K, we take $k_1 = \log(K/S_0) - \frac{N}{2}\Delta v$. Substituting (27) and (28) in (26) yields

$$\int_0^A g_{k_u}(v)\mathrm{d}v \approx \Delta v \left[\sum_{n=1}^{N} e^{-i[(n-1)\Delta v][k_1+(u-1)\Delta k]}\hat{c}_\alpha(v_n) - \frac{1}{2}[g_{k_u}(v_1) + g_{k_u}(v_N)]\right]$$

$$= \Delta v \left[\sum_{n=1}^{N} e^{-i\Delta v \Delta k (n-1)(u-1)} g_{k_1}(v_n) - \frac{1}{2}[g_{k_u}(v_1) + g_{k_u}(v_N)]\right].$$

By setting

$$\Delta v\ \Delta k = \frac{2\pi}{N},$$

we have

$$\int_0^A g_{k_u}(v)\mathrm{d}v \approx \Delta v \left[\sum_{n=1}^{N} \omega_N^{(n-1)(u-1)} g_{k_1}(v_n) - \frac{1}{2}[g_{k_u}(v_1) + g_{k_u}(v_N)] \right]. \tag{29}$$

The above sum in (29) takes the form of (24) with $f_n = g_{k_1}(v_n)$. Hence FFT can be applied to evaluate this sum. The final result for the Carr-Madan inversion is thus:

$$C(S_0 e^{k_u}) \approx \frac{e^{-\alpha k_u}}{\pi} \mathrm{Re} \left\{ \Delta v \left[\sum_{n=1}^{N} \omega_N^{(n-1)(u-1)} g_{k_1}(v_n)] - \frac{1}{2}[g_{k_u}(v_1) + g_{k_u}(v_N)] \right] \right\}.$$

Instead of the trapezoidal rule, we can apply the more accurate Simpson's rule. Along the same lines as above, one can easily show that in this case we have

$$C(S_0 e^{k_u}) \approx \frac{e^{-\alpha k_u}}{\pi} \mathrm{Re} \left\{ \frac{\Delta v}{3} \left[\sum_{n=1}^{N} \omega_N^{(n-1)(u-1)} g_{k_1}(v_n)(3 + (-1)^n - \delta_{n-1}) - [g_{k_u}(v_{N-1}) + 4 g_{k_u}(v_N)] \right] \right\}, \tag{30}$$

where δ_{j-1} denotes the Kronecker delta function that equals 1 whenever $j = 1$.

To apply the FFT algorithm, N must be a power of 2. For that, we fix $N = 4096$ and $\Delta v = 0.25$. This gives $\Delta k = 6.13 \cdot 10^{-3}$. We are interested in strikes around at the money $K = S_0$. We fix then $k_1 = -\frac{N}{2}\Delta k$.

6 Applications

Financial models with jumps fall into two categories. In the first category, called *jump-diffusion* models, the evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. Here the jumps represent rare events as crashes and large drawdowns. Such an evolution can be represented by modeling the log-price as a Lévy process with a nonzero Gaussian component and a jump part, which is a compound Poisson process with a finite number of jumps in every time interval. Examples of such models are the Merton jumps diffusion model with Gaussian jumps [32] and the Kou model with double exponential jumps [28]. In these models, the dynamical structure of the process is easy to understand and describe, since the distribution of jump sizes is known. The second category consists of models with an infinite number of jumps in every time interval, which called *infinite activity* models. In these models, one does not need to introduce a Brownian component since the dynamics of jumps is already rich enough to generate a nontrivial small time behavior [9] and it has been argued [9, 20] that such models give a more realistic description of the price process at various time scales. In addition, many models from this class can be constructed via Brownian subordination which gives them additional analytic tractability compared to jump-diffusion models. Two important examples of this category are the variance gamma model [10, 30] and the normal inverse Gaussian model [2, 3, 37]. However, since the real price process is observed on a discrete grid, it is difficult, and even impossible, to empirically observe to which category the price process belongs. The choice becomes rather a question of modeling convenience than an empirical one. In this section, we start by recalling the standard Black & Scholes model. Then, we apply the pricing method in a jump-diffusion example, the Merton model. We end by the variance gamma model: an infinite activity example. Unless otherwise stated, we use the parameters summarized in Table 1 for numerical applications.

Market	B&S	Merton	VG	FFT
$S_0 = 100$	$\mu = 0.145$	$\gamma = 0.1$	$\gamma = 0.1$	$N = 4096$
$r = 0.02$	$\sigma = 0.3$	$\sigma = 0.3$	$m = -0.01$	$\Delta v = 0.25$
$T = 0.5$		$\lambda = 1$	$\delta = 1$	$\Delta k = \frac{2\pi}{N\Delta v}$
		$m = -0.1$	$\kappa = 0.2$	$k_1 = -\frac{N}{2}\Delta k$
		$\delta = 0.2$		

Table 1: Summary of numerical values of different parameters used in Section 6.

6.1 Black & Scholes model

The Black & Scholes model, proposed by [7], is one of the most popular models in finance. The price process $(S_t)_{t\in[0,T]}$ is solution to the SDE

$$dS_t = \mu S_t dt + \sigma S_t dB_t; \; S_0 > 0, \tag{31}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $(B_t)_{t\in[0,T]}$ is a standard Brownian motion. By applying Itô's formula, we re-write $(S_t)_{t\in[0,T]}$ as an exponential Lévy process (2) where

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t, \tag{32}$$

for $t \in [0,T]$. The process $(X_t)_{t\in[0,T]}$ is a Lévy process with triplet $\left(\mu - \frac{\sigma^2}{2}, \sigma^2, 0\right)$. In fact, for each $t \in [0,T]$, as X_t follows a Gaussian distribution $\mathcal{N}((\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$, its characteristic function is given by

$$\Phi_t(u) = \mathbb{E}[e^{iuX_t}] = \exp\left(t\left(i\left(\mu - \frac{\sigma^2}{2}\right)u - \frac{\sigma^2}{2}u^2\right)\right).$$

In this model, the price process does not have jumps. So, we are here in the complete market case where there is only one equivalent martingale measure.

Esscher martingale measure. The exponential moment $\mathbb{E}[e^{\theta X_1}]$ is finite for all $\theta \in \mathbb{R}$. To prove the existence of the Esscher martingale measure $\mathbb{P}^*$, we look for a real $\theta \in \mathbb{R}$ solution to

$$\left(\mu - \frac{\sigma^2}{2}\right) + \sigma^2\theta + \frac{\sigma^2}{2} = r. \tag{33}$$

The solution is given by $\theta = (r-\mu)/\sigma^2$ and the measure $\mathbb{P}^*$ defined by the Esscher transform with respect to $(X_t)_{t\in[0,T]}$ and with the parameter θ is the unique equivalent martingale measure. The process $(X_t)_{t\in[0,T]}$ is again a Lévy process under $\mathbb{P}^*$ with triplet $(r - \frac{\sigma^2}{2}, \sigma, 0)$.

This means that $(X_t)_{t\in[0,T]}$ can be written as

$$X_t = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t,$$

where $(W_t)_{t\in[0,T]}$ given by $W_t = B_t + \frac{\mu - r}{\sigma}\, t$ is a standard Brownian motion under $\mathbb{P}^*$. The discounted price

$$\tilde{S}_t = S_0 \, e^{\sigma W_t - \frac{\sigma^2}{2} t}$$

solves the equation

$$d\tilde{S}_t = \tilde{S}_t dW_t; \quad S_0 > 0.$$

We found the same context of Black & Scholes modeling and the Girsanov theorem for measure changes.

Pricing with FFT. For a Black & Scholes model with parameters corresponding to the values in Table 1, the Esscher parameter that gives the Esscher martingale measure is $\theta = (r - \mu)/\sigma^2 = -1.39$. The numerical computation of the call price is presented in Figure 2 with respect to the strike K. We see that the option price approaches S_0 as $K \to 0$ and goes to 0 as $K \to +\infty$.

Otherwise, we have compared the closed-form price of Black & Scholes with the option price given by the Carr-Madan method. The absolute error of the numerical method is of order $6 \cdot 10^{-7}$ whatever the strike is.

6.2 Merton model

This model is proposed by [32]. The price is described by an exponential Lévy process (2) where

$$X_t = \gamma t + \sigma B_t + \sum_{i=1}^{N_t} Y_i, \tag{34}$$

with $\gamma \in \mathbb{R}$, $(B_t)_{t\in[0,T]}$ is a standard Brownian motion, $(N_t)_{t\in[0,T]}$ is a Poisson process with intensity λ and $(Y_i)_{i\geqslant 1}$ are i.i.d. Gaussian random variables with parameters m and δ^2. For each $0 \leqslant t \leqslant T$, the characteristic function of X is given by

$$\Phi_t(u) = \mathbb{E}[e^{iuX_t}] = e^{t\Psi(u)},$$

and

$$\Psi(u) = i\gamma u - \frac{\sigma^2}{2}u^2 + \lambda\left(\exp\left(imu - \frac{\delta^2}{2}u^2\right) - 1\right).$$

Define

$$\nu(x) = \lambda \times \frac{1}{\sqrt{2\pi}\delta}\exp\left(-\frac{(x-m)^2}{2\delta^2}\right), \; x \in \mathbb{R}$$

and

$$b = \gamma + \int_{|x|\leqslant 1} x \; \nu(\mathrm{d}x).$$

Then, Ψ can be written on the form

$$\begin{aligned}\Psi(u) &= i\gamma u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1\right) \nu(\mathrm{d}x) \\ &= ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|\leqslant 1}\right) \nu(\mathrm{d}x).\end{aligned}$$

We conclude that, under $\mathbb{P}$, $(X_t)_{t\in[0,T]}$ is a Lévy process with triplet (b, σ^2, ν). With this model, the market is incomplete. The compound Poisson process makes the risk uncontrolled and there exists an infinity of equivalent martingale measures.

Esscher martingale measure. The exponential moment $\mathbb{E}[e^{\theta X_1}]$ is finite for all $\theta \in \mathbb{R}$. To prove the existence of the Esscher martingale measure $\mathbb{P}^*$, we look for a real $\theta \in \mathbb{R}$ solution to

$$\gamma + \sigma^2\theta + \frac{\sigma^2}{2} + \int_{\mathbb{R}} e^{\theta x}\left(e^x - 1\right)\nu(\mathrm{d}x) = r,$$

or farther

$$\gamma + \sigma^2\theta + \frac{\sigma^2}{2} + \lambda\left(e^{m(\theta+1)+\frac{\delta^2}{2}(\theta+1)^2} - e^{m\theta+\frac{\delta^2}{2}\theta^2}\right) = r.$$

For numerical application with the parameters given in Table 1, we obtain $\theta \approx -0.352$.

The characteristic function of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}^*$ is given by

$$\Phi_t^*(u) = e^{t\Psi^*(u)},$$

where

$$\Psi^*(u) = i(\gamma + \sigma^2\theta)u + \frac{\sigma^2}{2}u^2 + \lambda\left(e^{m(\theta+iu)+\frac{\delta^2}{2}(\theta+iu)^2} - e^{m\theta+\frac{\delta^2}{2}\theta^2}\right).$$

We conclude that, with the Esscher martingale measure, we are able to provide analytic expression of the characteristic function under an equivalent martingale measure. This analytic form will be numerically inverted with the fast Fourier transform later to provide the option price.

Using Proposition 2.1, the process $(X_t)_{t\in[0,T]}$ is a Lévy process under $\mathbb{P}^*$ with triplet (b^*, σ^2, ν^*) defined by

$$b^* = \gamma + \sigma^2\theta + \int_{|x|\leqslant 1} x\nu^*(\mathrm{d}x)$$

and

$$\nu^*(x) = e^{\theta x}\nu(x) = \lambda^* \times \frac{1}{\sqrt{2\pi}\delta}\exp\left(x - (m + \delta^2\theta)\right)^2/(2\delta^2),$$

where $\lambda^* = \lambda\exp(m\theta + \frac{\delta^2\theta^2}{2})$. Hence, the Esscher martingale model $(X_t)_{t\in[0,T]}$ is once again a jump diffusion with compound Poisson jumps:

$$X_t = (\gamma + \sigma^2\theta)t + \sigma W_t + \sum_{i=1}^{N_t^*} Y_i^*,$$

where $(W_t)_{t\in[0,T]}$ is a $\mathbb{P}^*$-standard Brownian motion, $(N_t^*)_{t\in[0,T]}$ is a $\mathbb{P}^*$-Poisson process with intensity λ^* and $(Y_i^*)_{i\geqslant 1}$ are i.i.d. Gaussian random variables with parameters $m+\delta^2\theta$ and δ^2.

Minimal entropy martingale measure. The existence of the minimal entropy martingale measure $\mathbb{Q}^*$ is equivalent to the existence of a real β solution to

$$\gamma + \sigma^2\beta + \frac{\sigma^2}{2} + \int_{\mathbb{R}}\left((e^x - 1)e^{\beta(e^x-1)}\right)\nu(\mathrm{d}x) = r,$$

in the Merton model framework. With the same parameters of Table 1, we get numerically $\beta \approx -0.365$.

Using Theorem 3.2, $(X_t)_{t\in[0,T]}$ is a Lévy process under $\mathbb{Q}^*$ with triplet $(b^{\mathbb{Q}^*}, \sigma^2, \nu^{\mathbb{Q}^*})$ defined by

$$b^{\mathbb{Q}^*} = \gamma + \beta\sigma^2 + \int_{|x|\leqslant 1} x\nu^{\mathbb{Q}^*}(\mathrm{d}x)$$

and

$$\nu^{\mathbb{Q}^*}(x) = e^{\beta(e^x-1)}\nu(x) = \lambda^{\mathbb{Q}^*}\rho^{\mathbb{Q}^*}(x)$$

where $\lambda^{\mathbb{Q}^*} = \int \nu^{\mathbb{Q}^*}(\mathrm{d}x)$ is the intensity of the $\mathbb{Q}^*$-Poisson process and $\rho^{\mathbb{Q}^*}(x) = \frac{\nu^{\mathbb{Q}^*}(x)}{\int \nu^{\mathbb{Q}^*}(\mathrm{d}x)}$ is the jump size density under $\mathbb{Q}^*$.

The role of the damping factor $e^{\theta x}$ in the case of the Esscher martingale measure $\mathbb{P}^*$ and the dumping factor $e^{\beta(e^x-1)}$ in the case of the minimal entropy martingale measure $\mathbb{Q}^*$ is to mitigate the general trend of the log price. In the Merton model, the general trend is given by $\mathbb{E}[X_t] = (\gamma + \lambda m)t$. A positive expectation slope of the log price leads to a negative parameters θ and β which permit to give more weight to negative jumps and less weight to positive jumps in the Lévy densities ν^* and $\nu^{\mathbb{Q}^*}$, to rebalance the market in the risk-neutral world and to have the martingale property. Conversely, if the log price expectation is negative, the dumping factors will strongly reduce the left tail of the risk-neutral Lévy densities.

The initial Lévy density ν and the two risk-neutral Lévy densities ν^* and $\nu^{\mathbb{Q}^*}$, as well as the corresponding jump densities, are depicted in Figure 1 for different values of the log price expectation. We vary the jump size expectation $m = \{-0.1, -0.3, 0.3\}$ to have different signs of $\mathbb{E}[X_t]$ and therefore different signs of θ and β. The remaining parameters of the model are the ones given in Table 1. Depending on the Esscher parameter sign, we have a dumping of the right tail or the left tail of the risk-neutral Lévy density. Despite this dumping effect in the Lévy densities, we observe in the right hand figures that the jump size densities are not affected by this change of measures and the main effect is then an increase or decrease in the jump intensities λ^* and $\lambda^{\mathbb{Q}^*}$.

On the other hand, when we compare ν^* to $\nu^{\mathbb{Q}^*}$, we find that both functions are almost equal with variations that do not exceed 1.6×10^{-2}. We can explain this by the fact that X and $\hat{X}$ represent respectively the compound and the simple returns of the stock price. So, we consider models with small jump x for which $e^x - 1 \approx x$. Thus, the Esscher martingale measure and the minimal entropy martingale measure are almost the same.

While we have the triplet of $(X_t)_{t\in[0,T]}$ under the minimal entropy martingale measure, we still do not have an analytic expression of the characteristic function under this measure. For that, using the FFT to compute option prices under the minimal entropy martingale measure will be not possible.

Pricing with FFT under the Esscher martingale measure. In Figure 2, we compare the option price under a Black & Scholes model and a Merton model having the same expectation of $(X_t)_{t\in[0,T]}$. We choose models parameters such that $\mu - \frac{\sigma^2}{2} = \gamma + \lambda m$. The numerical computation of the call price is presented in Figure 2 with respect to the strike K. As the Merton model allows bigger negative jumps, the probability to finish in-the-money is smaller. Otherwise, the option price can always be written

$$C(K) = S_0\Pi_1 - Ke^{-rT}\mathbb{P}(S_T > K),$$

where Π_1 is the delta of the option. The call price is thus greater around the money in the Merton model.

Option price sensivity to jumps. To discuss the effects of jumps on the option price, we compare the results of the Merton model with respect to the jump intensity. Recall that the B&S and Merton models are exponential Lévy models with triplets $(\mu - \frac{\sigma^2}{2}, \sigma^2, 0)$

and $(\gamma + \int_{|x|\leqslant 1} x\nu(\mathrm{d}x), \sigma^2, \nu)$ respectively, where

$$\nu(x) = \lambda \times \frac{1}{\sqrt{2\pi}\delta} \exp\left((x-m)^2/(2\delta^2)\right).$$

Then, a B&S model is a Merton model with particular parameters $\gamma = \mu - \frac{\sigma^2}{2}$ et $\lambda = 0$. This model does not contain jumps as the jump intensity $\lambda = 0$. To see the effect of jumps on the option prices, we fix parameters such that $\gamma = \mu - \frac{\sigma^2}{2}$ and compare the B&S model to Merton models with different jump intensities. We take the parameters of Table 1 except for λ that varies in $\{2, 4, 6\}$. Note that this choice of parameters respects the condition $\gamma = \mu - \frac{\sigma^2}{2}$. The different option prices are presented in Figure 3. For all models, the option price approaches S_0 as $K \to 0$ and goes to 0 as $K \to +\infty$. The jump intensity modifies the option price only for strikes around at-the-money. Otherwise, as the intensity increases, we have more jumps in the period $[0, T]$ and the variance $\mathrm{Var}(X_t) = (\sigma^2 + \lambda(m^2 + \delta^2))\, t$ increases. Consequently, the option price increases because of greater risk that jumps brings to the seller. The same observation is found when the jump size volatility increases. We present it in Figure 5. The price with respect to the jump size mean is given by Figure 4. With a negative average of jumps size in the asset price, we have interesting out-the-money options, while in-the-money options is less interesting. A positive average of jumps size leads to reverse statement

6.3 Variance gamma model

The variance gamma process is proposed by [31] to describe stock price dynamics istead of the Brownian motion in the original Black & Scholes model. Two new parameters: m skewness and κ kurtosis are introduced in order to describe asymmetry and fat tails of real life distributions. A variance gamma process is obtained by evaluating a Brownian motion with a drift at a random time given by a gamma process.

Definition 6.1. *The variance gamma (VG) process $(Y_t)_{t\in[0,T]}$ with parameters (m, δ, κ) is defined as*

$$Y_t = m\gamma_t + \delta B_{\gamma_t},$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion and $(\gamma_t)_{t\in[0,T]}$ is a gamma process with unit mean rate and variance rate κ.

Proposition 6.1. *The VG process $(Y_t)_{t\in[0,T]}$ is a Lévy process with characteristic function Φ_t and characteristic triplet $(\int_{|x|\leqslant 1} x\nu(\mathrm{d}x), 0, \nu)$ where*

$$\mathbb{E}[e^{iuY_t}] = \left(\frac{1}{1 - im\kappa u + (\delta^2\kappa/2)u^2}\right)^{t/\kappa}$$

and

$$\nu(x) = \frac{1}{\kappa|x|} \exp\left(\frac{m}{\delta^2}x - \frac{\sqrt{\frac{m^2}{\delta^2} + \frac{2}{\kappa}}}{\delta}|x|\right).$$

From the expression of ν, there exists infinitely small jumps. Such a process is called an infinite activity process. It does not admit a distribution of jump size since jumps happen infinitely.

To describe the stock price, we just add a drift component to the VG process. A Brownian component is not necessary and the process moves essentially by jumps. The price process is defined by an exponential Lévy model (2) where

$$X_t = \gamma t + Y_t.$$

The process $(X_t)_{t\in[0,T]}$ is a Lévy process with characteristic triplet $\left(\gamma + \int_{|x|\leqslant 1} x\nu(\mathrm{d}x), 0, \nu\right)$ and characteristic ffunction given by

$$\Phi_t(u) = \mathbb{E}[e^{iuX_t}] = \frac{e^{i\gamma ut}}{\left(1 - im\kappa u + (\delta^2\kappa/2)u^2\right)^{t/\kappa}}.$$

Esscher martingale measure. By solving the inequality $1 - im\kappa u + (\delta^2\kappa/2)u^2 > 0$ with respect to u, we show that X_t for some t or, equivalently, for all t, possesses a moment generating function $u \mapsto \mathbb{E}[\exp(uX_t)]$ on

$$\left(-m/\delta^2 - \sqrt{m^2/\delta^2 + 2/\kappa}\ \delta, -m/\delta^2 + \sqrt{m^2/\delta^2 + 2/\kappa}\ \delta\right).$$

To apply Theorem 3.1, we assume that $2\sqrt{m^2/\delta^2 + 2/\kappa}\ \delta > 1$. Under this assumption, we look for $\theta \in \left(-m/\delta^2 - \sqrt{m^2/\delta^2 + 2/\kappa}\ \delta, -m/\delta^2 + \sqrt{m^2/\delta^2 + 2/\kappa}\ \delta - 1\right)$ solution to

$$\mathbb{E}\left[e^{(\theta+1)X_t}\right] = e^{rt}\mathbb{E}\left[e^{\theta X_t}\right].$$

For the parameters given in Table 1, the moment generating function of X_t is well defined on $(-3.15, 3.17)$. Numerically, we obtain $\theta = -0.57$.

The characteristic function of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}^*$ is given by

$$\Phi_t^*(u) = \frac{\mathbb{E}[e^{(\theta+iu)X_t}]}{\mathbb{E}[e^{\theta X_t}]} = \frac{e^{i\gamma ut}}{\left(1 - im^*\kappa u + (\delta^{*2}\kappa/2)u^2\right)^{t/\kappa}},$$

where $m^* = (m + \delta^2\theta)/A$, $\delta^* = \delta/\sqrt{A}$ and $A = 1 - m\kappa\theta - \frac{\delta^2\kappa}{2}\theta^2$. Hence, $(X_t)_{t\in[0,T]}$ is once again a variance gamma process under $\mathbb{P}^*$ with characteristic triplet $\left(\gamma + \int_{|x|\leqslant 1} x\nu^*(\mathrm{d}x), 0, \nu^*\right)$ where

$$\nu^*(x) = e^{\theta x}\nu(x) = \frac{1}{\kappa|x|}\exp\left(\frac{m^*}{\delta^{*2}}x - \frac{\sqrt{\frac{m^{*2}}{\delta^{*2}} + \frac{2}{\kappa}}}{\delta^*}|x|\right).$$

The initial Lévy density and the risk-neutral Lévy density are given in Figure 6. With our choice of parameters, the general trend $\mathbb{E}[X_t] = (\gamma + m)t$ is positive, which leads to a negative θ. The large positive jumps will be nearly irrelevant for pricing options, while large negative jumps will still contribute.

Pricing with FFT under the Esscher martingale measure. The explicit form of the characteristic function under the Esscher martingale measure $\mathbb{P}^*$ is applied to price call options. Figures 7 and 8 represent the sensivity of the call price to a variation of the parameters δ and κ respectively. We take the parameters of Table 1 except for δ that varies in $\{0.6, 0.8, 1\}$ in Figure 7 and κ that varies in $\{0.02, 0.2\}$ in Figure 8. The parameters δ and κ provide control over volatility and kurtosis respectively. Increasing δ leads to greater volatility which in turn increases the option price. In the other hand, when κ inscreases, the distribution tails of the asset price become fatter and the price of the out-the-money call increases, while the price of the at-the-money option decreases.

7 Conclusion

In this survey, we studied step by step how to price European option under an Exponential Lévy model. We used the Esscher transform technique to construct two equivalent martingale measures: the Esscher martingale measure and the minimal entropy martingale measure. From the observation that the compound return and the simple return of a stock price process are very close, we showed that these two measures are almost similar. However, the minimal entropy martingale measure is not adequate to the Carr-Madan pricing method as the characteristic function under this latter does not have an analytic expression, even in simple models. Under the Esscher martingale measure, we numerically computed the price of European option using the Carr-Madan method based on the fast Fourier transform. Numerical error of the Carr-Madan method is shown with comparison to the closed-form formula in the Black & Scholes context. Moreover, the sensivity of option price to jumps is presented in the Merton model and the variance gamma model with respect to jump parameters.

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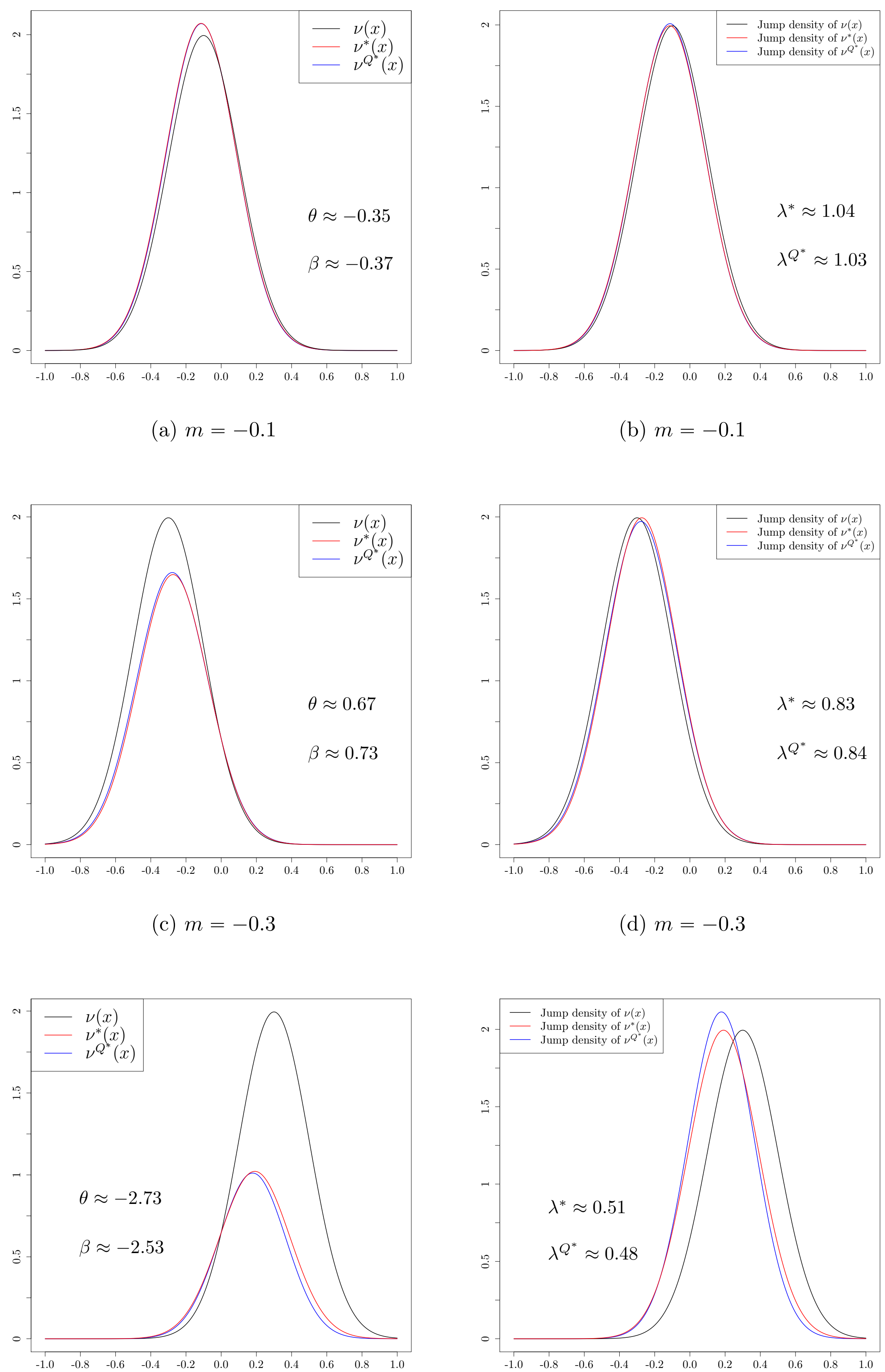

(a) $m = -0.1$

(b) $m = -0.1$

(c) $m = -0.3$

(d) $m = -0.3$

(e) $m = 0.3$

(f) $m = 0.3$

Figure 1: Lévy densities and jump size densities under the historical measure, the Esscher martingale measure and the minimal entropy martingale measure in the Merton model.

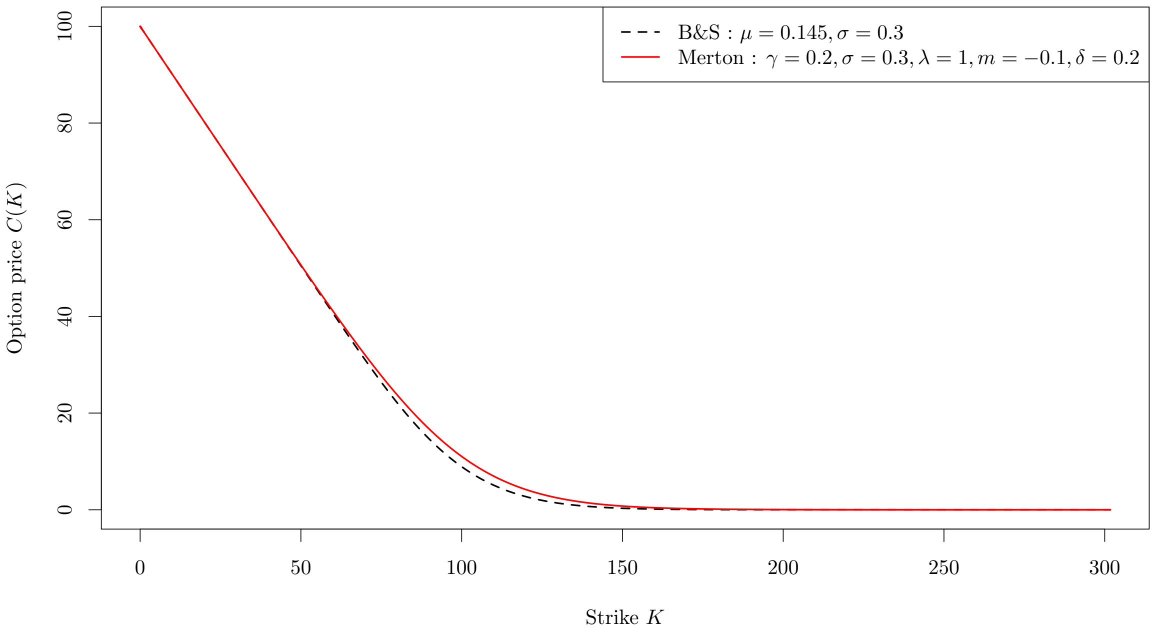

Figure 2: Option price with respect to the strike K. A comparison between a B&S model and a Merton model with the same expectation $\mu - \sigma^2/2 = \gamma + \lambda m$.

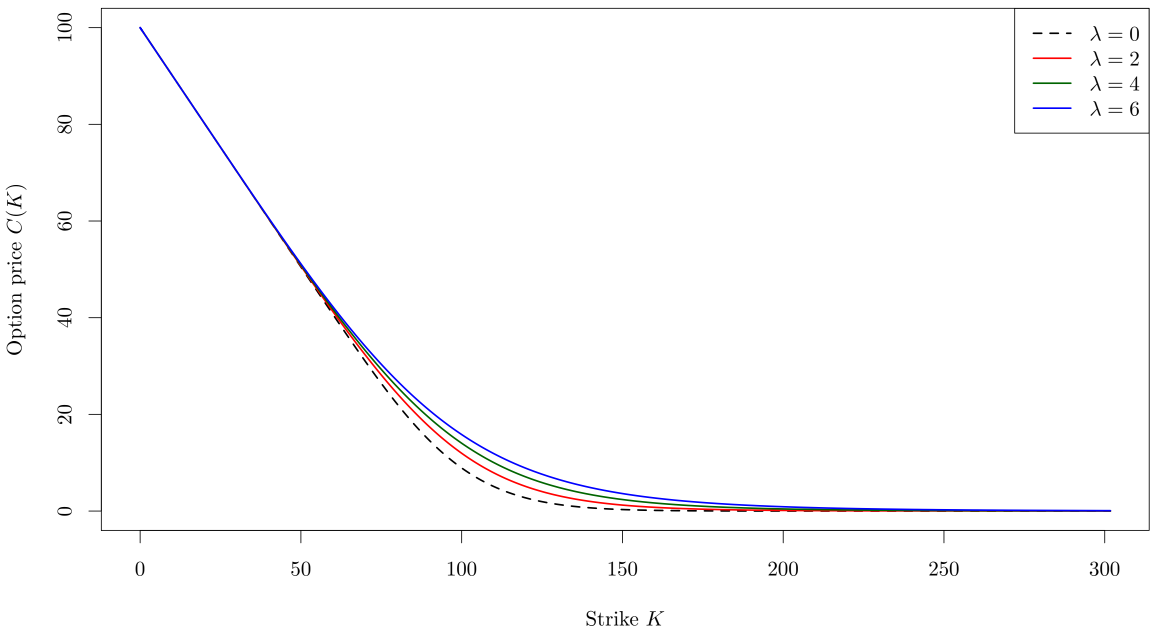

Figure 3: Call price sensivity to the jump intensity in Merton model.

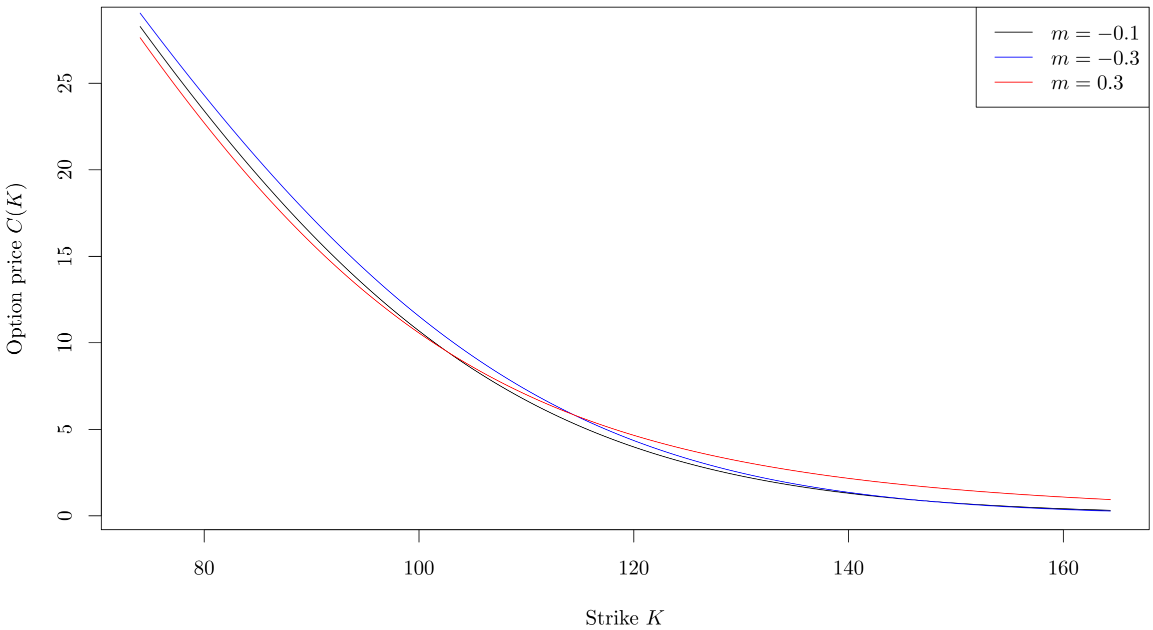

Figure 4: Call price sensivity to jump size mean in Merton in model.

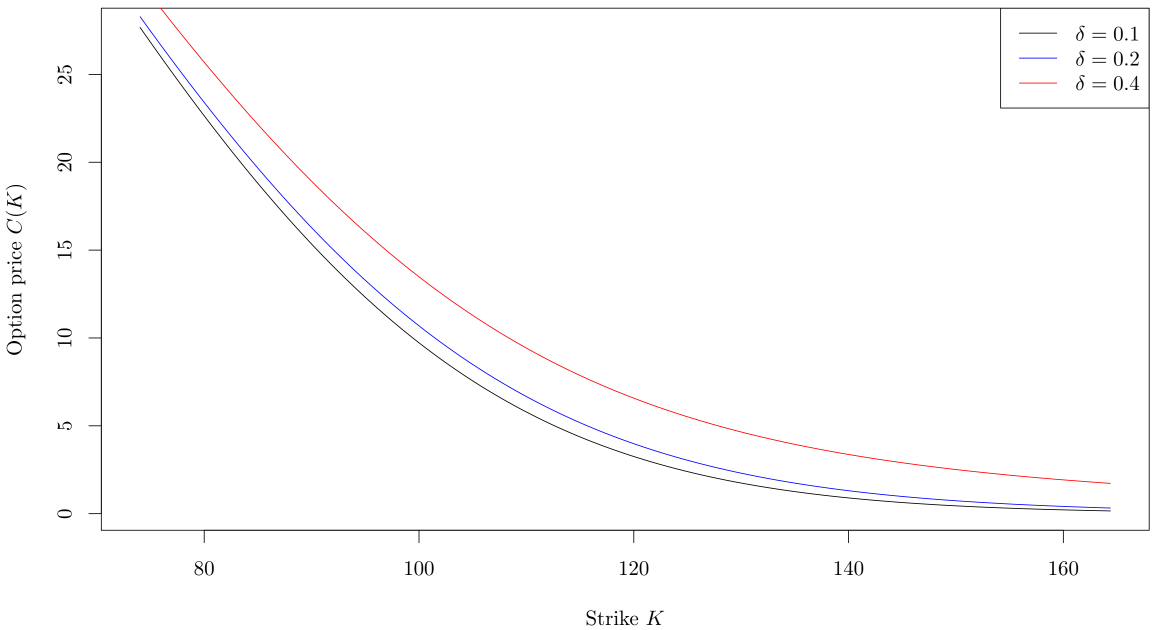

Figure 5: Call price sensivity to jump size variance in Merton model.

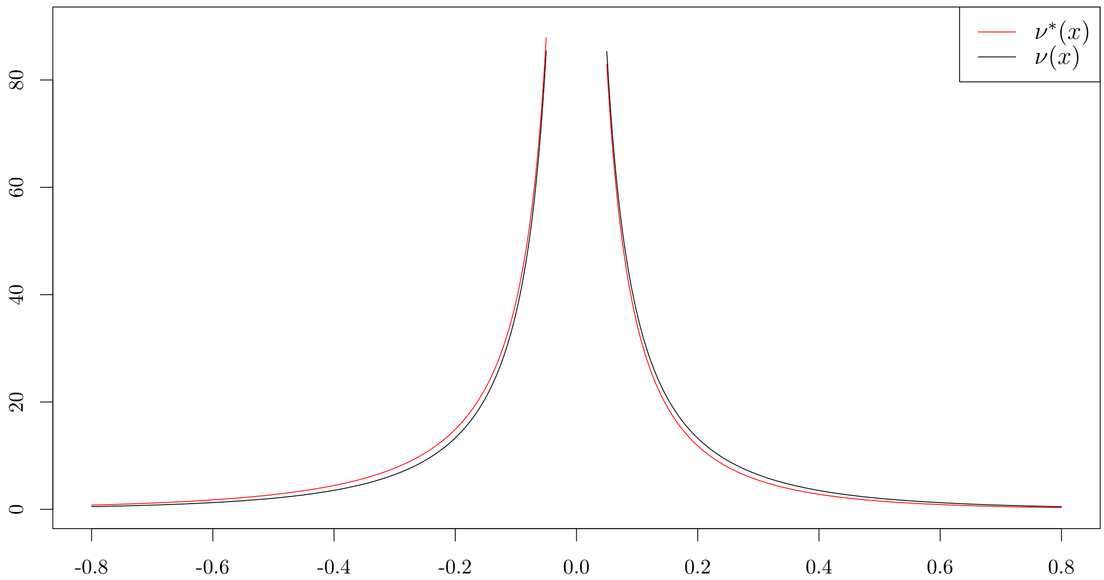

Figure 6: The risk neutral Lévy density $\nu^*(x)$ of the Esscher martingale measure in the variance gamma model, compared to the initial Lévy density $\nu(x)$.

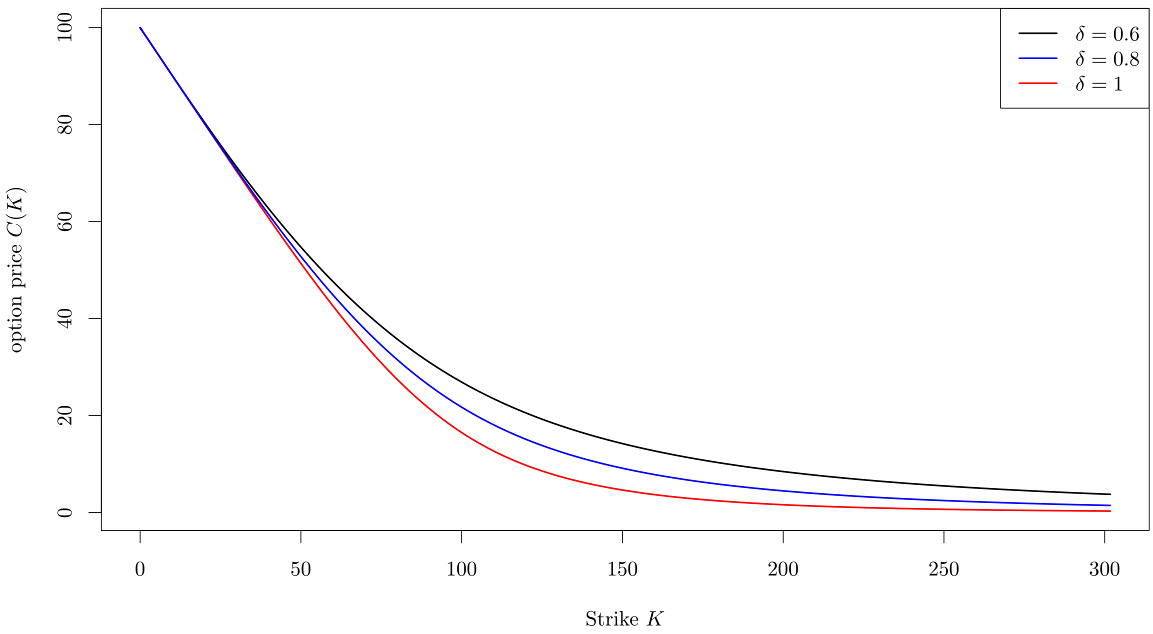

Figure 7: Call price sensivity to the volatility parameter in variance gamma model.

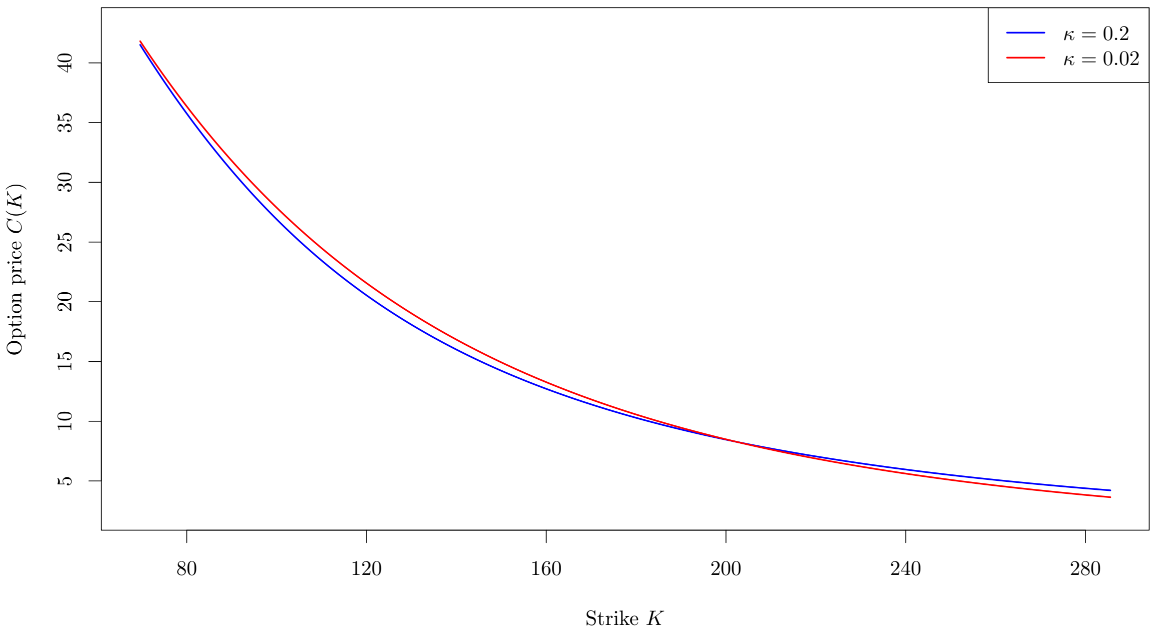

Figure 8: Call price sensivity to the kurtosis parameter in variance gamma model.