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Estimation of the bias parameter of the Skew Random Walk and application to the Skew Brownian motion

Antoine Lejay*

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Abstract

We study the asymptotic property of simple estimator of the parameter of a Skew Brownian Motion when one observes its positions on a fixed grid — or equivalently of a simple random walk with a bias at 0. This estimator, nothing more than the Maximum Likelihood Estimator, is based only on the number of passages of the random walk at 0. It is very simple to set up, is consistent and satisfies the local asymptotic mixed normality. We believe that this simplified framework is helpful to understand the asymptotic behavior of the maximum likelihood of the Skew Brownian Motion observed at discrete times which is studied in a companion paper.

Keywords

Skew random walk, Skew Brownian Motion, maximum likelihood estimator, local asymptotic mixed normality, local time, null recurrent process.

A flourishing literature is devoted to the stochastic diffusion processes with discontinuous coefficients or involving local times. This covers many fields of application, ranging from geophysics [3] to population ecology [1, 9] through molecular biology [7], or to study rank based statistics [17] as well as dealing with the tiled volatility model [35] or directional predictability [2] in mathematical finance.

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In this direction of research, the Skew Brownian motion (SBM) is a process of utter importance. It plays the role of the Brownian motion for diffusions with continuous coefficients. Informally, the SBM is similar to the Brownian motion, except that when it hits 0, its behavior depends on a parameter $\alpha \in (0, 1)$ which gives a trend to go on the positive or negative axis according to $\alpha > 1/2$ or $\alpha < 1/2$. As pointed out for example in [32], Lamperti's type transforms allow one to reduce one-dimensional diffusion processes generated by either $a\Delta$ or $\nabla(a\nabla\cdot)$ with a piecewise constant to processes which behave locally as a SBM.

Formally, the SBM of parameter α is the strong solution to the stochastic differential equation

$$X_t = B_t + \theta L_t, \quad \theta = \frac{1 + \alpha}{2} \in (-1, 1), \quad (1)$$

where B is a Brownian motion and L_t the symmetric local time of X at 0. Introduced through its infinitesimal generator by K. Itô and H.P. McKean (See *e.g.* [21, p. 220] or [22, Problem 1, p. 115]), there exists many ways to describe this process (See [31] for a survey of some possible constructions). The existence of a strong solution to (1) has been proved first by J.M. Harrison and L.A. Shepp in [19].

In the same way the Brownian motion is approximated by a simple random walk properly renormalized, the SBM is approximated by a Skew Random Walk (SRW) [19]. A SRW is a simple random walk $\{\zeta_k\}_{k \geq 0}$ with a bias at 0:

$$\mathbb{P}[\zeta_{k+1} = i | \zeta_k = j] = \begin{cases} 1/2 & \text{if } j \neq 0, i = j \pm 1, \\ \alpha = (1 + \theta)/2 & \text{if } j = 0, i = 1, \\ 1 - \alpha = (1 - \theta)/2 & \text{if } j = 0, i = -1, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Let us fix a parameter $h > 0$ which represents the size of a grid. The successive positions $\{\xi_k(h)\}_{k \geq 0}$ at which a SBM X passes through the grid $h\mathbb{Z}$, up to time T , are observed and recorded (the number of recorded points is random $n^*(h)$, yet close to T/h^2). By doing so, the embedded Markov chain $\{\xi_k(h)\}_{k \geq 0}$ is equal in distribution to the rescaled SRW $\{h\zeta_k\}_{k \geq 0}$.

Our main concern is to estimate $\alpha = (1 + \theta)/2$ from these observations.

From now, we denote by \mathbb{P}_α the distribution of the SBM of parameter $\alpha \in (0, 1)$.

Since $\mathbb{P}_\alpha[X_t \geq 0] = \alpha$ for any $t > 0$, $\mathbb{E}_\alpha[A_t] = t\alpha$ where A_t is the occupation time A_t above 0 up to time t . However, A_T/T is a meaningless estimator of α since by scaling, $A_T/T \stackrel{\text{law}}{=} A_1$ whatever $T > 0$ and A_1 follows

a variant of the Arc Sine distribution [27, Theorem 1]. Unlike most of the work in statistics of diffusion, ergodic theory could *not* be used here as the SBM is a null recurrent process.

The sole information useful for our estimator are the successive points $(\xi_k(h), \xi_{k+1}(h))$ for which $\xi_k(h) = 0$. The excursions of the SRW behave like the ones of a simple random walk. They carry no information about α , except from their sign.

Our estimator of α is then simply

$$\alpha(h) = \frac{\#\text{observed upcrossings before } T}{\#\text{observed passages to } 0 \text{ before } T}, \quad (3)$$

where a passage to 0 counts to 1 when $\xi_k(h) = 0$ and an upcrossing counts to 1 when $\xi_k(h) = 0$ and $\xi_{k+1}(h) = 1$.

This estimator (3) is also the empirical mean number of positive excursions. It is the Maximum Likelihood Estimator (MLE) of α for the SRW.

First, we show that $\alpha(h)$ is consistent, *i.e.* converges in probability to α as $h \rightarrow 0$ under \mathbb{P}_α for any $\alpha \in (0, 1)$, and that under \mathbb{P}_α ,

$$h^{-1/2}(\alpha(h) - \alpha) \text{ converges in distribution to } \sqrt{\alpha(1-\alpha)} \frac{W(L_T)}{L_T},$$

where W is a Brownian motion independent from B . Asymptotically, $\alpha(h)$ behaves like a mixed normal distribution. For n observations, the rate of convergence is not $n^{-1/2}$, but $n^{-1/4}$. The reason is that the number of passages to 0 is roughly of order \sqrt{n} . We then show that the distribution of the $\{\xi_k(h)\}_{k=0, \dots, n^*(h)}$ satisfies the Local Asymptotic Mixed Normal (LAMN) property [24, 25] — a generalization of the Local Asymptotic Normal (LAN) property [20, 29] — which carries some information about any asymptotically efficient estimator.

We are in a situation close to the one studied by D. Florens in [18] where non-parametric estimation of the diffusion coefficient of the solution of a stochastic differential equation observed at discrete time is performed through the number of crossings of levels. The kind of limit theorem we obtain here is typical for of null recurrent “regular” diffusion processes [14], although the approach shall be adapted to deal with distributions which are singular with respect to the Wiener measure [28].

In [33], we have studied the MLE estimator for the SBM observed at frequency T/n when $\alpha = 1/2$ (or $\theta = 0$). By using results from J. Jacod in [23], we obtained similar results, both to identify the limit and the rate of convergence. The consistency of the MLE estimator when $\alpha \neq 0$, which was left open, is now treated in the subsequent article [34]. However, following [23],

these articles use fine estimates on the semi-groups and martingales limit theorems.

As shown in [34], the MLE estimator α_n of α for the SBM observed at discrete times $\{iT/n\}_{i=0,\dots,n}$ solves

$$\alpha_n = \frac{\mathbb{E}_{\alpha_n}[U|\{X_{iT/n}\}_{i=0,\dots,n}]}{\mathbb{E}_{\alpha_n}[U + D|\{X_{iT/n}\}_{i=0,\dots,n}]} \quad (4)$$

where U (resp. D) is the number of intervals $J_i = [iT/n, (i+1)T/n]$ for which $X_t = 0$ for some $t \in J_i$ and $X_{(i+1)T/n} > 0$ (resp. $X_{(i+1)T/n} < 0$). With the above construction, the fact that the process crosses 0 or not is considered as some *hidden data* (actually, (4) was derived using some ideas of the Expectation-Minimization algorithm).

In our situation, the situation is much more simpler because the crossings are observed so that the number of up- and down-crossings are exactly counted. Our MLE estimator is nothing more than a binomial one with a random number of samples. The asymptotic error, given by a CLT-like result, stems from the Berry-Esseen theorem and the P. Lévy downcrossing theorem, which relates the local time as a limit of the renormalized number of crossings of the SRW.

We then believe that our results here explain with simple computations and elementary tools the results in [33, 34]. Also, it could be used for other statistics involving excursions such as the ones in [30].

Outline. Section 1 gives a quick account on the SBM and the SRW. The asymptotic of the estimator $\alpha(h)$ in (3) is studied in Section 2. In Section 3, the LAMN property is proved.

1. Approximation of the Skew Brownian motion by the Skew Random Walk

The *symmetric local time* of a semi-martingale Y is defined as

$$\mathcal{L}_t(Y) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{Y_s \in [-\epsilon, \epsilon]} d\langle Y \rangle_s, \quad t \geq 0. \quad (5)$$

All the results we need here on the local time could be found for example¹ in [22] or [39].

Let X be SBM of parameter $\alpha \in (0, 1)$ with distribution \mathbb{P}_α and $X_0 = 0$, which is the strong solution to (1) [19]. We denote by $L = \mathcal{L}(X)$ its symmetric local time at 0.

¹The reader shall be warned that different normalizations are used in different textbooks. For example, in [39, Chapter VI], the right local time is used. For the SBM, this distinction is crucial since $a \mapsto L_t^a(X)$ is discontinuous at 0 [41].

For the SBM, $\langle X \rangle_t = t$ and $L = \mathcal{L}(|X|)$ whatever $\alpha \in (0, 1)$.

Proposition 1 (Scaling property). *For any $c > 0$, with $X^c = cX_{./c^2}$, $(X^c, \mathcal{L}(X^c))$ is equal in distribution to (X, L) when $X_0 = 0$.*

Proof. With (5), $\mathcal{L}(X^c) \stackrel{\text{law}}{=} cL_{./c^2}$, so that $X_{t/c^2}^c = cB_{t/c^2} + \theta cL_{t/c^2}$ and then and then $X_{t/c^2}^c = W_t + \theta \mathcal{L}_t(X^c)$ for the Brownian motion $W = cB_{./c^2}$. The equality in distribution of X^c to X follows from the uniqueness to the solution to (1). \square

Hypothesis 1. We consider that $X_0 = 0$ so that $\xi_0(h) = 0$.

This hypothesis aims at simplifying the presentation of the results. Otherwise, we have to wait the random time τ at which the SBM hits first 0. The observations' timespan is then reduced from $[0, T]$ to $[\tau, T]$.

For $h > 0$, define iteratively $\tau_0(h) = 0$ and

$$\tau_{k+1}(h) = \inf\{t > \tau_k(h); |X_t - X_{\tau_k(h)}| = h\},$$

the successive passage times and positions of X to the grid $h\mathbb{Z}$. Also, set $\xi_k(h) = X_{\tau_k(h)}$, the normalized passage positions. Since the SBM is a strong Markov process, $(\xi_k(h))_{k \geq 0}$ is a Markov chain with values in $h\mathbb{Z}$.

The following result is an easy consequence of the results in [19, 41] on the scale function and the speed measure of the SBM, or from the construction from K. Itô and H.P. McKean [22] through its excursions.

Proposition 2 (Embedding a Markov chain into a SBM). *The discrete process $(\xi_k(h))_{k \geq 0}$ is a recurrent Markov chain with values in $h\mathbb{Z}$, and $(h^{-1}\xi_k(h))_{k \geq 0}$ has the same probability transition as the SRW $(\zeta_k)_{k \geq 0}$ whose transition kernel is defined by (2).*

In addition $(\tau_{k+1}(h) - \tau_k(h))_{k \geq 0}$ is a sequence of independent random variables, each distributed as $\tau_1(h)$. Besides, this sequence is independent from the successive positions $(\xi_k)_{k \geq 0}$.

Finally, the distribution of $\tau_1(h)$ does not depend on α and is the same as the one of $h^2\tau_1(1)$. The first two moments of $\tau_1(1)$ are $\mathbb{E}_\alpha[\tau_1(1)] = 1$ and $\mathbb{E}_\alpha[\tau_1(1)^2] = 5/3$.

Proof. The SBM is a recurrent, strong Markov process so that $(\xi_k(h))_{k \geq 0}$ is a recurrent Markov chain. Besides, if $\xi_k(h) \neq 0$, then the process behaves like a Brownian motion between the times $\tau_k(h)$ and $\tau_{k+1}(h)$. On the other hand, $\mathbb{P}_\alpha[\xi_{k+1}(h) = h | \xi_k(h) = 0]$ is the probability that the first excursion of X whose height is greater than k is positive, and this probability is equal to α , an easy consequence from the result from K. Itô and H.P. McKean on the SBM [22, p. 115].

The statement about the distributions of the $\tau_{k+1}(h) - \tau_k(h)$ follows from the strong Markov property. The distribution of $(\tau_{k+1}(h) - \tau_k(h))_{k \geq 0}$ does not depend on α as the one of $\tau_1(h)$ depends only on $|X|$ which is the same as a Reflected Brownian motion. The scaling property is a direct consequence of Proposition 1.

For the moments of $\tau_1(1)$, the Itô formula and the stopping theorem implies that for a Brownian motion B (case $\alpha = 1/2$),

$$1 = \mathbb{E}[B_{\tau_1(1)}^2] = \mathbb{E}[\tau_1(1)].$$

Again from the Itô formula, for any $t \geq 0$.

$$\begin{aligned} tB_t^2 &= \int_0^t B_s^2 ds + \int_0^t 2sB_s dB_s + \int_0^t s ds, \\ B_t^4 &= \int_0^t B_s^3 dB_s + 6 \int_0^t B_s^2 ds. \end{aligned}$$

Again from the stopping theorem,

$$\begin{aligned} 1 &= \mathbb{E}[B_{\tau_1(1)}^4] = 6\mathbb{E}\left[\int_0^{\tau_1(1)} B_s^2 ds\right], \\ 1 &= \mathbb{E}[\tau_1(1)B_{\tau_1(1)}^2] = \mathbb{E}\left[\int_0^{\tau_1(1)} B_s^2 ds\right] + \frac{1}{2}\mathbb{E}[\tau_1(1)^2]. \end{aligned}$$

Combining these two equations leads to the second moment of $\tau_1(1)$. \square

Generalizing the Donsker theorem, J.M. Harrison and L.A. Shepp [19] proved first that the SBM is the limit of a SWR (See also [16, 28]). Hence, the piecewise linear interpolation of $(\xi_k(h))_{k \geq 0}$ along the times $(\tau_k(h))_{k \geq 0}$ converges in probability to X .

2. An excursions based estimator

The data. We consider that given a path of a SBM X , we record the positions on the grid of the embedded chain $\xi(h) = (\xi_k(h))_{k=0, \dots, n^*(h)}$. From these observations, we construct an estimator $\alpha(h)$ of $\alpha \in (0, 1)$.

Indeed, using the scaling property, we may assume that we record either $(\xi_k(1))_{k=0, \dots, T/h^2}$ (long time horizon) or $\xi(h)$ for $\tau_k \leq T$ (fixed time horizon). We state only our result for the latter.

Embedding the family SRW into a SBM allows one to study the limit of $\alpha(h)$ as the parameter h decreases to 0. For this, the underlying probability space is the one of the SBM. Of course, our estimator is also suitable when

X_t	$\stackrel{\text{law}}{=} hX_{t/h^2}$	SBM, same scaling of for Brownian motion
L_t	$\stackrel{\text{law}}{=} hL_{t/h^2}$	Local time at 0, same scaling as for Brownian local time
$\tau_k(h)$	$\stackrel{\text{law}}{=} h^2\tau_k(1)$	Sum of iid rv, $\mathbb{E}_\alpha[\tau_1(1)] = 1$, $\mathbb{E}_\alpha[\tau_1(1)^2] = 5/3$
$\xi_k(h)$	$\stackrel{\text{law}}{=} h\xi_k(1)$	Biased random walk
$N_n(h)$	$\stackrel{\text{law}}{=} N_n(1)$	Number of zeros in n steps, $N_n(h)/\sqrt{n} \xrightarrow[n \rightarrow \infty]{\text{dist.}} Y \stackrel{\text{law}}{=} L_1$
$N_n^+(h)$	$\stackrel{\text{law}}{=} N_n^+(1)$	Number of upcrossings in n steps
$n^*(h)$	$\approx Th^{-2}$	Number of steps of the walk before T

Table 1: Main variables and their scaling properties.

one observes only a SRW. In this case, the estimation in long time horizon has to be considered, and the results are similar in nature (the local time has to be replaced by the limit of $n^{-1/2} \sum_{k=0}^n \mathbf{1}_{\xi_k(1)=0}$, which has the same distribution as the local time. See [38, Theorem 9.11, p. 99] and also Lemma 3 by noting that $L_n/\sqrt{n} \stackrel{\text{law}}{=} L_1$ and using Remark 6).

The estimator. For $n \geq 0$, we set

$$N_n(h) = \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k(h)=0} \text{ and } N_n^+(h) = \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k(h)=0, \xi_{k+1}(h)=1}.$$

As the random walk $(\xi_k(h))_{k \geq 0}$ is recurrent, $N_n(h) \rightarrow +\infty$ as $n \rightarrow \infty$ almost surely.

Finally, for $T > 0$, we set $n^*(h) = \sup\{k \geq 0 \mid \tau_k(h) \leq T\}$. From the Strong Law of Large Numbers, $\lim_{n \rightarrow \infty} \tau_n(h)/n = \mathbb{E}_\alpha[\tau_1(h)] = h^2$ almost surely. As $\tau_{n^*(h)}(h) \leq T < \tau_{n^*(h)+1}(h)$, $h^2 n^*(h)$ converges almost surely to T as $h \rightarrow 0$.

Table 1 summarizes the main variables and their scaling properties, which is justified by Proposition 1.

Our estimator of α is

$$\alpha(h) = \frac{N_{n^*(h)}^+(h)}{N_{n^*(h)}(h)}, \quad (6)$$

which is formalization of (3).

Remark 1. The likelihood is

$$\text{Lik}_\alpha(\boldsymbol{\xi}(h)) = \prod_{k=0}^{n^*(h)-1} p_\alpha(\xi_k(h), \xi_{k+1}(h)) = \alpha^{N_{n^*(h)}^+(h)} (1 - \alpha)^{N_{n^*(h)}(h) - N_{n^*(h)}^+(h)}$$

and the log-likelihood is

$$\text{Log-Lik}_\alpha(\boldsymbol{\xi}(h)) = N_{n^*(h)}^+(h) \log(\alpha) + (N_{n^*(h)}(h) - N_{n^*(h)}^+(h)) \log(1 - \alpha).$$

Solving $\partial_\alpha \text{Log-Lik}_{\alpha(h)}(\boldsymbol{\xi}(h)) = 0$ implies that $\alpha(h)$ given by (6) is the maximum likelihood estimator (MLE)

$$\alpha(h) = \operatorname{argmax}_\alpha \text{Log-Lik}_\alpha(\boldsymbol{\xi}) = \frac{N_{n^*(h)}^+(h)}{N_{n^*(h)}(h)}.$$

Theorem 1 (Convergence of the estimator $\alpha(h)$). *For any $\alpha \in (0, 1)$, under \mathbb{P}_α ,*

$$\alpha(h) \xrightarrow[h \rightarrow 0]{a.s.} \alpha, \tag{7}$$

implying in particular that $\alpha(h)$ is a consistent estimator of α . In addition, there exists a Brownian motion W independent from B such that

$$\frac{1}{\sqrt{h}} (\alpha(h) - \alpha) \xrightarrow[h \rightarrow 0]{dist.} \Upsilon_T = \sqrt{\alpha(1 - \alpha)} \frac{W(L_T)}{L_T}.$$

Moreover, for some constant C ,

$$\sup_{x \in \mathbb{R}} |\mathbb{P}[h^{-1/2}(\alpha_h - \alpha) \leq x] - \mathbb{P}[\Upsilon_T \leq x]| \leq Ch^{\frac{2}{9}}.$$

Remark 2. Using the scaling property on both the Brownian motion and the local time, $\Upsilon_T \stackrel{\text{law}}{=} T^{-1/4} \Upsilon_1$ for any $T > 0$. See [33] for more results about this random variable.

Remark 3. Since our estimator relies on the convergence of binomial random variables toward a Gaussian one, one could make use of the large literature on confidence intervals, even for small samples, which are better than the Wald confidence interval based on the Gaussian approximation: See *e.g.*, [6, 8, 40]. In particular, for $\alpha \neq 0$, these specific approaches provide non symmetric confidence interval around the estimator for a finite number of samples, although the limiting distribution of $h^{-1/2}(\alpha(h) - \alpha)$ is symmetric. This was observed empirically on the numerical tests in [32] on the MLE for the SBM.

Remark 4. While we record around $n \approx T/h^2$ points of the SBM, we really use $N_{n^*(h)}(h)$ points for the estimation, as the $\xi_k(h)$'s bring no information when the walk is away from 0. The number of points used for the estimation is of order $\sqrt{n} \approx \sqrt{T}/h$. Therefore, the rate of convergence in our CLT like result is $h^{1/2}$, that is of order $n^{-1/4}$. We are then not in the ‘‘usual’’ situation where the rate of convergence is $n^{-1/2}$. These are the same phenomena as the one of D. Florens [18] for estimators for stochastic differential equations based on crossings, and the results of J. Jacod [23].

The convergence of $\alpha(h)$ in (7) depends on the next lemma and the almost sure convergence of $n^*(h)$ to infinity.

Lemma 1 (Distribution of $N_n^+(h)$ given $N_n(h)$). *Under \mathbb{P}_α , the distribution of $N_n^+(h)$ given $N_n(h)$ is a binomial one with parameters $(\alpha, N_n(h))$, so that*

$$\frac{N_n^+(h)}{N_n(h)} \xrightarrow[n \rightarrow \infty]{} \alpha \text{ almost surely.} \quad (8)$$

The convergence of $h^{-1/2}(\alpha(h) - \alpha)$ follows from the Berry-Esseen theorem, whose application is immediate because of the conditional independence of $N_n^+(h)$ with respect to $N_n(h)$.

Lemma 2 (Application of the Berry-Esseen inequality [5, 15]). *Let Φ be the cumulative distribution of the Gaussian distribution $\mathcal{N}(0, 1)$. With $\sigma = \sqrt{\alpha(1 - \alpha)}$,*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}_\alpha \left[\frac{N_n^+(h) - N_n(h)\alpha}{\sigma \sqrt{N_n(h)}} \leq x \middle| N_n(h) \right] - \Phi(x) \right| \leq \frac{C\alpha}{\sigma^3 \sqrt{N_n(h)}}, \quad (9)$$

where C is a constant.

Remark 5. Many articles have been devoted to exhibit the best constant C in (9) for binomial distribution (See *e.g.* [36, 37] and the references within).

From now, the result stems from two ingredients: (i) the convergence of $hN_{n^*(h)}$ toward L_T , which is granted by the P. Lévy's downcrossing theorem (See *e.g.* [11, 22, 26] or [39, Exercice VI.1.19] and references within for refinements of the statement). (ii) A control on $\mathbb{E}_\alpha[1/\sqrt{N_{n^*(h)}(h)}]$ which follows from the convergence of $N_n(h)/\sqrt{n}$ (more precise statements are given in [10] which contains some almost sure control of $N_n(h)/\sqrt{n}$ for n large enough).

Lemma 3 (P. Lévy downcrossing theorem). *Almost surely, $hN_{n^*(h)}(h)$ converges to L_T as $h \rightarrow 0$.*

Besides, for some constant C ,

$$\mathbb{E}_\alpha \left[\left| \frac{N_n(1)}{\sqrt{n}} - \frac{L_n}{\sqrt{n}} \right|^2 \right]^{1/2} \leq \frac{C}{n^{1/4}}, \quad \forall n \geq 1. \quad (10)$$

Remark 6. The cumulative distribution function of the local time L_1 is $\mathbb{P}[L_1 \leq x] = \sqrt{2/\pi} \int_0^x \exp(-y^2/2) dy$ for $x \geq 0$. (See *e.g.* [38, Theorem 9.1, p. 101]), a consequence of the reflection principle and the identity in distribution between L_1 and the supremum of a Brownian motion on $[0, 1]$.

Remark 7. It is proved in [12] (see also [38, Consequence 10.3, p. 110]) that for any $\epsilon > 0$, $N_n(1) - L_n = o(n^{1/4+\epsilon})$ (See [4, 13] for more precise bounds). Thus, the rate of convergence of the L^2 -norm of $1/n^{1/4}$ in (10) is “nearly” optimal.

Proof. The first statement is the P. Lévy’s downcrossing theorem, so that we skip its proof. It is valid for the SBM as it depends only on the absolute value of X , which is a reflected Brownian motion in distribution.

We consider that $h = 1$. Thus, we drop any reference to h .

We write $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$ and $\text{sgn}(0) = 0$ since we use the symmetric local time. The Itô-Tanaka formula yields that

$$|X_t| = |X_s| + \int_s^t \text{sgn}(X_r) dX_r + L_t - L_s \text{ for } 0 \leq s \leq t.$$

Since $|a| - |b| \leq |a - b|$ for $a, b \in \mathbb{R}$,

$$|L_n - L_{\tau_n}| \leq |X_n - X_{\tau_n}| + \left| \int_{\tau_n}^n \text{sgn}(X_r) dX_r \right| \text{ for } n \geq 0.$$

Thus,

$$\mathbb{E}_\alpha[|L_n - L_{\tau_n}|^2] \leq 2\mathbb{E}_\alpha[|X_n - X_{\tau_n}|^2] + 2\mathbb{E}_\alpha[|\tau_n - n|].$$

Since $(X_t - X_r)^2 = 2 \int_r^t X_r dX_r + (t - r)$ and $\int_s^t X_r dL_r = \int_s^t X_r \mathbf{1}_{X_r=0} dL_r = 0$,

$$\frac{1}{n}\mathbb{E}_\alpha[|L_n - L_{\tau_n}|^2] \leq 4\mathbb{E}_\alpha \left[\left| \frac{\tau_n}{n} - 1 \right| \right] \leq \frac{8}{3n},$$

as τ_n is the sum of n independent random variables of mean 1 and variance $2/3$.

On the other hand, it is easily seen that for any $k \geq 0$,

$$|\xi_{k+1}| = |\xi_k| + \text{sgn}(\xi_k)(\xi_{k+1} - \xi_k) + \mathbf{1}_{\xi_k=0}. \quad (11)$$

Comparing (2) with (11), as $\xi_k = X_{\tau_k}$,

$$N_n = L_{\tau_n} + M_n \text{ with } M_n = \int_0^{\tau_n} (\text{sgn}(X_r) - \text{sgn}(X_{\pi(r)})) dX_r,$$

where $\pi(s) = \tau_k$ when $\tau_k \leq s < \tau_{k+1}$. It follows that

$$\begin{aligned} \frac{1}{n}\mathbb{E}_\alpha[M_n^2] &= \frac{1}{n}\mathbb{E}_\alpha \left[\int_0^{\tau_n} (\text{sgn}(X_r) - \text{sgn}(X_{\pi(r)}))^2 dr \right] \leq \frac{1}{n}\mathbb{E}_\alpha \left[\int_0^{\tau_n} \mathbf{1}_{X_r \in [0,1]} dr \right] \\ &\leq \frac{1}{n}\mathbb{E}_\alpha \left[\int_0^{\tau_n} \mathbf{1}_{X_r \in [-1,1]} dr \right] + \frac{1}{n}\mathbb{E}_\alpha[|\tau_n - n|]. \end{aligned}$$

Since $|X| \stackrel{\text{law}}{=} |B|$, $\mathbb{P}_\alpha[X_r \in [-1, 1]] = \mathbb{P}_\alpha[|B_r| \in [-1, 1]] \leq 2/\sqrt{2\pi r}$, so that

$$\frac{1}{n}\mathbb{E}_\alpha[M_n^2] \leq \frac{1}{n} + \frac{4}{\sqrt{2\pi n}}.$$

Eq. (10) is then proved. \square

Lemma 4. *For a constant C ,*

$$\mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_{n^*(h)}(h)}} \right] \leq Ch^{\frac{2}{9}} \xrightarrow{h \rightarrow \infty} 0. \quad (12)$$

Proof. Since $N_n(h) \geq 1$, $\mathbb{E}_\alpha[N_n(h)^{-1/2}] \leq 1$ for any $n \in \mathbb{N}$, $h > 0$.

From the scaling property, $L_n/\sqrt{n} \stackrel{\text{law}}{=} L_1$, and L_1 has the density $2\exp(-y^2/2)/\sqrt{2\pi}$ (See Remark 6).

We write N_n for $N_n(1)$. It holds that

$$\begin{aligned} \mathbb{P}_\alpha \left[\frac{N_n}{\sqrt{n}} \leq x \right] &\leq \mathbb{P}_\alpha \left[\frac{N_n}{\sqrt{n}} \leq x; \left| \frac{N_n}{\sqrt{n}} - \frac{L_n}{\sqrt{n}} \right| \leq \delta \right] + \mathbb{P}_\alpha \left[\frac{N_n}{\sqrt{n}} \leq x; \left| \frac{N_n}{\sqrt{n}} - \frac{L_n}{\sqrt{n}} \right| \geq \delta \right] \\ &\leq \mathbb{P}_\alpha \left[\frac{L_n}{\sqrt{n}} \leq x + \delta \right] + \frac{1}{\delta} \mathbb{E}_\alpha \left[\left| \frac{N_n}{\sqrt{n}} - \frac{L_n}{\sqrt{n}} \right|^2 \right]^{1/2} \\ &\leq \sqrt{\frac{2}{\pi}}(x + \delta) + \frac{C}{\delta n^{1/4}}. \end{aligned} \quad (13)$$

Let $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$ such that

$$\phi(n) \searrow_{n \rightarrow \infty} 0 \text{ and } n^{1/4}\phi(n) \nearrow_{n \rightarrow \infty} +\infty.$$

Since $N_n \geq 1$,

$$\begin{aligned} \mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_n}} \right] &= \mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_n}}; N_n < \sqrt{n}\phi(n) \right] + \mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_n}}; N_n \geq \sqrt{n}\phi(n) \right] \\ &\leq \mathbb{P}_\alpha[N_n < \sqrt{n}\phi(n)] + \frac{1}{\sqrt{n^{1/2}\phi(n)}}. \end{aligned}$$

Choosing $x = \delta = \phi(n)$ in (13), for some constant C ,

$$\mathbb{P}_\alpha \left[\frac{N_n}{\sqrt{n}} \leq \phi(n) \right] \leq C\phi(n) + \frac{C}{n^{1/4}\phi(n)}.$$

By balancing the expressions, $\phi(n) = 1/n^{1/8}$ is a suitable choice which ensures that

$$\mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_n}} \right] \leq \frac{C}{n^{1/8}} \xrightarrow{n \rightarrow \infty} 0. \quad (14)$$

For $\kappa(h) \xrightarrow[h \rightarrow 0]{} +\infty$ and $h^2\kappa(h) \xrightarrow[h \rightarrow 0]{} 0$,

$$\begin{aligned} \mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_{n^*(h)}(h)}} \right] &= \mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_{n^*(h)}(h)}}; n^*(h) \leq \kappa(h) \right] + \mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_{\kappa(h)}(h)}} \right] \\ &\leq \mathbb{P}_\alpha[n^*(h) \geq \kappa(h)] + \mathbb{E}_\alpha \left[\frac{1}{\sqrt{N_{\kappa(h)}(1)}} \right]. \end{aligned} \quad (15)$$

Using the Bienaymé-Tchebychev inequality,

$$\mathbb{P}_\alpha[n^*(h) \geq \kappa(h)] = \mathbb{P}_\alpha[\tau_{\kappa(h)-1}(h) \geq T] \leq \frac{1}{T} \mathbb{E}_\alpha[\tau_{\kappa(h)-1}(h)] \leq \frac{\kappa(h) - 1}{T} h^2.$$

Using (14), we take $\kappa(h) = h^{-16/9}$ to balance the rate of convergence in the two terms in the right-hand side of (15). This gives (12). \square

Proof of Theorem 1. Since $X_0 = 0$, $N_n(h) > 0$ and $L_T > 0$ almost surely.

The convergence of $\alpha(h) = N_{n^*(h)}^+(h)/N_{n^*(h)}(h)$ to α is a direct consequence of Lemma 1 and the convergence of $h^{-2}n^*(h)$ to T , since $n^*(h)$ is independent from $\{(N_n(h), N_n^+(h))\}_{n \geq 0}$.

From the Berry-Esseen theorem (See Lemma 2),

$$\begin{aligned} F_n(x) &= \mathbb{P}_\alpha \left[\frac{T^{1/4}}{\sqrt{h}} \left(\frac{N_n^+(h)}{N_n(h)} - \alpha \right) \leq x \right] \\ &= \mathbb{E}_\alpha \left[\mathbb{P}_\alpha \left[\frac{N_n^+(h) - \alpha N_n(h)}{\sigma \sqrt{N_n(h)}} \leq \frac{x \sqrt{h N_n(h)}}{\sigma T^{1/4}} \middle| N_n(h) \right] \right] \\ &= \mathbb{E}_\alpha \left[\Phi \left(\frac{x \sqrt{h N_n(h)}}{\sigma T^{1/4}} \right) \right] + \epsilon_n(h) \text{ with } |\epsilon_n(h)| \leq \mathbb{E}_\alpha \left[\frac{C\alpha}{\sigma^3 \sqrt{N_n(h)}} \right]. \end{aligned}$$

Hence,

$$\mathbb{E}_\alpha[F_{n^*(h)}(x)] = \mathbb{E}_\alpha \left[\Phi \left(\frac{x \sqrt{h N_{n^*(h)}(h)}}{\sigma T^{1/4}} \right) \right] + \mathbb{E}_\alpha[\epsilon_{n^*(h)}(h)].$$

The convergence of $h N_{n^*(h)}(h)$ to L_T and Lemma 4 imply the convergence in distribution of $h^{-1/2}(\alpha(h) - \alpha)$ to G/L_T for $G \sim \mathcal{N}(0, 1)$, where G is independent from L_T . The rate of convergence of the cumulative distribution function is given by Lemma 4.

Let us consider a Brownian motion W independent from B . Using the scaling property of the Brownian motion,

$$\mathbb{E}_\alpha \left[\Phi \left(\frac{x \sqrt{L_T}}{\sigma T^{1/4}} \right) \right] = \mathbb{P}_\alpha \left[\frac{\sigma T^{1/4} W(1)}{\sqrt{L_T}} \leq x \right] = \mathbb{P}_\alpha \left[\frac{\sigma T^{1/4} W(L_T)}{L_T} \leq x \right].$$

Thus, the limit in distribution of $h^{-1/2}(\alpha(h) - \alpha)$ could be written $\sigma W(L_T)/L_T$. \square

3. The Local Asymptotic Mixed Normal property

The Local Asymptotic Mixed Normal (LAMN) property [24] is an extension of the Local Asymptotic Normal property introduced by L. Le Cam [20, 29] when the limits are mixed normal distributions. Among other properties, one may deduce results on the convergence of estimators constructed from $\boldsymbol{\xi}(h)$ (See *e.g.* [25]).

Proposition 3 (The LAMN property). *For any $\gamma \in \mathbb{R}$ and $h > 0$,*

$$\log \frac{\text{Lik}_{\alpha+\gamma\sqrt{h}}(\boldsymbol{\xi}(h))}{\text{Lik}_{\alpha}(\boldsymbol{\xi}(h))} - \gamma\sqrt{Q(h)}G(h) - \frac{\gamma^2}{2}Q(h) + \epsilon(h, \gamma) \xrightarrow[h \rightarrow 0]{\mathbb{P}_{\alpha}} 0$$

where $G(h)$ and $Q(h)$ are $\mathcal{F}_{n^*(h)}$ -measurable,

$$(G(h), Q(h)) \xrightarrow[h \rightarrow 0]{\text{dist.}} (H, \sigma^{-2}L_T) \text{ with } H \sim \mathcal{N}(0, 1)$$

and H and L_T are independent. In other words, $(\mathbb{P}_{\alpha} \circ \boldsymbol{\xi}(h)^{-1})_{h>0}$ satisfies the LAMN property.

Proof. To simplify the notations, let us set $N(h) = N_{n^*(h)}$, $N^+(h) = N_{n^*(h)}$ and $N^-(h) = N(h) - N^+(h)$.

The logarithm of the ratio of the likelihood is, for any $\gamma \in \mathbb{R}$,

$$\begin{aligned} \log \frac{\text{Lik}_{\alpha+\gamma\sqrt{h}}(\boldsymbol{\xi})}{\text{Lik}_{\alpha}(\boldsymbol{\xi})} &= N^+(h) \log \left(1 + \frac{\gamma\sqrt{h}}{\alpha} \right) + N^-(h) \log \left(1 - \frac{\gamma\sqrt{h}}{1-\alpha} \right) \\ &= \gamma\sqrt{h} \left(\frac{N^+(h)}{\alpha} - \frac{N^-(h)}{1-\alpha} \right) - \frac{\gamma^2 h}{2} \left(\frac{N^+(h)}{\alpha^2} + \frac{N^-(h)}{(1-\alpha)^2} \right) + \epsilon(h, \gamma) \end{aligned}$$

with, from the Taylor development of the logarithm,

$$\epsilon(h, \gamma) \leq C\gamma^3 N(h)h^{3/2}$$

for some constant C . Since $hN(h)$ converges almost surely to L_T which is almost surely finite, $\epsilon(h, \gamma)$ converges almost surely to 0 for any $\gamma > 0$, hence in probability. Thanks to Lemma 1,

$$\begin{aligned} A(h) &= \frac{1}{N(h)} \left(\frac{N^+(h)}{\alpha} - \frac{N^-(h)}{1-\alpha} \right) \xrightarrow[h \rightarrow 0]{\text{a.s.}} 0 \\ \text{and } B(h) &= \frac{1}{N(h)} \left(\frac{N^+(h)}{\alpha^2} + \frac{N^-(h)}{(1-\alpha)^2} \right) \xrightarrow[h \rightarrow 0]{\text{a.s.}} \frac{1}{\sigma^2} \text{ with } \sigma = \sqrt{\alpha(1-\alpha)}. \end{aligned}$$

Theorem 1 implies that

$$\frac{A(h)}{\sqrt{h}} = \frac{1}{\sigma^2} \left(\frac{N^+(h)}{N(h)} - \alpha \right) \xrightarrow[h \rightarrow 0]{\text{dist.}} \frac{W(L_T)}{\sigma L_T}.$$

On the other hand, since $hN(h)$ converges almost surely to L_T ,

$$\sqrt{h}N(h)A(h) = hN(h) \frac{A(h)}{\sqrt{h}} \xrightarrow[h \rightarrow 0]{\text{dist.}} \sigma^{-1}W(L_T) \stackrel{\text{law}}{=} \sigma^{-1}\sqrt{L_T}W(1).$$

To summarize,

$$\log \frac{\text{Lik}_{\alpha+\gamma\sqrt{h}}(\boldsymbol{\xi})}{\text{Lik}_{\alpha}(\boldsymbol{\xi})} = \gamma\sqrt{h}N(h)A(h) - \gamma^2hN(h)B(h) + \epsilon(h, \gamma).$$

With

$$Q(h) = hN(h)B(h) \xrightarrow[h \rightarrow 0]{\text{a.s.}} \sigma^{-2}L_T$$

and $G(h) = A(h) \frac{\sqrt{N(h)}}{\sqrt{B(h)}} \xrightarrow[h \rightarrow 0]{\text{dist.}} W(1) \sim \mathcal{N}(0, 1).$

This proves the LAMN property. □

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