Some remarks on Bürgi's interpolations

Denis Roegel*

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Jost Bürgi's recently rediscovered *Fundamentum Astronomiæ* [3, 4, 9] describes a number of methods for the computation of tables of sines. One algorithm makes it possible to compute accurate values of the sines of a subdivision of the quadrant, but in practice this method can only be used for small subdivisions [12].

Another technique shown by Bürgi allows him to find a more accurate value of $\sin 1'$, given the value of $\sin 1^\circ$ and the value of the first third difference. However, this technique involves a circularity problem, in that knowing the third difference requires the knowledge of the values of several sines [13].

Bürgi also seems to allude to a technique of interpolation, which was described by Launert [5]. Bürgi seems to start with pivots and then to add up differences. This technique was apparently used to construct the table of sines for every minute of the quadrant, which we have recently reconstructed [14].

It is now interesting to find how many differences Bürgi has used, and with how many sexagesimal digits his computations were done, before the results were rounded. These questions are exactly those that Prony had to answer when he worked out the methods for the *Tables du cadastre* 200 years later [8]. We will not enter into the details of these computations now, but we can adopt an experimental approach, which may well be one that Bürgi used. That is, we can try to interpolate between 0° and 1°, by steps of $1'$. If we consider the third differences to be constant, then it will be easy to see that 7 (sexagesimal) digits add up to 1, 2, 49, 43, 10 after one degree, instead of 1, 2, 49, 43, 11 (60 units make up the *sinus totus*). So, 7 digits are not enough, even for the first interpolation. However, 8 digits will suffice, at least for that interpolation. Unfortunately, as one goes

*Denis Roegel, LORIA, BP 239, 54506 Vandœuvre-lès-Nancy cedex, France, roegel@loria.fr

further, 8 digits quickly are no longer enough. But doing the interpolation on 9, 10, or even more digits, will in fact not help.

The problem is that the third difference can not be considered constant and varies too much with respect to the range of the interpolation. This, Bürgi must have noticed, because his values are in general very accurate, with at most one unit of error in the last place, and only sometimes. It is therefore clear that Bürgi went beyond the third differences, even for the table with a $1'$ interval.

If the $2''$ sine table did really exist, it could have been computed using these techniques, with no more than third differences. For ease of reference, we have provided two complete sets of interpolations for that hypothetical table, one using only 9 sexagesimal places [11] and consequently not sufficient to attain eight correct sexagesimal digits, and another with the computations done on 10 sexagesimal places [10], and which is sufficient for the purpose.

In the sequel, we give some excerpts of the interpolations which may have been carried through.

Our first example (figure 1) is an interpolation of the first degree of the $1'$ sine table, using constant third differences and 10 sexagesimal digits. This interpolation is correct to 5 sexagesimal places. Reducing the number of digits to 8 (figure 2), still yields 5 correct places at the end of the interpolation.

If we now consider interpolations to span every half-degree, then we can do with less sexagesimal places, but depending how rounding is done, we can still end up with one unit of error in the last place, for instance between 10° and $10^\circ 30'$ (figure 3).

Towards the end of the quadrant, third differences are no longer sufficient, even for obtaining 5 correct places. For instance, using 10 digits and a constant difference between 87° and 88° only yields sines with 4 correct places at the end of the interpolation (figure 4). Between 89° and 90°, we also end up with only 4 correct places (figure 5).

But if we use the fourth differences, then 9 places are enough in the range 87° to 88° to produce 5 correct places (figure 6). If the computations between 89° and 90° are done on 10 places, we end up with 7 correct places (figure 7).

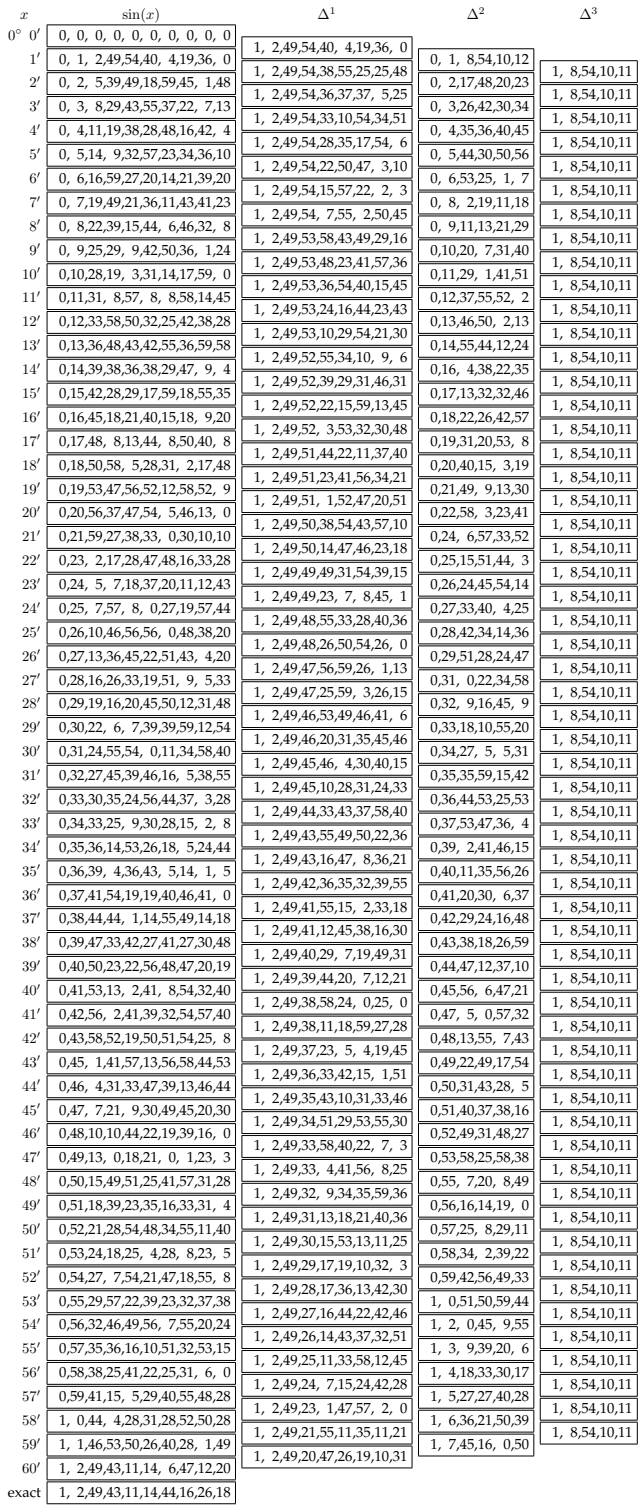
Figure 8 shows an interpolation for the $2''$ sine table. The use of a constant third difference and 10 sexagesimal places is sufficient in the considered range to secure 8 sexagesimal places at the end of the interpolation. However, as shown in the full 9-place interpolations [11], using only 9 places would not be sufficient.

Figure 9 shows an interpolation at the other end of the $2''$ table, with

only 9 sexagesimal places, and this time the last value is correct to 8 places, when rounded.

In the case of the $1'$ table, Bürgi gave the results to 6 and 7 places for the last two degrees, and it is possible that he used even higher differences than the fourth difference, though this is in fact not necessary. By a careful analysis of the errors in Bürgi's table, it may be possible to find exactly how many places were used in the actual computations, as well as the level of differences used, but we have not yet investigated this matter.

The fact that the calculations are done on a number of places, and then truncated, may also be a source of confusion, because the draft of the table could be taken for the actual final table. For instance, if someone had seen the untruncated table to $1'$, he could have taken it for a table to 8 (or more places), when it fact it wasn't. The same applies for the $2''$ table, where the actual computations were likely done on 10 places. It is hard to imagine that one man could do that, given the volume of that table, as shown in our reconstruction [15].



x	$\sin(x)$	Δ^1	Δ^2	Δ^3
$0^\circ\ 0'$	0, 0, 0, 0, 0, 0, 0, 0, 0, 0	1, 2,49,54,40, 4,19,36, 0		
$1'$	0, 1, 2,49,54,40, 4,19,36, 0	1, 2,49,54,38,55,25,25,48	0, 1, 8,54,10,12	1, 8,54,10,11
$2'$	0, 2, 5,39,49,18,59,45, 1,48	1, 2,49,54,36,37,37, 5,25	0, 2,17,48,20,23	1, 8,54,10,11
$3'$	0, 3, 8,29,43,55,37,22, 7,13	1, 2,49,54,33,10,54,34,51	0, 3,26,42,30,34	1, 8,54,10,11
$4'$	0, 4,11,19,38,28,48,16,42, 4	1, 2,49,54,28,35,17,54, 6	0, 4,35,36,40,45	1, 8,54,10,11
$5'$	0, 5,14, 9,32,57,23,34,36,10	1, 2,49,54,22,50,47, 3,10	0, 5,44,30,50,56	1, 8,54,10,11
$6'$	0, 6,16,59,27,20,14,21,39,20	1, 2,49,54,15,57,22, 2, 3	0, 6,53,25, 1, 7	1, 8,54,10,11
$7'$	0, 7,19,49,21,36,11,43,41,23	1, 2,49,54, 7,55, 2,50,45	0, 8, 2,19,11,18	1, 8,54,10,11
$8'$	0, 8,22,39,15,44, 6,46,32, 8	1, 2,49,53,58,43,49,29,16	0, 9,11,13,21,29	1, 8,54,10,11
$9'$	0, 9,25,29, 9,42,50,36, 1,24	1, 2,49,53,48,23,41,57,36	0,10,20, 7,31,40	1, 8,54,10,11
$10'$	0,10,28,19, 3,31,14,17,59, 0	1, 2,49,53,36,54,40,15,45	0,11,29, 1,41,51	1, 8,54,10,11
$11'$	0,11,31, 8,57, 8, 8,58,14,45	1, 2,49,53,24,16,44,23,43	0,12,37,55,52, 2	1, 8,54,10,11
$12'$	0,12,33,58,50,32,25,42,38,28	1, 2,49,53,10,29,54,21,30	0,13,46,50, 2,13	1, 8,54,10,11
$13'$	0,13,36,48,43,42,55,36,59,58	1, 2,49,52,55,34,10, 9, 6	0,14,55,44,12,24	1, 8,54,10,11
$14'$	0,14,39,38,36,38,29,47, 9, 4	1, 2,49,52,39,29,31,46,31	0,16, 4,38,22,35	1, 8,54,10,11
$15'$	0,15,42,28,29,17,59,18,55,35	1, 2,49,52,22,15,59,13,45	0,17,13,32,32,46	1, 8,54,10,11
$16'$	0,16,45,18,21,40,15,18, 9,20	1, 2,49,52, 3,53,32,30,48	0,18,22,26,42,57	1, 8,54,10,11
$17'$	0,17,48, 8,13,44, 8,50,40, 8	1, 2,49,51,44,22,11,37,40	0,19,31,20,53, 8	1, 8,54,10,11
$18'$	0,18,50,58, 5,28,31, 2,17,48	1, 2,49,51,23,41,56,34,21	0,20,40,15, 3,19	1, 8,54,10,11
$19'$	0,19,53,47,56,52,12,58,52, 9	1, 2,49,51, 1,52,47,20,51	0,21,49, 9,13,30	1, 8,54,10,11
$20'$	0,20,56,37,47,54, 5,46,13, 0	1, 2,49,50,38,54,43,57,10	0,22,58, 3,23,41	1, 8,54,10,11
$21'$	0,21,59,27,38,33, 0,30,10,10	1, 2,49,50,14,47,46,23,18	0,24, 6,57,33,52	1, 8,54,10,11
$22'$	0,23, 2,17,28,47,48,16,33,28	1, 2,49,49,49,31,54,39,15	0,25,15,51,44, 3	1, 8,54,10,11
$23'$	0,24, 5, 7,18,37,20,11,12,43	1, 2,49,49,23, 7, 8,45, 1	0,26,24,45,54,14	1, 8,54,10,11
$24'$	0,25, 7,57, 8, 0,27,19,57,44	1, 2,49,48,55,33,28,40,36	0,27,33,40, 4,25	1, 8,54,10,11
$25'$	0,26,10,46,56,56, 0,48,38,20	1, 2,49,48,26,50,54,26, 0	0,28,42,34,14,36	1, 8,54,10,11
$26'$	0,27,13,36,45,22,51,43, 4,20	1, 2,49,47,56,59,26, 1,13	0,29,51,28,24,47	1, 8,54,10,11
$27'$	0,28,16,26,33,19,51, 9, 5,33	1, 2,49,47,25,59, 3,26,15	0,31, 0,22,34,58	1, 8,54,10,11
$28'$	0,29,19,16,20,45,50,12,31,48	1, 2,49,46,53,49,46,41, 6	0,32, 9,16,45, 9	1, 8,54,10,11
$29'$	0,30,22, 6, 7,39,39,59,12,54	1, 2,49,46,20,31,35,45,46	0,33,18,10,55,20	1, 8,54,10,11
$30'$	0,31,24,55,54, 0,11,34,58,40	1, 2,49,45,46, 4,30,40,15	0,34,27, 5, 5,31	1, 8,54,10,11
$31'$	0,32,27,45,39,46,16, 5,38,55	1, 2,49,45,10,28,31,24,33	0,35,35,59,15,42	1, 8,54,10,11
$32'$	0,33,30,35,24,56,44,37, 3,28	1, 2,49,44,33,43,37,58,40	0,36,44,53,25,53	1, 8,54,10,11
$33'$	0,34,33,25, 9,30,28,15, 2, 8	1, 2,49,43,55,49,50,22,36	0,37,53,47,36, 4	1, 8,54,10,11
$34'$	0,35,36,14,53,26,18, 5,24,44	1, 2,49,43,16,47, 8,36,21	0,39, 2,41,46,15	1, 8,54,10,11
$35'$	0,36,39, 4,36,43, 5,14, 1, 5	1, 2,49,42,36,35,32,39,55	0,40,11,35,56,26	1, 8,54,10,11
$36'$	0,37,41,54,19,19,40,46,41, 0	1, 2,49,41,55,15, 2,33,18	0,41,20,30, 6,37	1, 8,54,10,11
$37'$	0,38,44,44, 1,14,55,49,14,18	1, 2,49,41,12,45,38,16,30	0,42,29,24,16,48	1, 8,54,10,11
$38'$	0,39,47,33,42,27,41,27,30,48	1, 2,49,40,29, 7,19,49,31	0,43,38,18,26,59	1, 8,54,10,11
$39'$	0,40,50,23,22,56,48,47,20,19	1, 2,49,39,44,20, 7,12,21	0,44,47,12,37,10	1, 8,54,10,11
$40'$	0,41,53,13, 2,41, 8,54,32,40	1, 2,49,38,58,24, 0,25, 0	0,45,56, 6,47,21	1, 8,54,10,11
$41'$	0,42,56, 2,41,39,32,54,57,40	1, 2,49,38,11,18,59,27,28	0,47, 5, 0,57,32	1, 8,54,10,11
$42'$	0,43,58,52,19,50,51,54,25, 8	1, 2,49,37,23, 5, 4,19,45	0,48,13,55, 7,43	1, 8,54,10,11
$43'$	0,45, 1,41,57,13,56,58,44,53	1, 2,49,36,33,42,15, 1,51	0,49,22,49,17,54	1, 8,54,10,11
$44'$	0,46, 4,31,33,47,39,13,46,44	1, 2,49,35,43,10,31,33,46	0,50,31,43,28, 5	1, 8,54,10,11
$45'$	0,47, 7,21, 9,30,49,45,20,30	1, 2,49,34,51,29,53,55,30	0,51,40,37,38,16	1, 8,54,10,11
$46'$	0,48,10,10,44,22,19,39,16, 0	1, 2,49,33,58,40,22, 7, 3	0,52,49,31,48,27	1, 8,54,10,11
$47'$	0,49,13, 0,18,21, 0, 1,23, 3	1, 2,49,33, 4,41,56, 8,25	0,53,58,25,58,38	1, 8,54,10,11
$48'$	0,50,15,49,51,25,41,57,31,28	1, 2,49,32, 9,34,35,59,36	0,55, 7,20, 8,49	1, 8,54,10,11
$49'$	0,51,18,39,23,35,16,33,31, 4	1, 2,49,31,13,18,21,40,36	0,56,16,14,19, 0	1, 8,54,10,11
$50'$	0,52,21,28,54,48,34,55,11,40	1, 2,49,30,15,53,13,11,25	0,57,25, 8,29,11	1, 8,54,10,11
$51'$	0,53,24,18,25, 4,28, 8,23, 5	1, 2,49,29,17,19,10,32, 3	0,58,34, 2,39,22	1, 8,54,10,11
$52'$	0,54,27, 7,54,21,47,18,55, 8	1, 2,49,28,17,36,13,42,30	0,59,42,56,49,33	1, 8,54,10,11
$53'$	0,55,29,57,22,39,23,32,37,38	1, 2,49,27,16,44,22,42,46	1, 0,51,50,59,44	1, 8,54,10,11
$54'$	0,56,32,46,49,56, 7,55,20,24	1, 2,49,26,14,43,37,32,51	1, 2, 0,45, 9,55	1, 8,54,10,11
$55'$	0,57,35,36,16,10,51,32,53,15	1, 2,49,25,11,33,58,12,45	1, 3, 9,39,20, 6	1, 8,54,10,11
$56'$	0,58,38,25,41,22,25,31, 6, 0	1, 2,49,24, 7,15,24,42,28	1, 4,18,33,30,17	1, 8,54,10,11
$57'$	0,59,41,15, 5,29,40,55,48,28	1, 2,49,23, 1,47,57, 2, 0	1, 5,27,27,40,28	1, 8,54,10,11
$58'$	1, 0,44, 4,28,31,28,52,50,28	1, 2,49,21,55,11,35,11,21	1, 6,36,21,50,39	1, 8,54,10,11
$59'$	1, 1,46,53,50,26,40,28, 1,49	1, 2,49,20,47,26,19,10,31	1, 7,45,16, 0,50	
$60'$	1, 2,49,43,11,14, 6,47,12,20			
exact	1, 2,49,43,11,14,44,16,26,18			

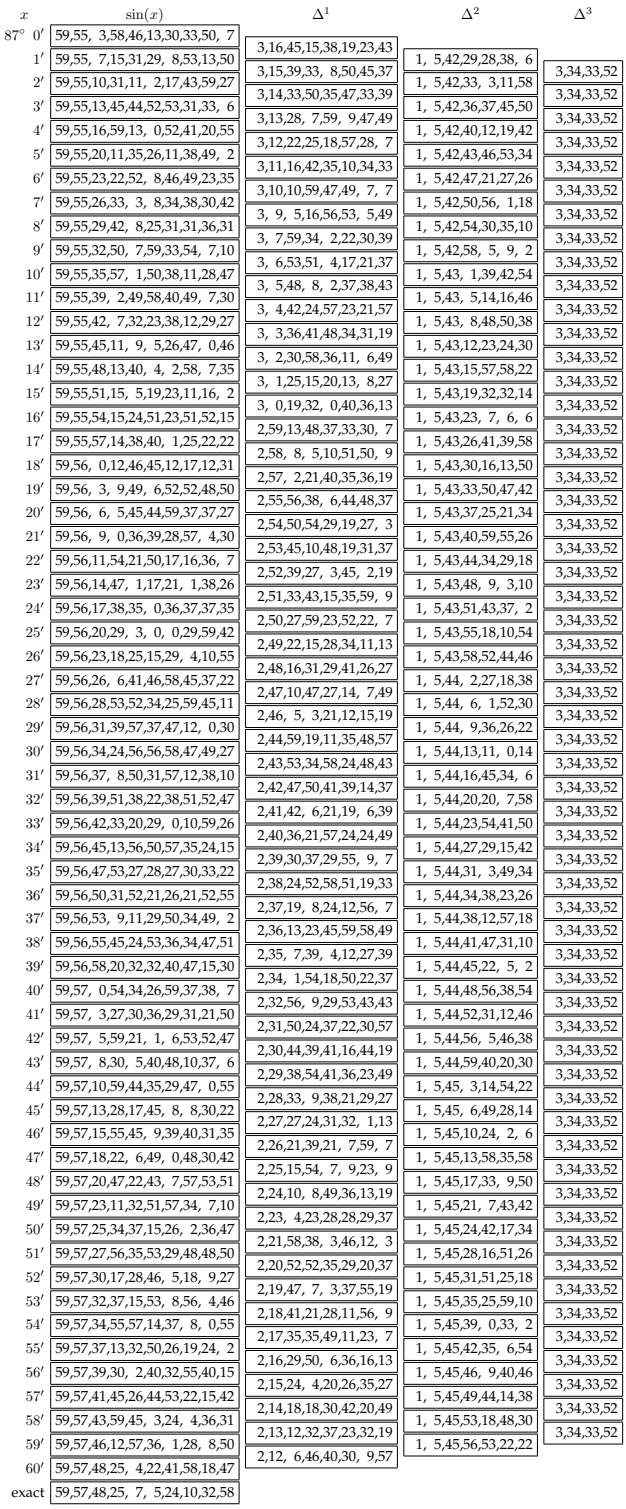
Figure 1: $1'$ interpolation from 0° to 1° with Δ^3 constant and 10 digits.

x	$\sin(x)$	Δ^1	Δ^2	Δ^3
$0^\circ\ 0'$	0, 0, 0, 0, 0, 0, 0, 0	1, 2,49,54,40, 4,20		
$1'$	0, 1, 2,49,54,40, 4,20	1, 2,49,54,38,55,26	0, 1, 8,54	1, 8,54
$2'$	0, 2, 5,39,49,18,59,46	1, 2,49,54,36,37,38	0, 2,17,48	1, 8,54
$3'$	0, 3, 8,29,43,55,37,24	1, 2,49,54,33,10,56	0, 3,26,42	1, 8,54
$4'$	0, 4,11,19,38,28,48,20	1, 2,49,54,28,35,20	0, 4,35,36	1, 8,54
$5'$	0, 5,14, 9,32,57,23,40	1, 2,49,54,22,50,50	0, 5,44,30	1, 8,54
$6'$	0, 6,16,59,27,20,14,30	1, 2,49,54,15,57,26	0, 6,53,24	1, 8,54
$7'$	0, 7,19,49,21,36,11,56	1, 2,49,54, 7,55, 8	0, 8, 2,18	1, 8,54
$8'$	0, 8,22,39,15,44, 7, 4	1, 2,49,53,58,43,56	0, 9,11,12	1, 8,54
$9'$	0, 9,25,29, 9,42,51, 0	1, 2,49,53,48,23,50	0,10,20, 6	1, 8,54
$10'$	0,10,28,19, 3,31,14,50	1, 2,49,53,36,54,50	0,11,29, 0	1, 8,54
$11'$	0,11,31, 8,57, 8, 9,40	1, 2,49,53,24,16,56	0,12,37,54	1, 8,54
$12'$	0,12,33,58,50,32,26,36	1, 2,49,53,10,30, 8	0,13,46,48	1, 8,54
$13'$	0,13,36,48,43,42,56,44	1, 2,49,52,55,34,26	0,14,55,42	1, 8,54
$14'$	0,14,39,38,36,38,31,10	1, 2,49,52,39,29,50	0,16, 4,36	1, 8,54
$15'$	0,15,42,28,29,18, 1, 0	1, 2,49,52,22,16,20	0,17,13,30	1, 8,54
$16'$	0,16,45,18,21,40,17,20	1, 2,49,52, 3,53,56	0,18,22,24	1, 8,54
$17'$	0,17,48, 8,13,44,11,16	1, 2,49,51,44,22,38	0,19,31,18	1, 8,54
$18'$	0,18,50,58, 5,28,33,54	1, 2,49,51,23,42,26	0,20,40,12	1, 8,54
$19'$	0,19,53,47,56,52,16,20	1, 2,49,51, 1,53,20	0,21,49, 6	1, 8,54
$20'$	0,20,56,37,47,54, 9,40	1, 2,49,50,38,55,20	0,22,58, 0	1, 8,54
$21'$	0,21,59,27,38,33, 5, 0	1, 2,49,50,14,48,26	0,24, 6,54	1, 8,54
$22'$	0,23, 2,17,28,47,53,26	1, 2,49,49,49,32,38	0,25,15,48	1, 8,54
$23'$	0,24, 5, 7,18,37,26, 4	1, 2,49,49,23, 7,56	0,26,24,42	1, 8,54
$24'$	0,25, 7,57, 8, 0,34, 0	1, 2,49,48,55,34,20	0,27,33,36	1, 8,54
$25'$	0,26,10,46,56,56, 8,20	1, 2,49,48,26,51,50	0,28,42,30	1, 8,54
$26'$	0,27,13,36,45,23, 0,10	1, 2,49,47,57, 0,26	0,29,51,24	1, 8,54
$27'$	0,28,16,26,33,20, 0,36	1, 2,49,47,26, 0, 8	0,31, 0,18	1, 8,54
$28'$	0,29,19,16,20,46, 0,44	1, 2,49,46,53,50,56	0,32, 9,12	1, 8,54
$29'$	0,30,22, 6, 7,39,51,40	1, 2,49,46,20,32,50	0,33,18, 6	1, 8,54
$30'$	0,31,24,55,54, 0,24,30	1, 2,49,45,46, 5,50	0,34,27, 0	1, 8,54
$31'$	0,32,27,45,39,46,30,20	1, 2,49,45,10,29,56	0,35,35,54	1, 8,54
$32'$	0,33,30,35,24,57, 0,16	1, 2,49,44,33,45, 8	0,36,44,48	1, 8,54
$33'$	0,34,33,25, 9,30,45,24	1, 2,49,43,55,51,26	0,37,53,42	1, 8,54
$34'$	0,35,36,14,53,26,36,50	1, 2,49,43,16,48,50	0,39, 2,36	1, 8,54
$35'$	0,36,39, 4,36,43,25,40	1, 2,49,42,36,37,20	0,40,11,30	1, 8,54
$36'$	0,37,41,54,19,20, 3, 0	1, 2,49,41,55,16,56	0,41,20,24	1, 8,54
$37'$	0,38,44,44, 1,15,19,56	1, 2,49,41,12,47,38	0,42,29,18	1, 8,54
$38'$	0,39,47,33,42,28, 7,34	1, 2,49,40,29, 9,26	0,43,38,12	1, 8,54
$39'$	0,40,50,23,22,57,17, 0	1, 2,49,39,44,22,20	0,44,47, 6	1, 8,54
$40'$	0,41,53,13, 2,41,39,20	1, 2,49,38,58,26,20	0,45,56, 0	1, 8,54
$41'$	0,42,56, 2,41,40, 5,40	1, 2,49,38,11,21,26	0,47, 4,54	1, 8,54
$42'$	0,43,58,52,19,51,27, 6	1, 2,49,37,23, 7,38	0,48,13,48	1, 8,54
$43'$	0,45, 1,41,57,14,34,44	1, 2,49,36,33,44,56	0,49,22,42	1, 8,54
$44'$	0,46, 4,31,33,48,19,40	1, 2,49,35,43,13,20	0,50,31,36	1, 8,54
$45'$	0,47, 7,21, 9,31,33, 0	1, 2,49,34,51,32,50	0,51,40,30	1, 8,54
$46'$	0,48,10,10,44,23, 5,50	1, 2,49,33,58,43,26	0,52,49,24	1, 8,54
$47'$	0,49,13, 0,18,21,49,16	1, 2,49,33, 4,45, 8	0,53,58,18	1, 8,54
$48'$	0,50,15,49,51,26,34,24	1, 2,49,32, 9,37,56	0,55, 7,12	1, 8,54
$49'$	0,51,18,39,23,36,12,20	1, 2,49,31,13,21,50	0,56,16, 6	1, 8,54
$50'$	0,52,21,28,54,49,34,10	1, 2,49,30,15,56,50	0,57,25, 0	1, 8,54
$51'$	0,53,24,18,25, 5,31, 0	1, 2,49,29,17,22,56	0,58,33,54	1, 8,54
$52'$	0,54,27, 7,54,22,53,56	1, 2,49,28,17,40, 8	0,59,42,48	1, 8,54
$53'$	0,55,29,57,22,40,34, 4	1, 2,49,27,16,48,26	1, 0,51,42	1, 8,54
$54'$	0,56,32,46,49,57,22,30	1, 2,49,26,14,47,50	1, 2, 0,36	1, 8,54
$55'$	0,57,35,36,16,12,10,20	1, 2,49,25,11,38,20	1, 3, 9,30	1, 8,54
$56'$	0,58,38,25,41,23,48,40	1, 2,49,24, 7,19,56	1, 4,18,24	1, 8,54
$57'$	0,59,41,15, 5,31, 8,36	1, 2,49,23, 1,52,38	1, 5,27,18	1, 8,54
$58'$	1, 0,44, 4,28,33, 1,14	1, 2,49,21,55,16,26	1, 6,36,12	1, 8,54
$59'$	1, 1,46,53,50,28,17,40	1, 2,49,20,47,31,20	1, 7,45, 6	
$60'$	1, 2,49,43,11,15,49, 0			
exact	1, 2,49,43,11,14,44,16			

Figure 2: $1'$ interpolation from 0° to 1° with Δ^3 constant and 8 digits.

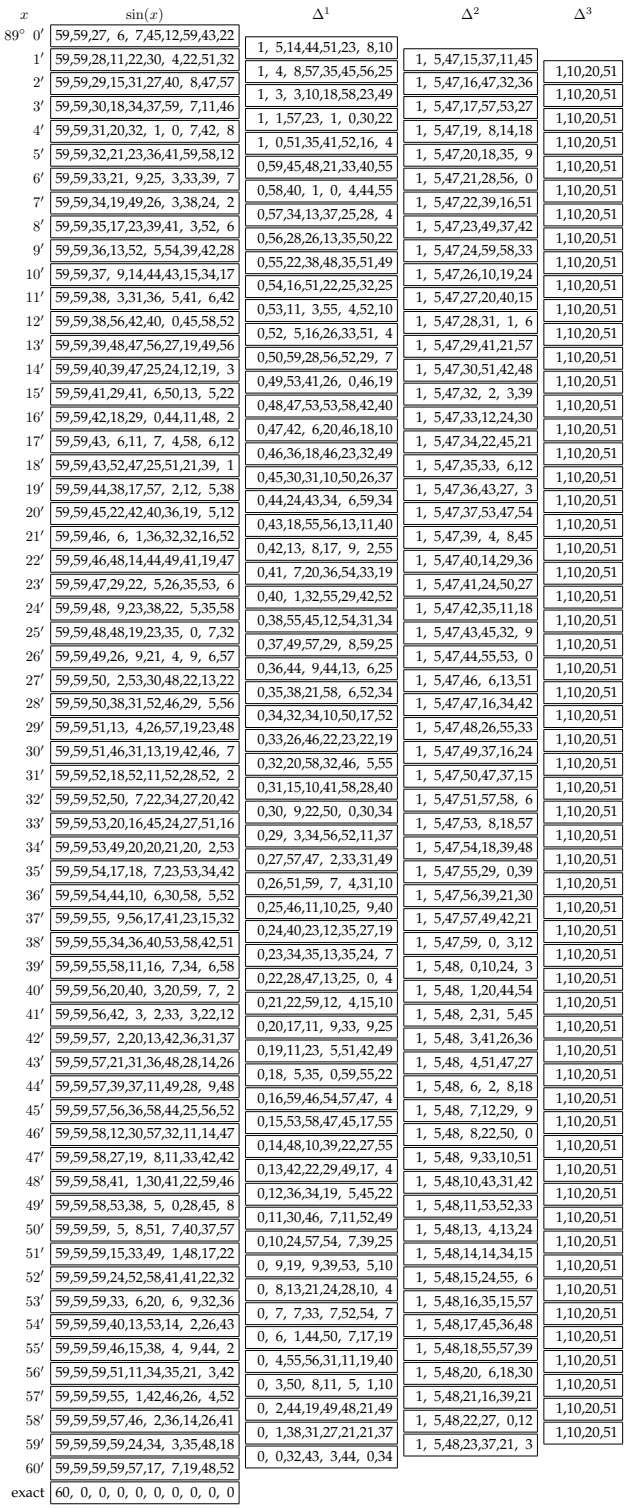
x	$\sin(x)$	Δ^1	Δ^2	Δ^3
$10^\circ\ 0'$	10,25, 8, 0,22,57,14			
		1, 1,52,32,32,59		
$1'$	10,26, 9,52,55,30,13		11,26,40	
		1, 1,52,21, 6,19		1, 8
$2'$	10,27,11,45,16,36,32		11,27,48	
		1, 1,52, 9,38,31		1, 8
$3'$	10,28,13,37,26,15, 3		11,28,56	
		1, 1,51,58, 9,35		1, 8
$4'$	10,29,15,29,24,24,38		11,30, 4	
		1, 1,51,46,39,31		1, 8
$5'$	10,30,17,21,11, 4, 9		11,31,12	
		1, 1,51,35, 8,19		1, 8
$6'$	10,31,19,12,46,12,28		11,32,20	
		1, 1,51,23,35,59		1, 8
$7'$	10,32,21, 4, 9,48,27		11,33,28	
		1, 1,51,12, 2,31		1, 8
$8'$	10,33,22,55,21,50,58		11,34,36	
		1, 1,51, 0,27,55		1, 8
$9'$	10,34,24,46,22,18,53		11,35,44	
		1, 1,50,48,52,11		1, 8
$10'$	10,35,26,37,11,11, 4		11,36,52	
		1, 1,50,37,15,19		1, 8
$11'$	10,36,28,27,48,26,23		11,38, 0	
		1, 1,50,25,37,19		1, 8
$12'$	10,37,30,18,14, 3,42		11,39, 8	
		1, 1,50,13,58,11		1, 8
$13'$	10,38,32, 8,28, 1,53		11,40,16	
		1, 1,50, 2,17,55		1, 8
$14'$	10,39,33,58,30,19,48		11,41,24	
		1, 1,49,50,36,31		1, 8
$15'$	10,40,35,48,20,56,19		11,42,32	
		1, 1,49,38,53,59		1, 8
$16'$	10,41,37,37,59,50,18		11,43,40	
		1, 1,49,27,10,19		1, 8
$17'$	10,42,39,27,27, 0,37		11,44,48	
		1, 1,49,15,25,31		1, 8
$18'$	10,43,41,16,42,26, 8		11,45,56	
		1, 1,49, 3,39,35		1, 8
$19'$	10,44,43, 5,46, 5,43		11,47, 4	
		1, 1,48,51,52,31		1, 8
$20'$	10,45,44,54,37,58,14		11,48,12	
		1, 1,48,40, 4,19		1, 8
$21'$	10,46,46,43,18, 2,33		11,49,20	
		1, 1,48,28,14,59		1, 8
$22'$	10,47,48,31,46,17,32		11,50,28	
		1, 1,48,16,24,31		1, 8
$23'$	10,48,50,20, 2,42, 3		11,51,36	
		1, 1,48, 4,32,55		1, 8
$24'$	10,49,52, 8, 7,14,58		11,52,44	
		1, 1,47,52,40,11		1, 8
$25'$	10,50,53,55,59,55, 9		11,53,52	
		1, 1,47,40,46,19		1, 8
$26'$	10,51,55,43,40,41,28		11,55, 0	
		1, 1,47,28,51,19		1, 8
$27'$	10,52,57,31, 9,32,47		11,56, 8	
		1, 1,47,16,55,11		1, 8
$28'$	10,53,59,18,26,27,58		11,57,16	
		1, 1,47, 4,57,55		1, 8
$29'$	10,55, 1, 5,31,25,53		11,58,24	
		1, 1,46,52,59,31		
$30'$	10,56, 2,52,24,25,24			
exact	10,56, 2,52,24,37,22			

Figure 3: $1'$ interpolation from 10° to $10^\circ 30'$ with Δ^3 constant and 7 digits.



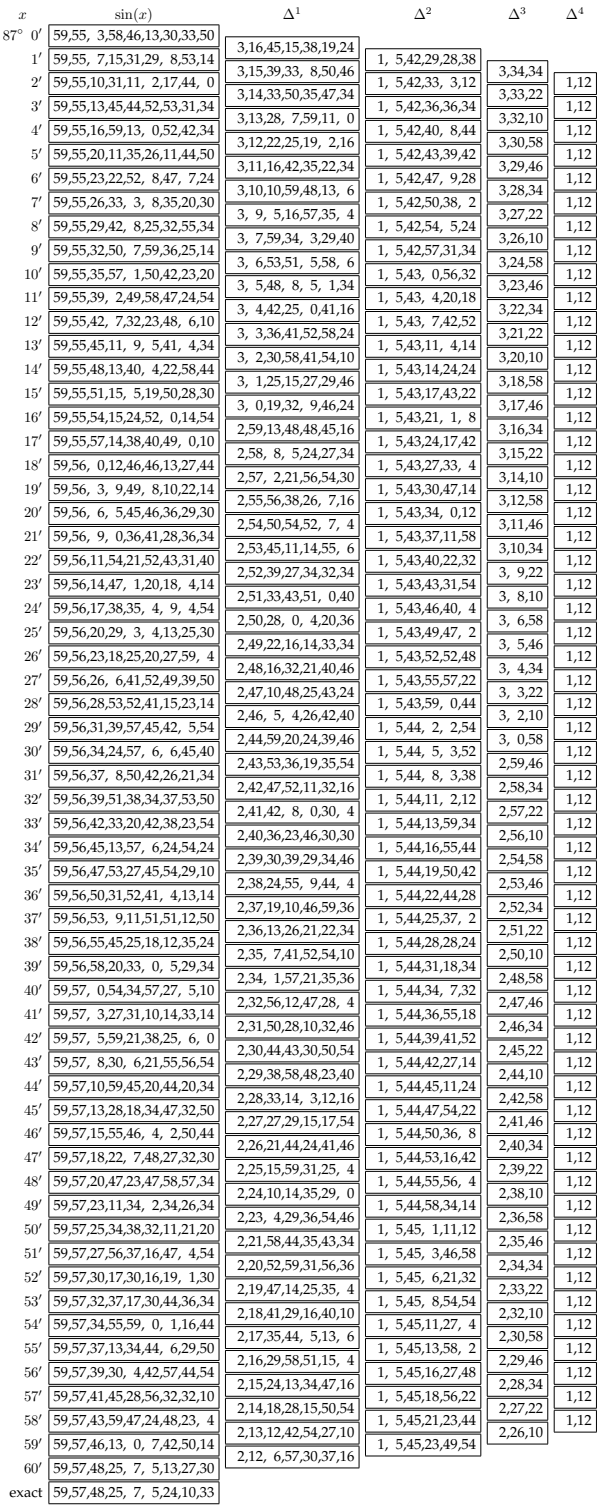
x	$\sin(x)$	Δ^1	Δ^2	Δ^3
87° 0′	59,55, 3,58,46,13,30,33,50, 7			
1′	59,55, 7,15,31,29, 8,53,13,50	3,16,45,15,38,19,23,43	1, 5,42,29,28,38, 6	3,34,33,52
2′	59,55,10,31,11, 2,17,43,59,27	3,15,39,33, 8,50,45,37	1, 5,42,33, 3,11,58	3,34,33,52
3′	59,55,13,45,44,52,53,31,33, 6	3,14,33,50,35,47,33,39	1, 5,42,36,37,45,50	3,34,33,52
4′	59,55,16,59,13, 0,52,41,20,55	3,13,28, 7,59, 9,47,49	1, 5,42,40,12,19,42	3,34,33,52
5′	59,55,20,11,35,26,11,38,49, 2	3,12,22,25,18,57,28, 7	1, 5,42,43,46,53,34	3,34,33,52
6′	59,55,23,22,52, 8,46,49,23,35	3,11,16,42,35,10,34,33	1, 5,42,47,21,27,26	3,34,33,52
7′	59,55,26,33, 3, 8,34,38,30,42	3,10,10,59,47,49, 7, 7	1, 5,42,50,56, 1,18	3,34,33,52
8′	59,55,29,42, 8,25,31,31,36,31	3, 9, 5,16,56,53, 5,49	1, 5,42,54,30,35,10	3,34,33,52
9′	59,55,32,50, 7,59,33,54, 7,10	3, 7,59,34, 2,22,30,39	1, 5,42,58, 5, 9, 2	3,34,33,52
10′	59,55,35,57, 1,50,38,11,28,47	3, 6,53,51, 4,17,21,37	1, 5,43, 1,39,42,54	3,34,33,52
11′	59,55,39, 2,49,58,40,49, 7,30	3, 5,48, 8, 2,37,38,43	1, 5,43, 5,14,16,46	3,34,33,52
12′	59,55,42, 7,32,23,38,12,29,27	3, 4,42,24,57,23,21,57	1, 5,43, 8,48,50,38	3,34,33,52
13′	59,55,45,11, 9, 5,26,47, 0,46	3, 3,36,41,48,34,31,19	1, 5,43,12,23,24,30	3,34,33,52
14′	59,55,48,13,40, 4, 2,58, 7,35	3, 2,30,58,36,11, 6,49	1, 5,43,15,57,58,22	3,34,33,52
15′	59,55,51,15, 5,19,23,11,16, 2	3, 1,25,15,20,13, 8,27	1, 5,43,19,32,32,14	3,34,33,52
16′	59,55,54,15,24,51,23,51,52,15	3, 0,19,32, 0,40,36,13	1, 5,43,23, 7, 6, 6	3,34,33,52
17′	59,55,57,14,38,40, 1,25,22,22	2,59,13,48,37,33,30, 7	1, 5,43,26,41,39,58	3,34,33,52
18′	59,56, 0,12,46,45,12,17,12,31	2,58, 8, 5,10,51,50, 9	1, 5,43,30,16,13,50	3,34,33,52
19′	59,56, 3, 9,49, 6,52,52,48,50	2,57, 2,21,40,35,36,19	1, 5,43,33,50,47,42	3,34,33,52
20′	59,56, 6, 5,45,44,59,37,37,27	2,55,56,38, 6,44,48,37	1, 5,43,37,25,21,34	3,34,33,52
21′	59,56, 9, 0,36,39,28,57, 4,30	2,54,50,54,29,19,27, 3	1, 5,43,40,59,55,26	3,34,33,52
22′	59,56,11,54,21,50,17,16,36, 7	2,53,45,10,48,19,31,37	1, 5,43,44,34,29,18	3,34,33,52
23′	59,56,14,47, 1,17,21, 1,38,26	2,52,39,27, 3,45, 2,19	1, 5,43,48, 9, 3,10	3,34,33,52
24′	59,56,17,38,35, 0,36,37,37,35	2,51,33,43,15,35,59, 9	1, 5,43,51,43,37, 2	3,34,33,52
25′	59,56,20,29, 3, 0, 0,29,59,42	2,50,27,59,23,52,22, 7	1, 5,43,55,18,10,54	3,34,33,52
26′	59,56,23,18,25,15,29, 4,10,55	2,49,22,15,28,34,11,13	1, 5,43,58,52,44,46	3,34,33,52
27′	59,56,26, 6,41,46,58,45,37,22	2,48,16,31,29,41,26,27	1, 5,44, 2,27,18,38	3,34,33,52
28′	59,56,28,53,52,34,25,59,45,11	2,47,10,47,27,14, 7,49	1, 5,44, 6, 1,52,30	3,34,33,52
29′	59,56,31,39,57,37,47,12, 0,30	2,46, 5, 3,21,12,15,19	1, 5,44, 9,36,26,22	3,34,33,52
30′	59,56,34,24,56,56,58,47,49,27	2,44,59,19,11,35,48,57	1, 5,44,13,11, 0,14	3,34,33,52
31′	59,56,37, 8,50,31,57,12,38,10	2,43,53,34,58,24,48,43	1, 5,44,16,45,34, 6	3,34,33,52
32′	59,56,39,51,38,22,38,51,52,47	2,42,47,50,41,39,14,37	1, 5,44,20,20, 7,58	3,34,33,52
33′	59,56,42,33,20,29, 0,10,59,26	2,41,42, 6,21,19, 6,39	1, 5,44,23,54,41,50	3,34,33,52
34′	59,56,45,13,56,50,57,35,24,15	2,40,36,21,57,24,24,49	1, 5,44,27,29,15,42	3,34,33,52
35′	59,56,47,53,27,28,27,30,33,22	2,39,30,37,29,55, 9, 7	1, 5,44,31, 3,49,34	3,34,33,52
36′	59,56,50,31,52,21,26,21,52,55	2,38,24,52,58,51,19,33	1, 5,44,34,38,23,26	3,34,33,52
37′	59,56,53, 9,11,29,50,34,49, 2	2,37,19, 8,24,12,56, 7	1, 5,44,38,12,57,18	3,34,33,52
38′	59,56,55,45,24,53,36,34,47,51	2,36,13,23,45,59,58,49	1, 5,44,41,47,31,10	3,34,33,52
39′	59,56,58,20,32,32,40,47,15,30	2,35, 7,39, 4,12,27,39	1, 5,44,45,22, 5, 2	3,34,33,52
40′	59,57, 0,54,34,26,59,37,38, 7	2,34, 1,54,18,50,22,37	1, 5,44,48,56,38,54	3,34,33,52
41′	59,57, 3,27,30,36,29,31,21,50	2,32,56, 9,29,53,43,43	1, 5,44,52,31,12,46	3,34,33,52
42′	59,57, 5,59,21, 1, 6,53,52,47	2,31,50,24,37,22,30,57	1, 5,44,56, 5,46,38	3,34,33,52
43′	59,57, 8,30, 5,40,48,10,37, 6	2,30,44,39,41,16,44,19	1, 5,44,59,40,20,30	3,34,33,52
44′	59,57,10,59,44,35,29,47, 0,55	2,29,38,54,41,36,23,49	1, 5,45, 3,14,54,22	3,34,33,52
45′	59,57,13,28,17,45, 8, 8,30,22	2,28,33, 9,38,21,29,27	1, 5,45, 6,49,28,14	3,34,33,52
46′	59,57,15,55,45, 9,39,40,31,35	2,27,27,24,31,32, 1,13	1, 5,45,10,24, 2, 6	3,34,33,52
47′	59,57,18,22, 6,49, 0,48,30,42	2,26,21,39,21, 7,59, 7	1, 5,45,13,58,35,58	3,34,33,52
48′	59,57,20,47,22,43, 7,57,53,51	2,25,15,54, 7, 9,23, 9	1, 5,45,17,33, 9,50	3,34,33,52
49′	59,57,23,11,32,51,57,34, 7,10	2,24,10, 8,49,36,13,19	1, 5,45,21, 7,43,42	3,34,33,52
50′	59,57,25,34,37,15,26, 2,36,47	2,23, 4,23,28,28,29,37	1, 5,45,24,42,17,34	3,34,33,52
51′	59,57,27,56,35,53,29,48,48,50	2,21,58,38, 3,46,12, 3	1, 5,45,28,16,51,26	3,34,33,52
52′	59,57,30,17,28,46, 5,18, 9,27	2,20,52,52,35,29,20,37	1, 5,45,31,51,25,18	3,34,33,52
53′	59,57,32,37,15,53, 8,56, 4,46	2,19,47, 7, 3,37,55,19	1, 5,45,35,25,59,10	3,34,33,52
54′	59,57,34,55,57,14,37, 8, 0,55	2,18,41,21,28,11,56, 9	1, 5,45,39, 0,33, 2	3,34,33,52
55′	59,57,37,13,32,50,26,19,24, 2	2,17,35,35,49,11,23, 7	1, 5,45,42,35, 6,54	3,34,33,52
56′	59,57,39,30, 2,40,32,55,40,15	2,16,29,50, 6,36,16,13	1, 5,45,46, 9,40,46	3,34,33,52
57′	59,57,41,45,26,44,53,22,15,42	2,15,24, 4,20,26,35,27	1, 5,45,49,44,14,38	3,34,33,52
58′	59,57,43,59,45, 3,24, 4,36,31	2,14,18,18,30,42,20,49	1, 5,45,53,18,48,30	3,34,33,52
59′	59,57,46,12,57,36, 1,28, 8,50	2,13,12,32,37,23,32,19	1, 5,45,56,53,22,22	3,34,33,52
60′	59,57,48,25, 4,22,41,58,18,47	2,12, 6,46,40,30, 9,57		
exact	59,57,48,25, 7, 5,24,10,32,58			

Figure 4: $1'$ interpolation from $87°$ to $88°$ with Δ^3 constant and 10 digits.



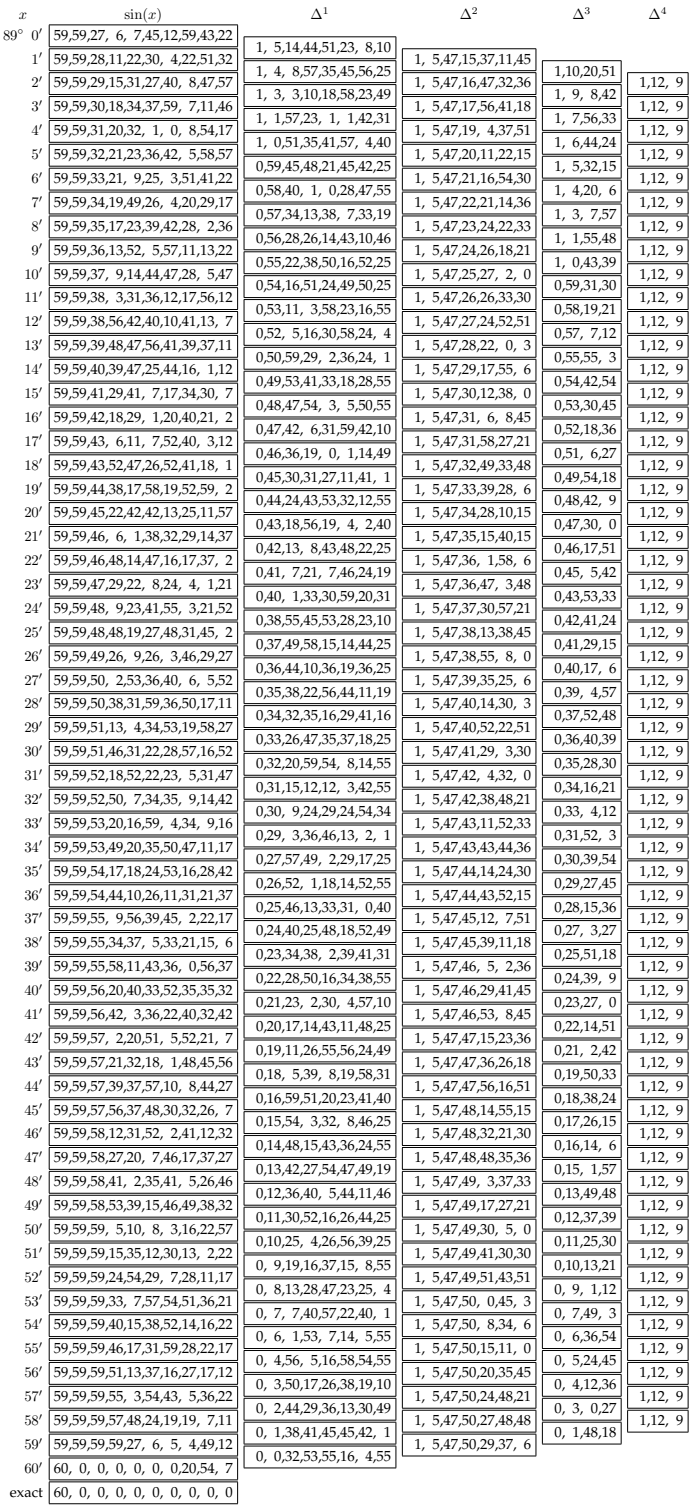
x	$\sin(x)$	Δ^1	Δ^2	Δ^3
89° 0′	59,59,27, 6, 7,45,12,59,43,22			
		1, 5,14,44,51,23, 8,10		
1′	59,59,28,11,22,30, 4,22,51,32		1, 5,47,15,37,11,45	
		1, 4, 8,57,35,45,56,25		1,10,20,51
2′	59,59,29,15,31,27,40, 8,47,57		1, 5,47,16,47,32,36	
		1, 3, 3,10,18,58,23,49		1,10,20,51
3′	59,59,30,18,34,37,59, 7,11,46		1, 5,47,17,57,53,27	
		1, 1,57,23, 1, 0,30,22		1,10,20,51
4′	59,59,31,20,32, 1, 0, 7,42, 8		1, 5,47,19, 8,14,18	
		1, 0,51,35,41,52,16, 4		1,10,20,51
5′	59,59,32,21,23,36,41,59,58,12		1, 5,47,20,18,35, 9	
		0,59,45,48,21,33,40,55		1,10,20,51
6′	59,59,33,21, 9,25, 3,33,39, 7		1, 5,47,21,28,56, 0	
		0,58,40, 1, 0, 4,44,55		1,10,20,51
7′	59,59,34,19,49,26, 3,38,24, 2		1, 5,47,22,39,16,51	
		0,57,34,13,37,25,28, 4		1,10,20,51
8′	59,59,35,17,23,39,41, 3,52, 6		1, 5,47,23,49,37,42	
		0,56,28,26,13,35,50,22		1,10,20,51
9′	59,59,36,13,52, 5,54,39,42,28		1, 5,47,24,59,58,33	
		0,55,22,38,48,35,51,49		1,10,20,51
10′	59,59,37, 9,14,44,43,15,34,17		1, 5,47,26,10,19,24	
		0,54,16,51,22,25,32,25		1,10,20,51
11′	59,59,38, 3,31,36, 5,41, 6,42		1, 5,47,27,20,40,15	
		0,53,11, 3,55, 4,52,10		1,10,20,51
12′	59,59,38,56,42,40, 0,45,58,52		1, 5,47,28,31, 1, 6	
		0,52, 5,16,26,33,51, 4		1,10,20,51
13′	59,59,39,48,47,56,27,19,49,56		1, 5,47,29,41,21,57	
		0,50,59,28,56,52,29, 7		1,10,20,51
14′	59,59,40,39,47,25,24,12,19, 3		1, 5,47,30,51,42,48	
		0,49,53,41,26, 0,46,19		1,10,20,51
15′	59,59,41,29,41, 6,50,13, 5,22		1, 5,47,32, 2, 3,39	
		0,48,47,53,53,58,42,40		1,10,20,51
16′	59,59,42,18,29, 0,44,11,48, 2		1, 5,47,33,12,24,30	
		0,47,42, 6,20,46,18,10		1,10,20,51
17′	59,59,43, 6,11, 7, 4,58, 6,12		1, 5,47,34,22,45,21	
		0,46,36,18,46,23,32,49		1,10,20,51
18′	59,59,43,52,47,25,51,21,39, 1		1, 5,47,35,33, 6,12	
		0,45,30,31,10,50,26,37		1,10,20,51
19′	59,59,44,38,17,57, 2,12, 5,38		1, 5,47,36,43,27, 3	
		0,44,24,43,34, 6,59,34		1,10,20,51
20′	59,59,45,22,42,40,36,19, 5,12		1, 5,47,37,53,47,54	
		0,43,18,55,56,13,11,40		1,10,20,51
21′	59,59,46, 6, 1,36,32,32,16,52		1, 5,47,39, 4, 8,45	
		0,42,13, 8,17, 9, 2,55		1,10,20,51
22′	59,59,46,48,14,44,49,41,19,47		1, 5,47,40,14,29,36	
		0,41, 7,20,36,54,33,19		1,10,20,51
23′	59,59,47,29,22, 5,26,35,53, 6		1, 5,47,41,24,50,27	
		0,40, 1,32,55,29,42,52		1,10,20,51
24′	59,59,48, 9,23,38,22, 5,35,58		1, 5,47,42,35,11,18	
		0,38,55,45,12,54,31,34		1,10,20,51
25′	59,59,48,48,19,23,35, 0, 7,32		1, 5,47,43,45,32, 9	
		0,37,49,57,29, 8,59,25		1,10,20,51
26′	59,59,49,26, 9,21, 4, 9, 6,57		1, 5,47,44,55,53, 0	
		0,36,44, 9,44,13, 6,25		1,10,20,51
27′	59,59,50, 2,53,30,48,22,13,22		1, 5,47,46, 6,13,51	
		0,35,38,21,58, 6,52,34		1,10,20,51
28′	59,59,50,38,31,52,46,29, 5,56		1, 5,47,47,16,34,42	
		0,34,32,34,10,50,17,52		1,10,20,51
29′	59,59,51,13, 4,26,57,19,23,48		1, 5,47,48,26,55,33	
		0,33,26,46,22,23,22,19		1,10,20,51
30′	59,59,51,46,31,13,19,42,46, 7		1, 5,47,49,37,16,24	
		0,32,20,58,32,46, 5,55		1,10,20,51
31′	59,59,52,18,52,11,52,28,52, 2		1, 5,47,50,47,37,15	
		0,31,15,10,41,58,28,40		1,10,20,51
32′	59,59,52,50, 7,22,34,27,20,42		1, 5,47,51,57,58, 6	
		0,30, 9,22,50, 0,30,34		1,10,20,51
33′	59,59,53,20,16,45,24,27,51,16		1, 5,47,53, 8,18,57	
		0,29, 3,34,56,52,11,37		1,10,20,51
34′	59,59,53,49,20,20,21,20, 2,53		1, 5,47,54,18,39,48	
		0,27,57,47, 2,33,31,49		1,10,20,51
35′	59,59,54,17,18, 7,23,53,34,42		1, 5,47,55,29, 0,39	
		0,26,51,59, 7, 4,31,10		1,10,20,51
36′	59,59,54,44,10, 6,30,58, 5,52		1, 5,47,56,39,21,30	
		0,25,46,11,10,25, 9,40		1,10,20,51
37′	59,59,55, 9,56,17,41,23,15,32		1, 5,47,57,49,42,21	
		0,24,40,23,12,35,27,19		1,10,20,51
38′	59,59,55,34,36,40,53,58,42,51		1, 5,47,59, 0, 3,12	
		0,23,34,35,13,35,24, 7		1,10,20,51
39′	59,59,55,58,11,16, 7,34, 6,58		1, 5,48, 0,10,24, 3	
		0,22,28,47,13,25, 0, 4		1,10,20,51
40′	59,59,56,20,40, 3,20,59, 7, 2		1, 5,48, 1,20,44,54	
		0,21,22,59,12, 4,15,10		1,10,20,51
41′	59,59,56,42, 3, 2,33, 3,22,12		1, 5,48, 2,31, 5,45	
		0,20,17,11, 9,33, 9,25		1,10,20,51
42′	59,59,57, 2,20,13,42,36,31,37		1, 5,48, 3,41,26,36	
		0,19,11,23, 5,51,42,49		1,10,20,51
43′	59,59,57,21,31,36,48,28,14,26		1, 5,48, 4,51,47,27	
		0,18, 5,35, 0,59,55,22		1,10,20,51
44′	59,59,57,39,37,11,49,28, 9,48		1, 5,48, 6, 2, 8,18	
		0,16,59,46,54,57,47, 4		1,10,20,51
45′	59,59,57,56,36,58,44,25,56,52		1, 5,48, 7,12,29, 9	
		0,15,53,58,47,45,17,55		1,10,20,51
46′	59,59,58,12,30,57,32,11,14,47		1, 5,48, 8,22,50, 0	
		0,14,48,10,39,22,27,55		1,10,20,51
47′	59,59,58,27,19, 8,11,33,42,42		1, 5,48, 9,33,10,51	
		0,13,42,22,29,49,17, 4		1,10,20,51
48′	59,59,58,41, 1,30,41,22,59,46		1, 5,48,10,43,31,42	
		0,12,36,34,19, 5,45,22		1,10,20,51
49′	59,59,58,53,38, 5, 0,28,45, 8		1, 5,48,11,53,52,33	
		0,11,30,46, 7,11,52,49		1,10,20,51
50′	59,59,59, 5, 8,51, 7,40,37,57		1, 5,48,13, 4,13,24	
		0,10,24,57,54, 7,39,25		1,10,20,51
51′	59,59,59,15,33,49, 1,48,17,22		1, 5,48,14,14,34,15	
		0, 9,19, 9,39,53, 5,10		1,10,20,51
52′	59,59,59,24,52,58,41,41,22,32		1, 5,48,15,24,55, 6	
		0, 8,13,21,24,28,10, 4		1,10,20,51
53′	59,59,59,33, 6,20, 6, 9,32,36		1, 5,48,16,35,15,57	
		0, 7, 7,33, 7,52,54, 7		1,10,20,51
54′	59,59,59,40,13,53,14, 2,26,43		1, 5,48,17,45,36,48	
		0, 6, 1,44,50, 7,17,19		1,10,20,51
55′	59,59,59,46,15,38, 4, 9,44, 2		1, 5,48,18,55,57,39	
		0, 4,55,56,31,11,19,40		1,10,20,51
56′	59,59,59,51,11,34,35,21, 3,42		1, 5,48,20, 6,18,30	
		0, 3,50, 8,11, 5, 1,10		1,10,20,51
57′	59,59,59,55, 1,42,46,26, 4,52		1, 5,48,21,16,39,21	
		0, 2,44,19,49,48,21,49		1,10,20,51
58′	59,59,59,57,46, 2,36,14,26,41		1, 5,48,22,27, 0,12	
		0, 1,38,31,27,21,21,37		1,10,20,51
59′	59,59,59,59,24,34, 3,35,48,18		1, 5,48,23,37,21, 3	
		0, 0,32,43, 3,44, 0,34		
60′	59,59,59,59,57,17, 7,19,48,52			
exact	60, 0, 0, 0, 0, 0, 0, 0, 0, 0			

Figure 5: $1'$ interpolation from $89°$ to $90°$ with Δ^3 constant and 10 digits.



x	$\sin(x)$	Δ^1	Δ^2	Δ^3	Δ^4
87° 0′	59,55, 3,58,46,13,30,33,50				
		3,16,45,15,38,19,24			
1′	59,55, 7,15,31,29, 8,53,14		1, 5,42,29,28,38		
		3,15,39,33, 8,50,46		3,34,34	
2′	59,55,10,31,11, 2,17,44, 0		1, 5,42,33, 3,12		1,12
		3,14,33,50,35,47,34		3,33,22	
3′	59,55,13,45,44,52,53,31,34		1, 5,42,36,36,34		1,12
		3,13,28, 7,59,11, 0		3,32,10	
4′	59,55,16,59,13, 0,52,42,34		1, 5,42,40, 8,44		1,12
		3,12,22,25,19, 2,16		3,30,58	
5′	59,55,20,11,35,26,11,44,50		1, 5,42,43,39,42		1,12
		3,11,16,42,35,22,34		3,29,46	
6′	59,55,23,22,52, 8,47, 7,24		1, 5,42,47, 9,28		1,12
		3,10,10,59,48,13, 6		3,28,34	
7′	59,55,26,33, 3, 8,35,20,30		1, 5,42,50,38, 2		1,12
		3, 9, 5,16,57,35, 4		3,27,22	
8′	59,55,29,42, 8,25,32,55,34		1, 5,42,54, 5,24		1,12
		3, 7,59,34, 3,29,40		3,26,10	
9′	59,55,32,50, 7,59,36,25,14		1, 5,42,57,31,34		1,12
		3, 6,53,51, 5,58, 6		3,24,58	
10′	59,55,35,57, 1,50,42,23,20		1, 5,43, 0,56,32		1,12
		3, 5,48, 8, 5, 1,34		3,23,46	
11′	59,55,39, 2,49,58,47,24,54		1, 5,43, 4,20,18		1,12
		3, 4,42,25, 0,41,16		3,22,34	
12′	59,55,42, 7,32,23,48, 6,10		1, 5,43, 7,42,52		1,12
		3, 3,36,41,52,58,24		3,21,22	
13′	59,55,45,11, 9, 5,41, 4,34		1, 5,43,11, 4,14		1,12
		3, 2,30,58,41,54,10		3,20,10	
14′	59,55,48,13,40, 4,22,58,44		1, 5,43,14,24,24		1,12
		3, 1,25,15,27,29,46		3,18,58	
15′	59,55,51,15, 5,19,50,28,30		1, 5,43,17,43,22		1,12
		3, 0,19,32, 9,46,24		3,17,46	
16′	59,55,54,15,24,52, 0,14,54		1, 5,43,21, 1, 8		1,12
		2,59,13,48,48,45,16		3,16,34	
17′	59,55,57,14,38,40,49, 0,10		1, 5,43,24,17,42		1,12
		2,58, 8, 5,24,27,34		3,15,22	
18′	59,56, 0,12,46,46,13,27,44		1, 5,43,27,33, 4		1,12
		2,57, 2,21,56,54,30		3,14,10	
19′	59,56, 3, 9,49, 8,10,22,14		1, 5,43,30,47,14		1,12
		2,55,56,38,26, 7,16		3,12,58	
20′	59,56, 6, 5,45,46,36,29,30		1, 5,43,34, 0,12		1,12
		2,54,50,54,52, 7, 4		3,11,46	
21′	59,56, 9, 0,36,41,28,36,34		1, 5,43,37,11,58		1,12
		2,53,45,11,14,55, 6		3,10,34	
22′	59,56,11,54,21,52,43,31,40		1, 5,43,40,22,32		1,12
		2,52,39,27,34,32,34		3, 9,22	
23′	59,56,14,47, 1,20,18, 4,14		1, 5,43,43,31,54		1,12
		2,51,33,43,51, 0,40		3, 8,10	
24′	59,56,17,38,35, 4, 9, 4,54		1, 5,43,46,40, 4		1,12
		2,50,28, 0, 4,20,36		3, 6,58	
25′	59,56,20,29, 3, 4,13,25,30		1, 5,43,49,47, 2		1,12
		2,49,22,16,14,33,34		3, 5,46	
26′	59,56,23,18,25,20,27,59, 4		1, 5,43,52,52,48		1,12
		2,48,16,32,21,40,46		3, 4,34	
27′	59,56,26, 6,41,52,49,39,50		1, 5,43,55,57,22		1,12
		2,47,10,48,25,43,24		3, 3,22	
28′	59,56,28,53,52,41,15,23,14		1, 5,43,59, 0,44		1,12
		2,46, 5, 4,26,42,40		3, 2,10	
29′	59,56,31,39,57,45,42, 5,54		1, 5,44, 2, 2,54		1,12
		2,44,59,20,24,39,46		3, 0,58	
30′	59,56,34,24,57, 6, 6,45,40		1, 5,44, 5, 3,52		1,12
		2,43,53,36,19,35,54		2,59,46	
31′	59,56,37, 8,50,42,26,21,34		1, 5,44, 8, 3,38		1,12
		2,42,47,52,11,32,16		2,58,34	
32′	59,56,39,51,38,34,37,53,50		1, 5,44,11, 2,12		1,12
		2,41,42, 8, 0,30, 4		2,57,22	
33′	59,56,42,33,20,42,38,23,54		1, 5,44,13,59,34		1,12
		2,40,36,23,46,30,30		2,56,10	
34′	59,56,45,13,57, 6,24,54,24		1, 5,44,16,55,44		1,12
		2,39,30,39,29,34,46		2,54,58	
35′	59,56,47,53,27,45,54,29,10		1, 5,44,19,50,42		1,12
		2,38,24,55, 9,44, 4		2,53,46	
36′	59,56,50,31,52,41, 4,13,14		1, 5,44,22,44,28		1,12
		2,37,19,10,46,59,36		2,52,34	
37′	59,56,53, 9,11,51,51,12,50		1, 5,44,25,37, 2		1,12
		2,36,13,26,21,22,34		2,51,22	
38′	59,56,55,45,25,18,12,35,24		1, 5,44,28,28,24		1,12
		2,35, 7,41,52,54,10		2,50,10	
39′	59,56,58,20,33, 0, 5,29,34		1, 5,44,31,18,34		1,12
		2,34, 1,57,21,35,36		2,48,58	
40′	59,57, 0,54,34,57,27, 5,10		1, 5,44,34, 7,32		1,12
		2,32,56,12,47,28, 4		2,47,46	
41′	59,57, 3,27,31,10,14,33,14		1, 5,44,36,55,18		1,12
		2,31,50,28,10,32,46		2,46,34	
42′	59,57, 5,59,21,38,25, 6, 0		1, 5,44,39,41,52		1,12
		2,30,44,43,30,50,54		2,45,22	
43′	59,57, 8,30, 6,21,55,56,54		1, 5,44,42,27,14		1,12
		2,29,38,58,48,23,40		2,44,10	
44′	59,57,10,59,45,20,44,20,34		1, 5,44,45,11,24		1,12
		2,28,33,14, 3,12,16		2,42,58	
45′	59,57,13,28,18,34,47,32,50		1, 5,44,47,54,22		1,12
		2,27,27,29,15,17,54		2,41,46	
46′	59,57,15,55,46, 4, 2,50,44		1, 5,44,50,36, 8		1,12
		2,26,21,44,24,41,46		2,40,34	
47′	59,57,18,22, 7,48,27,32,30		1, 5,44,53,16,42		1,12
		2,25,15,59,31,25, 4		2,39,22	
48′	59,57,20,47,23,47,58,57,34		1, 5,44,55,56, 4		1,12
		2,24,10,14,35,29, 0		2,38,10	
49′	59,57,23,11,34, 2,34,26,34		1, 5,44,58,34,14		1,12
		2,23, 4,29,36,54,46		2,36,58	
50′	59,57,25,34,38,32,11,21,20		1, 5,45, 1,11,12		1,12
		2,21,58,44,35,43,34		2,35,46	
51′	59,57,27,56,37,16,47, 4,54		1, 5,45, 3,46,58		1,12
		2,20,52,59,31,56,36		2,34,34	
52′	59,57,30,17,30,16,19, 1,30		1, 5,45, 6,21,32		1,12
		2,19,47,14,25,35, 4		2,33,22	
53′	59,57,32,37,17,30,44,36,34		1, 5,45, 8,54,54		1,12
		2,18,41,29,16,40,10		2,32,10	
54′	59,57,34,55,59, 0, 1,16,44		1, 5,45,11,27, 4		1,12
		2,17,35,44, 5,13, 6		2,30,58	
55′	59,57,37,13,34,44, 6,29,50		1, 5,45,13,58, 2		1,12
		2,16,29,58,51,15, 4		2,29,46	
56′	59,57,39,30, 4,42,57,44,54		1, 5,45,16,27,48		1,12
		2,15,24,13,34,47,16		2,28,34	
57′	59,57,41,45,28,56,32,32,10		1, 5,45,18,56,22		1,12
		2,14,18,28,15,50,54		2,27,22	
58′	59,57,43,59,47,24,48,23, 4		1, 5,45,21,23,44		1,12
		2,13,12,42,54,27,10		2,26,10	
59′	59,57,46,13, 0, 7,42,50,14		1, 5,45,23,49,54		
		2,12, 6,57,30,37,16			
60′	59,57,48,25, 7, 5,13,27,30				
exact	59,57,48,25, 7, 5,24,10,33				

Figure 6: $1'$ interpolation from $87°$ to $88°$ with Δ^4 constant and 9 digits.



x	$\sin(x)$	Δ^1	Δ^2	Δ^3	Δ^4
89° 0′	59,59,27, 6, 7,45,12,59,43,22	1, 5,14,44,51,23, 8,10	1, 5,47,15,37,11,45		
1′	59,59,28,11,22,30, 4,22,51,32	1, 4, 8,57,35,45,56,25	1, 5,47,16,47,32,36	1,10,20,51	
2′	59,59,29,15,31,27,40, 8,47,57	1, 3, 3,10,18,58,23,49	1, 5,47,17,56,41,18	1, 9, 8,42	1,12, 9
3′	59,59,30,18,34,37,59, 7,11,46	1, 1,57,23, 1, 1,42,31	1, 5,47,19, 4,37,51	1, 7,56,33	1,12, 9
4′	59,59,31,20,32, 1, 0, 8,54,17	1, 0,51,35,41,57, 4,40	1, 5,47,20,11,22,15	1, 6,44,24	1,12, 9
5′	59,59,32,21,23,36,42, 5,58,57	0,59,45,48,21,45,42,25	1, 5,47,21,16,54,30	1, 5,32,15	1,12, 9
6′	59,59,33,21, 9,25, 3,51,41,22	0,58,40, 1, 0,28,47,55	1, 5,47,22,21,14,36	1, 4,20, 6	1,12, 9
7′	59,59,34,19,49,26, 4,20,29,17	0,57,34,13,38, 7,33,19	1, 5,47,23,24,22,33	1, 3, 7,57	1,12, 9
8′	59,59,35,17,23,39,42,28, 2,36	0,56,28,26,14,43,10,46	1, 5,47,24,26,18,21	1, 1,55,48	1,12, 9
9′	59,59,36,13,52, 5,57,11,13,22	0,55,22,38,50,16,52,25	1, 5,47,25,27, 2, 0	1, 0,43,39	1,12, 9
10′	59,59,37, 9,14,44,47,28, 5,47	0,54,16,51,24,49,50,25	1, 5,47,26,26,33,30	0,59,31,30	1,12, 9
11′	59,59,38, 3,31,36,12,17,56,12	0,53,11, 3,58,23,16,55	1, 5,47,27,24,52,51	0,58,19,21	1,12, 9
12′	59,59,38,56,42,40,10,41,13, 7	0,52, 5,16,30,58,24, 4	1, 5,47,28,22, 0, 3	0,57, 7,12	1,12, 9
13′	59,59,39,48,47,56,41,39,37,11	0,50,59,29, 2,36,24, 1	1, 5,47,29,17,55, 6	0,55,55, 3	1,12, 9
14′	59,59,40,39,47,25,44,16, 1,12	0,49,53,41,33,18,28,55	1, 5,47,30,12,38, 0	0,54,42,54	1,12, 9
15′	59,59,41,29,41, 7,17,34,30, 7	0,48,47,54, 3, 5,50,55	1, 5,47,31, 6, 8,45	0,53,30,45	1,12, 9
16′	59,59,42,18,29, 1,20,40,21, 2	0,47,42, 6,31,59,42,10	1, 5,47,31,58,27,21	0,52,18,36	1,12, 9
17′	59,59,43, 6,11, 7,52,40, 3,12	0,46,36,19, 0, 1,14,49	1, 5,47,32,49,33,48	0,51, 6,27	1,12, 9
18′	59,59,43,52,47,26,52,41,18, 1	0,45,30,31,27,11,41, 1	1, 5,47,33,39,28, 6	0,49,54,18	1,12, 9
19′	59,59,44,38,17,58,19,52,59, 2	0,44,24,43,53,32,12,55	1, 5,47,34,28,10,15	0,48,42, 9	1,12, 9
20′	59,59,45,22,42,42,13,25,11,57	0,43,18,56,19, 4, 2,40	1, 5,47,35,15,40,15	0,47,30, 0	1,12, 9
21′	59,59,46, 6, 1,38,32,29,14,37	0,42,13, 8,43,48,22,25	1, 5,47,36, 1,58, 6	0,46,17,51	1,12, 9
22′	59,59,46,48,14,47,16,17,37, 2	0,41, 7,21, 7,46,24,19	1, 5,47,36,47, 3,48	0,45, 5,42	1,12, 9
23′	59,59,47,29,22, 8,24, 4, 1,21	0,40, 1,33,30,59,20,31	1, 5,47,37,30,57,21	0,43,53,33	1,12, 9
24′	59,59,48, 9,23,41,55, 3,21,52	0,38,55,45,53,28,23,10	1, 5,47,38,13,38,45	0,42,41,24	1,12, 9
25′	59,59,48,48,19,27,48,31,45, 2	0,37,49,58,15,14,44,25	1, 5,47,38,55, 8, 0	0,41,29,15	1,12, 9
26′	59,59,49,26, 9,26, 3,46,29,27	0,36,44,10,36,19,36,25	1, 5,47,39,35,25, 6	0,40,17, 6	1,12, 9
27′	59,59,50, 2,53,36,40, 6, 5,52	0,35,38,22,56,44,11,19	1, 5,47,40,14,30, 3	0,39, 4,57	1,12, 9
28′	59,59,50,38,31,59,36,50,17,11	0,34,32,35,16,29,41,16	1, 5,47,40,52,22,51	0,37,52,48	1,12, 9
29′	59,59,51,13, 4,34,53,19,58,27	0,33,26,47,35,37,18,25	1, 5,47,41,29, 3,30	0,36,40,39	1,12, 9
30′	59,59,51,46,31,22,28,57,16,52	0,32,20,59,54, 8,14,55	1, 5,47,42, 4,32, 0	0,35,28,30	1,12, 9
31′	59,59,52,18,52,22,23, 5,31,47	0,31,15,12,12, 3,42,55	1, 5,47,42,38,48,21	0,34,16,21	1,12, 9
32′	59,59,52,50, 7,34,35, 9,14,42	0,30, 9,24,29,24,54,34	1, 5,47,43,11,52,33	0,33, 4,12	1,12, 9
33′	59,59,53,20,16,59, 4,34, 9,16	0,29, 3,36,46,13, 2, 1	1, 5,47,43,43,44,36	0,31,52, 3	1,12, 9
34′	59,59,53,49,20,35,50,47,11,17	0,27,57,49, 2,29,17,25	1, 5,47,44,14,24,30	0,30,39,54	1,12, 9
35′	59,59,54,17,18,24,53,16,28,42	0,26,52, 1,18,14,52,55	1, 5,47,44,43,52,15	0,29,27,45	1,12, 9
36′	59,59,54,44,10,26,11,31,21,37	0,25,46,13,33,31, 0,40	1, 5,47,45,12, 7,51	0,28,15,36	1,12, 9
37′	59,59,55, 9,56,39,45, 2,22,17	0,24,40,25,48,18,52,49	1, 5,47,45,39,11,18	0,27, 3,27	1,12, 9
38′	59,59,55,34,37, 5,33,21,15, 6	0,23,34,38, 2,39,41,31	1, 5,47,46, 5, 2,36	0,25,51,18	1,12, 9
39′	59,59,55,58,11,43,36, 0,56,37	0,22,28,50,16,34,38,55	1, 5,47,46,29,41,45	0,24,39, 9	1,12, 9
40′	59,59,56,20,40,33,52,35,35,32	0,21,23, 2,30, 4,57,10	1, 5,47,46,53, 8,45	0,23,27, 0	1,12, 9
41′	59,59,56,42, 3,36,22,40,32,42	0,20,17,14,43,11,48,25	1, 5,47,47,15,23,36	0,22,14,51	1,12, 9
42′	59,59,57, 2,20,51, 5,52,21, 7	0,19,11,26,55,56,24,49	1, 5,47,47,36,26,18	0,21, 2,42	1,12, 9
43′	59,59,57,21,32,18, 1,48,45,56	0,18, 5,39, 8,19,58,31	1, 5,47,47,56,16,51	0,19,50,33	1,12, 9
44′	59,59,57,39,37,57,10, 8,44,27	0,16,59,51,20,23,41,40	1, 5,47,48,14,55,15	0,18,38,24	1,12, 9
45′	59,59,57,56,37,48,30,32,26, 7	0,15,54, 3,32, 8,46,25	1, 5,47,48,32,21,30	0,17,26,15	1,12, 9
46′	59,59,58,12,31,52, 2,41,12,32	0,14,48,15,43,36,24,55	1, 5,47,48,48,35,36	0,16,14, 6	1,12, 9
47′	59,59,58,27,20, 7,46,17,37,27	0,13,42,27,54,47,49,19	1, 5,47,49, 3,37,33	0,15, 1,57	1,12, 9
48′	59,59,58,41, 2,35,41, 5,26,46	0,12,36,40, 5,44,11,46	1, 5,47,49,17,27,21	0,13,49,48	1,12, 9
49′	59,59,58,53,39,15,46,49,38,32	0,11,30,52,16,26,44,25	1, 5,47,49,30, 5, 0	0,12,37,39	1,12, 9
50′	59,59,59, 5,10, 8, 3,16,22,57	0,10,25, 4,26,56,39,25	1, 5,47,49,41,30,30	0,11,25,30	1,12, 9
51′	59,59,59,15,35,12,30,13, 2,22	0, 9,19,16,37,15, 8,55	1, 5,47,49,51,43,51	0,10,13,21	1,12, 9
52′	59,59,59,24,54,29, 7,28,11,17	0, 8,13,28,47,23,25, 4	1, 5,47,50, 0,45, 3	0, 9, 1,12	1,12, 9
53′	59,59,59,33, 7,57,54,51,36,21	0, 7, 7,40,57,22,40, 1	1, 5,47,50, 8,34, 6	0, 7,49, 3	1,12, 9
54′	59,59,59,40,15,38,52,14,16,22	0, 6, 1,53, 7,14, 5,55	1, 5,47,50,15,11, 0	0, 6,36,54	1,12, 9
55′	59,59,59,46,17,31,59,28,22,17	0, 4,56, 5,16,58,54,55	1, 5,47,50,20,35,45	0, 5,24,45	1,12, 9
56′	59,59,59,51,13,37,16,27,17,12	0, 3,50,17,26,38,19,10	1, 5,47,50,24,48,21	0, 4,12,36	1,12, 9
57′	59,59,59,55, 3,54,43, 5,36,22	0, 2,44,29,36,13,30,49	1, 5,47,50,27,48,48	0, 3, 0,27	1,12, 9
58′	59,59,59,57,48,24,19,19, 7,11	0, 1,38,41,45,45,42, 1	1, 5,47,50,29,37, 6	0, 1,48,18	
59′	59,59,59,59,27, 6, 5, 4,49,12	0, 0,32,53,55,16, 4,55			
60′	60, 0, 0, 0, 0, 0, 0,20,54, 7				
exact	60, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0				

Figure 7: $1'$ interpolation from $89°$ to $90°$ with Δ^4 constant and 10 digits.

x	$\sin(x)$	Δ^1	Δ^2	Δ^3
$0^\circ\ 0'\ 0''$	0, 0, 0, 0, 0, 0, 0, 0, 0, 0	2, 5,39,49,20,31,35,44		
$2''$	0, 0, 2, 5,39,49,20,31,35,44	2, 5,39,49,20,31,26,33	0, 9,11	9,11
$4''$	0, 0, 4,11,19,38,41, 3, 2,17	2, 5,39,49,20,31, 8,11	0,18,22	9,11
$6''$	0, 0, 6,16,59,28, 1,34,10,28	2, 5,39,49,20,30,40,38	0,27,33	9,11
$8''$	0, 0, 8,22,39,17,22, 4,51, 6	2, 5,39,49,20,30, 3,54	0,36,44	9,11
$10''$	0, 0,10,28,19, 6,42,34,55, 0	2, 5,39,49,20,29,17,59	0,45,55	9,11
$12''$	0, 0,12,33,58,56, 3, 4,12,59	2, 5,39,49,20,28,22,53	0,55, 6	9,11
$14''$	0, 0,14,39,38,45,23,32,35,52	2, 5,39,49,20,27,18,36	1, 4,17	9,11
$16''$	0, 0,16,45,18,34,43,59,54,28	2, 5,39,49,20,26, 5, 8	1,13,28	9,11
$18''$	0, 0,18,50,58,24, 4,25,59,36	2, 5,39,49,20,24,42,29	1,22,39	9,11
$20''$	0, 0,20,56,38,13,24,50,42, 5	2, 5,39,49,20,23,10,39	1,31,50	9,11
$22''$	0, 0,23, 2,18, 2,45,13,52,44	2, 5,39,49,20,21,29,38	1,41, 1	9,11
$24''$	0, 0,25, 7,57,52, 5,35,22,22	2, 5,39,49,20,19,39,26	1,50,12	9,11
$26''$	0, 0,27,13,37,41,25,55, 1,48	2, 5,39,49,20,17,40, 3	1,59,23	9,11
$28''$	0, 0,29,19,17,30,46,12,41,51	2, 5,39,49,20,15,31,29	2, 8,34	9,11
$30''$	0, 0,31,24,57,20, 6,28,13,20	2, 5,39,49,20,13,13,44	2,17,45	9,11
$32''$	0, 0,33,30,37, 9,26,41,27, 4	2, 5,39,49,20,10,46,48	2,26,56	9,11
$34''$	0, 0,35,36,16,58,46,52,13,52	2, 5,39,49,20, 8,10,41	2,36, 7	9,11
$36''$	0, 0,37,41,56,48, 7, 0,24,33	2, 5,39,49,20, 5,25,23	2,45,18	9,11
$38''$	0, 0,39,47,36,37,27, 5,49,56	2, 5,39,49,20, 2,30,54	2,54,29	9,11
$40''$	0, 0,41,53,16,26,47, 8,20,50	2, 5,39,49,19,59,27,14	3, 3,40	9,11
$42''$	0, 0,43,58,56,16, 7, 7,48, 4	2, 5,39,49,19,56,14,23	3,12,51	9,11
$44''$	0, 0,46, 4,36, 5,27, 4, 2,27	2, 5,39,49,19,52,52,21	3,22, 2	9,11
$46''$	0, 0,48,10,15,54,46,56,54,48	2, 5,39,49,19,49,21, 8	3,31,13	9,11
$48''$	0, 0,50,15,55,44, 6,46,15,56	2, 5,39,49,19,45,40,44	3,40,24	9,11
$50''$	0, 0,52,21,35,33,26,31,56,40	2, 5,39,49,19,41,51, 9	3,49,35	9,11
$52''$	0, 0,54,27,15,22,46,13,47,49	2, 5,39,49,19,37,52,23	3,58,46	9,11
$54''$	0, 0,56,32,55,12, 5,51,40,12	2, 5,39,49,19,33,44,26	4, 7,57	9,11
$56''$	0, 0,58,38,35, 1,25,25,24,38	2, 5,39,49,19,29,27,18	4,17, 8	9,11
$58''$	0, 1, 0,44,14,50,44,54,51,56	2, 5,39,49,19,25, 0,59	4,26,19	
$60''$	0, 1, 2,49,54,40, 4,19,52,55			
exact	0, 1, 2,49,54,40, 4,19,36, 0			

Figure 8: $2''$ interpolation from $0^\circ 0'$ to $0^\circ 1'$, with Δ^3 constant and 10 digits.

x	$\sin(x)$	Δ^1	Δ^2	Δ^3
$89°59'\ 0''$	59,59,59,59,27, 6, 4,44,53			
		2, 9,24, 5,19		
$2''$	59,59,59,59,29,15,28,50,12		4,23,11,22	
		2, 5, 0,53,57		0
$4''$	59,59,59,59,31,20,29,44, 9		4,23,11,22	
		2, 0,37,42,35		0
$6''$	59,59,59,59,33,21, 7,26,44		4,23,11,22	
		1,56,14,31,13		0
$8''$	59,59,59,59,35,17,21,57,57		4,23,11,22	
		1,51,51,19,51		0
$10''$	59,59,59,59,37, 9,13,17,48		4,23,11,22	
		1,47,28, 8,29		0
$12''$	59,59,59,59,38,56,41,26,17		4,23,11,22	
		1,43, 4,57, 7		0
$14''$	59,59,59,59,40,39,46,23,24		4,23,11,22	
		1,38,41,45,45		0
$16''$	59,59,59,59,42,18,28, 9, 9		4,23,11,22	
		1,34,18,34,23		0
$18''$	59,59,59,59,43,52,46,43,32		4,23,11,22	
		1,29,55,23, 1		0
$20''$	59,59,59,59,45,22,42, 6,33		4,23,11,22	
		1,25,32,11,39		0
$22''$	59,59,59,59,46,48,14,18,12		4,23,11,22	
		1,21, 9, 0,17		0
$24''$	59,59,59,59,48, 9,23,18,29		4,23,11,22	
		1,16,45,48,55		0
$26''$	59,59,59,59,49,26, 9, 7,24		4,23,11,22	
		1,12,22,37,33		0
$28''$	59,59,59,59,50,38,31,44,57		4,23,11,22	
		1, 7,59,26,11		0
$30''$	59,59,59,59,51,46,31,11, 8		4,23,11,22	
		1, 3,36,14,49		0
$32''$	59,59,59,59,52,50, 7,25,57		4,23,11,22	
		0,59,13, 3,27		0
$34''$	59,59,59,59,53,49,20,29,24		4,23,11,22	
		0,54,49,52, 5		0
$36''$	59,59,59,59,54,44,10,21,29		4,23,11,22	
		0,50,26,40,43		0
$38''$	59,59,59,59,55,34,37, 2,12		4,23,11,22	
		0,46, 3,29,21		0
$40''$	59,59,59,59,56,20,40,31,33		4,23,11,22	
		0,41,40,17,59		0
$42''$	59,59,59,59,57, 2,20,49,32		4,23,11,22	
		0,37,17, 6,37		0
$44''$	59,59,59,59,57,39,37,56, 9		4,23,11,22	
		0,32,53,55,15		0
$46''$	59,59,59,59,58,12,31,51,24		4,23,11,22	
		0,28,30,43,53		0
$48''$	59,59,59,59,58,41, 2,35,17		4,23,11,22	
		0,24, 7,32,31		0
$50''$	59,59,59,59,59, 5,10, 7,48		4,23,11,22	
		0,19,44,21, 9		0
$52''$	59,59,59,59,59,24,54,28,57		4,23,11,22	
		0,15,21, 9,47		0
$54''$	59,59,59,59,59,40,15,38,44		4,23,11,22	
		0,10,57,58,25		0
$56''$	59,59,59,59,59,51,13,37, 9		4,23,11,22	
		0, 6,34,47, 3		0
$58''$	59,59,59,59,59,57,48,24,12		4,23,11,22	
		0, 2,11,35,41		
$60''$	59,59,59,59,59,59,59,59,53			
exact	60, 0, 0, 0, 0, 0, 0, 0, 0			

Figure 9: $2''$ interpolation from $89°59'$ to $90°$, with Δ^3 (and in fact also Δ^2) constant and 9 digits.

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