Kernel estimation of extreme risk measures for all domains of attraction

by

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in collaboration with

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CIRM Workshop "Extremes - Copulas - Actuarial science"
Marseille February 2016

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The Value-at-Risk

- Let $Y \in \mathbb{R}$ be a random loss variable. The Value-at-Risk of level $\alpha \in (0,1)$ denoted by $\mathrm{VaR}(\alpha)$ is defined by

$$\mathrm{VaR}(\alpha) := \overline{F}^{\leftarrow}(\alpha) = \inf\{y, \overline{F}(y) \leq \alpha\},$$
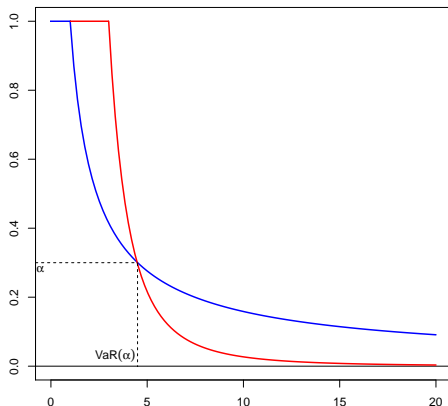
where $\overline{F}^{\leftarrow}$ is the generalized inverse of the survival function $\overline{F}(y) = \mathbb{P}(Y \geq y)$ of Y.
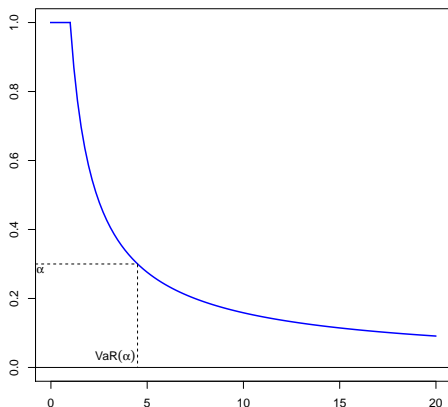
- The $\mathrm{VaR}(\alpha)$ is the quantile of level α of the survival function of the *r.v.* Y.

Drawbacks of the Value-at-Risk

- Let us consider Y_1 and Y_2 two loss *r.v.* with associated survival function $\overline{F}_1$ and $\overline{F}_2$.

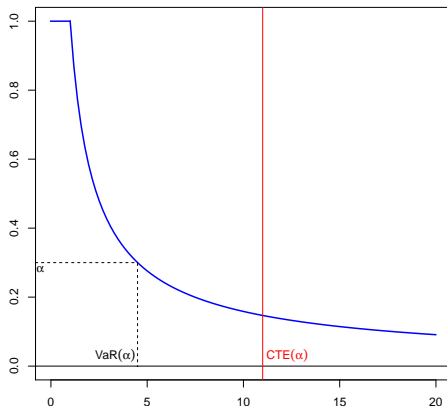$\implies$ Random variables with light tail probabilities and with heavy tail probabilities may have the same VaR(α). This is one of the main criticism against VaR as a risk measure (Embrechts *et al.* [1997]).

- The Conditional Tail Expectation of level $\alpha \in (0, 1)$ denoted CTE(α) is defined by

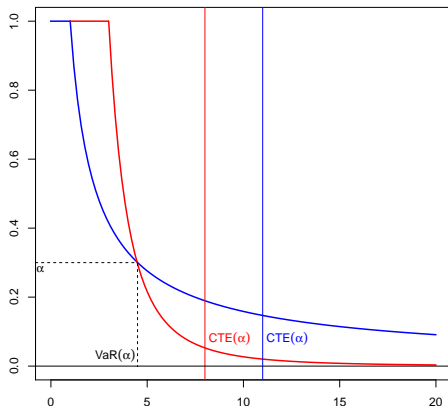$$\mathrm{CTE}(\alpha) := \mathbb{E}(Y | Y > \mathrm{VaR}(\alpha)).$$

$\implies$ The CTE(α) takes into account the whole information contained in the upper part of the tail distribution.

- The Conditional Tail Expectation of level $\alpha \in (0, 1)$ denoted $\mathrm{CTE}(\alpha)$ is defined by

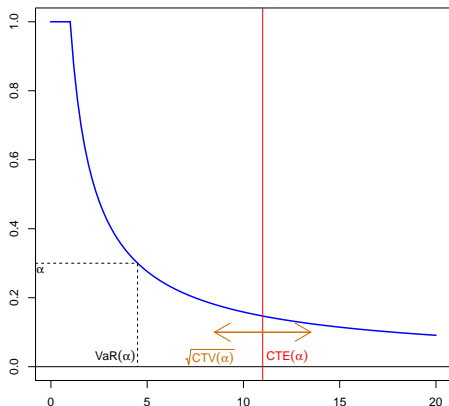$$\mathrm{CTE}(\alpha) := \mathbb{E}(Y | Y > \mathrm{VaR}(\alpha)).$$

$\implies$ The CTE(α) takes into account the whole information contained in the upper part of the tail distribution.

The Conditional Tail Variance

- The Conditional Tail Variance of level $\alpha \in (0,1)$ denoted $\mathrm{CTV}(\alpha)$ and introduced by Valdez [2005] is defined by

$$\mathrm{CTV}(\alpha) := \mathbb{E}((Y - \mathrm{CTE}(\alpha))^2 | Y > \mathrm{VaR}(\alpha)).$$

$\implies$ The $\mathrm{CTV}(\alpha)$ measures the conditional variability of Y given that $Y > \mathrm{VaR}(\alpha)$ and indicates how far away the events deviate from $\mathrm{CTE}(\alpha)$.

- The Conditional Tail Skewness of level $\alpha \in (0,1)$ denoted $\mathrm{CTS}(\alpha)$ and introduced by Hong and Elshahat [2010] is defined by

$$\mathrm{CTS}(\alpha) := \frac{\mathbb{E}(Y^3 | Y > \mathrm{VaR}(\alpha))}{(\mathrm{CTV}(\alpha))^{3/2}}$$

$\Longrightarrow$ We can unify the definitions of the previous risk measures using the Conditional Tail Moment introduced by El Methni *et al.* [2014].

Definition

The Conditional Tail Moment of level $\alpha \in (0,1)$ is defined by

$$\mathrm{CTM}_b(\alpha) := \mathbb{E}(Y^b | Y > \mathrm{VaR}(\alpha)),$$

where $b \geq 0$ is such that the moment of order b of Y exists.

All the previous risk measures of level α can be rewritten as

Risk Measure	Rewritten risk measure
$\text{CTE}(\alpha) = \mathbb{E}(Y\|Y > \text{VaR}(\alpha))$	$\text{CTM}_1(\alpha)$
$\text{CTV}(\alpha) = \mathbb{E}((Y - \text{CTE}(\alpha))^2\|Y > \text{VaR}(\alpha))$	$\text{CTM}_2(\alpha) - \text{CTM}_1^2(\alpha)$
$\text{CTS}(\alpha) = \mathbb{E}(Y^3\|Y > \text{VaR}(\alpha))/(\text{CTV}(\alpha))^{3/2}$	$\text{CTM}_3(\alpha)/(\text{CTV}(\alpha))^{3/2}$

$\implies$ All the risk measures depend on the CTM_b.

$\implies$ Our contributions consist in adding two difficulties in the framework of the estimation of risk measures.

1. First we add the presence of a random covariate $X \in \mathbb{R}^p$.

- Y is a positive random variable and $X \in \mathbb{R}^p$ a random vector of regressors recorded simultaneously with Y.

- In what follows, it is assumed that (X, Y) is a continuous random vector.

- The probability density function (p.d.f.) of X is denoted by $g(\cdot)$.

- The conditional p.d.f. of Y given $X = x$ is denoted by $f(\cdot|x)$.

For any $x \in \mathbb{R}^p$ such that $g(x) \neq 0$, the conditional distribution of Y given $X = x$ is characterized by the conditional survival function

$$\overline{F}(\cdot|x) = \mathbb{P}(Y > \cdot|X = x)$$

or, equivalently, by the Regression Value at Risk defined for $\alpha \in (0, 1)$ by

$$\mathrm{RVaR}(\alpha|x) := \overline{F}^{\leftarrow}(\alpha|x) = \inf\{t, \overline{F}(t|x) \leq \alpha\}.$$

The Regression Value at Risk of level α is a generalization to a regression setting of the Value at Risk.

The Regression Conditional Tail Moment of order b is defined by

$$\mathrm{RCTM}_b(\alpha|x) := \mathbb{E}(Y^b|Y > \mathrm{RVaR}(\alpha|x), X = x),$$

where $b \geq 0$ is such that the moment of order b of Y exists.

● Second we are interested in the estimation of risk measures in the case of extreme losses.

$\implies$ To this end, we replace the fixed order $\alpha \in (0,1)$ by a sequence $\alpha_n \to 0$ as the sample size $n \to \infty$.

$$
\begin{aligned}
\mathrm{RVaR}(\alpha_n|x) &:= \overline{F}^{\leftarrow}(\alpha_n|x) \\
\mathrm{RCTM}_b(\alpha_n|x) &:= \mathbb{E}(Y^b|Y > \mathrm{RVaR}(\alpha_n|x), X = x)
\end{aligned}
$$

$\implies$ All the risk measures depend on the RCTM_b.

$$
\begin{aligned}
\mathrm{RCTE}(\alpha_n|x) &= \mathrm{RCTM}_1(\alpha_n|x), \\
\mathrm{RCTV}(\alpha_n|x) &= \mathrm{RCTM}_2(\alpha_n|x) - \mathrm{RCTM}_1^2(\alpha_n|x), \\
\mathrm{RCTS}(\alpha_n|x) &= \mathrm{RCTM}_3(\alpha_n|x)/(\mathrm{RCTV}(\alpha_n|x))^{3/2}.
\end{aligned}
$$

Starting from n independent copies $(X_1, Y_1), \ldots, (X_n, Y_n)$ of the random vector (X, Y), we address here the estimation of the Regression Conditional Tail Moment of level α_n and order $b \geq 0$ given by

$$\mathrm{RCTM}_b(\alpha_n | x) := \frac{1}{\alpha_n} \mathbb{E} \left(Y^b \mathbb{I} \{ Y > \mathrm{RVaR}(\alpha_n | x) \} | X = x \right),$$

where b is such that the moment of order b of Y exits and $\mathbb{I}\{\cdot\}$ is the indicator function.

$\Longrightarrow$ We want to estimate all the above mentioned risk measures.

To do it, we need the asymptotic joint distribution of

$$\left\{ \left(\widehat{\mathrm{RCTM}}_{b_j, n}(\alpha_n | x), \ j = 1, \ldots, J \right) \right\},$$

with $0 \leq b_1 < \ldots < b_J$ and where J is an integer.

Estimator of the RVaR

The estimator of the Regression Value at Risk of level α_n considered is given by

$$\widehat{\mathrm{RVaR}}_n(\alpha_n|x) = \inf\{t, \ \hat{\bar{F}}_n(t|x) \leq \alpha_n\}$$

with

$$\hat{\bar{F}}_n(y|x) = \frac{\sum_{i=1}^n \mathcal{K}_{k_n}(x - X_i)\mathbb{I}\{Y_i > y\}}{\sum_{i=1}^n \mathcal{K}_{k_n}(x - X_i)}.$$

- The bandwidth (k_n) is a non random sequence converging to 0 as $n \to \infty$.

- It controls the smoothness of the kernel estimator.

- For $z > 0$, we have also introduced the notation $\mathcal{K}_z(\cdot) = z^{-p}\mathcal{K}(\cdot/z)$ where $\mathcal{K}(\cdot)$ is a density on $\mathbb{R}^p$.

- The estimation of the $\mathrm{RVaR}(\alpha_n|x)$ has been addressed for instance by Daouia *et al.* [2013].

The estimator of the Regression Conditional Tail Moment of level α_n and order b is given by

$$\widehat{\mathrm{RCTM}}_{b,n}(\alpha_n|x) = \frac{1}{\alpha_n} \frac{\sum_{i=1}^n \mathcal{K}_{h_n}(x - X_i) Y_i^b \mathbb{I}\{Y_i > \widehat{\mathrm{RVaR}}_n(\alpha_n|x)\}}{\sum_{i=1}^n \mathcal{K}_{h_n}(x - X_i)}$$

where

$$\widehat{\mathrm{RVaR}}_n(\alpha_n|x) = \inf\{t, \ \hat{\bar{F}}_n(t|x) \leq \alpha_n\}$$

with

$$\hat{\bar{F}}_n(y|x) = \frac{\sum_{i=1}^n \mathcal{K}_{k_n}(x - X_i)\mathbb{I}\{Y_i > y\}}{\sum_{i=1}^n \mathcal{K}_{k_n}(x - X_i).}$$

- The bandwidths (h_n) and (k_n) are non random sequences converging to 0 as $n \to \infty$.
- They control the smoothness of the kernel estimators. In what follows, the dependence on n for these two sequences is omitted.
- For the sake of simplicity we have chosen the same kernel $\mathcal{K}(\cdot)$.

To obtain the asymptotic property of the Regression Conditional Tail Moment estimator, an assumption on the right tail behavior of the conditional distribution of Y given $X = x$ is required. In the sequel, we assume that,

(F) The function $\mathrm{RVaR}(\cdot|x)$ is differentiable and

$$\lim_{\alpha \to 0} \frac{\mathrm{RVaR}'(t\alpha|x)}{\mathrm{RVaR}'(\alpha|x)} = t^{-(\gamma(x)+1)},$$

locally uniformly in $t \in (0, \infty)$.

$\implies$ In other words :

$-\mathrm{RVaR}'(\cdot|x)$ is said to be regularly varying at 0 with index $-(\gamma(x) + 1)$

The condition **(F)** entails that the conditional distribution of Y given $X = x$ is in the maximum domain of attraction of the extreme value distribution with extreme value index $\gamma(x)$.

Conditional extreme-value index

The unknown function $\gamma(x)$ is referred as the conditional extreme-value index.

It controls the behaviour of the tail of the survival function and by consequence the behaviour of the extreme values.

$\Longrightarrow$ if $\gamma(x) < 0$, $F(.|x)$ belongs to the domain of attraction of Weibull. It contains distributions with finite right tail, *i.e.* short-tailed.

$\Longrightarrow$ if $\gamma(x) = 0$, $F(.|x)$ belongs to the domain of attraction of Gumbel. It contains distributions with survival function exponentially decreasing, *i.e.* light-tailed.

$\Longrightarrow$ if $\gamma(x) > 0$, $F(.|x)$ belongs to the domain of attraction of Fréchet. It contains distributions with survival function polynomially decreasing, *i.e.* heavy-tailed.

The case $\gamma(x) > 0$ has already been investigated by El Methni *et al.* [2014].

The asymptotic normality of $\widehat{\mathrm{RCTM}}_{b,n}(\alpha_n|x)$ is obtained under additional assumptions.

First, a Lipschitz condition on the probability density function g of X is required. For all $(x, x') \in \mathbb{R}^p \times \mathbb{R}^p$, denoting by $d(x, x')$ the distance between x and x', we suppose that

(L) There exists a constant $c_g > 0$ such that $|g(x) - g(x')| \leq c_g d(x, x')$.

The next assumption is devoted to the kernel function $\mathcal{K}(\cdot)$.

(K) $\mathcal{K}(\cdot)$ is a bounded density on $\mathbb{R}^p$, with support S included in the unit ball of $\mathbb{R}^p$.

Before stating our main result, some further notations are required.

For $\xi > 0$, the largest oscillation at point $(x, y) \in \mathbb{R}^p \times \mathbb{R}_*^+$ associated with the Regression Conditional Tail Moment of order $b \in [0, 1/\gamma_+(x))$ is given by

$$\omega\left(x, y, b, \xi, h\right) = \sup\left\{\left|\frac{\varphi_b(z|x)}{\varphi_b(z|x')} - 1\right| \text{ with } \left|\frac{z}{y} - 1\right| \leq \xi \text{ and } x' \in B(x, h)\right\},$$

where $\varphi_b(\cdot|x) := \overline{F}(\cdot|x) RCTM_b(\overline{F}(\cdot|x)|x)$ and $B(x, h)$ denotes the ball centred at x with radius h.

Theorem 1

Suppose **(F)**, **(L)** and **(K)** hold. For $x \in \mathbb{R}^p$ such that $g(x) > 0$, let $\alpha_n \to 0$ such that

$$nk^p\alpha_n \to \infty \quad \text{as} \quad n \to \infty$$

If there exists $\xi > 0$ such that

$$nk^p\alpha_n \left(k \vee \omega(x, \mathrm{RVaR}(\alpha_n|x), 0, \xi, k)\right)^2 \to 0,$$

then

$$(nk^p\alpha_n^{-1})^{1/2}f(\mathrm{RVaR}(\alpha_n|x)|x)\left(\widehat{\mathrm{RVaR}}_n(\alpha_n|x) - \mathrm{RVaR}(\alpha_n|x)\right) \overset{d}{\longrightarrow} \mathcal{N}\left(0, \frac{\|\mathcal{K}\|_2^2}{g(x)}\right).$$

$\implies$ We thus find back the result established in Daouia *et al.* [2013] under weaker assumptions.

Asymptotic joint distribution of our estimators

Theorem 2

Suppose **(F)**, **(L)** and **(K)** hold. For $x \in \mathbb{R}^p$ such that $g(x) > 0$:

- Let $0 \leq b_1 \leq \ldots \leq b_J < 1/(2\gamma_+(x))$,
- $\underline{\ell} = h \wedge k$ and $\overline{\ell} = h \vee k$.
- Let $\alpha_n \to 0$ such that $n\underline{\ell}^p \alpha_n \to \infty$ as $n \to \infty$.
- If there exists $\xi > 0$ such that

$$n\overline{\ell}^p \alpha_n \left(\overline{\ell} \vee \max_b \omega(x, \mathrm{RVaR}(\alpha_n|x), b, \xi, \overline{\ell}) \right)^2 \to 0,$$

then, if

$$h/k \to 0 \quad \text{or} \quad k/h \to 0$$

the random vector

$$(n\underline{\ell}^p \alpha_n)^{1/2} \left\{ \left(\frac{\widehat{\mathrm{RCTM}}_{b_j,n}(\alpha_n|x)}{\mathrm{RCTM}_{b_j}(\alpha_n|x)} - 1 \right) \right\}_{j \in \{1,\ldots,J\}}$$

is asymptotically Gaussian, centred, with a $J \times J$ covariance matrix.

Covariance matrix two cases

In what follows, $(\cdot)_+$ (resp. $(\cdot)_-$) denotes the positive (resp. negative) part function.

1. If $k/h \to 0$ then the covariance matrix is given by

$$\frac{\|\mathcal{K}\|_2^2 \Sigma^{(1)}(x)}{g(x)}$$

   where for $(i,j) \in \{1, \ldots, J\}^2$,

$$\Sigma_{i,j}^{(1)}(x) = (1 - b_i\gamma_+(x))(1 - b_j\gamma_+(x)).$$

2. If $h/k \to 0$ then the covariance matrix is given by

$$\frac{\|\mathcal{K}\|_2^2 \Sigma^{(2)}(x)}{g(x)}$$

   where for $(i,j) \in \{1, \ldots, J\}^2$,

$$\Sigma_{i,j}^{(2)}(x) = \frac{(1 - b_i\gamma_+(x))(1 - b_j\gamma_+(x))}{1 - (b_i + b_j)\gamma_+(x)} = \frac{\Sigma_{i,j}^{(1)}(x)}{1 - (b_i + b_j)\gamma_+(x)}$$

Covariance matrix two cases

Recall that

$$\Sigma_{i,j}^{(1)}(x) = (1 - b_i\gamma_+(x))(1 - b_j\gamma_+(x)) \quad \text{and} \quad \Sigma_{i,j}^{(2)}(x) = \frac{\Sigma_{i,j}^{(1)}(x)}{1 - (b_i + b_j)\gamma_+(x)}$$

- Note that if $\gamma(x) \leq 0$, asymptotic covariance matrices do not depend on $\{b_1, \ldots, b_J\}$ and thus the estimators share the same rate of convergence.

- Conversely, when $\gamma(x) > 0$, asymptotic variances are increasing functions of the RCTM order.

- Moreover, in this case, note that for all $i \in \{1, \ldots, J\}$

$$\Sigma_{i,i}^{(2)}(x) > \Sigma_{i,i}^{(1)}(x)$$

$\implies$ Taking $k/h \to 0$ leads to more efficient estimators than $h/k \to 0$.

Corollary

Suppose that the asumptions of Theorem 1 hold with $\gamma(x) < 1/2$. If there exists $\xi > 0$ such that

$$n\overline{\ell}^p \alpha_n \left(\overline{\ell} \vee \max \omega(x, \mathrm{RVaR}(\alpha_n|x), 1, \xi, \overline{\ell})\right)^2 \to 0,$$

then, if $h/k \to 0$ or $k/h \to 0$, the random variable

$$(n\underline{\ell}^p \alpha_n)^{1/2} \left(\frac{\widehat{\mathrm{RCTE}}_n(\alpha_n|x)}{\mathrm{RCTE}(\alpha_n|x)} - 1 \right),$$

is asymptotically Gaussian, centred with variance

$$\frac{\vartheta_{\mathrm{RCTE}} \|\mathcal{K}\|_2^2}{g(x)}$$

where

- If $k/h \to 0$ then
$$\vartheta_{\mathrm{RCTE}} = (1 - \gamma_+(x))^2$$

- If $h/k \to 0$ then
$$\vartheta_{\mathrm{RCTE}} = \frac{(1 - \gamma_+(x))^2}{(1 - 2\gamma_+(x))}$$

Under **(F)**, the Regression Conditional Tail Moment or order b is asymptotically proportional to the Regression Value at Risk to the power b.

Proposition

Under **(F)**, *for all* $b \in [0, 1/\gamma_+(x))$,

$$\lim_{\alpha \to 0} \frac{\mathrm{RCTM}_b(\alpha|x)}{[\mathrm{RVaR}(\alpha|x)]^b} = \frac{1}{1 - b\gamma_+(x)},$$

and $\mathrm{RCTM}_b(\cdot|x)$ *is regularly varying with index* $-b\gamma_+(x)$.

In particular, the Proposition is an extension to a regression setting of the result established in Hua and Joe [2011] for the Conditional Tail Expectation ($b = 1$) in the framework of heavy-tailded distributions ($\gamma = \gamma(x) > 0$).

Let us note $y_F(x) = \bar{F}^{\leftarrow}(0|x) \in (0, \infty]$ the endpoint of Y given $X = x$

Two cases :

1. If the endpoint $y_F(x)$ is infinite :

$$y_F(x) = \infty \quad \text{then} \quad \gamma(x) \geq 0$$

$\Longrightarrow$ We can make risk measure estimation.

$\Longrightarrow$ We propose an application in pluviometry in the case $\gamma(x) > 0$.

Y : daily rainfall measured in mm. $X = \{\text{longitude, latitude, altitude}\}$. 1958 $\Longrightarrow$ 2000.
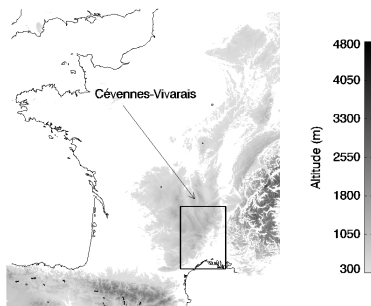
The Cévennes-Vivarais region

Y : daily rainfall measured in mm. $X = \{$longitude, latitude, altitude$\}$. 1958 $\Longrightarrow$ 2000.

The Cévennes-Vivarais region

Aim $\Longrightarrow$ to obtain maps of estimated extreme risk measures in all points of the region.

Y : daily rainfall measured in mm. $X = \{$longitude, latitude, altitude$\}$. 1958 $\Longrightarrow$ 2000.

The Cévennes-Vivarais region

The 523 stations of interest
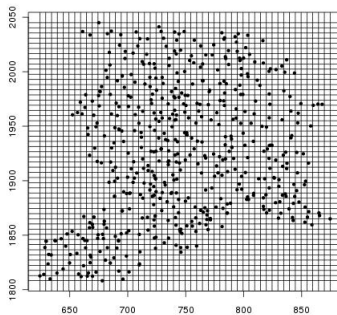
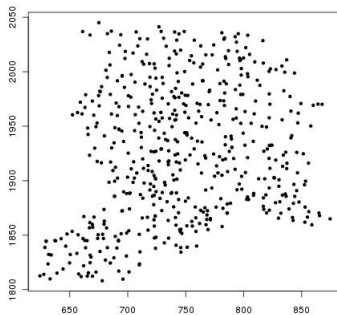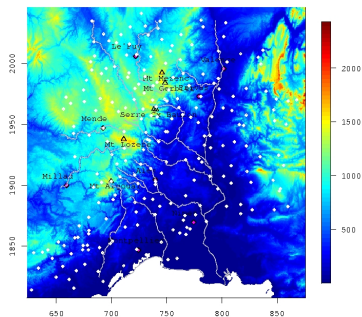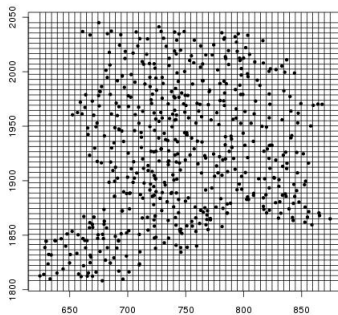Aim $\Longrightarrow$ to obtain maps of estimated extreme risk measures in all points of the region.

Application in pluviometry

Y : daily rainfall measured in mm. $X = \{$longitude, latitude, altitude$\}$. 1958 $\implies$ 2000.

The Cévennes-Vivarais region

Grid 200 $\times$ 200 points

Aim $\implies$ to obtain maps of estimated extreme risk measures in all points of the region.

- Daouia *et al.* [2011] have established the asymptotic normality of an extrapolated version of the $\mathrm{RVaR}(\beta_n|x)$ with β_n arbitrary small.

- El Methni *et al.* [2014] have established the asymptotic normality of an extrapolated version of the $\mathrm{RCTM}_b(\beta_n|x)$ with β_n arbitrary small.

- As a consequence, replacing $\widehat{\mathrm{RVaR}}_n$ and $\widehat{\mathrm{RCTM}}_{b,n}$ by theirs extrapolated versions provides estimators for all risk measures considered in this presentation adapted to arbitrary small levels.

$\Longrightarrow$ In particular we want to obtain maps of risk measures of daily rainfall corresponding to an amount of rain which is exceeded on average once in 100 years.

$\Longrightarrow$ It corresponds to a level of risk $\beta = 1/(100 \times 365.25)$

Y : daily rainfall measured in mm. $X = \{$longitude, latitude, altitude$\}$. 1958 $\implies$ 2000.

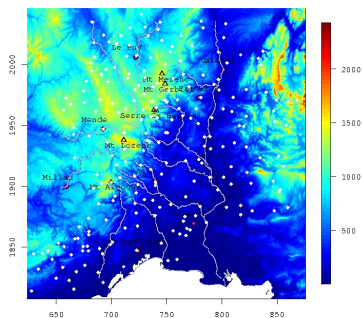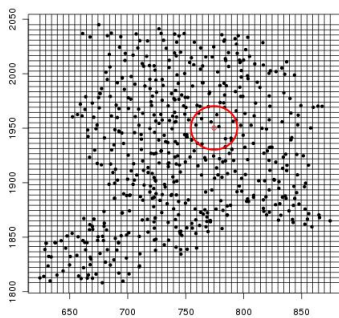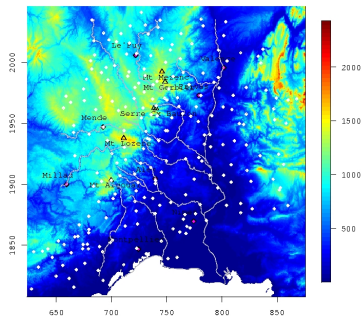The Cévennes-Vivarais region

The 523 stations of interest

Aim $\implies$ to obtain maps of estimated extreme risk measures in all point of the region.

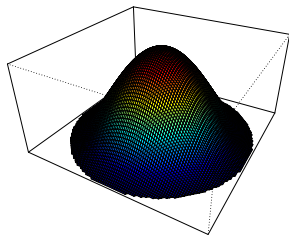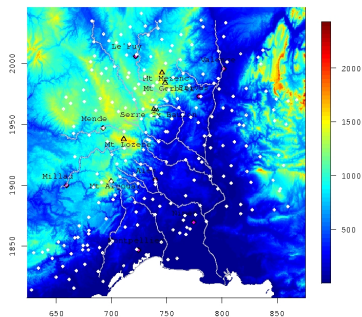# Application in pluviometry

Y : daily rainfall measured in mm. $X = \{$longitude, latitude, altitude$\}$. 1958 $\Longrightarrow$ 2000.

The Cévennes-Vivarais region

Grid 200 × 200 points

Aim $\Longrightarrow$ to obtain maps of estimated extreme risk measures in all point of the region.

Y : daily rainfall measured in mm. $X = \{$longitude, latitude, altitude$\}$. 1958 $\implies$ 2000.

The Cévennes-Vivarais region

Work in $B(x, h_n)$

Aim $\implies$ to obtain maps of estimated extreme risk measures in all point of the region.

Y : daily rainfall measured in mm. $X = \{$longitude, latitude, altitude$\}$. 1958 $\implies$ 2000.

The Cévennes-Vivarais region

Bi-quadratic kernel

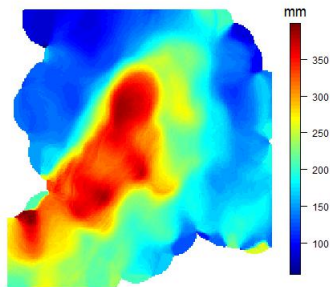Aim $\implies$ to obtain maps of estimated extreme risk measures in all point of the region.

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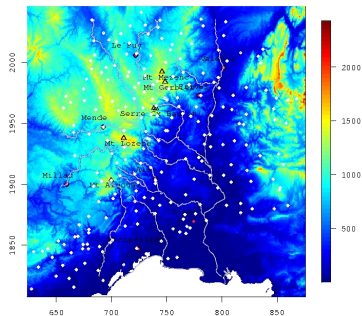

Example : $\widehat{\text{RVaR}}_n$ with $\gamma(x) > 0$

Aim $\Longrightarrow$ to obtain maps of estimated extreme risk measures in all point of the region.
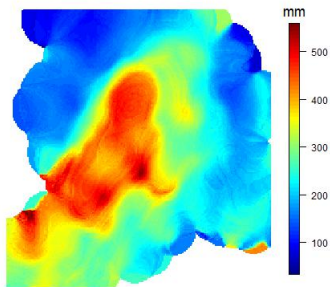
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The Cévennes-Vivarais region

Example : $\widehat{\text{RCTE}}_n$ with $\gamma(x) > 0$

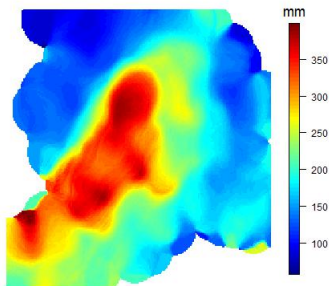Aim $\implies$ to obtain maps of estimated extreme risk measures in all point of the region.
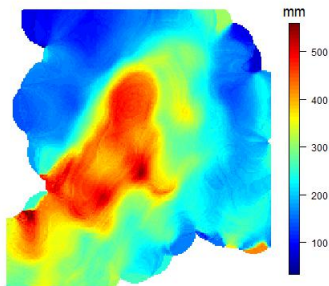
Application in pluviometry

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Aim $\Longrightarrow$ to obtain maps of estimated extreme risk measures in all point of the region.

2. If the endpoint $y_F(x)$ is finite, the risk measures do not have sense :

$$y_F(x) < \infty \quad \text{then} \quad \gamma(x) \leq 0 \Longrightarrow \gamma_+(x) = 0$$

As a consequence of the Proposition

$$\lim_{\alpha \to 0} \text{RCTM}_b(\alpha|x) = y_F^b(x)$$

$\Longrightarrow$ We can use our Proposition to make frontier estimation.

$\Longrightarrow$ We propose an application in nuclear reactor reliability.

Application in nuclear reactor reliability

- The dataset comes from the US Electric Power Research Institute and consists of $n = 254$ toughness results obtained from non-irradiated representative steels.

- An accurate knowledge of the change in fracture toughness of the reactor pressure vessel materials as a function of the temperature is of prime importance in a nuclear power plant's lifetime programme.
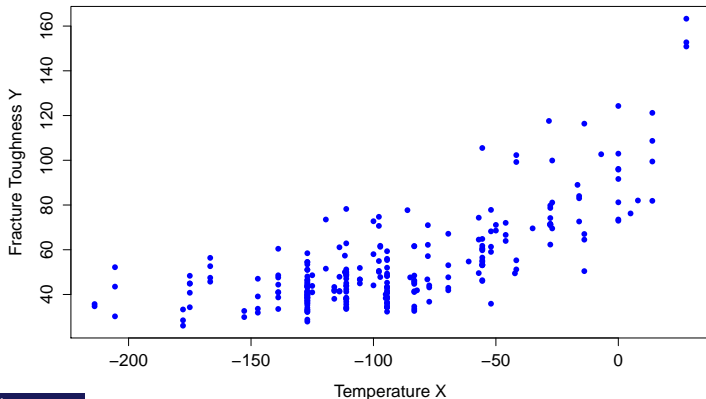
Frontier estimation

- As the temperature decreases, the steel fissures more easily.

- Here, it is important to know the upper limit of fracture toughness of each material as a function of temperature.

- This translates into estimating the optimal support upper boundary.
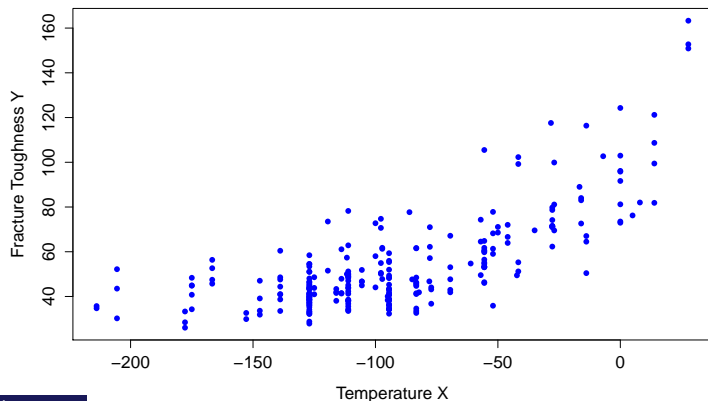
Commentaries

+ New tool for the prevention of risk and frontier estimation.
+ Theoretical properties similar to the univariate case (extreme or not) and with or without a covariate.
+ Our theoretical results are similar to those obtained by Daouia *et al.* [2013] and El Methni *et al.* [2014].
+ Capable to estimate all risk measures based on conditional moments of the *r.v.* of losses given that the losses are greater than $RVaR(\alpha)$ for short, light and heavy-tailed distributions.

Short-term perspectives

- Application to the nuclear data set.
- Tuning parameter selection procedure to choose (α_n, h).

Long-term perspectives

- Curse of dimensionality.

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Thank you for your attention