Maximum degree in minor-closed classes of graphs

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Omer Giménez [*] Dieter Mitsche [†] Marc Noy [‡]

Abstract

Given a class of graphs $\mathcal{G}$ closed under taking minors, we study the maximum degree Δ_n of random graphs from $\mathcal{G}$ with n vertices. We prove several lower and upper bounds that hold with high probability. Among other results, we find classes of graphs providing orders of magnitude for Δ_n not observed before, such us $\log n / \log\log\log n$ and $\log n / \log\log\log\log n$.

1 Introduction

A class of labelled graphs $\mathcal{G}$ is minor-closed if whenever a graph G is in $\mathcal{G}$ and H is a minor of G, then H is also in $\mathcal{G}$. A basic example is the class of planar graphs or, more generally, the class of graphs embeddable in a fixed surface.

All graphs in this paper are labelled. Let $\mathcal{G}_n$ be the graphs in $\mathcal{G}$ with n vertices. By a random graph from $\mathcal{G}$ of size n we mean a graph drawn with uniform probability from $\mathcal{G}_n$. We say that an event A in the class $\mathcal{G}$ holds with high probability (w.h.p.) if the probability that A holds in $\mathcal{G}_n$ tends to 1 as $n \to \infty$. Let Δ_n be the random variable equal to the maximum vertex degree in random graphs from $\mathcal{G}_n$. We are interested in events of the form

$$\Delta_n \leq f(n) \qquad \text{w.h.p.}$$

and of the form

$$\Delta_n \geq f(n) \qquad \text{w.h.p.}$$

Typically $f(n)$ will be of the form $c \log n$ for some constant c, or some related functions. We say that $f(n) = O(g(n))$ if there exists an integer n_0 and a constant $c > 0$ such that $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$, $f(n) = \Omega(g(n))$, if $g(n) = O(f(n))$, and finally $f(n) = \Theta(g(n))$, if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold. Also, $f(n) = \omega(g(n))$, if $\lim_{n\to\infty} |f(n)|/|g(n)| = \infty$, and $f(n) = o(g(n))$, if $g(n) = \omega(f(n))$. Throughout this paper $\log n$ refers to the natural logarithm.

A classical result says that for labelled trees Δ_n is of order $\log n / \log\log n$ (see [13]). In fact, much more precise results are known in this case, in particular that (see [2])

$$\frac{\Delta_n}{\log n / \log\log n} \to 1 \qquad \text{in probability.}$$

[*]Google Inc., Palo Alto. E-mail: omer.gimenez@gmail.com

[†]Laboratoire J.A. Dieudonné, Université de Nice-Sophia Antipolis, France. E-mail: dmitsche@unice.fr

[‡]Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain. E-mail: marc.noy@upc.edu

Many more results about the distribution of maximum degree, its concentration, and several different models of randomly generated trees can be found in the survey of [9].

McDiarmid and Reed [11] show that for the class of planar graphs there exist constants $0 < c_1 < c_2$ such that

$$c_1 \log n < \Delta_n < c_2 \log n \qquad \text{w.h.p.}$$

More recently this result has been strengthened using subtle analytic and probabilistic methods [5], by showing the existence of a computable constant c such that

$$\frac{\Delta_n}{\log n} \to c \qquad \text{in probability.}$$

For planar maps (planar graphs with a given embedding), more precise results on the distribution of Δ_n can be found in [7].

Analogous results have been proved for series-parallel and outerplanar graphs [4], with suitable constants. Using the framework of Boltzmann samplers, results about the degree distribution of subcritical graph classes such as outerplanar graphs, series-parallel graphs, cactus graphs and clique graphs can also be found in [1]. This paper also contains conjectures of the exact values of c_{OP} (c_{SP} , respectively) so that the maximum degree in outerplanar graphs (series-parallel graphs, respectively) will be roughly $c_{OP} \log n$ ($c_{SP} \log n$, respectively).

The goal in this paper is to analyze the maximum degree in additional minor-closed classes of graphs. Our main inspiration comes from the work of McDiarmid and Reed mentioned above. The authors develop proof techniques based on double counting that assume only mild conditions on the classes of graphs involved. We now explain the basic principle.

Let $\mathcal{G}$ be a class of graphs and suppose we want to show that a property P holds in $\mathcal{G}$ w.h.p. Let $\mathcal{B}_n$ the graphs in $\mathcal{G}_n$ that do not satisfy P (the 'bad' graphs). Suppose that for a constant fraction $\alpha > 0$ of graphs in $\mathcal{B}_n$ we have a rule producing at least $C(n)$ graphs in $\mathcal{G}_n$ (the 'construction' function). A graph in $\mathcal{G}_n$ can be produced more than once, but assume every graph in $\mathcal{G}_n$ is produced at most $R(n)$ times (the 'repetition' function). By double counting we have

$$\alpha |\mathcal{B}_n| C(n) \leq |\mathcal{G}_n| R(n),$$

hence

$$\alpha \frac{|\mathcal{B}_n|}{|\mathcal{G}_n|} \leq \frac{R(n)}{C(n)}.$$

If the procedure is such that $C(n)$ grows faster than $R(n)$, that is $R(n) = o(C(n))$, then we conclude that $|\mathcal{B}_n| = o(|\mathcal{G}_n|)$, that is, the proportion of bad graphs goes to 0. Equivalently, property P holds w.h.p. We often use the equivalent formulation $C(n)/R(n) \to \infty$.

We will apply this principle in order to obtain lower and upper bounds on the maximum degree for several classes. In this context, lower bounds are easier to obtain, and only in some cases we are able to prove matching upper bounds. The proof of the upper bound for planar graphs in [11] depends very strongly on planarity, and it seems difficult to adapt it to general situations; however we obtain such a proof for outerplanar graphs. On the other hand, we develop new tools for proving upper bounds based on the decomposition of a connected graph into 2-connected components.

Here is a summary of our main results. We denote by $\mathrm{Ex}(H)$ the class of graphs not containing H as a minor. All the claims hold w.h.p. in the corresponding class, and c, c_1 and c_2 are suitable positive constants. The fan graph F_n consists of a path with $n-1$ vertices plus a vertex adjacent to all the vertices in the path.

- In $\mathrm{Ex}(C_4)$ we have, for all $\epsilon > 0$,
$$(2-\epsilon)\frac{\log n}{\log\log n} \le \Delta_n \le (2+\epsilon)\frac{\log n}{\log\log n}.$$
- In $\mathrm{Ex}(C_5)$ we have, for all $\epsilon > 0$,
$$(1-\epsilon)\frac{\log n}{\log\log\log n} \le \Delta_n \le (1+\epsilon)\frac{\log n}{\log\log\log n}.$$
- In $\mathrm{Ex}(C_6)$ we have
$$c_1\frac{\log n}{\log\log\log n} \le \Delta_n \le c_2\frac{\log n}{\log\log\log n}.$$
- In $\mathrm{Ex}(C_7)$ we have
$$c_1\frac{\log n}{\log\log\log\log n} \le \Delta_n \le c_2\frac{\log n}{\log\log\log\log n}.$$
- If H is 2-connected and contains $C_{2\ell+1}$ as a minor, then in $\mathrm{Ex}(H)$ we have
$$\Delta_n \ge c\frac{\log n}{\log^{(\ell+1)} n},$$
where $\log^{(\ell+1)} n = \log\cdots\log n$, iterated $\ell+1$ times.
- If H is 2-connected and is not a minor of F_n for any n, then in $\mathrm{Ex}(H)$ we have
$$\Delta_n \ge c\log n.$$

The results on $\mathrm{Ex}(H)$ also hold when forbidding more than one graph as a minor, as discussed in the next section.

Organization of the paper. In Section 2 we prove the lower bounds for the maximum degree. In Section 3 we determine the structure of 2-connected graphs in the classes $\mathrm{Ex}(C_5)$, $\mathrm{Ex}(C_6)$ and $\mathrm{Ex}(C_7)$. This is quite technical and based on case analysis. The reason we undertake this analysis is to exemplify our technique for proving upper bounds and to show that different asymptotic estimates for the maximum degree are indeed possible. The proofs for the upper bound are contained in Section 4. We conclude with some remarks and several conjectures and open problems.

2 Lower bounds

A *pendant* vertex is a vertex of degree one. The following lemma follows from [12].

Lemma 1. *Let $H_1,\dots,H_k$ be 2-connected graphs and let $\mathcal{G} = \mathrm{Ex}(H_1,\dots,H_k)$. Then there is a constant $\alpha > 0$ such that a graph in $\mathcal{G}_n$ contains at least αn pendant vertices w.h.p.*

To illustrate our proof technique, we reprove the following well-known result (see [13], and see [2] for more precise results, as mentioned above), but without the need of enumerative tools.

Lemma 2. *Let $\epsilon > 0$ be any constant. In the class of forests, w.h.p.*

$$(1-\epsilon)\frac{\log n}{\log\log n} \le \Delta_n.$$

Proof. Let $\mathcal{G}$ be the class of forests, and $\mathcal{G}_n$ the class of forests with exactly n vertices. Let $\epsilon > 0$ be any constant and let $\mathcal{B}_n \subseteq \mathcal{G}_n$ denote the set of bad graphs with $\Delta_n < (1-\epsilon)\frac{\log n}{\log\log n}$, and suppose for contradiction that $|\mathcal{B}_n| \ge \mu|\mathcal{G}_n|$ for some $\mu > 0$, infinitely often. Our goal is to show that we can obtain $\omega(|\mathcal{B}_n|)$ new graphs in $\mathcal{G}_n$, or equivalently, $C(n)/R(n) \to \infty$, contradicting $|\mathcal{B}_n| \ge \mu|\mathcal{G}_n|$. Consider the subclass $\mathcal{B}'_n \subseteq \mathcal{B}_n$ of graphs in $\mathcal{B}_n$ with at least αn pendant vertices. By Lemma 1, $|\mathcal{B}'_n| = (1+o(1))|\mathcal{B}_n|$. Let G be a graph in $\mathcal{B}'_n$. Choose from the pendant vertices a subset of size $s+1$, where $s = \lceil (1-\epsilon)\frac{\log n}{\log\log n} \rceil$, and delete all their pendant edges. Among those choose a vertex, call it v_1, and make it adjacent to all other s vertices. Finally, choose a vertex u different from the $s+1$ chosen vertices, and make u adjacent to v_1 (we have at least $n - s \ge n/2$ choices for u). In this way one can construct at least $\binom{\alpha n}{s+1}(s+1)\frac{n}{2}$ graphs. From how many graphs G may the newly constructed graph G' come? We identify v_1 as the only vertex with largest degree in G' and u as the only non-pendant neighbor of v_1. In order to reconstruct G completely we only need to reattach the $s+1$ vertices in all possible ways, which can be done in at most n^{s+1} ways. Hence

$$\frac{C(n)}{R(n)} \ge \frac{\binom{\alpha n}{s+1}(s+1)n}{2n^{s+1}} \ge \frac{n(\alpha/2)^{s+1}}{2s!}.$$

Taking logarithms, this gives

$$\log\frac{C(n)}{R(n)} \ge \log n - s\log s + O(s) = \log n - (1-\epsilon)(1+o(1))\log n,$$

which tends to infinity. Hence, $|\mathcal{B}_n| = o(|\mathcal{G}_n|)$, and thus w.h.p. $(1-\epsilon)\frac{\log n}{\log\log n} \le \Delta_n$, and the result follows. □

Now we are ready to state new results that can be obtained using our techniques. In order to prove a lower bound for Δ_n in a class $\mathcal{G}_n$, the basic idea is to generalize the previous proof. Take a graph G in $\mathcal{G}_n$ whose maximum degree is too small (a bad graph), take enough pendant vertices and make with them a special graph S rooted at a special vertex v (in the previous proof a star rooted at its center), and attach S to G through a single edge, producing a new graph G' in $\mathcal{G}_n$. Then v becomes the unique vertex of maximum degree $s = |S|$, and G can be reconstructed from G' easily by reattaching the vertices in S, which are neighbors of v in G'. Double counting is then used to show that the proportion of bad graphs goes to 0 as n goes to infinity.

Theorem 3. *The following claims refer to the class* $\mathrm{Ex}(H_1, \ldots, H_k)$.

1. *Let c be a positive constant satisfying $c < \frac{1}{\log(2/\alpha)}$. If all the H_i are 2-connected and none of them is a minor of a fan graph F_n, then*

   $$\Delta_n \ge c\log n \qquad \text{w.h.p.}$$

   This holds in particular if the H_i are 3-connected or not outerplanar.

2. *If all the H_i are 2-connected and contain C_4 as a minor (that is, all the H_i are not C_3), then for every $\epsilon > 0$,*

   $$\Delta_n \ge (2-\epsilon)\frac{\log n}{\log\log n} \qquad \text{w.h.p.}$$

3. If all the H_i are 2-connected and contain C_5 as a minor, then for every $\epsilon > 0$,

$$\Delta_n \geq (1-\epsilon)\frac{\log n}{\log\log\log n} \qquad \textit{w.h.p.}$$

4. For $\ell \geq 3$, let $c = c(\ell)$ be a positive constant satisfying $c < 1/\ell$. If all the H_i are 2-connected and contain $C_{2\ell+1}$ as a minor for some $\ell \geq 3$, then

$$\Delta_n \geq c\frac{\log n}{\log^{(\ell+1)} n} \qquad \textit{w.h.p.}$$

Note that if all the H_i are 2-connected, since every 2-connected graph contains C_3 as a minor, the bound $\Delta_n \geq c \log n/\log\log n$ always holds for $c < 1$.

Proof. Throughout the proof we will assume for contradiction that there is some constant $\mu > 0$ such that for each item and its corresponding graphs in $\mathcal{B}_n$, we have $|\mathcal{B}_n| \geq \mu|\mathcal{G}_n|$ infinitely often. Our goal is to show that we can obtain $\omega(|\mathcal{B}_n|)$ new graphs in $\mathcal{G}_n$, or equivalently, $C(n)/R(n) \to \infty$, contradicting $|\mathcal{B}_n| \geq \mu|\mathcal{G}_n|$. Since, by assumption, $|\mathcal{B}_n| \geq \mu|\mathcal{G}_n|$, as before, by Lemma 1, the subclass $\mathcal{B}'_n \subseteq \mathcal{B}_n$ of graphs with at least αn pendant vertices satisfies $|\mathcal{B}'_n| = (1 + o(1))|\mathcal{B}_n|$, and we will in all cases below consider a graph of $\mathcal{B}'_n$, where the definition of $\mathcal{B}_n$, and thus of $\mathcal{B}'_n$, changes from case to case.

1. Let $\mathcal{G} = \mathrm{Ex}(H_1, \ldots, H_k)$ and let $\mathcal{B}_n \subseteq \mathcal{G}_n$ the graphs with $\Delta_n < c \log n$, where c is a positive constant satisfying $c < \frac{1}{\log(2/\alpha)}$, and let $h = \lceil c \log n \rceil$. Let G be a graph in $\mathcal{B}'_n \subseteq \mathcal{B}_n$. Choose an ordered list $v_1, \ldots, v_h$ of h pendant vertices in G, delete the edges joining the v_i to the rest of the graph, and make a copy of F_h with a path $v_2, \ldots, v_h$ and v_1 adjacent to all of them. Select a vertex u of G different from the v_i and make it adjacent to v_1. The graph G' constructed in this way belongs to $\mathcal{G}_n$, since the H_i are 2-connected and none of them is a minor of a fan graph, and has the same number of vertices as G.

The number of graphs constructed in this way is at least (where $(m)_k$ denotes a falling factorial)

$$(\alpha n)_h(n-h) \geq \left(\frac{\alpha n}{2}\right)^h n,$$

the last inequality being true for n large enough; we use the fact that $h = \lceil c \log n \rceil$ is small compared with n.

How many times a graph G' can be constructed in this way? Since $G \in \mathcal{B}_n$, v_1 can be identified as the only vertex of degree h. Vertices $v_2, \ldots, v_h$ can be identified as the neighbors of v_1 inducing a path (among the neighbors of v_1, u is the only cut-vertex, and hence it can be identified easily). In order to recover G, we delete all the edges among the v_i and the edge $v_1 u$, and make $v_1, \ldots, v_h$ adjacent to one of the remaining vertices through a single edge. The number of possibilities is at most

$$(n-h)^h \leq n^h.$$

Summarizing, we can take $C(n) = (\alpha/2)^h n^{h+1}$ and $R(n) = n^h$. Then

$$\frac{C(n)}{R(n)} \geq n(\alpha/2)^{c \log n},$$

which tends to infinity if $c < \frac{1}{\log(2/\alpha)}$. This finishes the proof.

2. Assume now that the H_i contain C_4 as a minor, that is, they all contain a cycle of length at least four. As before, let $\mathcal{G} = \mathrm{Ex}(H_1, \ldots, H_k)$, let $\mathcal{B}_n \subseteq \mathcal{G}_n$ be the graphs with $\Delta_n < (2-\epsilon)\log n/\log\log n$, and let $s = \lceil (2-\epsilon)\log n/\log\log n \rceil$. Let G be a graph in $\mathcal{B}'_n$. Choose an (unordered) set of $s+1$ pendant vertices $v_1, \ldots, v_{s+1}$ in G, and delete the edges joining the v_i to the rest of the graph. Among those choose one of them, say v_1, and make it adjacent to all others. The other s vertices are paired up, and vertices of pairs are made adjacent (if s is odd, one vertex remains unpaired). Finally, another pendant vertex u is chosen and made adjacent to v_1. Note that there are at least $\alpha n/2$ choices for u. There are thus at least $\binom{\alpha n}{s+1}(s+1)\,((s-1)!!)\,(\alpha n/2)$ constructions, where $(2k-1)!! = 1 \cdot 3 \cdots (2k-1)$. The graph G' constructed in this way belongs to $\mathcal{G}_n$, and has the same number of vertices as G. When reconstructing G, v_1 can be identified as the unique vertex of maximum degree, and u is identified as the only neighbor of v_1 adjacent to a vertex which is not a neighbor of v_1. Thus, only the $s+1$ chosen vertices have to be reattached, and there are at most n^{s+1} choices. Hence,

$$\frac{C(n)}{R(n)} \geq \frac{\binom{\alpha n}{s+1}(\frac{1}{2}\alpha n)\,((s+1)!!)}{n^{s+1}} \geq \frac{(\frac{1}{2}\alpha)^{s+2}\,((s+1)!!)\,n}{(s+1)!}.$$

Using $(2g-1)!! = (2g)!/(2^g g!)$ and taking logarithms we obtain

$$\log\frac{C(n)}{R(n)} \geq \log n - (s/2)\log s + O(s) = \log n - (1-(\epsilon/2))(1+o(1))\log n,$$

which tends to infinity, as desired.

3. Now we may assume that the H_i contain C_5 as a minor. As before, Let $\mathcal{G} = \mathrm{Ex}(H_1, \ldots, H_k)$ and let $\mathcal{B}_n \subseteq \mathcal{G}_n$ be the graphs with $\Delta_n < (1-\epsilon)\log n/\log\log\log n$, and let $s' = \lceil (1-\epsilon)\log n/\log\log\log n \rceil$.

Let $F_{n,m}$ be the following graph: take m disjoint copies of $K^+_{2,n-1}$ (the complete bipartite graph $K_{2,n-1}$ plus an edge joining the two vertices in the part of size two), and glue them together by identifying a vertex of degree $n-1$ in each copy. Notice that the longest cycle in $F_{n,m}$ is C_4, and that $F_{n,m}$ has $mn+1$ vertices. Let G be a graph in $\mathcal{B}'_n$. For an integer $s < s'$ to be made precise below, choose a set of $s+1$ pendant vertices $v_1, \ldots, v_{s+1}$ in G, delete the edges joining the v_i to the rest of the graph, and make a copy of $F_{r,s/r}$ with the v_i, where r is an integer to be determined later. Let v_1 be the vertex chosen to be adjacent to all other v_i (there are $s+1$ choices for this vertex). Select a vertex u of G different from the v_i and make it adjacent to v_1. The graph G' constructed in this way belongs to $\mathcal{G}_n$, since the H_i are 2-connected and have no cycle of length more than four, and has the same number of vertices as G.

The number of graphs constructed in this way is at least

$$\frac{\binom{\alpha n}{s+1}(s+1)\binom{s}{r,\ldots,r}r^{s/r}\frac{n}{2}}{(s/r)!},$$

where the first binomial is for the choice of the pendant vertices; $(s+1)$ is for the choice of the center vertex v_1, the multinomial coefficient divided by $(s/r)!$ stands for a lower bound on the number of partitions of the s vertices into groups of size r; the factor $r^{s/r}$ for the choice of the vertices of degree r in each group; and finally $n/2$ is a lower bound for the choices of the target vertex u. The number of ways such a graph G' can be constructed is at most n^{s+1}, the argument is the same as before. Therefore, for n large enough, we have

$$\frac{C(n)}{R(n)} \geq \frac{\binom{\alpha n}{s+1}(s+1)\binom{s}{r,\ldots,r}r^{s/r}\frac{n}{2}}{(s/r)!\,n^{s+1}} \geq \frac{(\frac{\alpha}{2})^{s+1}\frac{n}{2}r^{s/r}}{(r!)^{s/r}(s/r)!}.$$

Taking logarithms in the last expression we obtain

$$(1+o(1))\left((s+1)\log\frac{\alpha}{2} + \log\frac{1}{2} + \log n + \frac{s}{r}\log r - s\log r - \frac{s}{r}\log\frac{s}{r}\right).$$

For the choices

$$s' = \lceil (1-\epsilon)\frac{\log n}{\log\log\log n} \rceil, \qquad r = \lfloor \frac{2\log s'}{\epsilon \log\log s'} \rfloor$$

and s to be the largest integer smaller or equal to s' with the property of being divisible by r (note that $s = (1+o(1))s'$), we can safely ignore the term $(s+1)\log(\alpha/2) + \log(1/2) + (s/r)\log r$. Plugging in these values of s and r into the remaining term, we obtain

$$\begin{aligned}
&(1+o(1))\left(\log n - s\log r - \frac{s}{r}\log\frac{s}{r}\right) \\
\geq &(1+o(1))\left(\log n - s(\log\log s - \log\log\log s) - \frac{\epsilon}{2} s\log\log s\right) \\
\geq &(1+o(1))\left(\log n - \left(1+\frac{\epsilon}{2}\right) s\log\log s\right) \\
\geq &(1+o(1))\left(\log n - \left(1+\frac{\epsilon}{2}\right)(1-\epsilon)\log n\right),
\end{aligned}$$

which tends to infinity, since $(1+\frac{\epsilon}{2})(1-\epsilon) < 1$.

4. As before, assume that the H_i contain $C_{2\ell+1}$ as a minor, and let $\mathcal{G} = \text{Ex}(H_1, \ldots, H_k)$. Let $\mathcal{B}_n \subseteq \mathcal{G}_n$ be the graphs with $\Delta_n < c\log n/\log^{(\ell+1)} n$ (where c is a small enough constant), and let $s = \lceil c\log n/\log^{(\ell+1)} n \rceil$.

Let G be a graph in $\mathcal{B}'_n \subseteq \mathcal{B}_n$. Choose a set of $s+1$ pendant vertices $v_1, \ldots, v_{s+1}$ in G, delete the edges joining the v_i to the rest of the graph, and make a copy of the following graph F with the v_i: first, as before, choose one special vertex, call it v_1, and make it adjacent to all other v_i. Group the remaining v_i (all except for v_1) into groups of size $r_1 = \log s/\log^{(\ell)} s$ (we ignore rounding issues, taking care of them below). Choose in each of the s/r_1 groups a center vertex. Call all center vertices to be vertices at level 1. Iteratively, for $i = 1, \ldots, \ell-2$, do the following: group each group of size $r_i - 1$ (from each group we eliminate the center vertices at level i) into subgroups of size $r_{i+1} = \log^{(i+1)} s/\log^{(\ell)} s$. Choose in each subgroup a new center vertex, and call all center vertices chosen in this step to be vertices at level $i+1$. Connect each center vertex at level i with all center vertices at level $i+1$ resulting from subgroups of the group of vertex i. Connect all center vertices at level $\ell-1$ with the remaining vertices of its corresponding subgroup (those vertices not chosen as centers).

Observe that the graph F does not contain a $C_{2\ell+1}$, since in the construction we add a forest of maximum path length $2(\ell-1)$ to a star centered at v_1, and thus the maximum cycle length is 2ℓ.

Next select a vertex u of G different from the v_i and make it adjacent to v_1. The graph G' constructed in this way belongs to $\mathcal{G}_n$, and has the same number of vertices as G. As before, we count the number of different graphs obtained by applying this construction to one graph of $\mathcal{B}'_n$. We obtain at least

$$\frac{\frac{n}{2}\binom{\alpha n}{s+1}(s+1)\binom{s}{r_1,\ldots,r_1}\prod_{i=1}^{\ell-2}\binom{r_i-1}{r_{i+1},\ldots,r_{i+1}}^{s/r_i(1+\beta_i)}(r_i-1)^{s/r_i(1+\beta_i)}}{(\frac{s}{r_1})!\prod_{i=1}^{\ell-2}((\frac{r_i-1}{r_{i+1}})!)^{s/r_i(1+\beta_i)}}$$

many graphs, where the $\beta_i = o(1)$ take into the account rounding issues and also the fact that in the ith step only $r_i - 1$ vertices are split into subgroups of size r_{i+1} (for example, we approximate $\frac{s(r_1-1)}{r_1 r_2}$ by $\frac{s}{r_2}$; β_2 accounts for the difference. Indeed, even for the last term $\beta_{\ell-2}$ the error term is bounded from above by $\sum_{i=1}^{\ell-3}\frac{1}{r_i} = o(1)$. By the same argument as in the proof of **2.**, a new graph can have at most n^{s+1} preimages. Thus, for n sufficiently large (the factors r_i^{s/r_i} in the denominator are a lower bound corresponding to the fact that the factors r_i in the numerator do not exactly cancel), we have

$$\frac{C(n)}{R(n)} \geq \frac{\frac{1}{2}n(\frac{1}{2}\alpha)^{s+1}((r_1-1)!)^{s/r_1}}{(r_1!)^{s/r_1}(\frac{s}{r_1})!(r_{\ell-1})!^{s/r_{\ell-1}(1+\beta_{\ell-1})}\prod_{i=2}^{\ell-2} r_i^{s/r_i}\prod_{i=1}^{\ell-2}((\frac{r_i-1}{r_{i+1}})!)^{s/r_i(1+\beta_i)}}.$$

Taking logarithms, we obtain

$$(1+o(1))\left(\log n + s\log(r_1-1) - s\log r_1 - \frac{s}{r_1}\log\frac{s}{r_1} - s\log r_{\ell-1} - \sum_{i=1}^{\ell-2}\frac{s}{r_i}\frac{r_i-1}{r_{i+1}}\log\frac{r_i-1}{r_{i+1}}\right).$$

Using $s\log(r_1-1) = s\log(r_1) + s\log(1-1/r_1)$ and $\frac{s}{r_i}\frac{r_i-1}{r_{i+1}}\log\frac{r_i-1}{r_{i+1}} \leq \frac{s}{r_{i+1}}\log r_i$, we get that this expression is at least

$$(1+o(1))\left(\log n - \frac{s}{r_1}\log\frac{s}{r_1} - s\log r_{\ell-1} - \sum_{i=1}^{\ell-2}\frac{s}{r_{i+1}}\log r_i\right). \tag{1}$$

Plugging in the values $r_i = \log^{(i)} s/\log^{(\ell)} s$, all but the first term are $(1+o(1))s\log^{(\ell)} s$, and thus, plugging in the value $s = c\log n/\log^{(\ell+1)} n$, for $c < 1/\ell$, the expression in (1) tends to infinity. □

Remark. The 2-connected graphs which are a minor of some F_n consist just of a cycle and some chords, all of them incident to the same vertex. In particular, if we forbid the graph consisting of a cycle of length six $v_1, v_2, v_3, v_4, v_5, v_6$ and the chords v_1v_3 and v_4v_6, the condition of part 1 of Theorem 3 still holds, and the conclusion that w.h.p. $\Delta_n \geq c\log n$ follows. The same also holds when forbidding the 6-cycle together with the chords v_1v_3, v_3v_5, v_5v_1.

3 Characterization of 2-connected graphs in $\mathrm{Ex}(C_5), \mathrm{Ex}(C_6)$ and $\mathrm{Ex}(C_7)$

In this section we determine all 2-connected graphs in the classes $\mathrm{Ex}(C_5), \mathrm{Ex}(C_6)$ and $\mathrm{Ex}(C_7)$. This is an essential ingredient for the proofs in the next section.

As usual, $K_{2,n}$ is the complete bipartite graph with partite sets of size 2 and n. Recall that $K_{2,n}^+$ denotes the graph obtained from $K_{2,n}$ by adding an edge between the two vertices of degree n. We have the following:

Lemma 4. *The only 2-connected graphs in* $\mathrm{Ex}(C_5)$ *are* K_3, K_4, $K_{2,m}$ *and* $K_{2,m}^+$, *for* $m \geq 2$.

Proof. Let G be a 2-connected graph in $\mathrm{Ex}(C_5)$. If G has at most three vertices, then it has to be K_3. Otherwise, if G has exactly four vertices, then it is either C_4, K_4 minus one edge, or K_4. Otherwise, suppose that G has at least 5 vertices. Let v, v_1, v_2, v_3 be the vertices in cyclic order of a C_4 in G. Assume without loss of generality that v has another neighbor different from v_1 and v_3, and also different from v_2. Observe that a cannot be adjacent to v_1 or v_3, since this would create a C_5. By 2-connectivity, there must exist a path from a to v_2 containing none of v, v_1, v_3. Since G is in $\mathrm{Ex}(C_5)$, it follows that a is adjacent to v_2. This holds for all neighbors of v different from v_2. Thus, they must form an independent set, and we obtain a copy of $K_{2,m}$. The only edge that can be added while staying in $\mathrm{Ex}(C_5)$ is the edge vv_2, giving rise to $K_{2,m}^+$. □

For $s, t \geq 0$, define the graph $H_{2,s,t}$, obtained by identifying a vertex v of degree $s+1$ in $K_{2,s+1}$ and a vertex of degree $t+1$ in $K_{2,t+1}$, and by adding an edge between the other vertices v_2 and v_3 of degree $s+1$ and $t+1$, respectively. Note that v has thus at least one common neighbor with v_2, call it v_1, and at least one common neighbor with v_3, call it v_4 (see Figure 1). We denote by $H^*_{2,s,t}$ any graph obtained from $H_{2,s,t}$ by adding a subset of the edges between vertices x and y with $x, y \in \{v, v_1, v_2, v_3, v_4\}$, unless they are creating a cycle of length 6 or longer (see Figure 1). Observe that the subset of edges allowed depends on the fact whether s or t are different from 0 or not; only in the case $s = t = 0$ all edges between special vertices can be added, yielding K_5.

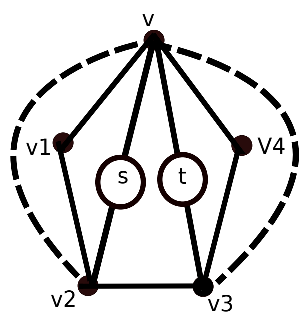

Figure 1: The graph $H_{2,s,t}$ with the notation as in Lemma 5, and with two optional edges (dashed)

Lemma 5. *The only 2-connected graphs in* Ex(C_6) *are those in* Ex(C_5)*, the graphs* $H_{2,s,t}$*, and any graph of the form* $H^*_{2,s,t}$*, for* $s, t \geq 0$.

Proof. Let G be a 2-connected graph in Ex(C_6). If G is in Ex(C_5), we apply the previous lemma. If G contains C_5 and has exactly 5 vertices, then G is either $H_{2,0,0}$ or $H^*_{2,0,0}$. Otherwise let v, v_1, v_2, v_3, v_4 be the vertices in cyclic order of a C_5 in G (see Figure 1). Call these vertices *special.* Observe that except for possible edges between neighbors of v that are both special vertices, $N(v)$ is an independent set. Consider a non-special neighbor a of v. As in the proof of Lemma 4, by 2-connectivity, a is adjacent to either v_2 or v_3, but not both. Let $A = N(v) \cap N(v_2) - \{v_1\}$, $B = N(v) \cap N(v_3) - \{v_4\}$, $s = |A|$, and $t = |B|$. With this notation, it can be checked that G is either $H_{2,s,t}$ or is in $H^*_{2,s,t}$, possibly with v_3 or v_4 playing the role of v. □

Remark. When later we refer to graphs $H_{2,s,t}$ or in $H^*_{2,s,t}$, with v_3 or v_4 playing the role of v, they will be denoted as $\widetilde{H}_{2,s,t}$ and $\widetilde{H}^*_{2,s,t}$.

Define the graph $S_{s,t,u,w}$ to be the graph constructed as follows: start with a 6-cycle whose vertices in cyclic order are $v, v_1, v_2, v_3, v_4, v_5$, and call these vertices *special.* In addition there are $w \geq 0$ vertices connecting v_2 and v_4, $s \geq 0$ vertices connecting v with v_2, $t \geq 0$ vertices connecting v with v_4, and $u \geq 0$ vertices connecting v with both v_2 and v_4 (in all cases excluding special vertices). Define then by $S^*_{s,t,u,w}$

any graph obtained by possibly adding any of the edges between special vertices without creating a cycle of length 7 or more, see Figure 2.

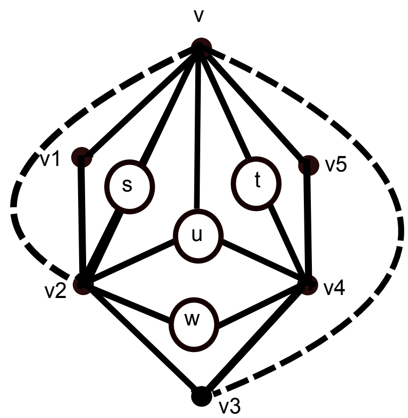

Figure 2: The graph $S_{s,t,u,w}$ with the notation as in Lemma 6, and with two optional edges (dashed)

Finally, let $V_{s,t,E}$ the following class of graphs: start with a 6-cycle $v, v_1, v_2, v_3, v_4, v_5$, again called special vertices. There is a set A of $s \geq 0$ vertices connecting v with v_2, and a set B of $t \geq 0$ vertices connecting v with v_4 (always excluding special vertices).

In addition, there is the following set of connections between v and v_3 (not including vertices in A or B or special vertices) specified by $K = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. There are $e_1 \geq 0$ vertices connecting v with v_3, and e_2 pairs of vertices which are adjacent to each other, and both are adjacent to both v and v_3. Furthermore, there are e_3 disjoint graphs K_{2,q_i} (for $i = 1, \ldots, e_3$) emanating from v_3, and the other vertex of degree q_i is connected to v. For e_4, the construction is the same, except that for these graphs also the edge between v_3 and the other vertex of degree q_i is present. Finally, there are e_5 and e_6 disjoint graphs K_{2,q_i} which are as the graphs e_3 and e_4, but with the roles of v_3 and v exchanged. For further reference, call the graphs of group e_3 and e_4 *double stars* of degree q_i emanating from v_3 (for $i = 1, \ldots, e_3$), and those of group e_5 and e_6 double stars emanating from v of degree q_i. All vertices appearing in any of the six groups are disjoint and we refer to them as *external* vertices. Finally, $V^*_{s,t,E}$ is the class of graphs obtained by possibly adding any of the edges between special vertices without creating a cycle of length 7 or more (see Figure 3 for an example).

Lemma 6. *The only 2-connected graphs in* $\mathrm{Ex}(C_7)$ *are those in* $\mathrm{Ex}(C_6)$*, the graphs* $S_{s,t,u,w}$*,* $V_{s,t,E}$ *and the corresponding graphs* $S^*_{s,t,u,w}$*,* $V^*_{s,t,E}$*.*

Proof. Let G be a 2-connected graph in $\mathrm{Ex}(C_7)$. If G is in $\mathrm{Ex}(C_6)$, we apply the previous lemma. If G contains C_6 and has exactly 6 vertices, then $G = S_{0,0,0,0}$ or $G = S^*_{0,0,0,0}$. Otherwise, let $v, v_1, v_2, v_3, v_4, v_5$ be the vertices in cyclic order of a C_6 in G, again called special. We distinguish two cases now. In the sequel all new vertices considered are not special vertices.
Case 1: There is no other vertex a with the property that there are two internally vertex-disjoint paths of length three from v to a. We distinguish between two subcases.
Case 1.1: Suppose first that there exist $u \geq 1$ vertices $a \in N(v)$ that are adjacent to both v_2 and v_4. Observe that the existence of such a vertex a implies that no external vertex e can be present in G, as otherwise one

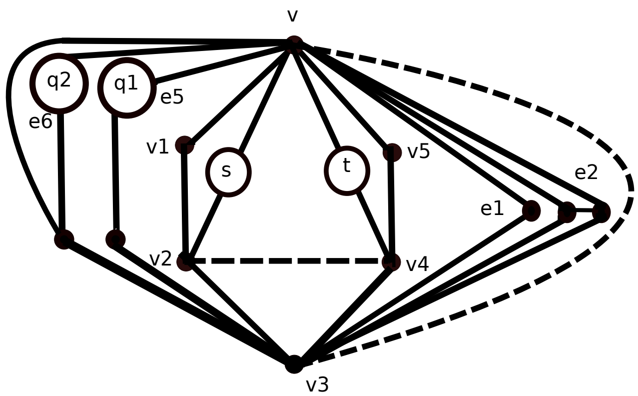

Figure 3: The graph $V_{s,t,E}$ with the notation as in Lemma 6 ($e_1 = e_2 = 1$, $e_3 = e_4 = 0$, $e_5 = e_6 = 1$ with corresponding degrees q_1 and q_2), and with two optional edges (dashed)

would have a cycle of length at least 7 (namely, v,v_1,v_2,a,v_4,v_3,e,v). Hence, all non-special neighbors of v can be partitioned into three sets A, B and C, where A is the set of $s \geq 0$ vertices connected only with v_2, B is the set of $t \geq 0$ vertices connected only v_4, and C is the set of $u \geq 1$ vertices connected to both v_2 and v_4. This corresponds exactly to the graph $S_{s,t,u,w}$ with $w = 0$. It is easy to check that except for edges yielding a graph in $S^*_{s,t,u,0}$, no edge can be added, as otherwise a 7-cycle would be generated (see Figure 2).
Case 1.2: Suppose that there is no vertex $a \in N(v)$ adjacent to both v_2 and v_4. Let A be the neighbors of v connecting v with v_2, and let B be the neighbors of v connecting v with v_4. Let $s = |A|$ and $t = |B|$. External vertices connecting v with v_3 are now possible. Note first that none of them can be adjacent to a special vertex except for v,v_2 in the case of A and except for v,v_4 in the case of B, neither to another vertex in A or B. There can be e_1 vertices connecting v with v_3, and e_2 pairs of vertices, adjacent to each other, both adjacent to v and v_3. Also, we might have e_3 (e_4, respectively) double stars of degree $q_i \geq 0$ emanating from v_3, where the other vertex of degree q_i is also adjacent to v (in the case of the e_4 vertices, the edge between v_3 and the other vertex of degree q_i is also present). Also, the roles of v_3 and v can be interchanged, yielding e_5 double stars (e_6, respectively) of degree q_i emanating from v (in the case of the e_6 stars, the edge between v and the other vertex of degree q_i is added as well; observe that in the case of the e_5 double stars we may assume $q_i \geq 2$, as otherwise these vertices appear already among the e_3 stars). The six groups are disjoint and there can be no other edge between external vertices. Thus, denoting $K = \{e_1, \ldots, e_6\}$, we obtain a graph in $V_{s,t,E}$. As before, no other edge except for edges yielding a graph in $V^*_{s,t,E}$ can be added (see Figure 3).
Case 2: There exists at least one more vertex a such that there are two internally vertex-disjoint paths of length three from v to a. These paths must be of the form v, v_1, v_2, a and v, v_5, v_4, a (if for example instead of the edge vv_1 there would be an edge vz for some other vertex z, there would be a path of length 6 going from z,v,$v_5, \ldots, v_1$, which, by 2-connectivity, would give a cycle of length at least 7). We suppose there are $w \geq 1$ such vertices a with such paths. Observe that the existence of such a vertex a implies that no external vertex e can be present in G, as otherwise one would have a cycle of length at least 7 (namely, the cycle v,v_1,v_2,a,v_4,v_3,e,v). All non-special neighbors of v can thus be partitioned into three sets A, B, and C, where A are those connected only with v_2, B those connected only with v_4, and C those connected both with v_2 and v_4. We let $s = |A|$, $t = |B|$, $u = |C|$. Let W be the vertices which are neither neighbors of v nor special vertices, and $w = |W|$. Again it can be checked that they all are such that there are two internally vertex-disjoint paths of length three from v to them, thus yielding a graph in $S_{s,t,u,w}$. As before, except for edges yielding a graph in $S^*_{s,t,u,w}$, no other edge can be added. □

Remark. When later we refer to graphs in $S_{s,t,u,w}$ or $V_{s,t,E}$ (or to the corresponding graphs in $S^*_{s,t,u,w}$ or $V^*_{s,t,E}$), where either v_2, v_3, v_4 or any of the external vertices of high degree play the role of v, they will be denoted as $\widetilde{S}_{s,t,u,w}$ and $\widetilde{V}_{s,t,E}$ ($\widetilde{S}^*_{s,t,u,w}$ and $\widetilde{V}^*_{s,t,E}$, respectively).

4 Upper bounds

We make repeated use of the following well-known lemma, whose proof is standard and therefore omitted.

Lemma 7. *Let $n_1, \ldots, n_r$ be positive integers such that $\sum_i n_i = N$ for some constant N. Then $\sum_i n_i \log n_i$ is minimized when all n_i are equal to $\lceil N/r \rceil$ or $\lfloor N/r \rfloor$.*

Also, we need the following lemma, whose proof is a straightforward generalization of Lemma 2.2 from [11].

Lemma 8. *Let $\mathcal{G} = \mathrm{Ex}(H_1, \ldots, H_k)$, where the H_i are 2-connected. Then w.h.p. each vertex in a graph in $\mathcal{G}_n$ is adjacent to at most $2 \log n / \log \log n$ pendant vertices.*

As in Section 2, we illustrate our technique to reprove in a simpler way the following known result (see [13, 2]), complementing Lemma 2.

Lemma 9. *Let $\epsilon > 0$ be any constant. In the class of forests, w.h.p.*

$$\Delta_n \leq (1+\epsilon)\frac{\log n}{\log \log n}.$$

Proof. Let $\mathcal{G}$ be the class of forests and $\mathcal{G}_n$ the class of forests with n vertices. Let $\mathcal{B}_n \subseteq \mathcal{G}_n$ now denote the set of bad graphs with

$$\Delta_n > (1+\epsilon)\frac{\log n}{\log \log n},$$

and suppose for contradiction that $|\mathcal{B}_n| \geq \mu |\mathcal{G}_n|$ for some $\mu > 0$, infinitely often. Let $\mathcal{B}'_n \subseteq \mathcal{B}_n$ be the class of graphs that has at least αn pendant vertices, and which is such that every vertex is adjacent to at most $2 \log n / \log \log n$ pendant vertices. By Lemma 1 and Lemma 8, $|\mathcal{B}'_n| = (1 + o(1))|\mathcal{B}_n|$. Let G be a graph in $\mathcal{B}'_n$, and let v be a vertex with degree $k > (1+\epsilon)\frac{\log n}{\log \log n}$. Since $G \in \mathcal{B}'_n$, there are at least $(\alpha n - 2 \log n / \log \log n) \geq 2\alpha n/3$ pendant vertices not adjacent to v. Let $c = \min(\frac{\epsilon/2}{1+\epsilon}, \alpha/3)$ and choose a set of $\lceil ck \rceil \leq 2\alpha n/3$ pendant vertices not adjacent to v and delete their adjacent edges. Maintain vertex v and delete all its adjacent edges. Attach the $\lceil ck \rceil$ chosen vertices to v, and construct many new graphs by attaching the former k neighbors of v in all possible ways to any of the previously added $\lceil ck \rceil$ vertices. More precisely, a fixed new graph is obtained by choosing for each of the former k neighbors of v, its corresponding vertex among the $\lceil ck \rceil$ vertices previously added, and then connecting to it by an edge. Observe that the new vertices have been added in a tree-like way, and hence the new graph is still in $\mathcal{G}_n$. Since we are interested in an asymptotic result, we may ignore ceilings from now on. The number of graphs constructed in this way is at least $\binom{2\alpha n/3}{ck}(ck)^k$. From how many graphs may the newly constructed graph G' come? One has to guess v, and then reattach the ck pendant vertices, giving rise to at most n^{ck+1} choices. Hence,

$$\frac{C(n)}{R(n)} \geq \frac{\binom{2\alpha n/3}{ck}(ck)^k}{n^{ck+1}} \geq \frac{(\alpha/3)^{ck}(ck)^k}{n(ck)!}.$$

Note that $(ck)^k/(ck)! > (ck)^{(1-c)k}$. Taking logarithms, this gives

$$\log \frac{C(n)}{R(n)} \geq (1-c)k\log k - \log n + O(k) \geq (1-c)(1+\epsilon)(1+o(1))\log n - \log n,$$

which tends to infinity by our choice of c. Hence, $|\mathcal{B}_n| = o(|\mathcal{G}_n|)$, and thus w.h.p. $\Delta_n > (1+\epsilon)\frac{\log n}{\log\log n}$, and the result follows. $\square$

Recall that a block H is a maximal connected subgraph without having a cut-vertex. Note that if H is a block, either H is 2-connected or H has at most 2 vertices.

Now we proceed to prove new results. In order to prove an upper bound for Δ_n in a class $\mathcal{G}_n$, the basic idea is to generalize the previous proof. Take a graph G in $\mathcal{G}_n$ whose maximum degree is too large (a bad graph), and let v be a vertex with large degree. Consider the blocks containing v and their contribution to the degree of v: the lemmas in Section 3 tell us all possible 2-connected components that can occur, which therefore, together with isolated vertices and isolated edges, tell us all blocks that can occur. We classify the blocks according to whether this contribution is larger or smaller than a suitable threshold. If B is a block with a vertex b of large degree t, remove the edges connecting b to its neighbors $b_1, \ldots, b_t$, take ct pendant vertices (where $c < 1$ is a suitable constant), isolate them and make them adjacent to v, and connect arbitrarily each of the b_i to any of the new ct vertices. Whatever was attached to the b_i remains untouched. When necessary, we add a few extra vertices and edges to ensure unique reconstruction. Blocks with small degree are not dismantled. This construction guarantees that we stay in $\mathcal{G}_n$. Double counting is used again to show that the proportion of bad graphs goes to 0 as n goes to infinity.

In the next proof we do not need all the power of this method, since blocks in $\mathrm{Ex}(C_4)$ have bounded degree, but already in the class $\mathrm{Ex}(C_5)$ there are blocks of arbitrary high degree.

Lemma 10. *Let $\epsilon > 0$ be any constant. In the class $\mathrm{Ex}(C_4)$, w.h.p.*

$$\Delta_n \leq (2+\epsilon)\frac{\log n}{\log\log n}.$$

Proof. We first observe that the only blocks in $\mathrm{Ex}(C_4)$ are isolated vertices, edges and triangles. Let $\mathcal{G} = \mathrm{Ex}(C_4)$ and let $\mathcal{B}_n \subseteq \mathcal{G}_n$ now denote the set of bad graphs with

$$\Delta_n > (2+\epsilon)\frac{\log n}{\log\log n}.$$

As before, let $\mathcal{B}'_n \subseteq \mathcal{B}_n$ be the class of graphs that has at least αn pendant vertices, and which is such that every vertex is adjacent to at most $2\log n/\log\log n$ pendant vertices. Once again, by Lemma 1 and Lemma 8, $|\mathcal{B}'_n| = (1+o(1))|\mathcal{B}_n|$. Let G be a graph in $\mathcal{B}'_n$ and let v be a vertex with degree $k > (2+\epsilon)\frac{\log n}{\log\log n}$. As before, there are at least $(\alpha n - 2\log n/\log\log n) \geq 2\alpha n/3$ pendant vertices not adjacent to v. Let $c = \min(\frac{\epsilon/3}{1+(\epsilon/2)}, \alpha/3)$. Let r be the number of blocks incident to v and observe that $(k/2) \leq r \leq k$, since the only blocks are edges and triangles. Choose a set of $\lceil cr \rceil \leq 2\alpha n/3$ pendant vertices not adjacent to v and delete their adjacent edges. Maintain vertex v and delete all its adjacent edges. Attach the $\lceil cr \rceil$ chosen vertices to v, and construct, as before, new graphs by attaching the roots of all r blocks in all possible ways to any of the previously added $\lceil cr \rceil$ vertices. Ignoring ceilings, the counting is as before: the number of graphs constructed in this way is at least $\binom{2\alpha n/3}{cr}(cr)^r$, and for recovering G, one has to guess v, then reattach the cr pendant vertices, giving rise to at most n^{cr+1} choices. Hence,

$$\frac{C(n)}{R(n)} \geq \frac{\binom{2\alpha n/3}{cr}(cr)^r}{n^{cr+1}} \geq \frac{(\frac{1}{3}\alpha)^{cr}(cr)^r}{n(cr)!}.$$

Note that $(cr)^r/(cr)! > (cr)^{(1-c)r}$. Thus, taking logarithms, this gives

$$\log\frac{C(n)}{R(n)} \geq (1-c)r\log r - \log n + O(r) \geq (1-c)(k/2)\log k - \log n + O(k),$$

which again tends to infinity by our choice of c. Hence, $|\mathcal{B}_n| = o(|\mathcal{G}_n|)$. ☐

Theorem 11. *Let $\epsilon > 0$ be any constant. In the class* $\mathrm{Ex}(C_5)$*, w.h.p.*

$$\Delta_n \leq (1+\epsilon)\frac{\log n}{\log\log\log n}.$$

Proof. Let $\mathcal{G} = \mathrm{Ex}(C_5)$ and let $\mathcal{B}_n \subseteq \mathcal{G}_n$ the graphs with

$$\Delta_n > (1+\epsilon)\log n/\log\log\log n.$$

Assume for contradiction that there is some constant μ such that $|\mathcal{B}_n| \geq \mu|\mathcal{G}_n|$ infinitely often. Once more, let $\mathcal{B}'_n \subseteq \mathcal{B}_n$ be the class of graphs that has at least αn pendant vertices, and which is such that every vertex is adjacent to at most $2\log n/\log\log n$ pendant vertices. Again, we have $|\mathcal{B}'_n| = (1+o(1))|\mathcal{B}_n|$. Let G be a graph in $\mathcal{B}'_n$ and let v be a vertex of G such that $k = \deg(v) > (1+\epsilon)\log n/\log\log\log n$. Since $G \in \mathcal{B}'_n$, at least $(\alpha n - 2\log n/\log\log n) \geq 2\alpha n/3$ pendant vertices are not adjacent to v. The strategy of the proof is as follows. We partition the blocks incident with v according to their type and to their contribution to the degree of v. Those with degree smaller than a threshold can be safely ignored for the asymptotics. Those of large degree, which by Lemma 4 are isomorphic to either $K_{2,t}$ or $K^+_{2,t}$, are used to produce many new graphs as in the proofs for the lower bounds. Then a double counting argument is used again to show that $|\mathcal{B}_n|/|\mathcal{G}_n| \to 0$. The strategy for $\mathrm{Ex}(C_6)$ and $\mathrm{Ex}(C_7)$ is very similar but there are more types of blocks to consider, making the situation a bit cumbersome.

Let us proceed with the proof. We partition the blocks attached to v. Using Lemma 4, they can be partitioned into the following classes:

1. blocks contributing to $\deg(v)$ at most $\frac{\log k}{\log\log k}$. That is, these are blocks whose root degree is at most $\frac{\log k}{\log\log k}$.
2. blocks of type $K_{2,t}$ with $t > \frac{\log k}{\log\log k}$
3. blocks of type $K^+_{2,t'}$ with $t' > \frac{\log k}{\log\log k}$

Let r_i be the number of blocks of class i and denote by k_i the total contribution of edges belonging to a block of class i to $\deg(v)$. Clearly, $k = k_1 + k_2 + k_3$, and also observe that $r_1 \geq \frac{k_1\log\log k}{\log k}$ and that $r_i < \frac{k_i\log\log k}{\log k}$ for $i = 2, 3$.

In order not to run out of pendant vertices, let now $c = \min(\frac{\epsilon/2}{1+\epsilon}, \frac{1}{4}\alpha)$. From G we construct now a class of graphs, as follows.

- Choose a set U of h (h will be determined below) pendant vertices and delete their adjacent edges. Maintain vertex v and delete all its adjacent edges. Choose three vertices from U, eliminate them from U and make them neighbors of v. Call them w.l.o.g. v_1, v_2, v_3 and assume that their labels are sorted increasingly. Choose $\lceil cr_1 \rceil$ vertices from U, eliminate them from U and make them neighbors of v_1. Attach the roots of all blocks of class 1 in all possible ways to any of the previously added $\lceil cr_1 \rceil$ vertices.

- Choose r_2 vertices from U, eliminate them from U (each of them representing a block of class 2) and make them neighbors of v_2. For each block of class 2 of type K_{2,t_i} ($i = 1, \ldots, r_2$) choose $1 + \lceil ct_i \rceil$ vertices from U, eliminate them from U, and connect all of them to the previously added vertex that represents the i-th block of this class. Let x_i be the vertex with smallest label among the $1 + \lceil ct_i \rceil$ vertices added ($i = 1, \ldots, r_2$). For each block K_{2,t_i} of G, define z_i^0 to be the other vertex apart from v of degree t_i, and let $z_i^1, \ldots, z_i^{t_i}$ the vertices of degree 2. In our construction, we delete all edges belonging to the original block and we add the following edges: z_i^0 is connected with x_i, and we connect each of the vertices z_i^j ($j \geq 1$) in all possible ways to any of the previously added $\lceil ct_i \rceil$ vertices excluding x_i.

- For blocks of class 3, do the analogous steps as for blocks of type 2.

Observe that the new vertices have been added in a tree-like way in this construction, that is, we have not created any cycle that did not exist in the original graph. In particular, if $G \in Ex(C_5)$, so are all the newly constructed graphs. Also observe that the number of pendant vertices h used satisfies $h \leq ck(1 + o(1)) < \alpha n/3$.

We proceed to count the number of different graphs we obtain by applying this construction to one graph of $\mathcal{B}'_n$. To simplify notation, we will ignore ceilings. We obtain at least

$$\binom{2\alpha n/3}{h} \binom{h}{cr_1, r_2, ct_1 + 1, \ldots, ct_{r_2} + 1, r_3, ct'_1 + 1, \ldots, ct'_{r_3} + 1, 3} r_2! r_3! \times$$
$$(cr_1)^{r_1} \left(\prod_{i=1}^{r_2} (ct_i)^{t_i} \right) \left(\prod_{i=1}^{r_3} (ct'_i)^{t'_i} \right) \tag{2}$$

many graphs, since there are at least $\binom{2\alpha n/3}{h}$ ways to choose h pendant vertices not incident to v, which then have to be partitioned into the different groups explained before (yielding the multinomial coefficient). The factors $r_2!$ and $r_3!$ come from the fact that blocks of class 2 and 3 are distinguishable because of their labels, hence any permutation of the r_2 and r_3 vertices will give rise to different graphs. The last group of three vertices in the multinomial coefficient corresponds to the vertices v_1, v_2, v_3 (there is no 3!, since the roles of these vertices are determined by their labels). The remaining factors count the possible ways to do the connections between the r_1 vertices and the added cr_1 vertices, between the added the t_i vertices and the added ct_i vertices, and between the t'_i and the ct'_i.

Since different original graphs may give rise to the same new graph, we have to divide the total number of constructions by the number of preimages of a new graph. This number is as before at most $n \cdot n^h$, since we first must guess the vertex v of the original graph (this gives the factor n) and then we have to redistribute the h newly added vertices as pendant vertices (for those we have at most n^h choices).

Our goal is to show that the total number of newly constructed graphs divided by the number of preimages of a new graph tends to infinity as n increases, hence contradicting the assumption that $|\mathcal{B}_n| \geq \mu |\mathcal{G}_n|$ for infinitely many values of n.

Note that the following expression is a lower bound of (2).

$$(1/2)^{k\log\log k/\log k}\left(\frac{1}{3}(\alpha-c)n\right)^h (cr_1)^{(1-c)r_1}\prod_{i=1}^{r_2}(ct_i)^{(1-c)t_i}\prod_{i=1}^{r_3}(ct'_i)^{(1-c)t'_i},$$

where we have used that $h=ck(1+o(1))$, $k<n$ so that $\frac{1}{6}(\frac{2}{3}\alpha n)!/(\frac{2}{3}\alpha n-h)!$ is bounded from below by $(\frac{1}{3}(\alpha-c)n)^h$; we also used that for any $g>0$ it holds that $(cg)^g/(cg)!\geq(cg)^{(1-c)g}$, and that for any g such that $cg\geq 3$ it holds that $(cg)^g/(cg+1)!\geq(cg)^{(1-c)g}$, and for smaller values of cg, $(cg)^g/(cg+1)!\geq\frac{1}{2}(cg)^{(1-c)g}$, giving the additional $(1/2)^{k\log\log k/\log k}$ leading factor.

We now divide by the number of preimages $n\cdot n^h$, and then we take logarithms. Hence, noting that $k_2=\sum_{i=1}^{r_2}t_i$ and $k_3=\sum_{i=1}^{r_3}t'_i$, we obtain

$$-\log n+o(k)+O(h)+(1-c)r_1\log r_1+O(r_1)+(1-c)\sum_{i=1}^{r_2}t_i\log t_i+O(k_2)+(1-c)\sum_{i=1}^{r_3}t'_i\log t'_i+O(k_3).$$

By Lemma 7, $\sum_{i=1}^{r_2}t_i\log t_i$ is minimal when all t_i are equal, and the same applies to the t'_i. Hence, the previous expression is bounded from below by

$$-\log n+O(k)+(1-c+o(1))\left(r_1\log r_1+k_2\log\frac{k_2}{r_2}+k_3\log\frac{k_3}{r_3}\right). \tag{3}$$

Now, letting $k_i=\beta_i k$ for $i=1,2,3$, we obtain

$$r_1\geq\frac{k_1\log\log k}{\log k}=\beta_1\frac{k\log\log k}{\log k},$$

and thus

$$r_1\log r_1\geq\beta_1\frac{k\log\log k}{\log k}(\log k+o(\log k))=\beta_1 k\log\log k(1+o(1)).$$

Also, recall that $r_2\leq\frac{k_2\log\log k}{\log k}$, so that

$$\frac{k_2}{r_2}\geq\frac{\log k}{\log\log k},$$

and the term $k_2\log\frac{k_2}{r_2}$ in (3) is at least

$$k_2\log\frac{k_2}{r_2}\geq k_2\log\log k(1+o(1))=\beta_2 k\log\log k(1+o(1)).$$

By the same argument, $k_3\log\frac{k_3}{r_3}\geq\beta_3 k\log\log k(1+o(1))$. As $\beta_1+\beta_2+\beta_3=1$, one of the β_i has to be at least $\frac{1}{3}$, hence we can safely ignore the term $O(k)$ in (3). The expression in (3) is thus bounded from below by

$$(1+o(1))(1-c)k\log\log k-\log n,$$

which by our choice of c tends to infinity, as desired. □

Theorem 12. *Let $C>0$ be a sufficiently large constant. In the class $\mathrm{Ex}(C_6)$, w.h.p.*

$$\Delta_n\leq C\frac{\log n}{\log\log\log n}.$$

Proof. The proof starts as for $\text{Ex}(C_5)$. Let $\mathcal{G} = \text{Ex}(C_6)$ and let $\mathcal{B}_n \subseteq \mathcal{G}_n$ the class of graphs with

$$\Delta_n > C \log n / \log\log\log n.$$

We assume for contradiction that there is some constant μ such that $|\mathcal{B}_n| \geq \mu|\mathcal{G}_n|$ infinitely often. Let also $\mathcal{B}'_n \subseteq \mathcal{B}_n$ be the class of graphs that has at least αn pendant vertices, and which is such that every vertex is adjacent to at most $2 \log n / \log\log n$ pendant vertices. Again, we have $|\mathcal{B}'_n| = (1 + o(1))|\mathcal{B}_n|$. Let G be a graph in $\mathcal{B}'_n$ and let v be a vertex of G such that

$$k = \deg(v) > \frac{C \log n}{\log\log\log n}$$

for some constant C large enough, and since $G \in \mathcal{B}'_n$, there are at least $2\alpha n/3$ pendant vertices not incident to v. We partition the blocks attached to v into different classes (see Lemma 5):

1. blocks contributing to $\deg(v)$ at most $\frac{\log k}{\log\log k}$.
2. blocks of type $K_{2,s}$ and $K^+_{2,s}$ with $s > \frac{\log k}{\log\log k}$.
3. blocks of type $H_{2,s,t}$ or $H^*_{2,s,t}$.
4. blocks of type $\widetilde{H}_{2,s,t}$ or $\widetilde{H}^*_{2,s,t}$ (see the remark after Lemma 5).

Choose a set U of h pendant vertices not incident to v and delete their adjacent edges. Maintain vertex v and delete all its adjacent edges. We now have a bounded number N of subclasses represented by classes 1 to 4 and the possible cases in the definition of $H^*_{2,s,t}$, $\widetilde{H}_{2,s,t}$ and $\widetilde{H}^*_{2,s,t}$. For each subclass i, let r_i be the number of blocks of subclass i incident with v. For each i, take a pendant vertex w_i from U and make it adjacent to v, and sort the w_i in increasing order of the labels. For each i (except for class 1), take r_i pendant vertices from U and make them adjacent to w_i. Let $c = \min(1 - \frac{3N}{C}, \frac{1}{4}\alpha)$. Note that for $C < 3N$ the expression for c is negative, so the assumption that C is sufficiently large in particular also implies that $C \geq 3N$.

For blocks in classes 1 and 2 (they give rise to r_1,r_2, r_3), the r_i play the same role as in the proof of Theorem 11, and we append the same construction as there.

For blocks of type $H_{2,s,t}$ the construction is very similar; they behave like the graphs $K_{2,s}$, but with two sets, of size s and t, of vertices of degree two. For each block of type $H_{2,s,t}$, we add two new sorted vertices from U and make them adjacent to the vertex representing the block. Take $2 + cs$ and $2 + ct$ vertices from U (ignoring ceilings from now on) and connect them, respectively, to the two previously added vertices. Let x_0, x_1 and y_0, y_1, respectively, be the vertices with smallest labels (in this order) among the $2 + cs$ and the $2 + ct$ added vertices. Delete all edges belonging to the original block and attach the s vertices to the newly added cs vertices (excluding x_0 and x_1) in all possible ways, and do the same for the t vertices (excluding y_0 and y_1). Also, connect x_0 to v_1 (notation as in Lemma 5), x_1 to v_2, and y_0 to v_3, y_1 to v_4. For blocks of type $H^*_{2,s,t}$ the construction is exactly the same; the fact that the different subclasses are identified by the labels as well as the special role of v_1, v_2, v_3, v_4 guarantees unique reconstruction. Finally, consider blocks of type $\widetilde{H}_{2,s,t}$, and assume without loss of generality that v_2 plays the role of v. In this case we add only $cs+4$ vertices from U and make them adjacent to the vertex representing the block. Let x_0, x_1, x_2 and x_3 be the vertices with the four smallest labels (in this order). Delete all edges in $\widetilde{H}_{2,s,t}$ emanating from v and v_2, all edges between special vertices, and connect all the s non-special neighbors of v_2 to the cs vertices (excluding x_0,x_1,x_2 and x_3) in all possible ways. Connect x_0 to v, x_1 to v_3, x_2 to v_1, and x_3 to v_4. As before, the same construction is applied for $\widetilde{H}^*_{2,s,t}$ (the only difference being that all optional edges are deleted as well); as before, the special roles and the different labels of different subclasses provide all information for unique reconstruction.

Since no new cycle is created, given v and the new graph, we can uniquely determine the original graph it comes from. Observe also that we used only $h \leq ck(1+o(1))$ pendant vertices. As before, we count the number of different graphs we obtain by applying this construction, yielding similar multinomial coefficients and other factors. Dividing by the number of preimages of a new graph, which is at most n^{h+1}, and taking logarithms, we obtain

$$-\log n + o(k) + O(h) + (1-c)r_1 \log r_1 + O(r_1) + (1-c)\sum_{j=1}^{r_2}(s_j)_2 \log(s_j)_2 + O(k_2) +$$
$$(1-c)\sum_{j=1}^{r_3}(s_j)_3 \log(s_j)_3 + O(k_3) + (1-c)\sum_{i\geq 4, i\in\mathcal{T}}\sum_{j=1}^{r_i}(s_j)_i \log(s_j)_i +$$
$$(1-c)\sum_{i\geq 4, i\in\mathcal{T}'}\sum_{j=1}^{r_i}(t_j)_i \log(t_j)_i + O(\textstyle\sum_{i\geq 4} k_i),$$

where we denote by k_i the total contribution of blocks of subclass i to the degree of v, and by $(s_j)_i$ and $(t_j)_i$ the corresponding sizes of the jth block of subclass $i \geq 2$; both sets $\mathcal{T}$ and $\mathcal{T}'$ contain all indices of subclasses belonging to blocks $H_{2,s,t}$ or $H^*_{2,s,t}$, $\mathcal{T}$ contains in addition to this all indices of subclasses of blocks of type $\widetilde{H}_{2,s,t}$ and $\widetilde{H}^*_{2,s,t}$ with v_2 playing the role of v, and $\mathcal{T}'$ contains in addition to this all subclasses of blocks of type $\widetilde{H}_{2,s,t}$ and $\widetilde{H}^*_{2,s,t}$ with v_3 playing the role of v. This distinction is needed since in the cases of $\widetilde{H}_{2,s,t}$ and $\widetilde{H}^*_{2,s,t}$ only one of the two sums above has to be counted. Suppose w.l.o.g. that the contribution of all $(t_j)_i$ to the degree of v is at most $k/2$, and we may ignore this contribution above. Define then for $1 \leq i \leq 3$, $k'_i = k_i$, and for $i \geq 4$, let $k'_i = \sum_{j=1}^{r_i}(s_j)_i$. Note that k'_i counts the contribution of the i-th subclass to the degree of v coming from the $(s_j)_i$ only. Clearly, $k'_i \leq k_i$ and by assumption $\sum_{i\geq 1} k'_i \geq k/2$. By Lemma 7, the previous expression is at least

$$-\log n + O(k) + (1-c+o(1))(r_1 \log r_1 + \sum_{i\geq 2} k'_i \log \frac{k'_i}{r_i}). \tag{4}$$

Since $\sum_{i\geq 1} k'_i \geq k/2$, there exists some $1 \leq i \leq N$ with $k'_i \geq k/(2N)$. If this is true for $i = 1$, then

$$r_1 \log r_1 \geq \frac{k'_1 \log\log k}{\log k}(\log k_1 + o(\log k_1)) = \frac{k \log\log k}{2N}(1+o(1)).$$

Otherwise, if $i \geq 2$, since $\frac{k'_i}{r_i} \geq \frac{\log k}{2N \log\log k}$, as before, $k'_i \log \frac{k'_i}{r_i} \geq \frac{k \log\log k}{2N}(1+o(1))$. Thus, by our choice of c, for C sufficiently large, (4) tends to infinity as desired. □

In the class $\mathrm{Ex}(C_7)$, the right order of magnitude of the expected maximum degree changes, compared to $\mathrm{Ex}(C_5)$ and $\mathrm{Ex}(C_6)$. Before going into the proof, we give some intuition about the different behavior in $\mathrm{Ex}(C_7)$. The existence of a component $V_{s,t,E}$ as described in Lemma 6, and in particular the existence of t stars of different degrees q_i inside one block, gives rise to new constructions. In order to ensure many constructions, both for the number of stars t (call this the first level), as well as for their degrees q_i (call this the second level), choices have to be made: if there were few stars of a high degree, only on the second level many choices can be made, but if, however, there are many stars of small degree, on the first level many choices can be made. For a medium number of stars with medium degree, on both levels some choices can be made. These two choices imply that the definition of *small* has to be changed, and the trade-off between the contribution of small blocks and other larger blocks (which give different types of contributions in the proofs) gives rise to an additional application of the logarithm. We now state the result for this class.

Theorem 13. *Let $C > 0$ be a sufficiently large constant. In the class* $\mathrm{Ex}(C_7)$*, w.h.p.*

$$\Delta_n \leq C \frac{\log n}{\log\log\log\log n}.$$

Proof. Let $\mathcal{G} = \mathrm{Ex}(C_7)$. The proof starts as for $\mathrm{Ex}(C_5)$ and $\mathrm{Ex}(C_6)$. Let $\mathcal{B}_n \subseteq \mathcal{G}_n$ the graphs with

$$\Delta_n > C \log n / \log\log\log\log n.$$

We assume once more for contradiction that there is some constant μ such that $|\mathcal{B}_n| \geq \mu|\mathcal{G}_n|$ infinitely often. As in the previous proofs, let $\mathcal{B}'_n \subseteq \mathcal{B}_n$ be the class of graphs that has at least αn pendant vertices, and which is such that every vertex is adjacent to at most $2 \log n / \log \log n$ pendant vertices. Again, $|\mathcal{B}'_n| = (1+o(1))|\mathcal{B}_n|$. Let G be a graph in $\mathcal{B}'_n$ and let v be a vertex of G such that

$$k = \deg(v) > \frac{C \log n}{\log\log\log\log n}$$

for some constant C large enough, and since $G \in \mathcal{B}'_n$, there are at least $2\alpha n/3$ pendant vertices not incident to v. As before, we partition the blocks attached to v into different classes. Using Lemma 6, whose notation is used in the following (see also the remark following Lemma 6), we may partition them into

1. blocks contributing to $\deg(v)$ at most $\frac{\log k}{\log\log\log k}$
2. blocks of type $K_{2,s}$, $K^+_{2,s}$, $H_{2,s,t}$, $H^*_{2,s,t}$, $\widetilde{H}_{2,s,t}$ and $\widetilde{H}^*_{2,s,t}$
3. blocks of type $S_{s,t,u,w}$, $V_{s,t,E}$, and the corresponding graphs $S^*_{s,t,u,w}$, $V^*_{s,t,E}$
4. blocks of type $\widetilde{S}_{s,t,u,w}$, $\widetilde{V}_{s,t,E}$, and the corresponding graphs $\widetilde{S}^*_{s,t,u,w}$, $\widetilde{V}^*_{s,t,E}$

Choose a set of U of h pendant vertices not incident to v and delete their adjacent edges. Maintain vertex v and delete all its adjacent edges. We still have a bounded number of subclasses N represented by the different classes and the possible optional edges. For each subclass i, let r_i be the number of blocks of subclass i incident with v. For each i, take a pendant vertex w_i from U and make it adjacent to v, and sort the w_i in increasing order of the labels. For each i (except for those subclasses belonging to class 1), take r_i pendant vertices from U and make them adjacent to w_i. Let $c = \min(1 - \frac{9N}{C}, \frac{1}{4}\alpha)$. As before, we assume that C is large enough and in particular $C \geq 9N$, so that c is positive.

For blocks in classes 1 and 2, we proceed as in the proof of Theorem 11 and Theorem 12. We ignore ceilings and justify after the constructions that they may be safely disregarded. For blocks $S_{s,t,u,w}$ and $S^*_{s,t,u,w}$ the construction is very similar as before: for the new vertex b (among the r_i added ones) representing a block of such a subclass, take three sorted vertices b_1, b_2, b_3 from U and make them adjacent to b. Take $5 + cs$, (ct, cu, respectively) vertices from U, and attach them to the first of these sorted vertices (second and third, respectively). Denote by x_1,x_2,x_3,x_4,x_5 the vertices with smallest labels (in this order) of the first group. Delete all edges from the original block except for the edges incident to the w vertices (excluding v,v_1,v_3,v_5, if the edges are present) that are connected with both v_2 and v_4. Append the special vertices v_1, v_2, v_3,

v_4 and v_5 to the vertices x_1,x_2, x_3, x_4, x_5 in this order. Connect then the s vertices (which originally were adjacent to v and v_2) to the cs vertices of the first group (excluding $x_1, \ldots, x_5$) in all possible ways, and do the analogous construction for the t and u vertices. Note that this time we might construct cycles of length 6 (of the type $b_1, x_2, v_2, a, v_4, x_4$), where a is one of the w vertices connecting v_2 and v_4, but by the special roles of special vertices unique reconstruction is still guaranteed.

For blocks of type $V_{s,t,E}$ and $V^*_{s,t,E}$, and its corresponding vertex b representing the block, take eight sorted vertices $b_1, \ldots, b_8$ from U and make them adjacent to b. Take ce_1, ce_2, ce_3, ce_4 elements from U and add them to b_1, b_2, b_3, b_4, respectively. Take $5 + cs$ elements from U (call the vertices with the 5 smallest labels $x_1, \ldots, x_5$, in this order, as before), make them adjacent to b_7, and take ct elements from U and make them adjacent to b_8. From the original block delete all edges emanating from v, v_2, v_4, all edges between special vertices, all edges going between v_3 and any of the e_1, e_2 vertices of the first and second group of E. For the e_3 graphs of the third group of E, for any i, $1 \leq i \leq e_3$, the edges between the vertices of degree q_i (different from v_3) and its q_i neighbors of degree 2 are retained, and all others are deleted, and analogously for the e_4 graphs of the forth group. For the e_5 and e_6 graphs of the fifth and sixth group of E, all edges of it are deleted if the vertex of degree q_i (different from v) satisfies $q_i > \frac{\log\log\log n}{\log\log\log\log n}$, otherwise all edges going between the vertex of degree q_i (different from v) and its q_i neighbors different from v and v_3 are retained and the others are deleted. Now, connect $v_1, \ldots, v_5$ with $x_1, \ldots, x_5$. For the e_1 vertices originally connecting v and v_3, connect them to the ce_1 vertices (which were attached to b_1) in all possible ways. For the e_2 pairs adjacent to each other and both connecting v and v_3, connect the one with smaller label in all possible ways to the ce_2 vertices attached to b_2 (recall that the edge connecting such a pair is not deleted). For the e_3 and e_4 double stars K_{2,q_i}, connect all vertices of degree q_i (different from v_3) and its pending q_i neighbors with the ce_3 and ce_4 vertices attached to b_3 and b_4, respectively, in all possible ways. For the e_5 graphs K_{2,q_i} emanating from v (of degrees $q_1, \ldots, q_{e_5}$), take $\frac{c}{2}e_5$ vertices from U, attach them to b_5, and connect each of the e_5 vertices $z_1, \ldots, z_{e_5}$ of degree $q_1, \ldots, q_{e_5}$ to the $\frac{c}{2}e_5$ vertices in all possible ways. Then, for each of the z_i ($1 \leq i \leq e_5$), do the following: if $q_i \leq \frac{\log\log\log n}{\log\log\log\log n}$, do nothing (recall that the neighbors of z_i are still pending). Otherwise, take $\frac{c}{2}q_i$ vertices from U and make them adjacent to z_i. Connect each of the q_i vertices (originally neighbors of z_i) in all possible ways to the newly attached $\frac{c}{2}q_i$ vertices. The analogous construction is done for e_6 (with b_6 instead of b_5). Finally, connect the s vertices originally connecting v and v_2 (excluding special vertices) with the group of cs new vertices (excluding $x_1, \ldots, x_5$) attached to b_7 in all possible ways. Similarly, connect the t vertices originally connecting v and v_4 with the group of ct new vertices attached to b_8 in all possible ways. Here, the graph constructed is always a tree, and reconstruction is unique.

For blocks of type $\widetilde{S}_{s,t,u,w}$ and $\widetilde{S}^*_{s,t,u,w}$, the strategy is similar as before. Assume without loss of generality that v_2 plays the role of v. In this case we take three vertices from U (sorted) and make them adjacent to the vertex representing this block. Take $5 + cs$ new vertices from U, make them adjacent to the first one, then cu further ones, make them adjacent to the second one, and finally another cw, which are made adjacent to the third one. All edges are deleted except for edges between v_4 and its t non-special neighbors that were also connected with v. The 5 vertices of the first group with smallest labels are connected to special vertices, and the s, u and w neighbors of v_2 (except for special vertices) are, as before, connected in all possible ways with the cs, cu and cw vertices of the respective groups. Observe that the constructed graph is a tree.

For blocks of type $\widetilde{V}_{s,t,E}$ and $\widetilde{V}^*_{s,t,E}$, either of v_2, v_3, v_4 or any of the external vertices in double stars of degree $q \geq \frac{C \log n}{\log\log\log\log n}$ arising in the groups e_3, e_4, e_5, e_6 may play the role of v. In all cases, edges between special vertices are always deleted. If v_3 plays the role of v, all edges between v_2 and its s neighbors that are connected with v, and all edges between v_4 and its t neighbors that are connected with v are retained;

for the others deletion is as for $V_{s,t,E}$ and $V^*_{s,t,E}$ with v_3 playing the role of v. In more detail, for each vertex b representing a block of such a subclass, six sorted vertices $b_1, \ldots, b_6$ from U are added, 5 extra vertices taking care of special vertices are attached to b_1, say, and the previous constructions restricted to $b_1, \ldots, b_6$ with v_3 playing the role of v is performed. If v_2 or v_4 (assume v_2 without loss of generality) plays the role of v, all edges emanating from a neighbor of v_2 are deleted (in particular, all edges emanating from v_3). For each block of such a subclass with b the newly attached vertex representing the block of this subclass, add $5 + cs$ vertices to b. The 5 vertices with smallest labels (in order) are attached to v,v_1,v_3,v_4 and v_5, respectively. The s vertices originally connected to v_2 are then connected in all possible ways to the cs vertices excluding the 5 special vertices. Note that cycles of length 6 such as $b, x_2, v, a, v_4, x_4, b$ with a being one of the t vertices connecting v with v_4 can occur and x_2 (x_4, respectively) one of the special neighbors of b to which v (v_4, respectively) is attached, but the special role of the special vertices still guarantees unique reconstruction. For the s edges emanating from v_2 the usual reconstruction is performed (again with 5 special vertices assuring unique reconstruction, and another cs vertices to which the s vertices may connect in all possible ways). If any of the external vertices a of degree q plays the role of v, say w.l.o.g. a neighbor of v, the procedure is very similar: all edges emanating from a neighbor of a, in particular all edges emanating from v, are deleted. Then, $5 + cq$ vertices are taken from U, and the q neighbors of a are connected in all possible ways to the cq new vertices (the 5 vertices take care of special vertices). Note that again cycles of length 6 can be constructed (in case a is a neighbor of v_3), but by the special roles of special vertices reconstruction is still unique.

Observe that the largest cycle created is of length at most 6, and in all cases the special vertices guarantee unique reconstruction. Observe also that the number of pendant vertices used is at most $h = ck(1+o(1))$: for contributions of type e_5 and e_6 in components $V_{s,t,E}$,$V^*_{s,t,E}$ (and of type e_3 and e_4 in components $\widetilde{V}_{s,t,E}$,$\widetilde{V}^*_{s,t,E}$ with v_3 playing the role of v), at the first level $\frac{c}{2}e_5$ vertices are used, and at the second level, at most $\frac{c}{2}\sum_{i=1}^{e_5} q_i(1+o(1))$ (note that ceilings may be safely disregarded, as only for sufficiently large q_i these vertices are chosen), and since $e_5 \le \sum_{i=1}^{e_5} q_i$, the total number is at most $c\sum_{i=1}^{e_5} q_i$. For the other contributions it is obvious. As before, for each case we count the number of different graphs we obtain by applying this construction, yielding similar multinomial coefficients and other factors as before. Then we divide by the number of preimages of a new graph, which is at most n^{h+1}, and take logarithms. Similar calculations as before show that the most negative term is $-\log n$, coming from the choice of the vertex v. Recall that N is the total number of subclasses.

Now, if at least k/N of the degree of v is in blocks of size at most $\frac{\log k}{\log\log\log k}$, then the number of such blocks r_1 is at least $\frac{k \log\log\log k}{N \log k}$. By the same arguments as before, the constructions of these blocks give a term $r_1 \log r_1 \ge \frac{k \log\log\log k}{N \log k}(\log k + o(\log k)) = \frac{k}{N}\log\log\log k(1 + o(1))$, and for c as chosen and C large enough this is bigger than the (negative) term $\log n$. Otherwise, suppose that at least k/N of the degree of v results from any fixed subclass of blocks excluding $V_{s,t,E}$ or $V^*_{s,t,E}$ (and also excluding $\widetilde{V}_{s,t,E}$ and $\widetilde{V}^*_{s,t,E}$ with v_3 playing the role of v). Letting r_j denote the number of such blocks, by similar calculations as before, as there is only one level of choice, we obtain a positive term $\Theta(k \log \frac{k}{r_j})$. Since $r_j \le \frac{k \log\log\log k}{\log k}$,

$$\Theta(k \log \frac{k}{r_j}) = \Omega(k \log\log k),$$

which is asymptotically bigger than $\log n$.

Hence, assume that k/N of the degree of v comes from a subclass j in $V_{s,t,E}$ or $V^*_{s,t,E}$ (or $\widetilde{V}_{s,t,E}$ and $\widetilde{V}^*_{s,t,E}$ with v_3 playing the role of v), and assume without loss of generality that it belongs to the class $V_{s,t,E}$. Let again be $r_j \le \frac{k \log\log\log k}{\log k}$ the number of blocks of this subclass. If at least $k/(2N)$ of the total degree comes from contributions of the groups of s, t, e_1, e_2, e_3, e_4 in the blocks of $V_{s,t,E}$, then, as before, only considering those terms, as there is one level of choice, we obtain a term $\Theta(k \log \frac{k}{r_j}) = \omega(\log n)$.

Hence, we may assume without loss of generality that $k/(4N)$ of the total degree comes from contributions of group e_5. Once more, we split this into two subcases: if at least $k/(8N)$ of the total degree comes from double stars $K_{2,q}$ with $q \leq \frac{\log\log\log n}{\log\log\log\log n}$, then at least $z \geq \frac{k \log\log\log\log n}{8N \log\log\log n}$ such double stars $K_{2,q}$ are needed. Denote by z_i the number of double stars inside the ith block to z, for $1 \leq i \leq r_j$. Each such block gives a term $z_i \log z_i$, and the total contribution is by Lemma 7 minimized when the number of double stars is equally split among all blocks. Assuming the worst case of $r_j = \frac{k \log\log\log k}{\log k}$ and $z = \frac{k \log\log\log\log n}{8N \log\log\log n}$, the total contribution is thus at least

$$(1+o(1))(1-c)\left(z \log \frac{z}{r_j}\right) = (1+o(1))(1-c)\left(z \log\log\log n\right) = (1+o(1))(1-c)\frac{C \log n}{8N},$$

which for our choice of c and C large enough is bigger than $\log n$. If on the other hand at least $k/(8N)$ of the total degree comes from double stars $K_{2,q}$ with $q > \frac{\log\log\log n}{\log\log\log\log n}$, then first observe that the number z of double stars $K_{2,q}$ contributing to the total degree satisfies $z \leq \frac{k \log\log\log\log n}{8N \log\log\log n}$. Recall that q_i denotes the degree of the ith double star for $1 \leq i \leq z$. Clearly, $\sum_{i=1}^{z} q_i \geq k/(8N)$. Each such double star gives on the second level of choice rise to a term $(1-c)q_i \log q_i$. Assume again the worst case $\sum_{i=1}^{z} q_i = k/(8N)$ and $z = \frac{k \log\log\log\log n}{8N \log\log\log n}$. This contribution is, once more by Lemma 7, minimized if the contribution is split evenly, that is, $q_i = \frac{k}{8Nz}$, and in this case we obtain

$$(1+o(1))(1-c)\left(\frac{k}{8N} \log \frac{k}{8Nz}\right) = (1+o(1))(1-c)\left(\frac{k}{8N} \log\log\log\log n\right) = (1+o(1))(1-c)\frac{C \log n}{8N},$$

which for our choice of c and C large enough again is bigger than $\log n$. Hence, in all cases, $C(n)/R(n) \to \infty$, as desired, and the proof is finished.

$\square$

5 Conclusion and open problems

Our work suggests several conjectures and open problems.

1. We conjecture that the lower bound
$$\Delta_n \geq c \frac{\log n}{\log^{(\ell+1)} n}$$
for the class $\mathrm{Ex}(C_{2\ell+1})$ is of the right order of magnitude. The proofs for $\mathrm{Ex}(C_5)$ and $\mathrm{Ex}(C_7)$ seem difficult to adapt for arbitrary ℓ.

2. We conjecture that the asymptotic behaviour of Δ_n is the same for $\mathrm{Ex}(C_{2\ell})$ as for $\mathrm{Ex}(C_{2\ell-1})$. We have shown this is the case for $\ell = 2$ and $\ell = 3$.

3. We conjecture an upper bound of the form
$$\Delta_n \leq c \log n$$

for the class $\mathrm{Ex}(H_1, \ldots, H_k)$, whenever the H_i are 2-connected (see also the concluding remarks of [12], where this question was also asked). Examples show that this is not true for arbitrary H (see the discussion below). Using analytic methods, this upper bound can be proved for so-called subcritical classes of graphs (see [6]), which include outerplanar and series-parallel graphs.

4. Which are the possible orders of magnitude of Δ_n when forbidding a 2-connected graph? Assuming the truth of the conjecture in item 1, are there other possibilities besides $\log n$ and $\log n / \log^{(k+1)} n$?

5. Which are the possible orders of magnitude of Δ_n for arbitrary minor-closed classes of graphs? Besides those discussed above, examples show that it can be constant (forbidding a star) and it can be linear (forbidding two disjoint triangles). The last statement follows from [10], where it is shown that the class $\mathrm{Ex}(C_3 \cup C_3)$ is asymptotically the same as the class of graphs G having a vertex v such that $G - v$ is a forest.

6. Is it true that if H consists of a cycle and some chords, all of them incident to the same vertex, then $\Delta_n = o(\log n)$ holds in $\mathrm{Ex}(H)$ w.h.p.? These are the 2-connected graphs that are a minor of some fan F_n, so that the proof of the first part in Theorem 3 does not hold.

7. Prove an upper bound $\Delta_n \leq c \log n$ for series-parallel graphs without using the analysis of generating functions as in [4]. More generally, prove such a bound for graphs of bounded tree-width (series-parallel graphs are those with tree-width at most two). For outerplanar graphs this is easy, but we decided to leave out the proof of this result.

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