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► **To cite this version:**

Audric Drogoul, Romain Veltz. Exponential stability of stationary distributions of a mean field of spiking neural network. [Research Report] RR-8899, INRIA Sophia Antipolis - Méditerranée. 2016. hal-01290264v1

**HAL Id: hal-01290264**

**<https://inria.hal.science/hal-01290264v1>**

Submitted on 8 Apr 2016 (v1), last revised 13 Apr 2018 (v2)

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# Exponential stability of stationary distributions for some nonlocal transport equations: application to neural dynamics

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**RESEARCH  
REPORT**

**N° 8899**

April 2016

Project-Team Mathneuro





## Exponential stability of stationary distributions for some nonlocal transport equations: application to neural dynamics

Audric Drogoul\*, Romain Veltz\*

Project-Team Mathneuro

Research Report n° 8899 — April 2016 — 16 pages

**Abstract:** In this work, we study the exponential stability of the stationary distribution for a McKean-Vlasov equation recently derived in [N. Fournier and E. Löcherbach. *Ann. Inst. H. Poincaré Probab. Statist.*, accepted]. We complete the convergence result proved in their paper. Our proof relies on two principal arguments. First, the linearized semigroup is positive which allows to precisely pinpoint the spectrum of the infinitesimal generator ; this is a remarkable property. Second, we use a time rescaling argument to transform the original quasilinear equation into another problem more easy to study and which enable to characterise the first one. Interestingly, this convergence result can be interpreted as the existence of a locally exponentially attracting center manifold.

**Key-words:** McKean–Vlasov equations, stationary distribution, nonlinear stability, center manifold.

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\* Inria, Mathneuro team, Sophia Antipolis, France

**RESEARCH CENTRE  
SOPHIA ANTIPOLIS – MÉDITERRANÉE**

2004 route des Lucioles - BP 93  
06902 Sophia Antipolis Cedex

## **Stabilité exponentielle de distributions stationnaires pour une équation de transport non-locale: application à la dynamique de neurones**

**Résumé :** Dans ce travail, nous montrons la stabilité exponentielle de la distribution stationnaire pour une équation de McKean-Vlasov récemment introduite dans [N. Fournier and E. Löcherbach. *Ann. Inst. H. Poincaré Probab. Statist.*, accepted] . On complète la convergence prouvée dans leur papier. Notre preuve repose sur deux arguments principaux. Premièrement, le semi-groupe linéarisé est positif ce qui permet de déterminer précisément le spectre du générateur infinitésimal; ceci est une propriété remarquable. Deuxièmement, nous utilisons un changement de variable en temps pour transformer l'équation quasi-linéaire originale en un problème plus facile à étudier et qui permet de caractériser le premier. Fait intéressant, cette convergence peut être interprétée comme l'existence d'une variété centrale exponentiellement attractive.

**Mots-clés :** équation de McKean–Vlasov, distribution stationnaire, stabilité non linéaire, variété centrale

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## Notations

- $a(\phi) = \int_0^\infty f(x)\phi(x) + \lambda x\phi(x)dx$
- $I(\phi) = \int_0^\infty \phi$
- $\mathcal{X} = L^1(\mathbb{R}_+)$
- $\mathcal{X}_f = \{\phi \in L^1(\mathbb{R}_+), f\phi \in L^1(\mathbb{R}_+)\}$
- $\hat{\mathcal{X}} = \{\phi \in \mathcal{X} \mid \int_0^\infty \phi = 0\}$
- $L_1^+(\mathbb{R}_+) = \{\phi \in L^1(\mathbb{R}_+), \phi \geq 0 \text{ a.e.}\}$
- $H$ : Heaviside function
- $\mathbb{C}_{\leq a} = \{z \in \mathbb{C} \mid \Re z \leq a\}$ ,  $\mathbb{R}_{\leq a} = \{z \in \mathbb{R} \mid \Re z \leq a\}$

## 1 Introduction

In [FL14], the authors derived mean field equations for a network of excitatory spiking neurons in the limit of a large number of neurons (see also [RT14]). It is based on a recently published model of simple neuron [DMGLP15] in which the spiking dynamics is modelled with a jump process rather than threshold crossing [GKNP14] or blow up [Izh07]. The distribution  $g$  of the limiting process solves:

$$\begin{cases} \frac{\partial}{\partial t} g = [\lambda x - a(g)] \partial_x g + [\lambda - f(x)] g \\ g(t, 0) = \frac{f(g)}{a(g)} \\ g(0, \cdot) \in L_1^+(\mathbb{R}_+) \end{cases}$$

where  $f(x) > 0$ , is the rate function, on  $\mathbb{R}_{>0}$ . Following [FL14], we make the following assumption concerning the rate function  $f$ :

*Hypothesis 1*  $f$  is convex increasing,  $f(0) = 0$ ,  $f(x) > 0$  for all  $x > 0$ ,  $\lim_{x \rightarrow \infty} f = \infty$  and  $f \in C^2(\mathbb{R}_+)$ . Further assume that  $f$  is convex and that  $\sup_{x \geq 1} \frac{f'(x)}{f(x)} + \frac{f''(x)}{f'(x)} < \infty$ .

*Remark 1* Grant Hypothesis 1 we have:

- (i) There is  $c > 0$  such that  $f(x) \geq cx$  for all  $x \geq 1$ .
- (ii) For all  $A > 0$ , there exists  $C_A > 0$  such that for all  $x \geq 0$ ,  $f(x + A) \leq C_A(1 + f(x))$ .
- (iii) There is  $C > 0$  such that  $f(x) \leq C \exp(Cx)$  for all  $x \geq 0$ .

In addition to the derivation of the mean field equations, the authors of [FL14] computed an analytical formula for the stationary distributions of the mean field equations. In the case  $\lambda = 0$ , they were able to prove that

$$\|g(t) - g_\infty\|_{L^1} \rightarrow 0$$

where  $g_\infty$  is the unique stationary distribution of the equations (note that it has a density) for some initial conditions. In the case where  $f(x) \geq cx^\xi$  for all  $x \in [0, 1]$  where  $c > 0, \xi \geq 1$ , they show that the above convergence is  $O((1+t)^{-1/\xi})$ .

The type of equations considered here is well known in the population dynamics literature [GM74, Pru83, Web85, BA06, Per07] but a complete analogy with (1) would require to introduce infinite birth / death rates of the species which is very little studied for modeling reasons. Another noticeable difference lies in the fact the equations are considered on a non compact domain here.

**The main result of this paper deals with the case  $\lambda = 0$ , we will restrict to this case latter.** This has the advantage of removing the pre-factor  $\lambda x$  which allows us to use a time rescaling to change the type of equation from a quasilinear one to a semilinear one [Paz83]. It also removes the boundary condition. The equation reads in this case

$$\begin{cases} \partial_t g(t, x) = -a(g)\partial_x g(t, x) - f(x)g(t, x) \\ g(t, 0) = 1. \end{cases} \quad (1)$$

In this work, we revisit the convergence to the stationary distribution from a dynamical system point of view in order to prove that the convergence is exponential. There is a 1d family of stationary solutions and there is a unique stationary distribution (of integral one). The family is given by:

$$g_\alpha(x) = \exp\left(-\frac{1}{\alpha} \int_0^x f\right), \int_0^\infty f g_\alpha = \alpha > 0. \quad (2)$$

The existence of this family implies that zero is in the spectrum of the linearized equation: the principle of linearized stability does not apply. There are several strategies to prove the nonlinear stability of  $g_\alpha$  in this case apart from entropy methods [Per07].

The first relies on the existence of a center manifold because zero is in the spectrum. The analysis of the spectrum (see proposition 2.2) shows that the center manifold should be one dimensional. Hence, the center manifold would be in fact the family  $(g_\alpha)_\alpha$ . To sum up, to prove the nonlinear stability, one would need to prove the existence of the center manifold, identify it with the family  $(g_\alpha)_\alpha$  and prove that this family is exponentially attracting [HG11]. Unfortunately, it is difficult to achieve such program as it relies heavily on the fact that the linear flow must be regularizing, which in the case of transport equations, requires to use very regular initial conditions.

The second idea, which we shall rely on, starts with the observation that the flow of (1) conserves the mass. Hence, the nonlinear flow is foliated by the mapping  $g \rightarrow \int_0^\infty g$ . The dynamics on each hyperplane possesses a unique equilibrium which does not contain zero in its spectrum anymore. Hence, one can hope proving nonlinear stability by simpler means in this case.

Using the second idea, we prove the existence of an exponentially attracting center manifold which is transverse to the hyperplanes associated to the linear form  $g \rightarrow \int_0^\infty g$ . This is remarkable as such general result is not known for transport equations in general and for quasilinear equations. It is for example well known for delay differential equations [Hal14, DVLvGW95, vGJKV12, VF13] which are a kind of transport equation with nonlinear boundary condition.

The plan of the paper is as follows. We first study the linearized equation around a stationary point  $g_\alpha$  in the space  $L^1(\mathbb{R}_+)$ . Then, we transform the nonlinear equation and reduce the state space to  $\mathcal{Y}$  so the the equation is of semilinear type for which nonlinear stability results are well-known.



## 2 Linear analysis

Let us consider the unique [FL14] stationary point  $g_\infty$  of the family  $(g_\alpha)$  such that  $\int_0^\infty g_\infty = 1$ . If we write  $g(t, x) = g_\infty(x) + \phi(t, x)$ , we find:

$$\begin{cases} \partial_t \phi(t, x) + a_\infty \partial_x \phi(t, x) + f(x) \phi(t, x) = -a(\phi) g'_\infty(x) - a(\phi) \partial_x \phi(t, x) \\ \phi(t, 0) = 0. \end{cases} \quad (3)$$

Let us define  $\mathcal{X} = L^1(\mathbb{R}_+)$  and the following unbounded linear operator on  $\mathcal{X}$ :

$$\mathbf{A}_0 \phi = -a_\infty \phi' - f \phi, \quad D(\mathbf{A}_0) = \{\phi \in \mathcal{X}, \phi' \in \mathcal{X}, f \phi \in \mathcal{X}, \phi(0) = 0\}.$$

We prove the following result concerning the well-posedness of the linear equation. It is based on fairly standard tools in semigroup theory,

**Proposition 2.1** *Let us define a semigroup  $(\mathbf{T}_0(t))_{t \geq 0}$  by the formula*

$$\mathbf{T}_0(t) \phi(x) = \exp\left(-\frac{1}{a_\infty} \int_{x-a_\infty t}^x f\right) \phi(x - a_\infty t) H(x - a_\infty t) \quad (4)$$

Then we have the following properties:

1.  $(\mathbf{T}_0(t))_{t \geq 0}$  is a positive contraction  $C_0$ -semigroup on  $\mathcal{X}$ ,
2. Its infinitesimal generator is given by  $(\mathbf{A}_0, D(\mathbf{A}_0))$ ,
3.  $\Sigma(\mathbf{A}_0) = \emptyset$  and the growth bound of  $(\mathbf{T}_0(t))_{t \geq 0}$  is  $\omega_0 = -\infty$ ,
4.  $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}_0 + \mathbf{B}$  where  $\mathbf{B} \phi \stackrel{\text{def}}{=} -a(\phi) g'_\infty$ , generates a positive strongly continuous semigroup on  $\mathcal{X}$  and  $D(\mathbf{A}) = D(\mathbf{A}_0)$ . Moreover,  $(\mathbf{T}(t))_{t \geq 0}$  is eventually norm continuous on  $\mathcal{X}$ .

*Proof.*

1. The semigroup/positivity properties are clear. Let us show that  $\mathbf{T}_0(t)$  leaves  $\mathcal{X}$  invariant. By definition

$$\begin{aligned} \|\mathbf{T}_0(t) \phi\|_{\mathcal{X}} &= \int_{a_\infty t}^\infty \exp\left(-\frac{1}{a_\infty} \int_{x-a_\infty t}^x f\right) |\phi(x - a_\infty t)| dx \\ &= \int_0^\infty \exp\left(-\frac{1}{a_\infty} \int_x^{x+a_\infty t} f\right) |\phi(x)| dx \leq \|\phi\|_{\mathcal{X}}. \end{aligned}$$

This shows that  $\mathbf{T}_0$  is a contraction. Let us now show the strong continuity (with  $a_\infty = 1$  for simplicity):

$$\|\mathbf{T}_0(t) \phi - \phi\|_{\mathcal{X}} \leq \int_0^\infty \left| \exp\left(-\int_x^{x+t} f\right) \phi(x) - \phi(x+t) \right| dx + \int_0^t |\phi(x)| dx$$

The two integrals tends to zeros when  $t \rightarrow 0$  by dominated convergence. This allows to finish the proof of item 1.

2. As  $\mathbf{T}_0$  is a contraction semigroup, we know that the resolvent of the infinitesimal generator satisfies  $R(\mu, A_0) = \int_0^\infty e^{-\mu t} \mathbf{T}_0(t) dt$  for  $\Re \mu > 0$  and that  $R(\mu, A_0) \mathcal{X} = D(\mathbf{A}_0)$ . We find that

$$\psi \equiv R(\mu, \mathbf{A}_0) \phi(x) = \frac{g_\infty(x)}{e^{\mu x / a_\infty}} \int_0^x \frac{e^{\mu t / a_\infty}}{g_\infty(t)} \phi(t) \frac{dt}{a_\infty}.$$

It follows that  $\psi \in W_{loc}^1(\mathbb{R}_+)$ ,  $\psi(0) = 0$ . Finally, we find using Fubini theorem that for all  $\mu \in \mathbb{C}_{\geq 0}$ :

$$\begin{aligned} a_\infty \|f\psi\|_{\mathcal{X}} &\leq \int_{\mathbb{R}_+^2} dx dy 1(y \leq x) f(x) \frac{g_\infty(x)}{g_\infty(y)} e^{-\Re\mu(x-y)/a_\infty} |\phi(y)| \\ &= \int_0^\infty dy |\phi(y)| \left[ \int_y^\infty f(x) \frac{g_\infty(x)}{g_\infty(y)} e^{-\Re\mu(x-y)/a_\infty} dx \right] \\ &\leq \int_0^\infty dy |\phi(y)| \left[ \int_y^\infty f(x) \frac{g_\infty(x)}{g_\infty(y)} dx \right] \leq a_\infty \|\phi\|_{\mathcal{X}}. \end{aligned}$$

For the last equality, we used  $\int_y^\infty f(x) \frac{g_\infty(x)}{g_\infty(y)} dx = \left[ -a_\infty \exp\left(-\frac{1}{a_\infty} \int_y^x f\right) \right]_y^\infty \leq a_\infty$ . This means

$$a(R(\mu, \mathbf{A}_0)\phi) \leq \|\phi\|_{\mathcal{X}}.$$

It follows that  $f\psi \in \mathcal{X}$ . Also:

$$a_\infty \psi'(x) + f(x)\psi(x) = \phi(x) - \mu\psi(x) \in \mathcal{X} \quad (5)$$

Hence, we have shown that  $\psi' \in \mathcal{X}$  using (5). Finally, (5) shows that  $\mathbf{A}_0\phi = -a_\infty\phi' - f\phi$ . It follows that

$$D(\mathbf{A}_0) \subset \{\phi \in \mathcal{X}, f\phi \in \mathcal{X}, \phi' \in \mathcal{X}, \phi(0) = 0\}.$$

Reciprocally, if we consider  $\psi \in \{\phi \in \mathcal{X}, f\phi \in \mathcal{X}, \phi' \in \mathcal{X}, \phi(0) = 0\}$  and take  $\phi = a_\infty\psi'(x) + f(x)\psi(x) + \mu\psi(x)$ , we find that  $\phi \in \mathcal{X}$  and  $\psi = R(\mu, \mathbf{A}_0)\phi$ . This concludes item 2.

3. From the definition of the growth bound  $\omega_0 = \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|\mathbf{T}(t)\| \leq Me^{\omega t} \forall t \geq 0\}$ , it is well known that  $\omega_0 \geq \sup\{\Re\mu, \mu \in \Sigma(\mathbf{A}_0)\} \equiv s(\mathbf{A}_0)$ . From

$$\|\mathbf{T}_0(t)\phi\|_{L_+^1} = \int_0^\infty \exp\left(-\frac{1}{a_\infty} \int_x^{x+a_\infty t} f\right) |\phi(x)| dx$$

we get that

$$\|\mathbf{T}_0(t)\|_{\mathcal{L}(\mathcal{X})} = \sup_{x \geq 0} \left[ \exp\left(-\frac{1}{a_\infty} \int_x^{x+a_\infty t} f\right) \right] = \exp\left(-\frac{1}{a_\infty} \int_0^{a_\infty t} f\right).$$

It gives  $\omega_0 = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\mathbf{T}_0(t)\| = -\infty$ .

4. We first note that  $D(\mathbf{A}_0) \subset D(\mathbf{B})$ . We compute for all  $\mu \in \mathbb{C}_{\geq 0}$

$$\|\mathbf{B}R(\mu, \mathbf{A}_0)\phi\|_{\mathcal{X}} = \|g'_\infty\|_{\mathcal{X}} \cdot |a(R(\mu, \mathbf{A}_0)\phi)| \stackrel{\text{see 2.}}{\leq} \|g'_\infty\|_{\mathcal{X}} \|\phi\|_{\mathcal{X}} \quad (6)$$

which shows that

$$\|\mathbf{B}\phi\|_{\mathcal{X}} \leq \|g'_\infty\|_{\mathcal{X}} \|(\mu - \mathbf{A}_0)\phi\|_{\mathcal{X}}, \forall \phi \in D(\mathbf{A}_0).$$

Similarly, one can show that  $\|\mathbf{A}_0\mathbf{B}\phi\|_{\mathcal{X}} = O(\|(\mu - \mathbf{A}_0)\phi\|_{\mathcal{X}})$ . Hence,  $\mathbf{B}$  is continuous on the Sobolev space  $\mathcal{X}_1^{\mathbf{A}_0}$ , i.e.  $\mathbf{B} \in \mathcal{L}(\mathcal{X}_1^{\mathbf{A}_0})$ , defined as the space  $(D(\mathbf{A}_0), \|(\mu - \mathbf{A}_0)\cdot\|_{\mathcal{X}})$ . It is then well known that  $\mathbf{A}$  generates a  $C_0$ -semigroup on  $\mathcal{X}_1^{\mathbf{A}_0}$  as a bounded perturbation of an infinitesimal generator. By extrapolation, this is also true on  $\mathcal{X}_0^{\mathbf{A}_0} \stackrel{\text{def}}{=} \mathcal{X}$ .

The semigroup  $\mathbf{T}_0$  is positive and  $\mathbf{B}$  is positive, hence [Are86] shows that  $\mathbf{T}$  is positive on  $\mathcal{X}_1^{\mathbf{A}}$ , hence on  $\mathcal{X}$  by extrapolation.

As  $\mathbf{B}$  is compact and  $\mathbf{T}_0$  is norm continuous (see point 3.), we can conclude that  $\mathbf{T}$  is eventually norm continuous on  $\mathcal{X}_1^{\mathbf{A}}$  by using [EN00][III.1.14]. By extrapolation, it is norm continuous on  $\mathcal{X}$ .

*Remark 2* Using more advanced perturbation results of Desch-Schappacher, we can prove that  $\mathbf{A}$  generates a positive strongly continuous semigroup on  $\mathcal{X}$ . From [Mä08], it follows that the semigroup is immediately norm continuous.

The previous results show that we can give a meaning to the linearized equation on  $\mathcal{X}$ . To investigate the stability of the equilibrium  $g_\infty$ , we need to understand the properties of  $\Sigma(\mathbf{A})$ . This is achieved in the following proposition.

**Proposition 2.2** *We have the following spectral properties for the semigroup  $(\mathbf{T}(t))_{t \geq 0}$  on  $\mathcal{X}$  and its generator:*

- the spectral mapping holds for the semigroup, namely

$$\Sigma(\mathbf{T}(t)) \setminus \{0\} = e^{\Sigma(\mathbf{A})t},$$

- the spectrum of  $(\mathbf{A}, D(\mathbf{A}))$  is composed of isolated eigenvalues solutions of

$$\Delta(\mu) \stackrel{\text{def}}{=} 1 - \frac{1}{a_\infty^2} \int_0^\infty f(x)g(x) \int_0^x f(y)e^{-\frac{\mu}{a_\infty}(x-y)} dy = 0,$$

- 0 is a simple eigenvalue of  $\mathbf{A}$  and  $s(\mathbf{A}) = 0 \in \Sigma(\mathbf{A})$ , hence  $\Sigma(\mathbf{A}) \subset \mathbb{C}_{\leq 0}$ .
- $\Sigma(\mathbf{A}) \cap \mathbb{B}\mathbb{R} = \{0\}$

*Proof.*

- $(\mathbf{T}(t))_{t \geq 0}$  is eventually norm continuous (prop 2.2) which gives the spectral mapping.
- We know that  $\mathbf{A}_0$  is invertible, hence solving  $(\mu - \mathbf{A})\phi = \psi$  for  $\psi \in \mathcal{X}$  is equivalent to solving  $\phi - R(\mu, \mathbf{A}_0)\mathbf{B}\phi = R(\mu, \mathbf{A}_0)\psi$ . It follows that  $\phi$  exists if and only if  $1 + a(R(\mu, \mathbf{A}_0)g'_\infty) \neq 0$  which gives  $\Sigma(\mathbf{A}) = \{\mu, 1 + a(R(\mu, \mathbf{A}_0)g'_\infty) = 0\}$ . When  $\mu \notin \Sigma(\mathbf{A})$ , the solution is  $\phi = R(\mu, \mathbf{A}_0)\psi - \frac{a(R(\mu, \mathbf{A}_0)\psi)}{1 + a(R(\mu, \mathbf{A}_0)g'_\infty)} R(\mu, \mathbf{A}_0)g'_\infty$ . Finally, the spectrum is composed of eigenvalues and the eigenvectors are given by  $R(\mu, \mathbf{A}_0)g'_\infty$ . It is easy to check that  $\Delta(\mu) = 1 + a(R(\mu, \mathbf{A}_0)g'_\infty)$ .
- The semigroup  $\mathbf{T}$  being positive, we know that the spectral bound  $s(\mathbf{A})$  belongs to the spectrum of  $\mathbf{A}$ :  $s(\mathbf{A}) \in \Sigma(\mathbf{A}) \cap \mathbb{R}$ . Hence, we are looking for  $s(\mathbf{A})$  as the maximal real eigenvalue. It is then easy to check that  $\Delta$  is strictly increasing on  $\mathbb{R}$  and that  $\Delta(0) = 0$ . Indeed:

$$\Delta(0) = 1 - \frac{1}{a_\infty} \int g'(y) \int_0^y f \stackrel{IBP}{=} 1 - \frac{1}{a_\infty} \int gf \stackrel{\text{def}}{=} 0.$$

Finally  $\Delta'(0) \neq 0$  hence 0 is a simple eigenvalue.

- Since 0 is a pole of the resolvent and  $\mathbf{T}$  is eventually continuous, we conclude the item with [Are86][C-III, Corollary 2.13]

The fact that  $0 \in \Sigma(\mathbf{A})$  is easily seen from the existence of the family of equilibria (2) parametrized by  $\alpha$ . The flow associated to (1), stemming from the distribution of a stochastic process, conserves the integral of  $g$ . In fact, it is straightforward to show that this property also holds for the semigroup  $\mathbf{T}(t)$ . Hence, it is convenient to define

$$\hat{\mathcal{X}} = \left\{ \phi \in \mathcal{X} \mid \int_0^\infty \phi = 0 \right\}.$$

**Lemma 2.1** *The set  $\hat{\mathcal{X}} \subset \mathcal{X}$  is invariant by  $(\mathbf{T}(t))_{t \geq 0}$ . The generator of its restriction  $\mathbf{T}_\uparrow$  is the restriction of  $\mathbf{A}$  to  $\hat{\mathcal{X}}$ . Hence,  $D(\mathbf{A}_\uparrow) = D(\mathbf{A}) \cap \hat{\mathcal{X}}$  and the range of  $\mathbf{A}_\uparrow$  is in  $\hat{\mathcal{X}}$ . Finally, the spectrum is given by*

$$\Sigma(\mathbf{A}_\uparrow) = \{\mu \in \mathbb{C}, I(R(\mu, \mathbf{A}_0)g'_\infty) = 0\}.$$

*Proof.* As  $\hat{\mathcal{X}}$  is closed in  $\mathcal{X}$ , the first statement follows easily. Then we show that

$$\forall \mu \in \mathbb{C}, \quad a(R(\mu, \mathbf{A}_0)\phi) = \mu I(R(\mu, \mathbf{A}_0)\phi) + I(\phi). \quad (7)$$

The resolvent  $R(\mu, \mathbf{A})\psi = \phi$  solves  $\phi + a(\phi)R(\mu, \mathbf{A}_0)g'_\infty = R(\mu, \mathbf{A}_0)\psi$  and it gives  $I(R(\mu, \mathbf{A}_0)a(\phi)) = I(R(\mu, \mathbf{A}_0)\psi)$ . If  $I(R(\mu, \mathbf{A}_0)g'_\infty) \neq 0$ , then we have a candidate solution

$$\phi = R(\mu, \mathbf{A}_0)\psi - \frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} R(\mu, \mathbf{A}_0)g'_\infty.$$

We now check it works:

$$\begin{aligned} (\mu - \mathbf{A})\phi &= (Id - \mathbf{B}R(\mu, \mathbf{A}_0)) \left[ \psi - \frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} g'_\infty \right] \\ &= \psi + \left[ -\frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} + a(R(\mu, \mathbf{A}_0)\psi) - \frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} a(R(\mu, \mathbf{A}_0)) \right] g'_\infty \\ &\stackrel{(7)}{=} \psi + \left[ -\frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} + \mu I(R(\mu, \mathbf{A}_0)\psi) - \frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} (\mu I(R(\mu, \mathbf{A}_0)g'_\infty) + I(g'_\infty)) \right] g_\infty \\ &= \psi + \left[ -\frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} - \frac{I(R(\mu, \mathbf{A}_0)\psi)}{I(R(\mu, \mathbf{A}_0)g'_\infty)} (-1) \right] g_\infty = \psi. \end{aligned}$$

This shows that  $\Sigma(\mathbf{A}_\uparrow) = \{\mu \in \mathbb{C}, I(R(\mu, \mathbf{A}_0)g'_\infty) = 0\}$ .

**Corollary 2.1**  *$(\mathbf{T}_\uparrow(t))_{t \geq 0}$  is exponentially stable.*

*Proof.* We first check that  $0 \notin \Sigma(\mathbf{A}_\uparrow)$  because of the sign of  $g'_\infty$ . Using the identity  $1 + a(R(\mu, \mathbf{A}_0)g'_\infty) = \mu I(R(\mu, \mathbf{A}_0)g'_\infty)$  proved in the previous lemma, we conclude that  $\Sigma(\mathbf{A}) \setminus \{0\} = \Sigma(\mathbf{A}_\uparrow)$ . From proposition 2.2, we find that  $\Sigma(\mathbf{A}_\uparrow) \cap i\mathbb{R} = \emptyset$ . The zeros of  $\mu \rightarrow I(R(\mu, \mathbf{A}_0)g'_\infty) = 0$  being isolated, we conclude that  $\sup \Re \Sigma(\mathbf{A}_\uparrow) < 0$ . The spectral mapping theorem provides the conclusion to the proof.

### 3 Nonlinear stability

This section establishes the local exponential convergence of the solution  $g$  (1) to  $g_\infty$  (2), for all initial conditions  $g_0 = g_\infty + \phi$  with  $\phi$  in a neighborhood of zero and of integral zero. This result improves on some points those shown in [FL14]: authors have shown that  $\|g(t) - g_\infty\|_{L^1} \rightarrow 0$  as  $O((1+t)^{-1/\xi})$  if  $f(x) \geq x^\xi$  for all  $x \in [0, 1]$  where  $c > 0, \xi \geq 1$ , and for some initial condition  $g_0$  of integral one.

#### 3.1 Difficulties and notations

Let us rewrite (3) as

$$\begin{cases} \frac{d}{dt}\phi(t) = \mathbf{A}\phi(t) + \mathbf{R}_0(\phi(t)) \\ \phi(t, 0) = 0 \end{cases} \quad (8)$$

with  $\mathbf{R}_0(\phi) = -a(\phi)\phi'$ . As said previously, the main disadvantage of this formulation is the lost of regularity of the nonlinearity  $\mathbf{R}_0$  which is only continuous from  $D(\mathbf{A})$  into  $\mathcal{X}$ . Let us note that this nonlinearity stems from the state dependent speed  $a(\phi)$  of the transport term. To overcome this problem, inspired by [GH90], we perform a change of time variable in time in (1). Roughly speaking we set  $h(\tau_g(t), x) = g(t, x)$  with  $\tau_g = \int_0^t a(g)(s)ds$ . This change of variable is possible only if  $\tau_g(t)$  is invertible or equivalently if  $a(g)(t)$  is strictly positive.

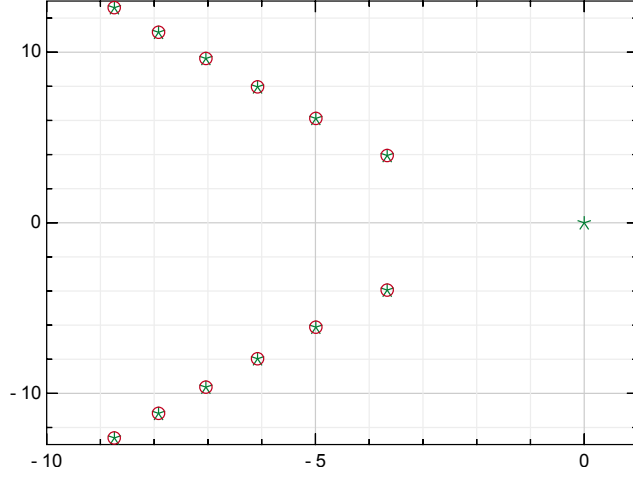


Figure 1: Rightmost part of the spectrum of  $\mathbf{A}$  (green) and  $\mathbf{A}_1$  (red) for  $f(x) = x^2$ . Computed using collocation methods provided by the *Julia* package *ApproxFun.jl*.

### 3.2 Time rescaling

In order to do a time rescaling, we introduce the following cut-off type velocity, strictly positive and locally identical to  $a(g)$  if this latter is close enough to  $a_\infty$ :

$$\tilde{a}(g) = a_\infty + \rho(a(g) - a_\infty), \quad \text{with} \quad \begin{cases} \rho(x) = x & \text{if } |x| \leq a_\infty - 2\eta \\ \rho(x) > -a_\infty + \eta & \forall x \in \mathbb{R} \\ \rho \in C^2(\mathbb{R}) \text{ non decreasing} & \text{and bounded in norm } C^2(\mathbb{R}) \end{cases} \quad (9)$$

where  $\eta$  is a constant such that  $0 < \eta < \frac{a_\infty}{2}$  (see Figure 2). By replacing  $a(g)$  by  $\tilde{a}(g)$  in (1) and by formally performing the time rescaling [GH90]:

$$h(\tau_g(t), x) = g(t, x) \quad \text{with} \quad \tau_g(t) = \int_0^t \tilde{a}(g(s, \cdot)) ds,$$

the function  $h(\tau, x)$  solves

$$\begin{cases} \partial_\tau h(\tau, x) + \partial_x h(\tau, x) = -\frac{fh}{\tilde{a}(h)} \\ h(\tau, 0) = 1. \end{cases} \quad (10)$$

By setting  $h = g_\infty + \phi$ , we obtain

$$\begin{cases} \frac{d}{d\tau} \phi(\tau) = \frac{1}{a_\infty} (\mathbf{A}\phi(\tau) + \mathbf{R}(\phi(\tau))) \\ \phi(\tau, 0) = 0. \end{cases} \quad (11)$$

with

$$\begin{aligned} \mathbf{R}(\phi) &= \frac{\rho(a(\phi))}{\tilde{a}(\phi)} [f\phi + \rho(a(\phi))g'_\infty] - [\rho(a(\phi)) - a(\phi)]g'_\infty \\ &= h(\phi)f\phi + R_1(\phi) \end{aligned}$$

with  $|h(\phi)| = |\rho(a(\phi))/\tilde{a}(\phi)| \leq C\|\phi\|_{\mathcal{X}_1^{\mathbf{A}}}$  and  $\|R_1(\phi)\|_{\mathcal{X}_1^{\mathbf{A}}} \stackrel{\text{prop. 2.1}}{\leq} C\|\phi\|_{\mathcal{X}_1^{\mathbf{A}}}^2$ .

Let us remark that

$$\mathbf{R}(\phi) = \frac{a(\phi)}{a_\infty + a(\phi)} (f\phi + a(\phi)g'_\infty), \quad \text{for } |\phi| \leq a_\infty - 2\eta$$

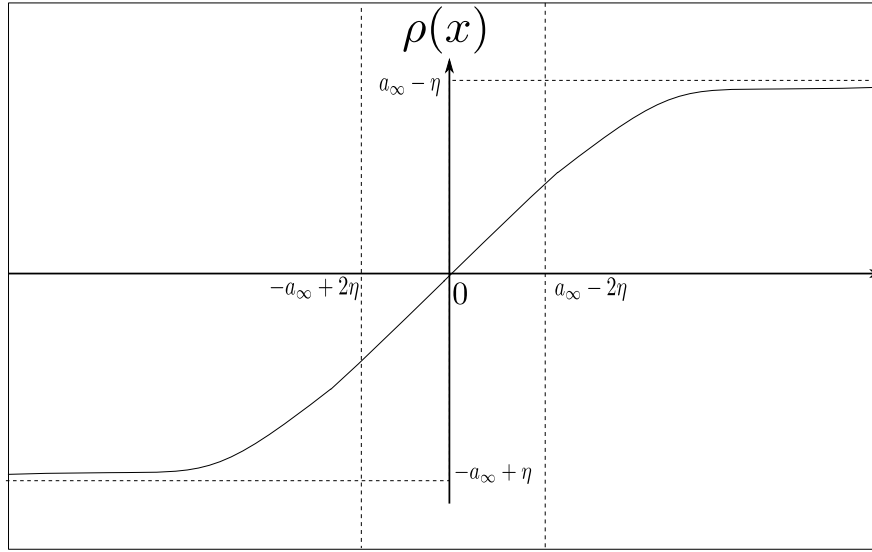


Figure 2: The cut-off function

### 3.3 Analysis of the rescaled equation

From [FL14] we can define the non linear semigroup  $\mathbf{S}_g(t)$  associated to (1)[Theorem 5] which is defined from a neighbourhood of  $g_\infty$  in  $\mathcal{X}_1^A$ . Hence,  $\mathbf{S}_g(t)g_0$  is the orbit of (1) starting at  $g_0$ . Let us write  $\mathbf{S}_h(\tau)$  the nonlinear semigroup associated to (10) and defined by  $\mathbf{S}_h(\tau) = \mathbf{S}_g(t_h(\tau))$  with  $t_h(\tau) = \int_0^\tau (\tilde{a}(h)(s))^{-1} ds$ . We will show that:

- for all  $\tau \geq 0$ ,  $\mathbf{S}_h(\tau)$  is  $C^1$  on a  $\mathcal{X}_1^A$  neighborhood of  $g_\infty$  with values in  $\mathcal{X}_1^A$ ,
- for all  $\tau \geq 0$ ,  $d\mathbf{S}_h(\tau)(g_\infty) = \frac{\mathbf{T}(\tau)}{a_\infty}$ ,
- $g_\infty$  is locally exponentially stable under the dynamic described by  $\mathbf{S}_h(\tau)$  and  $\mathbf{S}(t)$  on a subset of  $\mathcal{X}_1^A$ .

By using the method of characteristics and assuming for a while that  $\tau \rightarrow \tilde{a}(\tau)$  is known, the solution of (10) reads:

$$h(\tau, x) = \exp\left(-\int_{\tau-x}^\tau \frac{f(s-\tau+x)}{\tilde{a}(s)} ds\right) \mathbf{1}_{x \leq \tau} + g_0(x - \tau) \exp\left(-\int_0^\tau \frac{f(s-\tau+x)}{\tilde{a}(s)} ds\right) \mathbf{1}_{x > \tau} \quad (12)$$

where we recall that  $\tilde{a}(s) = a_\infty + \rho(a(s) - a_\infty)$ . By integrating against  $f$ , we obtain a closed formula defining  $a(\tau)$ :

$$a(\tau) = \int_0^\tau f(\tau - z) \exp\left(-\int_z^\tau \frac{f(s-z)}{\tilde{a}(s)} ds\right) dz + \int_0^\infty f(z + \tau) \exp\left[-\int_0^\tau \frac{f(z+u)}{\tilde{a}(u)} du\right] g_0(z) dz. \quad (13)$$

**Lemma 3.1** *There exists  $\delta > 0$  and a  $C^1$  mapping*

$$\mathcal{A}_h : \begin{array}{l} B\mathcal{X}_f(g_\infty, \delta) \longrightarrow C^0([0, T], \mathbb{R}_+) \\ g_0 \longrightarrow (\tau \rightarrow a(\tau)) \end{array} \quad (14)$$

such that  $\mathcal{A}_h(g_\infty) = a_\infty$ .

*Proof.* We write the right hand side of (13)  $\bar{F}(a, g_0)$  and  $F = a - \bar{F}$ . The mapping  $F$  belongs to  $C^1(C^0([0, T]), \mathcal{X}_f)$  and  $F(a_\infty, g_\infty) = 0$ . From Lemma A.1 and Lemma A.2 we show that  $d_1 F(a_\infty, g_\infty)$  is invertible and a positive operator. By the implicit functions theorem we deduce that there exists  $\delta > 0$  and  $\mathcal{A}_h : B_{\mathcal{X}_f}(g_\infty, \delta) \xrightarrow{C^1} C^0([0, T], \mathbb{R}_+)$  such that  $F(\mathcal{A}_h(g_0), g_0) = 0$ .

**Proposition 3.1** *There exists  $\delta > 0$  such that the nonlinear semigroup  $\mathbf{S}_h(\tau)$  given by (12) belongs to  $C^1(B_{\mathcal{X}_1^A}(g_\infty, \delta), \mathcal{X}_1^A)$  and the Frechet-differential of  $\mathbf{S}_h(\tau)$  at  $g_\infty$  is the semigroup  $\frac{\mathbf{T}(t)}{a_\infty}$  restricted to  $\mathcal{X}_1^A$ .*

*Proof.* We easily deduce the local regularity of  $\mathbf{S}_h(\tau)$  by composition and by using Lemma 3.1 and the explicit formula (12) defining  $\mathbf{S}_h(\tau)$  as function of  $a = \mathcal{A}_h(g_0)$  and  $g_0$ .

We now find the differential of  $\mathbf{S}_h(\tau)$ . Let us write  $\delta$  a constant such that  $\mathbf{S}_h(\tau)$  is differentiable on  $B_{\mathcal{X}_1^A}(g_\infty, \delta)$ . Let  $\phi \in C_c^1(\mathbb{R}^+) \cap B_{\mathcal{X}_1^A}(0, \delta)$  we have from (10) (variation of constant formula)

$$a_\infty(\mathbf{S}_h(\tau)(g_\infty + \phi) - \underbrace{\mathbf{S}_h(\tau)g_\infty}_{g_\infty}) = \mathbf{T}(\tau)\phi + \int_0^\tau \mathbf{T}(\tau - s)\mathbf{R}(\mathbf{S}_h(s)(g_\infty + \phi) - g_\infty)ds \quad (15)$$

From Lemma 3.1, we can find a small constant still denoted  $\delta > 0$  such that  $\mathcal{A}_h B_{\mathcal{X}_1^A}(g_\infty, \delta) \in B_{C^0([0, T])}(a_\infty, a_\infty - 2\eta)$ . Hence from the definition of the cut-off function if  $\phi \in B_{\mathcal{X}_1^A}(0, \delta)$  we obtain the simpler formula for  $\mathbf{R}$ :

$$\mathbf{R}(\mathbf{S}_h(s)(\phi + g_\infty) - g_\infty) = \frac{a(\mathbf{S}_h(s)(\phi + g_\infty) - g_\infty)}{a(\mathbf{S}_h(s)(\phi + g_\infty))} (a(\mathbf{S}_h(s)(\phi + g_\infty) - g_\infty)g'_\infty + f(\mathbf{S}_h(s)(g_\infty + \phi) - g_\infty)).$$

Now, we bound this term in norm  $\mathcal{X}_1^A$ . By using the following inequalities

$$\begin{aligned} |a(\mathbf{S}_h(s)(\phi + g_\infty))| &= |\tilde{a}(\mathbf{S}_h(s)(\phi + g_\infty))| \geq 2\eta \\ \|\mathbf{S}_h(s)(g_\infty + \phi) - g_\infty\|_{\mathcal{X}_1^A} &\stackrel{\text{Frechet diff.}}{\leq} C(s) \|\phi\|_{\mathcal{X}_1^A} \\ |a(u)| &\stackrel{\text{prop. 2.1}}{\leq} c\|u\|_{\mathcal{X}_1^A}, \quad \forall u \in \mathcal{X}_1^A \\ \|g'_\infty\|_{\mathcal{X}_1^A} &< \infty \end{aligned}$$

and the fact that if  $\text{supp}(\phi) \subset [0, M]$  then  $\text{supp}(\mathbf{S}_h(s)(g_\infty + \phi) - g_\infty) \subset [0, M + s]$  we get

$$\begin{aligned} \|\mathbf{R}(\mathbf{S}_h(s)(\phi + g_\infty) - g_\infty)\|_{\mathcal{X}_1^A} &\leq \frac{C(s)\|\phi\|_{\mathcal{X}_1^A}}{2\eta} \left( cC(s)\|\phi\|_{\mathcal{X}_1^A}\|g'_\infty\|_{\mathcal{X}_1^A} + f(M + s)C(s)\|\phi\|_{\mathcal{X}_1^A} \right) \\ &\leq \gamma(s)\|\phi\|_{\mathcal{X}_1^A}^2 \quad (16) \end{aligned}$$

with  $\gamma(s) = \frac{C(s)}{2\eta}(cC(s)\|g'_\infty\|_{\mathcal{X}_1^A} + f(M + s)C(s))$ . Coming back to (15) we get

$$\left\| \int_0^\tau \mathbf{T}(\tau - s)\mathbf{R}(\mathbf{S}_h(s)\phi)ds \right\|_{\mathcal{X}_1^A} \leq \sup_{s \in [0, \tau]} \|\mathbf{T}(s)\|_{\mathcal{L}(\mathcal{X}_1^A)} \sup_{s \in [0, \tau]} \gamma(s)\|\phi\|_{\mathcal{X}_1^A}^2$$

We deduce that the Frechet differential of  $\mathbf{S}_h(\tau)$  at  $g_\infty$  is  $\frac{\mathbf{T}(\tau)}{a_\infty}$  for directional given by  $C^1$ -functions with compact support in  $\mathbb{R}^+$ . We conclude by uniqueness of the Frechet differential and density of such functions in the space  $\mathcal{X}_1^A$ .

*Remark 3* Another method for finding the differential of  $\mathbf{S}_h(\tau)$  is to differentiate the expression of the nonlinear semigroup given by the method of characteristics.

### 3.4 Conclusion

In this section we conclude with the main result concerning the nonlinear stability of (1).

The difficulty stems from the fact that an open invariant set is required to apply [DS86]. Given the results in [FL14][prop 9], we restrict ourselves to the second order Sobolev space  $\mathcal{X}_2^{\mathbf{A}}$  which is defined as  $D(\mathbf{A}^2)$  with the graph norm.

**Theorem 3.1** *Assume that there is an open set  $C \subset \mathcal{X}_2^{\mathbf{A}}$  which is  $\mathbf{S}_h$  invariant where  $\mathcal{X}_2^{\mathbf{A}}$  is the Sobolev space of order 2 associated to  $\mathbf{A}$ . Then, the stationary distribution  $g_\infty$  is locally exponentially stable under the dynamic described by  $\mathbf{S}_h(\tau)$  (12) on a neighbourhood of  $g_\infty$  in  $C \cap (g_\infty + \hat{\mathcal{X}})$ .*

*Proof.* We check the applicability of [DS86][theorem 2.1] which states the nonlinear stability result.

- $\mathcal{A}_h$  is differentiable from a neighborhood of  $g_\infty$  in  $\mathcal{X}_2^{\mathbf{A}}$  into  $C^0([0, T])$ . By adapting proposition 3.1 ( $\mathcal{X}_2^{\mathbf{A}}$  contains the condition to ensure that the derivative of the orbit at  $x = \tau$  is continuous), we deduce that  $\mathbf{S}_h(\tau)$  is differentiable on a neighborhood of  $g_\infty$  in  $\mathcal{X}_2^{\mathbf{A}}$ .
- Thanks to Proposition 3.1, for all  $\tau \geq 0$ ,  $\mathbf{S}_h(\tau)$  is Frechet differentiable on  $\mathcal{X}_2^{\mathbf{A}} \cap (\hat{\mathcal{X}} + g_\infty)$  at  $g_\infty$  with differential  $\frac{1}{a_\infty} \mathbf{T}(t)$ .
- By proposition 2.1, the semigroup  $(\mathbf{T}(t))_{t \geq 0}$  restricted to  $\mathcal{X}_2^{\mathbf{A}} \cap \hat{\mathcal{X}}$  is exponentially stable because the growth bounds does not depend on the Sobolev order space [EN00].

We conclude with the nonlinear stability by using the result by Desch-Shappacher [DS86].

**Corollary 3.1** *Still assuming that there is an open set  $C \subset \mathcal{X}_2^{\mathbf{A}}$  which is  $\mathbf{S}_h$  invariant, the stationary distribution  $g_\infty$  is locally exponentially stable under the dynamic described by  $\mathbf{S}_g(t)$  (1) on  $C \cap (g_\infty + \hat{\mathcal{X}})$ .*

*Proof.* It suffices to use the fact that there exists  $\delta > 0$  such that the mapping  $\mathcal{A}_g : g_0 \mapsto a(g)$  with  $g$  solution (1) is  $C^0$  on a  $\mathcal{X}_2^{\mathbf{A}}$  neighborhood of  $g_\infty$  and such that its range is in  $B_{C^0([0, T])}(a_\infty, a_\infty - 2\eta)$  (it can be done by a similar reasoning as done in Lemma 3.1). Then we deduce that  $\mathbf{S}_h(\tau_g(t)) = \mathbf{S}_g(t)$  and that  $\tau_g(t) \geq \eta t$  which enable to conclude by using Theorem 3.1.

## A Technical lemmas

This two technical lemmas used in the proof of Lemma 3.1, prove that  $d_1 \bar{F}(a_\infty, g_\infty)$  is a positively inversible on  $C^0([0, T]) \times \mathcal{X}_f$ , where  $\bar{F} = I - F$  with  $F$  the mapping characterizing  $\mathcal{A}_h$  (14) and such that  $F(a_\infty, g_\infty) = 0$ .

**Lemma A.1** *There exists two continuous positive kernels  $K_1(z, \tau)$  and  $K_2(z, \tau, u)$  such that*

$$d_1 \bar{F}(a_\infty, g_\infty) \cdot h(\tau) = \int_0^\tau K_1(z, t) h(z) dz + \int_0^\tau \int_z^t K_2(z, t, u) h(u) du dz$$

*Proof.* We split  $\bar{F}(a, g_0)$  into the sum

$$\bar{F}(a, g_0) = \bar{F}_1(a) + \bar{F}_2(a, g_0)$$

where

$$\begin{aligned} \bar{F}_1(a)[\tau] &= \int_0^\tau f(\tau - z) \exp\left(-\int_z^\tau \frac{f(s-z)}{a(s)} ds\right) dz \\ \bar{F}_2(a, g_0)[\tau] &= \int_0^\infty f(z + \tau) \exp\left[-\int_0^\tau \frac{f(z+u)}{a(u)} du\right] g_0(z) dz \end{aligned}$$



Formal derivation at  $(a_\infty, g_\infty)$  gives the positive kernels

$$\begin{aligned} a_\infty^2 K_1(s, \tau) &= \int_0^\infty g_\infty(z) f(z + \tau) \exp\left(-\frac{1}{a_\infty} \int_0^\tau f(z + u) du\right) f(z + s) dz \\ a_\infty^2 K_2(z, \tau, s) &= \exp\left(-\frac{1}{a_\infty} \int_z^\tau f(u - z) du\right) f(s - z) f(\tau - z) \end{aligned}$$

**Lemma A.2**  $d_1 F(a_\infty, g_\infty)$  is invertible from  $C^0([0, T], \mathbb{R}_+)$  into itself.

*Proof.* The two positive kernels  $K_1$  and  $K_2$  are continuous and bounded for  $\tau \in [0, T]$ . Let be  $M_1$  and  $M_2$  the two bounds from above of respectively  $K_1(z, t)$  and  $K_2(z, t, u)$  on their compact domains depending on  $T$ . Let be  $\mathcal{T}_1$  and  $\mathcal{T}_2$  the two linear integral operators associated to  $K_1$  and  $K_2$  and  $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ . We will show that there exists  $n > 0$  such that  $\mathcal{T}^n$  is a contraction on  $C^0([0, T])$ . Let us show that there exist  $c_n$  such that  $|\mathcal{T}^n[h](t)| \leq c_n t^n$

$$|\mathcal{T}_1[h](t)| \leq M_1 t \|h\| \quad |\mathcal{T}_2[h](t)| \leq \frac{M_2}{2} t^2 \|h\|$$

we can take  $c_1 = (M_1 + M_2 T)$ . By induction assume that  $|\mathcal{T}^n[h](t)| \leq c_n t^n$  then  $\mathcal{T}^{n+1}[h](t) = \mathcal{T}^n[\tilde{h}](t)$  with  $|\tilde{h}(t)| \leq c_n t^n$ . Simple computations give that  $|\mathcal{T}_1[\tilde{h}](t)| \leq \frac{M_1 c_n}{n+1} t^{n+1}$  and  $|\mathcal{T}_2[\tilde{h}](t)| \leq \frac{M_2 c_n T}{n+1} t^{n+1}$ . The property holds by setting  $c_{n+1} = \frac{M_1 + M_2 T}{n+1} c_n \stackrel{\text{def}}{=} \frac{M}{n+1} c_n$ . We deduce that  $c_n = \frac{M^n}{n!}$  which tends to 0 when  $n \rightarrow +\infty$ . By Picard theorem we deduce that  $\mathcal{T}[h] = d_1 F(a_\infty, g_\infty)$  is invertible. The positiveness property is direct.

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ISSN 0249-6399