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# Extraction of cylinders and cones from minimal point sets

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## Abstract

We propose new algebraic methods for extracting cylinders and cones from minimal point sets, including oriented points. More precisely, we are interested in computing efficiently cylinders through a set of three points, one of them being oriented, or through a set of five simple points. We are also interested in computing efficiently cones through a set of two oriented points, through a set of four points, one of them being oriented, or through a set of six points. For these different interpolation problems, we give optimal bounds on the number of solutions. Moreover, we describe algebraic methods targeted to efficiently solve these problems.

*Keywords:* Mixed set of 3D points, cylinders, cones, interpolation.

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## 1. Introduction

The extraction of geometric primitives from 3D point clouds is an important problem in reverse engineering. These 3D point clouds are typically obtained by means of accurate 3D scanners and there exist several methods for performing 3D geometric primitives extraction [1]. An important category among these methods are based on the RANSAC approach [4, 7, 1]. For such methods, the primitives are directly extracted from the input point cloud. The basic idea is to extract a particular elementary type of shape, such as planes, spheres, cylinders, cones or tori, from the smallest possible set of points and then to judge if this extracted primitive is relevant to the full point cloud. Therefore, for this category of methods it is very important to compute a particular type of shape through the smallest possible number of points, including normals if available. If the extraction of planes and spheres is easy to treat, the cases of cylinders and cones are more involved. In this paper we provide new methods for extracting these geometric primitives from the smallest possible number of points, counting multiplicities of oriented points (i.e. points given with their normal vector). As far as we know, and surprisingly, these problems have not appeared in the existing literature with the exception of [3]. Instead, the classical approaches to these interpolation problems usually extract, actually we should say estimate, these geometric primitives from an overdetermined number of points, counting multiplicities (e.g. [6]).

An *oriented point* is a couple of a point and a nonzero vector. A surface is said to interpolate an oriented point if the point belongs to the surface and its associated vector is colinear to the normal of the surface at this point, we do not assume that the orientation of the normal of the

point is the same as the orientation of the surface; since often in the data sets normals are unoriented. Moreover, it is important to deal with inhomogeneous data, that is to say some points are oriented but not all, in order to take into account the estimated accuracy of oriented point clouds that are obtained by means of normal estimation algorithms. A set of data made of points and oriented points will be called a *mixed set of points*.

We emphasize that interpolating at a point imposes a single algebraic condition on a given shape whereas interpolating at an oriented 3D point imposes three algebraic conditions. Typically, a 3D plane is uniquely defined either by three distinct points or by one oriented point. A sphere is uniquely defined either by four points or by one oriented point and an additional point. In these two cases, it turns out that there is a unique shape that interpolates a mixed set of points corresponding to the number of parameters of this shape (a plane is determined by three parameters and the sphere is determined by four parameters). In this paper, we will treat interpolation of two other basic shapes, namely cylinders and cones for which the situation is more involved.

Our approach is inspired by effective methods in algebraic geometry. We consider two families of unknowns: first, the parameters needed to describe all features of the targeted surface (e.g. the radius and axis of a circular cylinder) hence their equation; second, auxiliaries unknowns (e.g. such as a special point on that axis) which permits to describe a collection of geometric constructions. These constructions are designed to establish a complete link between the input and the first family of unknowns. Then, we translate algebraically the collection of constraints attached to these geometric constructions into a system of polynomial equations that we further analyze and

simplify, discarding spurious solutions if needed. Since the input and output are (and should be) real approximate data, we designed efficient algorithms to compute very accurate real solutions of these systems of equations. Indeed, in all the considered cases, we were able to express the results as the solutions of (generalized) eigenvalues problems together with close formulas. This allows to rely on classical matrix computations software and achieve accuracy and efficiency. Prototypes of our algorithms were implemented in the computer algebra system MAPLE, and we provide some statistics and timings (which are quite satisfactory).

## 2. Interpolation of cylinders

A cylinder (more precisely a right circular cylinder) is defined as the set of points in the three-dimensional affine space  $\mathbb{R}^3$  located at a fixed distance (called the *radius* of the cylinder) of a given straight line (called the *axis* of the cylinder). It is hence defined by means of five parameters : four parameters describe a line in  $\mathbb{R}^3$  and an additional parameter measures the radius.

A popular determination of a cylinder is done by interpolation of two points with normals, which imposes six conditions (instead of five). So, a priori no cylinder interpolates these datas; therefore some approximations are necessary. In this section, we will give new methods to compute the cylinders using just five independent conditions. There are two possible such minimal configurations, either an oriented point and two other distinct points, or five distinct points.

### 2.1. Cylinders through a mixed minimal point set

We seek for the cylinders that interpolate a given mixed minimal set of points  $\mathcal{P}$ . Since a cylinder is given by 5 parameters,  $\mathcal{P}$  is assumed to be composed of an oriented point  $p_1$  with its normal vector  $n_1$  and two other distinct points  $p_2, p_3$  in  $\mathbb{R}^3$ .

First, by a linear change of coordinates, one can assume that  $p_1 = (0, 0, 0)$  and  $n_1 = (0, 0, 1)$  and we set  $p_2 = (x_2, y_2, z_2)$  and  $p_3 = (x_3, y_3, z_3)$ . Then, the axis of a cylinder interpolating  $\mathcal{P}$  must be orthogonal to the  $z$ -axis and must intersect it. It follows that a normal plane  $\Pi$  contains the  $z$ -axis, hence is given by an equation of the form  $lx + my = 0$  where  $t := (l, m, 0)$  is the corresponding direction of the axis. Observe that these directions are in correspondence with a projective line  $\mathbb{P}^1$ . For simplicity, we set  $\rho := \sqrt{l^2 + m^2} = \|t\| > 0$ .

Now, we compute the orthogonal projections  $q_1$  and  $q_2$  of  $p_2$  and  $p_3$ , respectively, onto the plane  $\Pi$ .  $\Pi$  contains the point  $p_1$  and is generated by the two orthogonal vectors  $n_1$  and  $v = n_1 \wedge t = (-m, l, 0)$ . The matrix

$$M = \begin{pmatrix} \frac{l}{\rho} & \frac{m}{\rho} & 0 \\ -\frac{m}{\rho} & \frac{l}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines the change of coordinates from the current coordinate system  $(x, y, z)$  to a new coordinate system  $(x', y', z')$ , with the same origin  $p_1$ , defined by the three vectors  $t/\rho$ ,  $v/\rho$ ,  $n_1$ , where  $\Pi$  has equation  $x' = 0$ . It follows that the coordinates  $(x'_i, y'_i, z'_i)$  of  $p_i$ ,  $i = 2, 3$ , in this new coordinate system are given by

$$(x'_i, y'_i, z'_i) = \left( x_i \frac{l}{\rho} + y_i \frac{m}{\rho}, -x_i \frac{m}{\rho} + y_i \frac{l}{\rho}, z_i \right).$$

Therefore, the coordinates of the orthogonal projections  $q_i$ ,  $i = 2, 3$  are given by

$$q_i = \left( -x_i \frac{m}{\rho} + y_i \frac{l}{\rho}, z_i \right) \in \Pi$$

in the basis  $v/\rho, n_1$ .

The existence of a cylinder interpolating  $\mathcal{P}$  is equivalent to the fact that the points  $p_1$ ,  $q_2$  and  $q_3$  all belongs to a circle whose center  $c$  is located on the  $z$ -axis, say  $c = (0, 0, r)$ . Such a circle has an equation of the form  $y'^2 + (z' - r)^2 = r^2$ , or equivalently  $y'^2 + z'^2 - 2rz' = 0$ . Therefore, this cocyclicity condition can be written as

$$0 = \begin{vmatrix} y_2'^2 + z_2'^2 & z_2 \\ y_3'^2 + z_3'^2 & z_3 \end{vmatrix} = \frac{1}{\rho^2} \begin{vmatrix} (-x_2m + y_2l)^2 + (l^2 + m^2)z_2^2 & z_2 \\ (-x_3m + y_3l)^2 + (l^2 + m^2)z_3^2 & z_3 \end{vmatrix}.$$

Since  $\rho > 0$ , the expansion of this latter determinant allows to rewrite this condition as a degree 2 homogeneous equation  $al^2 + blm + cm^2$  where the coefficients  $a, b, c$  are given by the following closed formulas

$$a := \begin{vmatrix} y_2^2 + z_2^2 & z_2 \\ y_3^2 + z_3^2 & z_3 \end{vmatrix}, \quad b := -2 \begin{vmatrix} x_2y_2 & z_2 \\ x_3y_3 & z_3 \end{vmatrix}, \\ c := \begin{vmatrix} x_2^2 + z_2^2 & z_2 \\ x_3^2 + z_3^2 & z_3 \end{vmatrix}.$$

Unless  $a = b = c = 0$ , this equation has two roots, counting multiplicities, in the field of complex numbers. If a real solution is found, that is to say the direction of a real cylinder interpolating  $\mathcal{P}$  (observe that one can impose  $\rho = 1$  since the condition is homogeneous in  $l, m$ ), then the remaining parameter  $r$  is uniquely determined by one of the formulas

$$2z_2r = y_2'^2 + z_2'^2 = \frac{1}{\rho^2} (-x_2m + y_2l)^2 + z_2^2, \quad (1)$$

$$2z_3r = y_3'^2 + z_3'^2 = \frac{1}{\rho^2} (-x_3m + y_3l)^2 + z_3^2, \quad (2)$$

depending whether  $z_2 \neq 0$  or  $z_3 \neq 0$ . We notice that if  $z_2 = z_3 = 0$  then  $a = b = c = 0$ .

**Theorem 1.** *Suppose given a mixed set of points  $\mathcal{P}$  composed of an oriented point  $p_1, n_1$  and two other points  $p_2, p_3$  such that*

- i)  $p_1, p_2, p_3$  are all distinct,
- ii)  $p_1, p_2, p_3$  do not belong to a same plane which is normal to  $n_1$ ,
- iii)  $p_2$  and  $p_3$  are not symmetric with respect to the line through  $p_1$  and generated by  $n_1$ ,

then there exist at most 2 real cylinders interpolating  $\mathcal{P}$ . Otherwise, there exist a cylinder (possibly "flat", i.e. with infinite radius) interpolating  $\mathcal{P}$  in any direction which is normal to  $n_1$ .

*Proof.* Following the above discussion, this theorem will be proved if we show that  $a = b = c = 0$  if and only if at least one of the three conditions i), ii), iii) holds. It is not hard to check that if one of the three latter conditions holds then  $a = b = c = 0$ . To prove the converse, we observe that by a linear change of coordinate, we can assume that  $x_2 = 0$  in addition to the fact that  $p_1 = (0, 0, 0)$  and  $n_1 = (0, 0, 1)$ . Then, we have  $b = 2x_3y_3z_2$  and hence three cases to analyze.

If  $x_3 = 0$ , then  $c = z_2z_3(z_2 - z_3)$  so that  $c = 0$  implies that  $z_2 = 0$ , or  $z_3 = 0$  or  $z_2 = z_3$ . If  $x_3 = z_2 = 0$ , then  $a = y_2^2z_3$  so that  $a = 0$  implies that i) or ii) hold. Similarly, if  $x_3 = z_3 = 0$  then  $a = y_3^2z_2$  so that  $a = 0$  implies that i) or ii) hold. Finally, if  $x_3 = 0$  and  $z_2 = z_3$  then  $a = z_2(y_2^2 - y_3^2)$  so that  $a = 0$  implies that i) or iii) hold.

The case corresponding to  $y_3 = 0$  can be treated exactly as the previous case  $x_3 = 0$ , exchanging  $x_2$  with  $x_3$  and  $y_2$  with  $y_3$  leaves unchanged  $b$  and permute  $a$  and  $c$ .

Finally, if  $z_2 = 0$ , then  $c = 0$  (recall  $x_2 = 0$ ) and  $a = y_2^2z_3$ . So  $a = 0$  if either  $y_2 = 0$  or  $z_3 = 0$ . But  $y_2 = 0$  means that  $p_1 = p_2$ , i.e. i) holds, and  $z_3 = 0$  means that ii) holds.  $\square$

When a mixed set of points  $\mathcal{P}$  satisfies conditions i), ii) and iii) in the above theorem, then  $a$ ,  $b$  and  $c$  are not all zero and hence the fact of having zero, one or two homogeneous real solutions to the equation  $al^2 + blm + cm^2 = 0$  is decided by means of the discriminant  $\Delta := b^2 - 4ac$  which depends on the coordinates of  $p_2$  and  $p_3$ . If  $\Delta < 0$  then there is no real homogeneous solution, if  $\Delta = 0$  then there exist a double homogeneous solution and if  $\Delta > 0$  then there exists two distinct homogeneous solutions. As we have already observed in the proof of Theorem 1, it is possible to assume, without loss of generality, that  $x_2 = 0$  in addition of  $p_1 = (0, 0, 0)$  and  $n_1 = (1, 0, 0)$ . Then, a straightforward computation shows that

$$\Delta = 4z_2z_3 \left( y_2^2 (x_3^2 + z_3^2) + z_2^2 (x_3^2 + y_3^2) - z_2z_3 (x_3^2 + y_2^2 + y_3^2 + (z_2 - z_3)^2) \right).$$

For instance, from this equation we see directly that there is no real cylinders interpolating  $\mathcal{P}$  if  $p_2$  and  $p_3$  and not on the same side of the plane through  $p_1$  and normal

to  $n_1$  (i.e.  $z_2$  and  $z_3$  have opposite signs). Another interesting case is to assume that  $p_2$  belongs to the plane through  $p_1$  and normal to  $n_1$  (i.e.  $z_2 = 0$ ). Indeed, in this case  $\Delta = 0$  so there is a unique cylinder (counted with multiplicity two) through  $\mathcal{P}$ : its direction is given by  $t = (-b, 2a, 0) = (2x_2y_2z_3, 2y_2^2z_3, 0)$  and  $r$  is still defined by (1) ( $z_3$  is assumed to be nonzero for otherwise  $a = b = c = 0$ ).

Finally, we notice that if  $a = 0$  then the directions are given by the equation  $blm + cm^2 = 0$ . It follows that these directions are given by  $(1, 0, 0)$  and  $(-c, b, 0)$ . Of course, if in addition  $b = 0$  then there is a single interpolation cylinder that appears with multiplicity 2 (the discriminant vanishes). The radius  $r_1$  and  $r_2$  are then deduced from (1). Similarly, if  $c = 0$  then the directions of the interpolating cylinders are given by  $(0, 1, 0)$  and  $(-b, a, 0)$ .

*Extraction algorithm.* From the above analysis, we deduce the following extraction algorithm of cylinders from a mixed point set  $\mathcal{P}$ :

1. Perform a linear change of coordinates so that the oriented point is at the origin and its normal vector is  $(0, 0, 1)$ . If  $z_2z_3 < 0$  then there is no solution.
2. Compute the quantities  $a$ ,  $b$  and  $c$ . If  $a = b = c = 0$  then stop because there are infinitely many interpolating cylinders.
3. If  $a = 0$  or  $c = 0$  then compute the interpolating cylinders as explained above and stop.
4. Otherwise, solve the equation  $al^2 + blm + cm^2 = 0$  and keep only those roots that are real numbers (up to numerical precision). For each such root, compute the corresponding radius by means of (1) and return the interpolating cylinder.

This algorithm has been implemented with the Maple software and all timings are measured on a Mac laptop equipped with Intel Core i7 CPU @ 2.8GHz, 16 GB memory. We observed that the computation of the cylinders through a random set of points takes in average 3.5ms and is almost constant independently of the point set. The proportion of the number of cylinders found through random point sets is given in Table 1 and an illustrative picture is given in Figure 1.

Number of cylinders	0	2
Proportion (%)	46.87	53.13

Table 1: Proportion of the number of cylinders found through a thousand random point sets.

## 2.2. Cylinders through five points

The problem of extracting a cylinder passing through five points has already been treated in the literature, see [3, 2, 5]. These works solve the problem by relying on a polynomial system solver. In this section, we briefly review

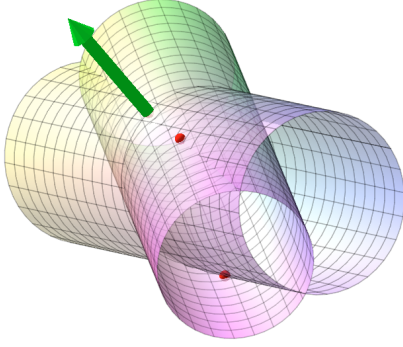


Figure 1: A general mixed set of points with the two interpolated cylinders.

the model already described in [3] and then we propose an improvement to gain efficiently. Our key ingredient removes some spurious solutions by means of additional algebraic manipulations.

### 2.2.1. Geometric analysis

We suppose given a set  $\mathcal{P}$  of five points  $p_1, p_2, p_3, p_4, p_5$  and we seek for cylinders through  $\mathcal{P}$ . First of all, by linear change of coordinates, we assume without loss of generality that the coordinate system  $(x, y, z)$  is such that

$$p_1 = (0, 0, 0), \quad p_2 = (x_2, 0, 0), \quad p_3 = (x_3, y_3, 0), \quad (3)$$

$$p_4 = (x_4, y_4, z_4), \quad p_5 = (x_5, y_5, z_5).$$

Let us pick a nonzero vector  $t = (l, m, n)$  (which will represent the direction of the axis) and denote by  $\Pi$  the plane through the origin which is orthogonal to  $t$ . Let us also denote by  $q_i$  the orthogonal projection of  $p_i$  onto  $\Pi$ , for all  $i = 1, \dots, 5$ . Then,  $p_1, \dots, p_5$  belongs to a cylinder of direction  $t$  if and only if the coplanar points  $q_1, \dots, q_5$  are cocyclic. In order to make explicit this latter condition, we consider a new system of coordinates  $(x', y', z')$  whose third axis is parallel to  $t$ . More precisely, we choose the coordinates system  $(x', y', z')$  which is obtained from the coordinate system  $(x, y, z)$  by the change of coordinates defined by the orthogonal matrix

$$M = \begin{pmatrix} \frac{\sqrt{m^2+n^2}}{\|t\|} & -\frac{lm}{\|t\|\sqrt{m^2+n^2}} & -\frac{ln}{\|t\|\sqrt{m^2+n^2}} \\ 0 & \frac{n}{\sqrt{m^2+n^2}} & -\frac{m}{\sqrt{m^2+n^2}} \\ \frac{l}{\|t\|} & \frac{m}{\|t\|} & \frac{n}{\|t\|} \end{pmatrix}.$$

Therefore, the coordinates of  $q_i$  in the system  $(x', y', z')$  are given by

$$\left( x_i \frac{\sqrt{m^2+n^2}}{\|t\|} - \frac{lm}{\|t\|\sqrt{m^2+n^2}} y_i - \frac{ln}{\|t\|\sqrt{m^2+n^2}} z_i, \right. \\ \left. \frac{n}{\sqrt{m^2+n^2}} y_i - \frac{m}{\sqrt{m^2+n^2}} z_i, 0 \right) =: (x'_i, y'_i, 0).$$

Now, the points  $q_1, q_2, q_3$  and  $q_4$  are cocyclic (or aligned, which corresponds to a circle of infinite radius) providing

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x'_1 & x'_2 & x'_3 & x'_4 \\ y'_1 & y'_2 & y'_3 & y'_4 \\ x'^2_1 + y'^2_1 & x'^2_2 + y'^2_2 & x'^2_3 + y'^2_3 & x'^2_4 + y'^2_4 \end{vmatrix} = 0.$$

Since  $x'_1 = y'_1 = 0$ ,  $x'_2 = \frac{\sqrt{m^2+n^2}}{\|t\|} x_2$ ,  $y'_2 = 0$  and

$$x'^2_i + y'^2_i = \|q_i\|^2 = \|p_i\|^2 - \frac{(\mathbf{t} \cdot p_i)^2}{\|t\|^2}$$

for  $i = 3, 4$ , we deduce that the points  $p_1, p_2, p_3, p_4$  all belong to a cylinder of direction  $\mathbf{t}$  if and only if

$$C_{p_1, p_2, p_3, p_4}(l, m, n) = 0$$

where, after some calculus,

$$C_{p_1, p_2, p_3, p_4}(l, m, n) := x'^2_2(m^2 + n^2) \begin{vmatrix} l & x_3 & x_4 \\ m & y_3 & y_4 \\ n & 0 & z_4 \end{vmatrix} \\ - x_2 \begin{vmatrix} m & y_3 & y_4 \\ n & 0 & z_4 \\ 0 & \|t\|^2 \|p_3\|^2 - (t \cdot p_3)^2 & \|t\|^2 \|p_4\|^2 - (t \cdot p_4)^2 \end{vmatrix}.$$

This condition is given by a homogeneous polynomial of degree 3 in  $l, m, n$ . Observe that it is satisfied for the six particular directions corresponding to the lines  $(p_i p_j)$ ,  $1 \leq i < j \leq 4$ . Then, to determine the cylinders through the five points  $p_1, \dots, p_5$ , we consider the polynomial system of equations

$$C_{p_1, p_2, p_3, p_4}(l, m, n) = C_{p_1, p_2, p_3, p_5}(l, m, n) = 0 \quad (4)$$

It corresponds geometrically to the intersection of two cubic curves, which is composed of nine points whenever this intersection is finite. As already noticed, the directions corresponding to the lines  $(p_1 p_2)$ ,  $(p_1 p_3)$  and  $(p_2 p_3)$  are roots of these two equations, but they do not imply the existence of a cylinder interpolating  $\mathcal{P}$ , they have to be removed once (if there is indeed a cylinder in this direction, then it will appear as a multiple solution). Moreover, when a direction of a cylinder through  $\mathcal{P}$  is found, then it is completely determined since its radius and a point on its axis are given by the center and the radius of the circle through the points  $q_1, q_2, q_3, q_4, q_5$ . Therefore, we have the following result.

**Theorem 2** ([3]). *If the algebraic system of equations (4) has a finite number of solutions, then there may be zero, two, four or six real cylinders through  $\mathcal{P}$ .*

In [3, 2, 5], various polynomial system solvers are used in order to extract the directions of the cylinders through  $\mathcal{P}$ . We will describe a new approach where we first simplify this algebraic system and then rely on eigen-computations.

To begin with, we observe that the three extraneous directions  $(p_1p_2)$ ,  $(p_1p_3)$  and  $(p_2p_3)$  which write

$$(x_2, 0, 0), (x_3, y_3, 0), (x_3 - x_2, y_3, 0),$$

are the common roots of both algebraic equations

$$F := nx_2 = 0, \quad G := (mx_3 - mx_2 - ly_3)(mx_3 - ly_3)mx_2 = 0.$$

Then a result in algebraic geometry says that the two polynomials  $C_{p_1, p_2, p_3, p_j}$ ,  $j = 4, 5$ , belong to the ideal generated by  $F$  and  $G$ ; and suggests to explicitly compute these membership relations. Computing the Euclidean (pseudo-)division of  $C_{p_1, p_2, p_3, p_4}$  and  $C_{p_1, p_2, p_3, p_5}$  by the polynomial  $x_2n$ , we get two homogeneous polynomials of degree 2,

$$\begin{aligned} D_j(l, m, n) = & (-y_j y_3^2 + y_j^2 y_3 + z_j^2 y_3) l^2 \\ & + ((2y_j y_3 x_3 - 2y_j x_j y_3) m + z_j (x_2 y_3 - 2x_j y_3) n) l + \\ & (x_2 y_j x_3 - x_2 x_j y_3 - y_j x_3^2 + x_j^2 y_3 + z_j^2 y_3) m^2 \\ & + z_j (-x_2 x_3 + x_3^2 + y_3^2 - 2y_j y_3) nm + \\ & (x_2 y_j x_3 - x_2 x_j y_3 - y_j x_3^2 + x_j^2 y_3 - y_j y_3^2 + y_j^2 y_3) n^2 \end{aligned}$$

where  $j = 4, 5$ , such that

$$\begin{pmatrix} D_4 & z_4 \\ D_5 & z_5 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} C_{p_1, p_2, p_3, p_4} \\ C_{p_1, p_2, p_3, p_5} \end{pmatrix}.$$

**Proposition 3.** *If  $z_4 \neq 0$  (resp.  $z_5 \neq 0$ ), then the directions of the cylinders interpolating  $\mathcal{P}$  are obtained as all the common roots of both equations  $C_{p_1, p_2, p_3, p_4}(l, m, n) = 0$  and  $\Delta(l, m, n) := z_5 D_4(l, m, n) - z_4 D_5(l, m, n) = 0$  (resp.  $C_{p_1, p_2, p_3, p_5}(l, m, n) = 0$  and  $\Delta(l, m, n) = 0$ ).*

*Proof.* If  $z_4 \neq 0$  then the system (4) is obviously equivalent to  $C_{p_1, p_2, p_3, p_4} = 0$  and

$$\begin{aligned} z_4 C_{p_1, p_2, p_3, p_5}(l, m, n) - z_5 C_{p_1, p_2, p_3, p_4}(l, m, n) = \\ nx_2(z_4 D_5 - z_5 D_4) = 0. \end{aligned}$$

The solutions corresponding to  $n = 0$  satisfy  $n = G = 0$  so that they define exactly the three extraneous directions  $(p_1p_2)$ ,  $(p_1p_3)$  and  $(p_2p_3)$ . Therefore, the remaining polynomial system

$$C_{p_1, p_2, p_3, p_4}(l, m, n) = 0, \quad \Delta(l, m, n) = 0 \quad (5)$$

gives exactly the six solutions of interest. The case  $z_5 \neq 0$  is treated similarly.  $\square$

We notice that the case where  $z_4 = z_5 = 0$  corresponds geometrically to five coplanar points. If these points are also aligned, then are infinitely many cylinders. If they belong to a conic, there are two non-degenerated cylinders thought these points. In all cases, they belong to a degenerated “flat” cylinder (infinite radius).

### 2.2.2. Solving via eigencomputations

Now, we describe a method that allows to compute efficiently the roots of the system of algebraic equations given in Proposition 3 as the eigenvalues of a pencil of matrices which is given in closed form. It is based on technics that are described in [2].

First, in order to treat apart the case  $n = 0$ , observe that the common roots of  $C_{p_1, p_2, p_3, p_5} = 0$  and  $\Delta = 0$  such that  $n = 0$  are easily computed since they are among the three directions  $(p_1p_2)$ ,  $(p_1p_3)$  and  $(p_2p_3)$  (i.e. the roots of  $n = G = 0$ ) and the decision is given by the evaluation of the equation  $\Delta(l, m, n)$  at these directions. For instance, the direction  $(1, 0, 0)$  is a solution if and only if

$$\begin{aligned} \Delta(1, 0, 0) = z_5 (-y_4 y_3^2 + y_4^2 y_3 + z_4^2 y_3) \\ - z_4 (-y_5 y_3^2 + y_5^2 y_3 + z_5^2 y_3) = 0. \end{aligned}$$

Therefore, from now on we set  $n = 1$ .

We consider the Sylvester Matrix  $S$  of the polynomials  $C_{p_1, p_2, p_3, p_5}(l, m, 1)$  and  $\Delta(l, m, 1)$  seen as univariate polynomials in the variable  $m$ . This matrix is a polynomial matrix of degree 2 in  $l$ . More precisely, it is of the form  $M_2 l^2 + M_1 l + M_0$  where each  $M_i$  is a  $5 \times 5$ -matrix whose coefficients are given in closed forms in terms of  $\mathcal{P}$ . For instance,

$$M_2 := \begin{pmatrix} 0 & 0 & x_2 y_3^2 z_4 & b & 0 \\ 0 & 0 & 0 & x_2 y_3^2 z_4 & b \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix}$$

where

$$\begin{aligned} a := -y_3^2 y_4 z_5 + y_3^2 z_4 y_5 + y_3 y_4^2 z_5 - y_3 z_4 y_5^2 \\ + y_3 z_4^2 z_5 - y_3 z_4 z_5^2 \end{aligned}$$

and

$$b := -x_2 y_3^2 y_4 + x_2 y_3 y_4^2 + x_2 y_3 z_4^2.$$

Now, following [2], we linearize this polynomial matrix by considering its companion matrices that are defined by

$$A = \begin{pmatrix} 0 & I \\ M_0^t & M_1^t \end{pmatrix}, \quad B = \begin{pmatrix} I & 0 \\ 0 & -M_2^t \end{pmatrix}$$

where  $I$  stands for the  $5 \times 5$ -identity matrix and  $M_i^t$  stands for the transpose of the matrix  $M_i$ . These matrices are of size  $10 \times 10$  and their main feature and importance is the fact that for all  $\lambda \in \mathbb{C}$  and all vector  $v \in \mathbb{C}^m$  we have

$$S^t(\lambda)v = 0 \Leftrightarrow (A - \lambda B) \begin{pmatrix} v \\ \lambda v \end{pmatrix} = 0.$$

In other words, the solutions to the system

$$C_{p_1, p_2, p_3, p_4}(l, m, 1) = 0, \quad \Delta(l, m, 1) = 0$$

can be computed from the eigenvalues and eigenvectors of the pencil  $A, B$  which is given in closed form in terms of the input data  $\mathcal{P}$ . We refer the reader to [2] for more details on these computations.

### 2.2.3. Extraction algorithm

From the above analysis, we deduce the following extraction algorithm of cylinders through a point set  $\mathcal{P}$ :

1. Perform a linear change of coordinates so that the five points are of the form (3).
2. If  $z_4 = z_5 = 0$  the points are coplanar and the algorithm stops here.
3. Assume  $z_4 \neq 0$ . Then, instantiate the pre-computed matrices  $M_0, M_1, M_2$ , and hence  $A, B$ , with the coordinates of the points  $p_i, i = 1, \dots, 5$ .
4. Compute the six finite eigenvalues of the pencil  $A, B$  (see [2] for details), and sort them in order to keep only those which are real numbers (up to a given precision).
5. For each real eigenvalue obtained in the previous step, compute the remaining coordinate of the direction by means of the associated eigenvectors (see [2] for details); keep those directions that are given by real numbers.
6. For each real direction of a cylinder through  $\mathcal{P}$ , compute  $q_1, q_2, q_3$  then the radius and the center of their circumcircle.

This algorithm has been implemented with the `Maple` software. We observed that the computation of the cylinders through a random set of points takes in average 15ms and is almost constant independently of the point set. The proportion of the number of cylinders found through random point sets is given in Table 2. Some illustrative configurations are presented in Figure 2.

Number of cylinders	0	2	4	6
Proportion (%)	22.7	53.9	21.4	2

Table 2: Proportion of the number of cylinders found through a thousand random point sets.

### 3. Interpolation of cones

A cone (more precisely a right circular cone) is a three-dimensional geometric shape that consists of the locus of all straight lines (the generatrices) joining a fixed point, called the *apex* to the points of a circle, the apex lying on the line, called the *axis*, passing through the center of the circle and normal to the plane containing this circle. It is determined by six parameters : four parameters for the axis, an additional parameter for the apex, which is a point on the axis and a last parameter for the angle of the cone, that is to say the angle made at the apex between the axis and any of the straight lines generating the cone.

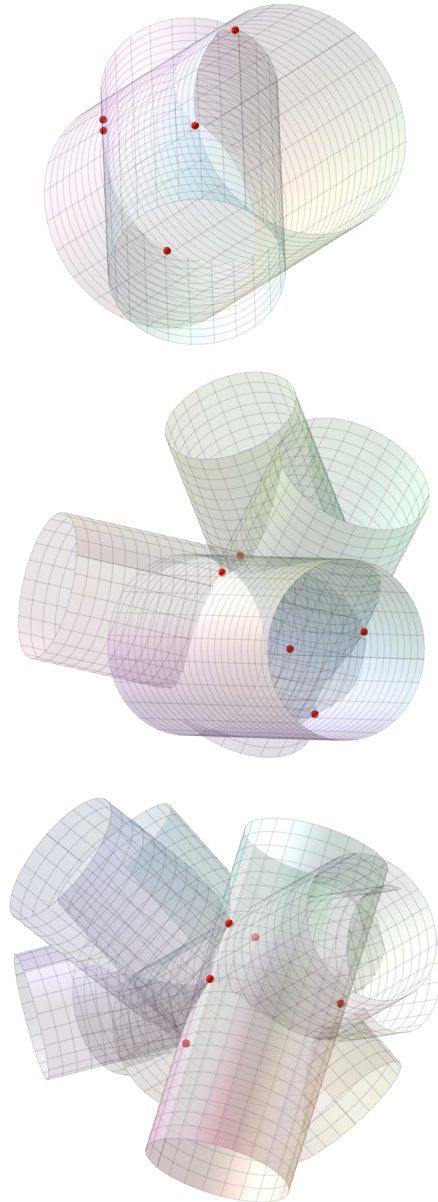


Figure 2: Three configurations of  $\mathcal{P}$  with 2,4 and 6 cylinders passing through  $\mathcal{P}$ .

It is classical to determine a cone from three oriented points. Indeed, each normal plane to each oriented point must contain the apex, so the apex can be determined as the intersection of three such planes. Nevertheless, this approach is overdetermined in the sense that it will not yield a cone in general. The reason is that three general oriented points give nine conditions whereas a cone has only six parameters. Hereafter, we provide methods for interpolating a cone through a mixed point set that define exactly six conditions. There are three possibilities of such mixed point sets : either two oriented points, or one oriented point and three other distinct points, or six distinct points.

### 3.1. Cones through two oriented points

We seek for cones through a set  $\mathcal{P}$  of two oriented points that we will denote by  $p_1, n_1$  and  $p_2, n_2$ . By a linear change of coordinates, one can assume without loss of generality that  $p_1 := (0, 0, 0)$  and  $n_1 := (0, 0, 1)$ . For the remaining data, we set  $p_2 = (x_2, y_2, z_2)$  and  $n_2 = (a_2, b_2, c_2)$ .

Assuming that  $n_2$  is not proportional to  $n_1$ , i.e.  $a_2 \neq 0$  or  $b_2 \neq 0$ , the normal plane to  $n_1$  through  $p_1$  and the normal plane to  $n_2$  through  $p_2$  intersect along a line  $\mathcal{L}$ . A key remark is that the apex of a cone through  $\mathcal{P}$  must be on  $\mathcal{L}$ . This line is defined by the equations

$$\mathcal{L} : \begin{cases} z = 0 \\ a_2x + b_2y + c_2z - p_2 \cdot n_2 = 0. \end{cases}$$

If  $n_2$  is proportional to  $n_1$ , then we get a degenerated situation. Indeed, either  $z_2 \neq 0$  and there is no cone through  $\mathcal{P}$ , or either  $z_2 = 0$  and there are infinitely many cones through  $\mathcal{P}$  (the line through  $p_1$  and  $p_2$  define a generatrix of the cone and then the apex can be freely chosen on this line, as well as the angle of the cone). So, from now on we assume that  $n_2$  is not proportional to  $n_1$ , i.e.  $a_2^2 + b_2^2 > 0$ .

The axis of a cone through  $\mathcal{P}$  must intersect the line  $\mathcal{L}_1$  through  $p_1$  with direction  $n_1$  and also intersect the line  $\mathcal{L}_2$  through  $p_2$  with direction  $n_2$ . So to characterize this axis, we pick a point on each of these lines, say  $q_1 = \lambda_1 n_1 \in \mathcal{L}_1$  and  $q_2 = p_2 + \lambda_2 n_2 \in \mathcal{L}_2$  where  $\lambda_1, \lambda_2 \in \mathbb{R}$ , and we consider the line  $\mathcal{A}$  through  $q_1$  and  $q_2$ . The only special case when  $\mathcal{A}$  is not well defined is when  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect and both  $q_1$  and  $q_2$  are located at this intersection point. For the sake of clarity, we will treat this case separately and we assume for the moment that  $\mathcal{L}_1$  and  $\mathcal{L}_2$  do not intersect.

The line  $\mathcal{A}$  will be the axis of a cone through  $\mathcal{P}$  providing the two following conditions hold:

- $\mathcal{A}$  intersect  $\mathcal{L}$ , in which case the intersection point is the apex of the cone,
- the angle between  $n_1$  and  $\mathcal{A}$  must be equal to the angle between  $n_2$  and  $\mathcal{A}$ .

$\mathcal{A}$  can be parameterized as the set of points  $(1 - \lambda)q_1 + \lambda q_2$  for  $\lambda \in \mathbb{R}$ . It is necessary to have  $z_2 + \lambda_2 c_2 - \lambda_1 \neq 0$  for otherwise  $\mathcal{A}$  is parallel to the plane  $z = 0$  and hence it does not intersect  $\mathcal{L}$ , unless  $\lambda_1 = z_2 + \lambda_2 c_2 = 0$  but in this case the apex would be  $p_1$  which is impossible (normal vector is not well defined at the apex). So, assuming  $z_2 + \lambda_2 c_2 - \lambda_1 \neq 0$ , the intersection point between  $\mathcal{A}$  and the plane  $z = 0$  is the point of coordinates

$$\left( -\lambda_1 \frac{x_2 + \lambda_2 a_2}{z_2 + \lambda_2 c_2 - \lambda_1}, -\lambda_1 \frac{y_2 + \lambda_2 b_2}{z_2 + \lambda_2 c_2 - \lambda_1}, 0 \right). \quad (6)$$

Reporting this point in  $\mathcal{L}$  we get the following first constraint on  $\lambda_1$  and  $\lambda_2$ :

$$\lambda_1 \lambda_2 (a_2^2 + b_2^2) - \lambda_1 z_2 c_2 + (p_2 \cdot n_2)(z_2 + \lambda_2 c_2) = 0. \quad (7)$$

Now, denote by  $q$  the vector  $q_2 - q_1$ , the second condition is obtained by imposing  $q \cdot n_1 = \pm q \cdot n_2 / \|n_2\|$ , the sign change being a consequence of the fact that the normals are supposed to be oriented with respect to the interpolating cone. More explicitly, we obtain the equation

$$\rho(z_2 + \lambda_2 c_2 - \lambda_1) = \varepsilon(-c_2 \lambda_1 + \rho^2 \lambda_2 + p_2 \cdot n_2), \quad (8)$$

where  $\rho = \|n_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2}$  and  $\varepsilon = \pm 1$ . This latter condition being linear in  $\lambda_1$  and  $\lambda_2$ , one can solve the system of equations (7) and (8) in closed form. More precisely, once  $\varepsilon$  is fixed, we get the two following solutions:

$$\begin{cases} \lambda'_1 = \frac{a_2^2 z_2 - a_2 c_2 x_2 + b_2^2 z_2 - b_2 c_2 y_2}{a_2^2 + b_2^2} \\ \lambda'_2 = -\frac{a_2 x_2 + b_2 y_2}{a_2^2 + b_2^2} \end{cases}, \quad \begin{cases} \lambda_1 = \frac{p_2 \cdot n_2}{c_2 + \varepsilon \rho} \\ \lambda_2 = -\frac{z_2}{c_2 + \varepsilon \rho} \end{cases}.$$

Observe that  $|c_2| \neq \rho$  since it is assumed that  $n_2$  is not parallel to  $n_1$ . Another important observation is that the solution  $(\lambda'_1, \lambda'_2)$  is independent of  $\varepsilon$  and moreover satisfies  $z_2 + \lambda'_2 c_2 - \lambda'_1 = 0$ . Therefore, it is not a valid solution. So, we are left with two solutions to our geometric interpolation problem, namely

$$\begin{cases} \lambda_1 = \frac{p_2 \cdot n_2}{c_2 - \rho} \\ \lambda_2 = -\frac{z_2}{c_2 - \rho} \end{cases}, \quad \begin{cases} \lambda_1 = \frac{p_2 \cdot n_2}{c_2 + \rho} \\ \lambda_2 = -\frac{z_2}{c_2 + \rho} \end{cases}. \quad (9)$$

Once such a solution is chosen, then one can determine a unique cone through  $\mathcal{P}$ : its apex is given by (6), its direction is given by  $q = q_2 - q_1$  and its angle is given by the angle between the vectors  $p_1 - apex$  and  $q$ .

It remains to treat the case where the lines  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect. Let us denote by  $\Pi$  the plane that contains these two lines, by  $\mathcal{N}_1$  the normal line to  $\mathcal{L}_1$  through  $p_1$ , by  $\mathcal{N}_2$  the normal line to  $\mathcal{L}_2$  through  $p_2$  and by  $\omega$  the intersection point between  $\mathcal{N}_1$  and  $\mathcal{N}_2$ . We notice that  $\omega$  is nothing but the intersection point between  $\Pi$  and  $\mathcal{L}$ .

First, we observe that there are always two cones through  $\mathcal{P}$ . These two cones have the same apex  $\omega$  and their axis are the perpendicular bisectors of the lines  $\mathcal{N}_1$  and  $\mathcal{N}_2$  through  $\omega$ . Therefore, these two cones are symmetric with respect to the plane  $\Pi$  and their intersection is composed of the two lines  $\mathcal{N}_1 \cup \mathcal{N}_2$ . They are no other cones through  $\mathcal{P}$  and whose axis is contained in  $\Pi$  (equivalently whose apex is  $\omega$ ).

Now, we are looking for cones through  $\mathcal{P}$  and whose apex  $\alpha$  is such that  $\alpha \in \mathcal{L} \setminus \{\omega\}$ . Since the axis of such a cone must intersect  $\mathcal{L}_1$  and  $\mathcal{L}_2$  and go through  $\alpha$  which is not in  $\Pi$ , then it must be the line through  $\alpha$  and  $q := \mathcal{L}_1 \cap \mathcal{L}_2 \in \Pi$  (recall that it is assumed that  $n_2$  and  $n_1$  are not proportional). Therefore, a necessary condition for the existence of a cone through  $\mathcal{P}$  and whose axis is the line through  $\alpha$  and  $q$  is that  $\|qp_1\| = \|qp_2\|$  (Euclidian distance) because  $p_1$  and  $p_2$  must be on the same circle (normal section) of the cone. But it turns out that this is also a



sufficient condition. More precisely, if  $\|qp_1\| = \|qp_2\|$  then for any point  $\alpha \in \mathcal{L} \setminus \{\omega\}$  there exists a cone through  $\mathcal{P}$  and whose axis is the line through  $q$  and  $\alpha$ . To prove this, we have to show that the angles  $(\alpha q, \alpha p_1)$  and  $(\alpha q, \alpha p_2)$  are the same. For  $i = 1, 2$ , we have that

$$\vec{\alpha q} \cdot \vec{\alpha p}_i = (\vec{\alpha \omega} + \vec{\omega q}) \cdot (\vec{\alpha \omega} + \vec{\omega p}_i) = \|\vec{\alpha \omega}\|^2 + \vec{\omega q} \cdot \vec{\omega p}_i$$

because  $\mathcal{L}$  is normal to  $\Pi$  by construction. But since  $\|qp_1\| = \|qp_2\|$ , the points  $q, p_1, p_2, \omega$  form an orthodiagonal quadrilateral and it follows that  $\vec{\omega q} \cdot \vec{\omega p}_1 = \vec{\omega q} \cdot \vec{\omega p}_2$ .

In summary, when the lines  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect there always exists two cones through  $\mathcal{P}$ , whose axis are contained in  $\Pi$ . If  $\|qp_1\| \neq \|qp_2\|$  then there are no more cones through  $\mathcal{P}$ , otherwise there are infinitely many cones through  $\mathcal{P}$ , more precisely a 1-dimensional family of cones which is parameterized by  $\mathcal{L}$ . A last observation is that if  $\|qp_1\| = \|qp_2\|$  then the two perpendicular bisectors of  $\mathcal{N}_1$  and  $\mathcal{N}_2$  through  $\omega$  goes through  $q$ , so that there is a continuity in this family of cones through  $\mathcal{P}$  when  $\alpha$  is moving on the line  $\mathcal{L}$ .

**Theorem 4.** *Suppose given an oriented set of two distinct points  $\mathcal{P}$  and denote by  $\mathcal{L}$  the line through  $p_1$  with direction  $n_1$  and by  $\mathcal{L}_2$  the line through  $p_2$  with direction  $n_2$ . If the two following conditions hold*

- i)  $n_1$  and  $n_2$  are not proportional,
- ii) if the lines  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect at a point  $q$  then  $\|qp_1\| \neq \|qp_2\|$ ,

then there are exactly two real cones through  $\mathcal{P}$ . Moreover, the intersection of these two cones consists only of the points  $p_1$  and  $p_2$ , except if they share the same apex in which case their intersection consists of the lines  $\mathcal{N}_1$  and  $\mathcal{N}_2$ .

If  $n_1$  is parallel to  $n_2$  then there is no cone through  $\mathcal{P}$  unless  $p_2$  belongs to the plane  $\mathcal{N}_1$ , in which case there are infinitely many cones through  $\mathcal{P}$ .

If the lines  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect at a point  $q$  such that  $\|qp_1\| = \|qp_2\|$ , then there are infinitely many cones through  $\mathcal{P}$ .

*Extraction algorithm.* From the above analysis, we deduce the following extraction algorithm of cones through a set of two oriented points  $\mathcal{P}$ .

1. Perform a linear change of coordinates so that we have  $p_1 = (0, 0, 0)$  and  $n_1 = (0, 0, 1)$ .
2. If  $n_1$  and  $n_2$  are proportional then check whether  $p_2$  belongs to  $\mathcal{N}_1$  and return that either there is no cone through  $\mathcal{P}$ , or there are infinitely many cones through  $\mathcal{P}$ .
3. If  $\mathcal{L}_1$  intersect  $\mathcal{L}_2$ , then if  $\|qp_1\| \neq \|qp_2\|$  return the two cones through  $\mathcal{P}$ , otherwise if  $\|qp_1\| = \|qp_2\|$  return that there are infinitely many cones through  $\mathcal{P}$ .

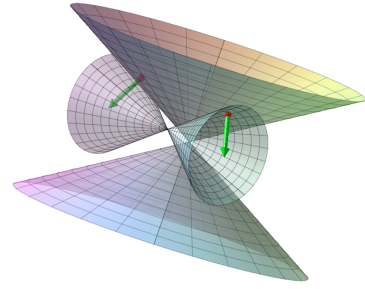


Figure 3: Two cones through a general set of two oriented points.

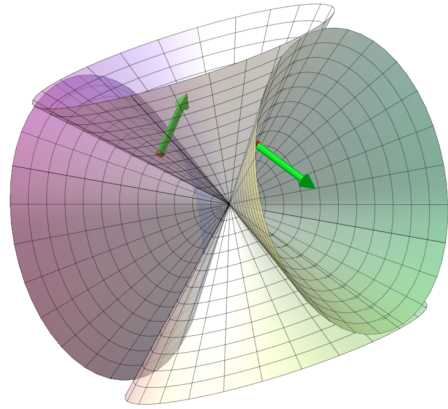


Figure 4: Two cones intersecting along two lines and sharing the same apex.

4. Now, since  $n_1$  is not proportional to  $n_2$  and  $\mathcal{L}_1 \cup \mathcal{L}_2 = \emptyset$ , there are exactly two cones through  $\mathcal{P}$  that are computed by means of the closed formulas (9).

This algorithm has been implemented with the Maple software. We observed that the computation of the cylinders through a random set of points takes in average 3.8ms and is almost constant independently of the point set. Some illustrative configurations are shown in Figure 3 and Figure 4.

### 3.2. Cones through a mixed minimal point set

We seek for cones through a set  $\mathcal{P}$  which is composed of an oriented point  $(p_1, n_1)$  and three other simple points  $p_2, p_3, p_4$ . By a linear change of coordinates, one can assume without loss of generality that  $p_1 := (0, 0, 0)$  and  $n_1 := (0, 0, 1)$ . For the remaining data, we set  $p_2 = (x_2, y_2, z_2)$ ,  $p_3 = (x_3, y_3, z_3)$  and  $p_4 = (x_4, y_4, z_4)$ .

The axis of a cone through  $\mathcal{P}$  must intersect the line generated by  $n_1$  through  $p_1$  in a point  $q = (0, 0, r)$ . Observe that  $r$  must be nonzero otherwise  $p_1$  would be the apex of such a cone, which is impossible since it is a non-singular point. In addition, the apex of a cone through  $\mathcal{P}$  must belong to the normal plane to  $n_1$  through the point  $p_1$  and we choose to represent the coordinates of this point by  $p = (ar, br, 0)$ . Again, observe for the same reason that  $a$  and  $b$  must be nonzero, i.e.  $a^2 + b^2 \neq 0$ .

Now, a point  $m = (x, y, z)$  belonging to a cone through  $\mathcal{P}$  must satisfy

$$(p\vec{m} \cdot p\vec{q})^2 = \|pm\|^2 \|pq\|^2 \kappa,$$

that is to say

$$r^2 (-a(x - ar) - b(y - br) - z)^2 = r^2 ((x - ar)^2 + (y - br)^2 + z^2) (a^2 + b^2 + 1) \kappa$$

where  $\kappa > 0$  is a constant that corresponds to the angle of the cone. The factor  $r^2$  can be cancelled out. Moreover, the cone goes through  $p_1$  so we deduce that

$$r^2 (a^2 + b^2)^2 = r^2 (a^2 + b^2) (a^2 + b^2 + 1) \kappa.$$

Reporting in the previous equation, we finally deduce that the point  $m = (x, y, z)$  belonging to a cone through  $\mathcal{P}$  must satisfy  $F(x, y, z) = 0$  where

$$F(x, y, z) = 2(a^2 + b^2) zr + (a^2 + b^2 - 1) z^2 - 2(ax + by) z + (bx - ay)^2.$$

The three parameters  $a, b$  and  $r$  are to be determined and we will use the three points  $p_2, p_3$  and  $p_4$  for that purpose.

Assume that one of the point  $p_2, p_3, p_4$ , say  $p_2$ , is not in the plane of equation  $z = 0$  (the case where all the points are in this plane will be discussed later). Hence  $z_2 \neq 0$ . Then, there exists a cone through  $\mathcal{P}$  if and only if

$$F(x_2, y_2, z_2) = 0, \quad z_3 F(x_2, y_2, z_2) - z_2 F(x_3, y_3, z_3) = 0, \\ z_4 F(x_2, y_2, z_2) - z_2 F(x_4, y_4, z_4) = 0.$$

It turns out that the two last equations are independent of  $r$ ; for  $i = 3, 4$  we have

$$z_i F(x_2, y_2, z_2) - z_2 F(x_i, y_i, z_i) = (y_2^2 z_i - y_i^2 z_2 + z_2^2 z_i - z_2 z_i^2) a^2 - (2x_2 y_2 z_i - 2x_i y_i z_2) ab - (2x_2 z_2 z_i - 2x_i z_2 z_i) a + (x_2^2 z_i - x_i^2 z_2 + z_2^2 z_i - z_2 z_i^2) b^2 - (2y_2 z_2 z_i - 2y_i z_2 z_i) b - z_2^2 z_i + z_2 z_i^2.$$

Since these two equations are quadratic in  $a$  and  $b$ , they have four common roots. Once  $a$  and  $b$  are determined then  $r$  is uniquely defined by the equation  $F(x_2, y_2, z_2) = 0$  which is linear in  $r$ . So, we have proved the following result.

**Theorem 5.** *Suppose given a general mixed point set  $\mathcal{P}$ , then there is 0, 2 or 4 cones through  $\mathcal{P}$ .*

*Extraction algorithm.* From the above analysis, the extraction algorithm of cones through a mixed point set  $\mathcal{P}$  composed of an oriented point and three other distinct points relies on the solving of two algebraic equations in two variables. To solve such systems, we proceed as in Section 2.2.2 by means of eigen-computations from a pencil of matrices which is in closed form in terms of the input data. This pencil  $A, B$  of  $8 \times 8$  matrices is obtained as the companion matrices of both equations  $z_i F(x_2, y_2, z_2) - z_2 F(x_i, y_i, z_i) = 0, i = 3, 4$ .

1. Perform a linear change of coordinates so that we have  $p_1 = (0, 0, 0)$  and  $n_1 = (0, 0, 1)$ .
2. Instantiate the pencil of matrices  $A, B$  with the coordinates of the input points.
3. Compute the eigenvalues and eigenvectors and deduce the real cones through  $\mathcal{P}$ .

This algorithm has been implemented with the `Maple` software. We observed that the computation of the cylinders through a random set of points takes in average 7.5ms and is almost constant independently of the point set. The proportion of the number of cylinders found through random point sets is given in Table 3 and some illustrative configurations are shown in Figure 5.

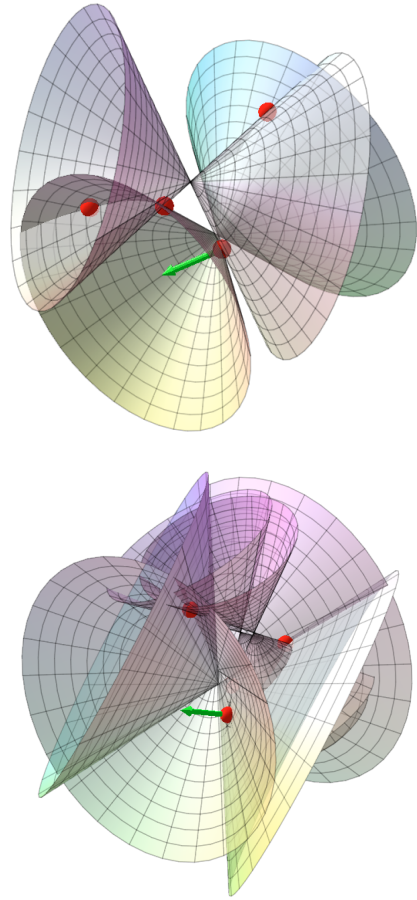


Figure 5: Two configurations of  $\mathcal{P}$  with 2 and 4 cones passing through  $\mathcal{P}$ .

Number of cones	0	2	4
Proportion (%)	6.9	85.9	7.2

Table 3: Proportion of the number of cylinders found through a thousand random point sets.

### 3.3. Cones through six points

We seek for cones through a set  $\mathcal{P}$  composed of six simple points  $p_1, p_2, p_3, p_4, p_5, p_6$ . By a linear change of coordinates, we can assume without loss of generality that

$$\begin{aligned} p_1 &= (0, 0, 0), \quad p_2 = (x_2, 0, 0), \quad p_3 = (x_3, y_3, 0), \\ p_4 &= (x_4, y_4, z_4), \quad p_5 = (x_5, y_5, z_5), \quad p_6 = (x_6, y_6, z_6). \end{aligned} \quad (10)$$

*Geometric analysis.* The general equation of a cone depends on 6 parameters:

$$\begin{aligned} F(x, y, z) &:= \\ (x-a)^2 + (y-b)^2 + (z-c)^2 - (l(x-a) + m(y-b) + n(z-c))^2 \end{aligned} \quad (11)$$

where  $(a, b, c)$  are the coordinates of the apex and the vector  $(l, m, n)$  encodes the direction of the axis as well as the angle of the cone ( $l^2 + m^2 + n^2 = 1/\cos^2(\theta)$ ,  $\theta$  being the angle of the cone).

We introduce the quantity  $k := al + bm + cn$ . Since  $p_1$  is the origin, we get the equation

$$F(p_1) = a^2 + b^2 + c^2 - k^2 = 0. \quad (12)$$

Moreover, we observe that the polynomial

$$\begin{aligned} F(x, y, z) - F(p_1) &= 2(lx + my + nz)k - 2(ax + by + cz) \\ &- 2lmxy - 2lnxz - 2mnyz - l^2x^2 - m^2y^2 - n^2z^2 + x^2 + y^2 + z^2 \end{aligned} \quad (13)$$

is linear in  $a, b, c$  and  $k$ . From the equation

$$F(p_2) - F(p_1) = 2lx_2k - 2x_2a - l^2x_2^2 + x_2^2 = 0$$

we get, assuming  $x_2 \neq 0$ , i.e.  $p_2 \neq p_1$ ,

$$a = lk + \frac{x_2}{2}(1 - l^2). \quad (14)$$

Now, we have that

$$\begin{aligned} F(p_3) - F(p_1) &= 2my_3k - 2y_3b + x_3l^2x_2 \\ &- l^2x_3^2 - 2lmx_3y_3 - m^2y_3^2 - x_3x_2 + x_3^2 + y_3^2 = 0 \end{aligned}$$

and we get, assuming  $y_3 \neq 0$ , i.e. assuming that  $p_1, p_2$  and  $p_3$  are not aligned,

$$\begin{aligned} b &= mk + \frac{1}{2y_3} (x_3l^2x_2 - l^2x_3^2 - 2lmx_3y_3 \\ &- m^2y_3^2 - x_3x_2 + x_3^2 + y_3^2). \end{aligned} \quad (15)$$

Continuing this way, we get the equation

$$\begin{aligned} F(p_4) - F(p_1) &= 2z_4nk - 2z_4c \\ &+ \frac{1}{y_3} (-l^2x_2x_3y_4 + l^2x_2x_4y_3 + l^2x_3^2y_4 - l^2x_4^2y_3 \\ &+ 2lmx_3y_3y_4 - 2lmx_4y_4y_3 - 2lnx_4z_4y_3 + m^2y_3^2y_4 \\ &- m^2y_4^2y_3 - 2mny_4z_4y_3 - n^2z_4^2y_3 + x_2x_3y_4 - x_2x_4y_3 \\ &- x_3^2y_4 + x_4^2y_3 - y_3^2y_4 + y_4^2y_3 + z_4^2y_3) = 0 \end{aligned}$$

we get, assuming that  $z_4 \neq 0$ , i.e. assuming that  $p_1, p_2, p_3$  and  $p_4$  are not coplanar,

$$\begin{aligned} c &= nk + \frac{1}{2y_3z_4} (-l^2x_2x_3y_4 + l^2x_2x_4y_3 + l^2x_3^2y_4 - l^2x_4^2y_3 \\ &+ 2lmx_3y_3y_4 - 2lmx_4y_4y_3 - 2lnx_4z_4y_3 + m^2y_3^2y_4 - m^2y_4^2y_3 \\ &- 2mny_4z_4y_3 - n^2z_4^2y_3 + x_2x_3y_4 - x_2x_4y_3 - x_3^2y_4 + x_4^2y_3 \\ &- y_3^2y_4 + y_4^2y_3 + z_4^2y_3). \end{aligned} \quad (16)$$

From here, we substitute  $a, b$  and  $c$  by the above quantity that we found in both equations  $F(p_i) - F(p_1) = 0$  with  $i = 5, 6$ . It is important to notice that these two equations do not depend on  $k$  because of the cancellation of the two terms  $2(lx + my + nz)k - 2(ax + by + cz)$  in (13) by this substitution. We get two equations  $H_i(l, m, n)$ ,  $i = 5, 6$ , that satisfy

$$\begin{aligned} y_3z_4H_i(l, m, n) &= \\ &(x_2x_3y_4z_i - x_2x_3y_iz_4 - x_2x_4y_3z_i + x_2x_iz_3y_4 \\ &- x_3^2y_4z_i + x_3^2y_iz_4 + x_4^2y_3z_i - x_i^2y_3z_4) l^2 \\ &+ (-2x_3y_3y_4z_i + 2x_3y_3y_iz_4 + 2x_4y_3y_4z_i - 2x_iz_3y_3z_4) lm \\ &+ (2x_4y_3z_4z_i - 2x_iz_3y_3z_4) ln \\ &+ (-y_3^2y_4z_i + y_3^2y_iz_4 + y_3y_4^2z_i - y_i^2y_3z_4) m^2 \\ &+ (2y_3y_4z_4z_i - 2y_3y_iz_4z_i) mn + (y_3z_4^2z_i - z_i^2y_3z_4) n^2 \\ &- x_2x_3y_4z_i + x_2x_3y_iz_4 + x_2x_4y_3z_i - x_2x_iz_3y_4 \\ &+ x_3^2y_4z_i - x_3^2y_iz_4 - x_4^2y_3z_i + x_i^2y_3z_4 + y_3^2y_4z_i \\ &- y_3^2y_iz_4 - y_3y_4^2z_i + y_i^2y_3z_4 - y_3z_4^2z_i + z_i^2y_3z_4 = 0. \end{aligned}$$

These equations  $H_5$  and  $H_6$  are polynomials of degree 2 in  $l, m, n$  and only the constant term and monomials of degree 2 have nonzero coefficients. The equation  $H_5$  depends on the coefficients of  $p_2, p_3, p_4$  and  $p_5$  whereas  $H_6$  depends on the coefficients of  $p_2, p_3, p_4$  and  $p_6$ . Now, applying the same substitutions and replacing  $k$  by its defining value in (12), we get another equation  $H_0(l, m, n)$  which is of degree 6 in  $l, m, n$  and such that only monomials of even degree have nonzero coefficients. Observe that  $H_0$  only depend on the coordinates of  $p_2, p_3$  and  $p_4$ . To give an idea, we have

$$\begin{aligned} 4y_3^2z_4^2(l^2 + m^2 + n^2 - 1)H_0 &= \\ &(-x_2^2x_3^2y_4^2 - x_2^2x_3^2z_4^2 + 2x_2^2x_3x_4y_3y_4 - x_2^2x_4^2y_3^2 \\ &+ 2x_2x_3^3y_4^2 + 2x_2x_3^3z_4^2 - 2x_2x_3^2x_4y_3y_4 \\ &- 2x_2x_3x_4^2y_3y_4 + 2x_2x_4^3y_3^2 - x_3^4y_4^2 - x_3^4z_4^2 \\ &+ 2x_3^2x_4^2y_3y_4 - x_4^4y_3^2) l^6 + \dots \end{aligned}$$

We notice that the quantity  $l^2 + m^2 + n^2 - 1$  can be assumed to be nonzero because cones such that  $l^2 + m^2 + n^2 = 1$  are degenerated to straight lines.

The common roots of the equations  $H_0, H_5$  and  $H_6$  yields exactly all the cones through  $\mathcal{P}$ . If finite, this number is at most  $2 \times 2 \times 6 = 24$  because of the Bézout theorem. However, each cone through  $\mathcal{P}$  yields two solutions

of this polynomial system  $H_0 = 0, H_5 = 0, H_6 = 0$ , namely  $(l_0, m_0, n_0)$  and  $(-l_0, -m_0, -n_0)$ . This latter property is a consequence of the fact that  $H_0, H_5, H_6$  can be written on monomials of even total degree in  $l, m, n$ . Therefore, we deduce that there is at most 12 cones through  $\mathcal{P}$ . Nevertheless, in order to devise an efficient algorithm for computing those cones through  $\mathcal{P}$ , we apply a transformation (called a  $\sigma$ -process or a blow-up in birational geometry) in order to break the above central symmetry and reduce the degree of the equations so that each cone will correspond to a unique root of the new polynomial system.

Consider the transformation

$$\sigma : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\ (l, m, n) \mapsto \left( u := \frac{l}{n}, v := \frac{m}{n}, w := n \right).$$

Since the equations  $H_0, H_5$  and  $H_6$  only contain even degree monomials, their transform under  $\sigma$  will be a polynomial in  $n^2$  (and not only a polynomial in  $n$ ). More precisely, setting  $w := n^2$ , we have

$$\begin{aligned} H_0(un, vn, n) &= w^3 P(u, v) + w^2 Q(u, v) + w R(u, v) + S, \\ H_5(un, vn, n) &= w H(u, v) + C, \\ H_6(un, vn, n) &= w K(u, v) + D, \end{aligned}$$

where  $H, K, P, Q, R$  are polynomials in  $u, v$  of degree 2, 2, 6, 4, 2 respectively, and where  $C, D, S$  are constant that only depends on the  $x_i$ 's,  $y_i$ 's and  $z_i$ 's. Now, since the two last equations are linear in  $w$  we can simplify this polynomial system by  $w$ , assuming  $w \neq 0$ , to get

$$\begin{aligned} E_0(u, v) &= C^3 P(u, v) - C^2 Q(u, v) H(u, v) \\ &\quad + CR(u, v) H(u, v)^2 - SH(u, v)^3, \\ E_1(u, v) &= DH(u, v) - CK(u, v), \\ E_2(u, v, w) &= wK(u, v) + D. \end{aligned} \quad (17)$$

The two equations  $E_0$  and  $E_1$  are bivariate polynomials in  $u, v$  of degree 6 and 2 respectively. Therefore, they define at most 12 solutions. In addition, for each solution the value of  $w$  can be computed from  $E_2$ . As a consequence, we have just proved the following result.

**Theorem 6.** *Suppose given a point set  $\mathcal{P}$  composed of six distinct points. Then, if finite, there is an even number of cones through  $\mathcal{P}$ , which is possibly 0 and at most 12.*

*Proof.* The solving of the polynomial system  $E_0(u, v) = E_1(u, v)$  leads to an even number of real solutions with a maximum of  $12 = 6 \times 2$  solutions. So the only thing to show is that there is a bijection between these solutions and the cones through  $\mathcal{P}$ . For that purpose, we first observe that the value of  $w$  is uniquely determined from each of those roots by equation  $E_2$ . And then the solutions  $(u, v, w)$  are uniquely pullback to a solution  $(l, m, n)$  under  $\sigma$ .

So, it remains to show that once a solution  $(l, m, n)$  is computed, then the parameters  $a, b$  and  $c$  are uniquely determined. This is indeed the case because  $a, b$  and  $c$  can

be actually given in closed form in terms of  $l, m, n$  and the coordinates of the input points. This is because the three equations (14), (15) and (16) yields a linear system in  $a, b, c$  after replacing  $k$  by its defining value  $al + bm + cn$ .  $\square$

In Figure 6 it is shown that all the possible number of solutions are reached in practice. Finally, observe that the particular case  $w = n^2 = 0$  can be treated independently in the same vain as above. Indeed, the remaining variables  $l, m$  can be computed from the equations  $H_5(l, m, 0) = 0$  and  $H_6(l, m, 0) = 0$ . Moreover, to solve efficiently this system one can perform a transformation similar to  $\sigma$  by considering the equations  $H_5(um, m, 0) = 0$  and  $H_6(um, m, 0) = 0$  that are linear equations in  $m^2$ . At the end, the computed solutions have to satisfied the equation  $H_0$  in order to validate that they correspond to cones through the given set of points  $\mathcal{P}$ .

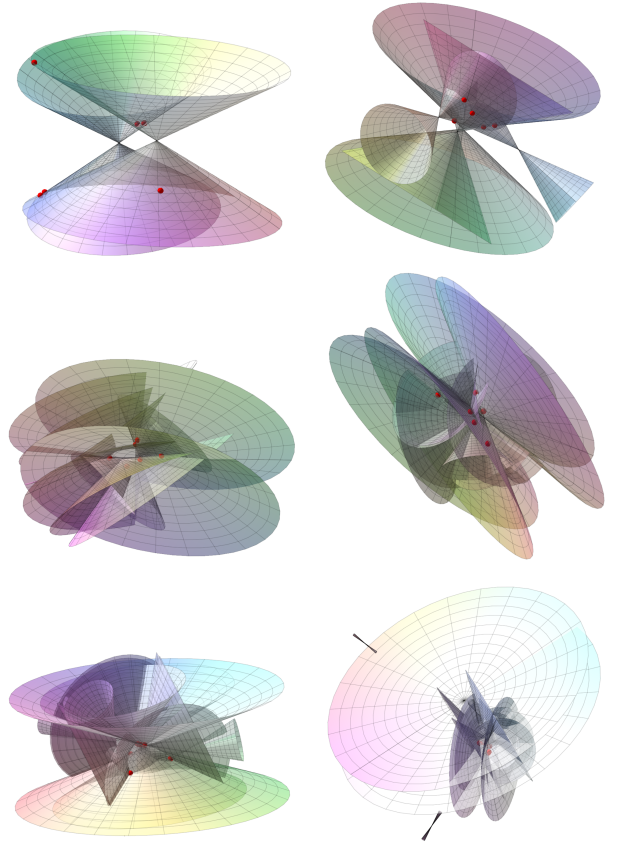


Figure 6: Configurations of points  $\mathcal{P}$  with 2, 4, 6, 8, 10 and 12 cones passing through  $\mathcal{P}$ .

*Extraction algorithm.* Summarizing the above analysis, the solving of the cones through  $\mathcal{P}$  can be done as follows.

1. Perform a linear change of coordinates so that the five points are of the form (10).
2. If  $x_2, y_3$  or  $z_4$  is equal to zero then we are in a particular configuration (see Section 3.3) then this algorithm stops here.

Number of cones	0	2	4	6
Proportion (%)	1	10,5	28,7	36,3
Number of cones	8	10	12	
Proportion (%)	18,7	3,9	0,9	

Table 4: Proportion of the number of cones found through a thousand random point sets.

3. Build the system of equations (17) which is given in closed form in terms of the coordinates of the input points.
4. Solve the system  $E_0(u, v) = E_1(u, v) = 0$  in the two variables  $u, v$  by means of eigenvectors and eigenvalues computations, as explained in Section 2.2.2 (see [2] for details). It returns a list of an even number between 0 and 12 of real solutions (up to a given precision).
5. For each above solution  $(u_i, v_i)$ , compute the corresponding value of  $w_i$  with the equation  $E_2(u_i, v_i, w_i) = 0$ . Then, pullback the solution  $(u_i, v_i, w_i)$  under  $\sigma$  to get the solutions  $(l_i = u_i n_i, m_i = v_i n_i, n_i = \sqrt{w_i})$  of the polynomial system  $H_0 = H_5 = H_6 = 0$ .
6. For each solution  $(l_i, m_i, n_i)$ , compute the corresponding  $(a_i, b_i, c_i)$  (as explained in the proof of Theorem 6) and return the corresponding cone through  $\mathcal{P}$ .

This algorithm has been implemented with the `Maple` software. We observed that the computation of the cones through a random set of points takes in average 80ms and is almost constant independently of the point set. In Table 4, we provide the repartition of the number of cones through a set of point  $\mathcal{P}$  for a random sample of a thousand point sets  $\mathcal{P}$ .

*Some particular configurations.* If any subset of four points in  $\mathcal{P}$  is coplanar then all the six points in  $\mathcal{P}$  are necessarily coplanar. In such a configuration, there exists a cone through  $\mathcal{P}$  if and only if  $\mathcal{P}$  can be interpolated by a conic section (parabola, hyperbola or ellipse). If this is the case, then there are infinitely many cones through  $\mathcal{P}$ . When a subset of five points in  $\mathcal{P}$  are coplanar then we get interesting particular configurations.

**Theorem 7.** *Suppose given a point set  $\mathcal{P}$  where five points are coplanar but not six. Then, there exists a cone through  $\mathcal{P}$  if these five points are located on a conic section. Moreover, if this is the case then there are at most four cones through  $\mathcal{P}$ .*

*More specifically, if the five points are co-circular, then there are two cones through  $\mathcal{P}$  and if they are on a parabola then there are at most 3 cones through  $\mathcal{P}$ .*

*Proof.* It is clear that the five coplanar points must be on a conic section if there exists a cone through  $\mathcal{P}$ . By a change of coordinate system, one can assume that this conic section in the plane  $z = 0$  and is in its canonical

form. We examine below the three cases that correspond to an ellipse, an hyperbola and a parabola.

*Ellipse.* The equation of the conic section is of the form

$$\alpha x^2 + \beta y^2 - 1 = 0, \quad \alpha, \beta > 0$$

and it must coincide with the intersection of the implicit equation of a cone (11) after the substitution  $z = 0$ . Now, we proceed by identification of the monomial coefficients of these two equations. the coefficient of the monomial  $xy$  yields the condition  $2lm = 0$  so that  $l = 0$  or  $m = 0$ . If  $l = 0$ , then the coefficients of  $y^2$  yields the condition  $m^2 = 1 - \beta/\alpha$  and if  $m = 0$  then the coefficients of  $x^2$  yields the condition  $l^2 = 1 - \alpha/\beta$ . Depending on the ratio  $\alpha/\beta$ , only one case leads to real solutions. From now on we assume that  $\alpha/\beta \geq 1$ , the other case can be treated similarly. So we have that

$$l = 0, \quad m^2 = 1 - \frac{\beta}{\alpha} \geq 0.$$

The coefficient of the monomial  $x$  yields the condition  $a = 0$  and the coefficient of  $y$  gives the condition

$$b = -\frac{cmn}{m^2 - 1} = \frac{\alpha}{\beta} cmn \quad (18)$$

(observe that  $m^2 - 1 \neq 0$ ). Finally, the coefficients of the monomial 1 yields the condition

$$(\alpha n^2 - \beta)c^2 - \frac{\beta}{\alpha} = 0. \quad (19)$$

Now, let  $(x_0, y_0, z_0)$  be the coordinates of the sixth point. The evaluation of (11) at this point together with the conditions  $l = 0, a = 0$  and (18) yields the condition

$$c^2 \left( n^2 - \frac{\beta}{\alpha} \right) - 2cz_0 \left( n^2 - \frac{\beta}{\alpha} \right) + \varphi_2(n) = 0 \quad (20)$$

(recall that  $\alpha, \beta > 0$  and in particular  $m^2 - 1 \neq 0$ ) where

$$\varphi_2(n) := (-m^2 z_0^2 + z_0^2) n^2 + (-2m^3 y_0 z_0 + 2m y_0 z_0) n - m^4 y_0^2 + m^2 x_0^2 + 2m^2 y_0^2 + m^2 z_0^2 - x_0^2 - y_0^2 - z_0^2$$

is a degree 2 polynomial in  $n$  which is independent of  $c$ . The cones through  $\mathcal{P}$  are in correspondence with the solutions of the equations (19) and (20) in the variables  $c$  and  $n$ . Using (19), Equation (20) becomes

$$2cz_0 (\alpha n^2 - \beta) = \alpha \varphi_2(n) + \frac{\beta}{\alpha}$$

and hence we get

$$\left( \alpha \varphi_2(n) + \frac{\beta}{\alpha} \right)^2 = 4c^2 z_0^2 (\alpha n^2 - \beta)^2 = 4z_0^2 \frac{\beta}{\alpha} (\alpha n^2 - \beta)$$

which is an equation of degree 4 in  $n$ . Therefore, once  $m$  is chosen from the condition  $m^2 = 1 - \beta/\alpha$ , we obtain



at most 4 values of  $n$  and all the others parameters are uniquely determined. We conclude that we have at most four cones through  $\mathcal{P}$ , as claimed.

If the conic section is a circle, i.e.  $\alpha = \beta$ , we have  $l = m = 0$ . In this case,  $\varphi_2(n) = z_0^2(n^2 - 1) - (x_0^2 + y_0^2)$  and hence the degree 4 equation in  $n$  is actually an equation in  $n^2$ . Therefore, the 4 solutions arises by pairs of opposite solutions and we deduce that we have two cones through  $\mathcal{P}$ .

*Hyperbola.* The equation of the conic section is of the form

$$\alpha x^2 - \beta y^2 - 1 = 0, \quad \alpha, \beta > 0$$

and we proceed similarly to the case of the ellipse. The identification of  $xy$  implies that  $2lm = 0$  so we have to cases to consider, namely  $l = 0$  and  $m = 0$ .

If  $l = 0$  then the coefficient of  $y^2$  yields  $m^2 = 1 + \beta/\alpha$ . The coefficient of  $x$  shows that  $a = 0$  and the coefficient of  $y$  gives

$$b = -\frac{\alpha}{\beta}cmn.$$

Then, the constant term yields the equality

$$b^2 + c^2 - (bm + cn)^2 = -\frac{1}{\alpha}$$

which becomes, after the substitution of  $b$ ,

$$c^2 \left( 1 + \frac{\alpha}{\beta} n^2 \right) = -\frac{1}{\alpha}.$$

Therefore, there are no real solutions in this case.

Now, if  $m = 0$ , then  $l^2 = 1 + \alpha/\beta$ . the coefficient of  $y$  shows that  $b = 0$  and the coefficient of  $x$  gives

$$a = -\frac{ncl}{l^2 - 1} = -\frac{\beta}{\alpha}ncl.$$

Then, the constant coefficient yields the equation

$$\frac{c^2(l^2 + n^2 - 1)}{l^2 - 1} = \frac{1}{\beta}.$$

From here, we use the sixth point as in the case of the ellipse and we get again 4 possible real cones through  $\mathcal{P}$ .

*Parabola.* The equation of the conic section is of the form

$$x^2 - \alpha y = 0, \quad \alpha > 0.$$

again, we inspect the coefficients of (11) after the substitution  $z = 0$ . The coefficient of  $x^2$  gives  $l = 0$  and the coefficient of  $y^2$  gives  $m^2 = 1$ . The coefficient of  $x$  gives  $a = 0$  and the coefficient of  $y$  gives  $2cmn = -\alpha$ . Finally, the constant coefficient yields

$$b = -\frac{c^2(1 - n^2)}{\alpha}.$$

Now, choosing  $m = 1$  and using the sixth point of coordinates  $(x_0, y_0, z_0)$ , the equation (11) gives an equation of degree 3 in  $n$ , namely

$$\begin{aligned} -2z_0^2 n^3 + (-\alpha z_0 - 4y_0 z_0) n^2 \\ + (-2y_0 \alpha + 2x_0^2 + 2z_0^2) n + \alpha z_0 = 0 \end{aligned}$$

and from here  $c$  can be uniquely determined.  $\square$

## 4. Conclusion

In this paper, we have presented several methods in order to extract efficiently cylinders and cones from minimal point sets. We have also provided a detailed analysis of these interpolation problems and we have given optimal bounds on the number of solutions. Our approach relies on closed algebraic formulas that have been computed and experimented with the help of a computer algebra system. In a near future, we plan to implement these methods into an efficient C++ library. Another future research direction will be the extraction of tori from a minimal point set, tori being described by means of seven parameters.

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## References

- [1] Matthew Berger, Andrea Tagliasacchi, Lee Seversky, Pierre Alliez, Joshua Levine, Andrei Sharf, and Claudio Silva. State of the Art in Surface Reconstruction from Point Clouds. In *Eurographics 2014 - State of the Art Reports*, volume 1 of *EUROGRAPHICS star report*, pages 161–185, Strasbourg, France, April 2014.
- [2] Laurent Busé, Houssam Khalil, and Bernard Mourrain. Resultant-based methods for plane curves intersection problems. In Victor G. Ganzha, Ernst W. Mayr, and Evgenii V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing (CASC)*, volume 3718, pages 75–92, Kalamata, Greece, September 2005. Springer Berlin / Heidelberg.
- [3] Olivier Devillers, Bernard Mourrain, Franco P. Preparata, and Philippe Trebuchet. Circular cylinders through four or five points in space. *Discrete & Computational Geometry*, 29(1):83–104, 2002.
- [4] Martin A. Fischler and Robert C. Bolles. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. *Commun. ACM*, 24(6):381–395, June 1981.
- [5] Daniel Lichtblau. Cylinders through five points: Computational algebra and geometry. *Journal of Mathematics Research*, 4(6), 2012.
- [6] Gabor Lukács, Ralph Martin, and Dave Marshall. Faithful least-squares fitting of spheres, cylinders, cones and tori for reliable segmentation. In Hans Burkhardt and Bernd Neumann, editors, *Computer Vision — ECCV'98*, volume 1406 of *Lecture Notes in Computer Science*, pages 671–686. Springer Berlin Heidelberg, 1998.
- [7] Ruwen Schnabel, Roland Wahl, and Reinhard Klein. Efficient ransac for point-cloud shape detection. *Computer Graphics Forum*, 26(2):214–226, June 2007.