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L^2 convergence rates of elliptic boundary value problems with singular parameterizations for Isogeometric analysis

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Abstract

In this paper, we discuss the impact of singular parameterizations on L^2 convergence rates of elliptic boundary value problems by deriving a priori estimate. Different from the traditional finite element method, for an elliptic boundary value problem with a non-smooth weak solution, the L^2 convergence rate is improved by controlling singularities of a parameterization in the framework of Isogeometric analysis. Several numerical experiments are presented.

1 Introduction

Isogeometric analysis [1] is a numerical method to solve geometric PDEs. Within this framework, the same basis functions are used to describe physical domains and Isogeometric analysis solutions obtained by numerical analysis. These basis functions are always chosen as splines, such as NURBS, B-Splines, T-splines [12], splines over T-meshes [13], LR-splines [11], THB-splines [10] and so on. A map to describe a physical domain is called as a parameterization. Parameterizations in Isogeometric analysis should have no self-intersection. Thus, when a parameterization is under consideration, the zero points of its Jacobian are concerned. These points are called singularities of this parameterization.

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For elliptic boundary value problems (1),

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

by finite element method or Isogeometric analysis-Gerlarkin method, weak solutions are always considered by the variational method, i.e., minimizing an associated error function. On the one hand, if a weak solution is smooth enough, then it is a classical solution. On the other hand, not all of the elliptic boundary value problems have classical solutions and this case appears in many physical processes, such as the stress analysis of elastic domains with cracks and/or corners [6] [7] [8] [9]. Let Ω be a physical domain and suppose that f in (1) is analytic on $\bar{\Omega}$. Based on Weyl's lemma [18], the weak solution u is analytic at the interior points of Ω . If $\partial\Omega$ is analytic, then u is analytic on Ω . For Ω with non-smooth boundaries, non-analytic points are among these non-smooth boundary points. Let Ω be the physical domain shown in Figure 1. There is a corner P . At P , the weak solution u of the elliptic boundary value problem (1) is non-analytic. If $\theta = \alpha\pi$, the weak solution u around the corner P has a singularity of type $O(r^{1/\alpha})$ and $u \in H^{1+\frac{1}{\alpha}-\epsilon}$ ($\forall \epsilon > 0$)[18], where $1/\alpha \notin \mathbb{N}$, $r = \sqrt{x^2 + y^2}$.

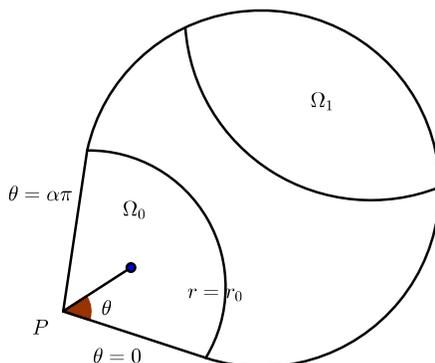


Figure 1: A physical domain Ω with a corner

In the traditional finite element method, the results of a priori estimate are completed. In the following, we collect these results from [18]. Let Ω be a polygonal domain and $f \in L^2(\Omega)$. If the weak solution $u \in \mathbf{H}^\sigma(\Omega)$ ($\sigma \geq 2$),

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{l+1} \|u\|_{\mathbf{H}^{l+1}(\Omega)}, \quad l = \min(k, \sigma - 1). \tag{2}$$

If $u \in \mathbf{H}^\sigma(\Omega)$ ($\sigma < 2$), the error bound is

$$\|u - u_h\|_{L^2(\Omega)} \leq C(h^{l+\sigma-1} + h^{2k}) \|u\|_{\mathbf{H}^{l+1}(\Omega)}, \quad l = \min(k, \sigma - 1), \tag{3}$$

where k is the degree of piecewise polynomial basis functions.

In Isogeometric analysis, priori estimates have been discussed in [19]. [19] assumes that there is no singularity of parameterizations. In some cases, singularities can be avoided via some constraints. For example, singularities are caused by distortions of regular parameterizations [14] [15] [16]. However, some singularities may stem essentially from the geometry of the physical domain and can not be avoided. For example, if a non-quadrangular physical domain (e.g., circular domains) is to be parameterized, singularities present necessarily[2] [3] for obtaining a compact representation, i.e., representing a non-quadrangular physical domain without splitting. In this paper, more effort is made to consider the impact of singularities of a parameterization on a priori L^2 -estimate and we improve L^2 convergence rates by controlling parameterization for elliptic boundary value problems with non-smooth weak solutions.

The structure of this paper is as follows. In Section 2, we present a prior estimate within the framework of Isogeometric analysis and singularities of parameterizations are taken into account. Based on this estimate and the property of non-smooth solutions of an elliptic boundary value problem, Examples 2.1 and 2.2 are presented to obtain optimal convergence rate by controlling singularities of parameterizations. In Section 3, by bicubic Hermite splines, convergence rates of the parameterizations in Isogeometric analysis with singularities are discussed. At last, in Section 4, we conclude this paper with future works.

2 Error bounds within parameterizations

Let Ω be a physical domain of an elliptic boundary problem (1). Its weak form reads:

$$\text{find } u \in \mathbb{V} : a(u, v) = l(v), \quad \forall v \in \mathbb{V},$$

where $\mathbb{V} \subset \mathbf{H}^1(\Omega)$ is a Hilbert space endowed with the norm of \mathbb{V} and whose functions satisfy the homogeneous counterpart of the essential boundary conditions. Moreover,

$$a(u, v) = \int_{\Omega} \nabla v^T \nabla u d\Omega, \quad l(v) = \int_{\Omega} f v d\Omega$$

where $a : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ is a continuous, (strongly) coercive, and bilinear form and $l : \mathbb{V} \rightarrow \mathbb{R}$ is a linear and continuous functional. In the framework of the finite-dimensional Galerkin method, the approximate solution u_h is obtained by solving the following problem:

$$\text{find } u_h \in \mathbb{V}^h : a(u_h, v_h) = l(v_h), \quad \forall v_h \in \mathbb{V}^h,$$

with $\mathbb{V}^h \subset \mathbb{V}$ a nontrivial and finite dimensional subspace.

In the following, suppose that

- $\mathbb{S}_p^h(\Sigma)$ is an Isogeometric analysis function space, i.e., splines in $\mathbb{S}_p^h(\Sigma)$ are used to represent a parameterization and numerical analysis on Σ .
 1. p is the algebraic accuracy of $\mathbb{S}_p^h(\Sigma)$, i.e., there is an operator $\Pi : \mathbf{H}^{p+1}(\Sigma) \rightarrow \mathbb{S}_p^h(\Sigma)$ such that $\Pi(f(s, t)) = f(s, t)$, $\forall f(s, t) \in \mathbb{P}_p(\Sigma)$, where $\mathbb{P}_p(\Sigma)$ is the polynomial space with degree p on Σ ;
 2. \mathcal{M}_h is a parametric rectangular mesh on Σ . h is the maximum diameter of its cells. Splines in $\mathbb{S}_p^h(\Sigma)$ are defined on \mathcal{M}_h .
- \mathcal{P} is a parameterization of Ω ,

$$\mathcal{P}(s, t) = \sum_{i=1}^n \mathbf{P}_i B_i(s, t),$$

where $\mathbf{P}_i \in \mathbb{R}^2$ and $\{B_1(s, t), B_2(s, t), \dots, B_n(s, t)\}$ is a set of basis functions of $\mathbb{S}_p^h(\Sigma)$. By $\mathcal{P}(s, t)$, the domain on Σ and cells of \mathcal{M}_h can be classified.

1. A cell of Σ is denoted by C_s if the Jacobian J of \mathcal{P} has a singularity on it. Cells not of this kind are called regular cells of Σ and denoted by C_r .
 2. $\Sigma_s = \cup C_s$, $\Sigma_r = \cup C_r$. $\Sigma = \Sigma_s \cup \Sigma_r$.
- $\mathbb{V}^h = \mathbb{S}_p^h(\Sigma) \circ \mathcal{P}^{-1}$, i.e., $\mathbb{V}^h = \text{span}\{B_1 \circ \mathcal{P}^{-1}(x, y), B_2 \circ \mathcal{P}^{-1}(x, y), \dots, B_n \circ \mathcal{P}^{-1}(x, y)\}$. We assume that $B_i \circ \mathcal{P}^{-1}(x, y) \in \mathbf{H}^1(\Omega)$ to guarantee $\mathbb{V}^h \subset \mathbb{V} \subset \mathbf{H}^1(\Omega)$.

Theorem 2.1. *Let Σ and Ω be a parametric domain and a physical domain and $\mathcal{P}|_{\partial\Sigma} = \partial\Omega$. u is the weak solution of the elliptic boundary value problem (1). $u \circ \mathcal{P}$ is a bounded function on Σ , where $u \circ \mathcal{P} \in \mathbf{H}^{k+1}(\Sigma_r)$ and $u \circ \mathcal{P} \in \mathbf{H}^{k'+1}(\Sigma_s)$. Then,*

$$\|u - u^h\|_{L^2(\Omega)} \leq C_1 h^{k+1} |u \circ \mathcal{P}|_{k+1, \Sigma_r} + C_2 h^{\sigma/2+k'+1} |u \circ \mathcal{P}|_{k'+1, \Sigma_s}, \quad (4)$$

where h is the maximum diameter of the cells of the parametric mesh \mathcal{M}_h on Σ . $\mathbb{S}_p^h(\Sigma)$ is chosen as the Isogeometric analysis function space where $p = \max(k, k')$. $|J|_{C_s} \sim h^{\sigma_{C_s}}$, $\sigma = \min\{\sigma_{C_s}\}$, u^h is the Isogeometric analysis solution of the elliptic boundary value problem (1) on Ω , C_1 and C_2 are positive constants. C_1 and C_2 are independent of the mesh size h and the solution u , but dependent on the shape of the domain Ω and its parameterization \mathcal{P} .

Proof. By Céa's lemma there is a const C such that,

$$\|u - u_h\|_{L^2(\Omega)} = C \inf_{v_h \in \mathbb{V}^h} \|u - v_h\|_{L^2(\Omega)},$$

i.e.

$$\begin{aligned}
\|u - u_h\|_{L^2(\Omega)}^2 &= C^2 \inf_{v_h \in \mathbb{V}^h} \|u - v_h\|_{L^2(\Omega)}^2 \\
&= C^2 \inf_{v_h \in \mathbb{V}^h} \int_{\Omega} (u - v_h)^2 dx dy \\
&= C^2 \inf_{v_h \in \mathbb{V}^h} \sum_{C \in \Sigma} \int_C (u \circ \mathcal{P} - v_h \circ \mathcal{P})^2 |J| ds dt \\
&\leq C^2 \sum_{C_r} \int_{C_r} (u \circ \mathcal{P} - \Pi(u \circ \mathcal{P}))^2 |J| ds dt + C^2 \sum_{C_s} \int_{C_s} (u \circ \mathcal{P} - \Pi(u \circ \mathcal{P}))^2 |J| ds dt \\
&\leq C_1 \sum_{C_r} \int_{C_r} (u \circ \mathcal{P} - \Pi(u \circ \mathcal{P}))^2 ds dt + C_2 h^\sigma \sum_{C_s} \int_{C_s} (u \circ \mathcal{P} - \Pi(u \circ \mathcal{P}))^2 ds dt,
\end{aligned}$$

Moreover, $u \circ \mathcal{P} \in \mathbf{H}^{k+1}(C_r)$ and $u \circ \mathcal{P} \in \mathbf{H}^{k'+1}(C_s)$, Π is an operator associated with $\mathbb{S}_p^h(\Sigma)$ that leaves invariant all polynomials of degree $\max(k, k')$. Thus, based on the error bound in [17],

$$\|u \circ \mathcal{P} - \Pi(u \circ \mathcal{P})\|_{L^2(C_r)} \leq C_3 h^{k+1} |u \circ \mathcal{P}|_{k+1, C_r},$$

$$\|u \circ \mathcal{P} - \Pi(u \circ \mathcal{P})\|_{L^2(C_s)} \leq C_4 h^{k'+1} |u \circ \mathcal{P}|_{k'+1, C_s},$$

and we get the error bound (4). □

Remark 2.1. 1. From Theorem 2.1, the convergence rate in Isogeometric analysis is determined by the regularity of $u \circ \mathcal{P}$ on Σ_r , $u \circ \mathcal{P}$ on Σ_s , the area of Σ_r , the area of Σ_s and the speed of J towards 0.

2. By the multivariate Faà di Bruno formula [20], a formula for derivatives of a composite function, the regularity property of $u \circ \mathcal{P}$ on Σ is not worse than u on Ω , i.e., if $u \in \mathbf{H}^{k+1}(\Omega)$, then $u \circ \mathcal{P} \in \mathbf{H}^{k+1}(\Sigma)$ at least. From the other hand, singularities may cause regularity problems for basis functions that are smooth on a parametric domain. Thus, there is a balance when choosing parameterizations. By locating properly singularities, the convergence rate can be improved.

3. In the following, the Isogeometric analysis function space $\mathbb{S}_p^h(\Sigma)$ is chosen as a bicubic Hermite spline space. There is a set of basis functions called bicubic Hermite bases and the Hermite interpolation operator Π satisfies $\Pi(f(s, t)) = f(s, t)$, $\forall f(s, t) \in \mathbb{P}_3(\Sigma)$, i.e., $p = 3$. Thus, in this case, the optimal convergence rate measured by L^2 norm is with order 4.

In [18], on Ω_0 (a neighbourhood of P shown in Figure 1) the weak solution u of the elliptic boundary problem (1),

$$u(r, \theta) = \sum_{j=1}^{\infty} \alpha_j r^{v_j} \phi_j(\theta) + \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} f_{j\ell} [(\ell+2)^2 - v_j^2]^{-1} \phi_j(\theta) r^{\ell+2}, \quad (5)$$

where,

$$\phi_j(\theta) = \sqrt{\frac{2}{\alpha\pi}} \sin v_j \theta, \quad v_j = j/\alpha; \quad f_j(r) = \int_0^{\alpha\pi} f \phi_j(\theta) d\theta, \quad f_j(r) = \sum f_{j\ell} r^\ell,$$

$$\Omega_0 = \{(r, \theta) | 0 < r < r_0, 0 < \theta < \alpha\pi\} \subset \Omega$$

The weak solution u is analytic at other points.

We can improve the convergence rate by introducing a natural parameterization with singularities at the corner points. This parametrization is $\mathcal{P}(s, t) = (s^\delta \cos t, s^\delta \sin t)$. As an application of Theorem 2.1, in Examples 2.1 and 2.2, by controlling δ , the optimal convergence rate is reached. Here the parameterization is

$$\begin{aligned} \mathcal{P}_\delta : \Sigma &\longrightarrow \Omega \\ (s, t) &\mapsto \left(\left(\frac{s}{2}\right)^\delta \cos\left(\frac{\alpha\pi}{2}t\right), \left(\frac{s}{2}\right)^\delta \sin\left(\frac{\alpha\pi}{2}t\right) \right) \end{aligned}$$

where the parametric domain Σ is $[0, 2] \times [0, 2]$. In order to avoid the error from describing the geometry, we choose fan shapes $\mathcal{P}_\delta(\Sigma)$ as physical domains, i.e., $\Omega = \mathcal{P}_\delta(\Sigma)$. Homogenous bicubic Hermite splines are chosen as test functions and on Σ because (1) is an elliptic problem with homogenous boundary condition. The following lemma assures all the test functions be \mathbf{H}^1 on Ω .

Lemma 2.2. *Let $H_1(s, t), H_2(s, t), \dots, H_n(s, t)$ be homogenous bicubic Hermite splines over Σ . Then*

$$H_i \circ \mathcal{P}_\delta^{-1} \in \mathbf{H}^1(\Omega), \quad \forall i = 1, 2, \dots, n.$$

The proof is shown in the appendix.

Example 2.1. [$\alpha = 2$] *The elliptic boundary value problem:*

$$\begin{aligned} -\Delta u &= \frac{3}{4}(x^2 + y^2)^{-\frac{3}{4}}(1 + \sqrt{x^2 + y^2}) \sin \theta \quad \text{in } \Omega_2, \\ u &= 0 \quad \text{on } \partial\Omega_2, \end{aligned} \quad (6)$$

where the physical domain shown in Figure 2(b) is

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 : x = r \cos(\theta), y = r \sin(\theta), 0 < r < 1, 0 < \theta < 2\pi\}.$$

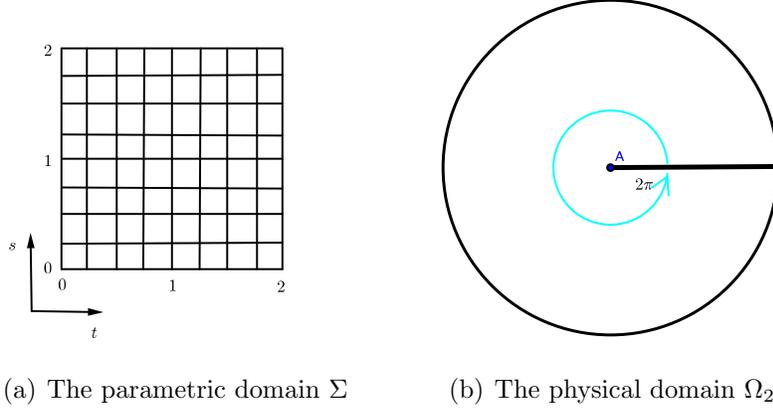


Figure 2:

The exact solution of this elliptic boundary value problem is

$$u(x, y) = (x^2 + y^2)^{1/4}(1 - \sqrt{x^2 + y^2}) \sin \theta.$$

With traditional finite element method, the convergence rate is with order 1. The parameterization $\mathcal{P}_\delta : \Sigma \rightarrow \Omega_2$ is singular on the line $s = 0$ when $\delta > 0$. Thus, the cells at the bottom row of the parametric domain shown in Figure 3(a) are singular cells. Based on Theorem 2.1 and the regularity of $u(x, y)$,

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)} &\leq C_1 h^4 + C_2 h^{\delta - \frac{1}{2}} h^{k'+1} \sqrt{\int_0^h ds \int_0^2 \sum_{0 \leq i+j \leq k'+1} \left| \frac{\partial^{i+j}(u \circ \mathcal{P})}{\partial s^i \partial t^j} \right|^2 dt} \\ &\leq C_1 h^4 + C_2 h^{\delta - \frac{1}{2}} h^{k'+1} (M h^{\frac{\delta}{2} - k' - \frac{1}{2}}) \quad (\text{where, } \delta - 2k' - 1 > 0) \\ &\leq C_1 h^4 + C_2 h^{\frac{3}{2}\delta}. \end{aligned}$$

So when $\delta \geq \frac{8}{3}$, the optimal convergence rate (with order 4) is reached. In order to avoid the machine error, in the following table, we present the numerical result by taking $\delta = 2.7$.

h (the size of cell)	The error ($\ e_h\ _{L^2}$)	The error order
1	0.0125708	
1/2	0.00108024	3.5407
1/4	9.57114e-05	3.4965
1/8	6.71963e-06	3.8322
1/16	4.33468e-07	3.9544
1/32	2.73128e-08	3.9883

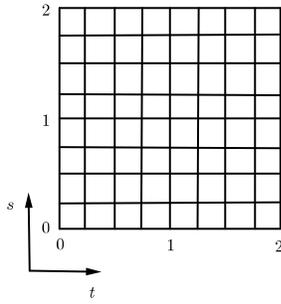
Example 2.2. ($\alpha = 3/2$) *The elliptic boundary value problem:*

$$-\Delta u = \left(\frac{32}{9}(x^2 + y^2)^{-2/3} - \frac{11}{9}(x^2 + y^2)^{-1/6}\right) \sin(2\theta) \quad \text{in } \Omega,$$

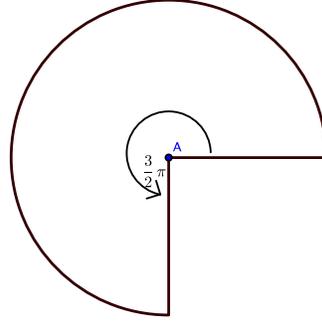
$$u = 0 \quad \text{on } \partial\Omega,$$

where the physical domain shown in Figure 3(b) is

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x = r \cos(\theta), y = r \sin(\theta), 0 < r < 1, 0 < \theta < \frac{3}{2}\pi\}.$$



(a) The parametric domain Σ



(b) The physical domain $\Omega_{\frac{3}{2}}$

Figure 3:

The exact solution of this elliptic boundary value problem is $u(x, y) = (x^2 + y^2)^{1/3}(1 - \sqrt{x^2 + y^2})\sin(2\theta)$. With traditional finite element method, the convergence rate is with order $4/3$. Based on Theorem 2.1,

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)} &\leq C_1 h^4 + C_2 h^{\delta - \frac{1}{2}} h^{k'+1} \sqrt{\int_0^h ds \int_0^2 \sum_{0 \leq i+j \leq k'+1} \left| \frac{\partial^{i+j}(u \circ \mathcal{P})}{\partial s^i \partial t^j} \right|^2 dt} \\ &\leq C_1 h^4 + C_2 h^{\delta - \frac{1}{2}} h^{k'+1} (M h^{\frac{2\delta}{3} - k' - \frac{1}{2}}) \quad (\text{where } \frac{4\delta}{3} - 2k' - 1 > 0) \\ &\leq C_1 h^4 + C_2 h^{\frac{5}{3}\delta}. \end{aligned}$$

i.e., when $\delta \geq 2.4$, the optimal convergence rate (with order 4) is reached. In the following table, we present the numerical result with $\delta = 2.4$.

h (the size of cell)	The error ($\ e_h\ _{L^2}$)	The error order
1	0.0380944	
1/2	0.00295726	3.6872
1/4	0.000314976	3.2309
1/8	2.4697e-05	3.6728
1/16	1.65221e-06	3.9019
1/32	1.05129e-07	3.9742

3 Parameterizations in Isogeometric analysis

In Isogeometric analysis, the geometric description and numerical analysis share the same basis functions. Thus, in this section, we will analyze the convergence rate with a singular parameterization that is represented by the same type of splines in numerical analysis. Here, bicubic Hermite splines are chosen as splines.

As the analysis in Section 2, the non-smooth boundary corner destroys the global regularity of the weak solution of the elliptic boundary problem. Thus, we design the parameterization such that it takes the non-smooth boundary corners with $\alpha > 1$ as its singularities, i.e. suppose that X is a singularity, a parameterization $\mathcal{P}(s, t)$ constructed satisfies:

$$\frac{\partial \mathcal{P}}{\partial s} \Big|_X = \mathbf{0}, \quad \frac{\partial \mathcal{P}}{\partial t} \Big|_X = \mathbf{0}.$$

Moreover, to guarantee all test functions be \mathbf{H}^1 , we assume that $\mathcal{P}(s, t)$ satisfies H^1 -integrability assumption (Assumption 5.1 in [4]). Thus, the Jacobian $|J| \sim h^2$ (i.e., $\sigma = 2$ in Theorem 2.1) on a singular cell of the parametric mesh. Based on Theorem 2.1 and suppose $u \circ \mathcal{P}$ is smooth enough on the regular cells,

$$\begin{aligned} \|u - u^h\|_{L^2(\Omega)} &\leq C_1 h^4 + C_2 h^{\sigma/2+k'+1} |u \circ \mathcal{P}|_{k'+1, \Sigma_s} \\ &\leq C_1 h^4 + C_2 h^{k'+2} \sqrt{\int \int_{\Sigma_s} \left(\sum_{i+j=k'+1} \frac{\partial^{i+j} u \circ \mathcal{P}}{\partial s^i \partial t^j} \right)^2 ds dt} \end{aligned}$$

where, u is the exact weak solution of the elliptic boundary problem (1). For Example 3.1, Σ_s consists of 4 cells. And for Example 3.2, Σ_s consists of 3 cells. Thus the area of $\Sigma_s \sim h^2$. Then there is a const M such that

$$\|u - u^h\|_{L^2(\Omega)} \leq C_1 h^4 + C_2 h^{k'+2} M h = C_1 h^4 + C_2 M h^{k'+3}, \quad (7)$$

where we suppose that $u \circ \mathcal{P} \in \mathbf{H}^{k'+1}(\Sigma_s)$. By the second point of Remark 2.1, the regularity of $u \circ \mathcal{P}$ on Σ is not worse than the regularity of u . Then, if $u \in \mathbf{H}^2(\Omega)$, based on the error bound (7), the optimal convergence rate (4) is reached. Note that to reach the optimal convergence rate in the traditional Finite element method $u \in \mathbf{H}^4(\Omega)$ should be assumed.

Example 3.1 ($\alpha = 2$). *The parametric mesh is shown in Figure 4(a). The parameterization is presented in Figure 4(b).*

In the following, we will solve the elliptic boundary value problem (1). The exact solution is $\frac{y(4-x^2)(4-y^2)}{(x^2+y^2)^{1/4}} \in \mathbf{H}^1(\Omega)$. Thus, for this example, based on (7), the convergence rate is 3 at least. With the parameterization in Figure 4(b), the table below gives the L^2 norm of e_h that is the error of the numerical solution compared to the exact solution and the convergence rate (error order) computed as $\log_2(e_h/e_{h/2})$.

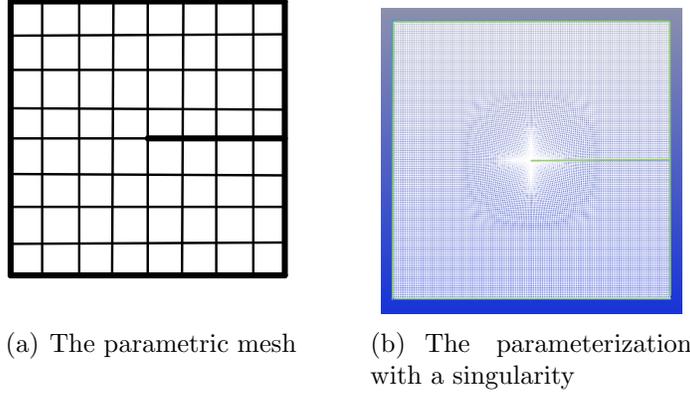


Figure 4:

h (the size of cell)	The error ($\ e_h\ _{L^2}$)	The error order
1	0.373774	
1/2	0.105402	1.8263
1/4	0.0193175	2.4479
1/8	0.00277067	2.8016
1/16	0.000364259	2.9272
1/32	4.66296e-05	2.9656

We will solve the elliptic boundary value problem (1). The exact solution is

$$y(x^2 + y^2)^{1/4}(4 - x^2)(4 - y^2) \in \mathbf{H}^2(\Omega).$$

h (the size of cell)	The error ($\ e_h\ _{L^2}$)	The error order
1	0.296035	
1/2	0.022368	3.7263
1/4	0.00202538	3.4652
1/8	0.000162817	3.6369
1/16	1.15173e-05	3.8214
1/32	7.62845e-07	3.9163

Example 3.2 ($\alpha = 3/2$). The parametric mesh is shown in Figure 5(a). The parameterization is presented in Figure 5(b)

In the following, we will solve the elliptic boundary value problem (1). The exact solution is

$$(1 - x^2)(1 - y^2)xy/(x^2 + y^2)^{2/3} \in \mathbf{H}^1(\Omega).$$

Based on (7), the convergence rate is 3 at least. With the parameterization in Figure 5(b), the table below gives the L^2 norm of e_h that is the error of the numerical solution compared to the exact solution and the convergence rate (error order) computed as $\log_2(e_h/e_{h/2})$.

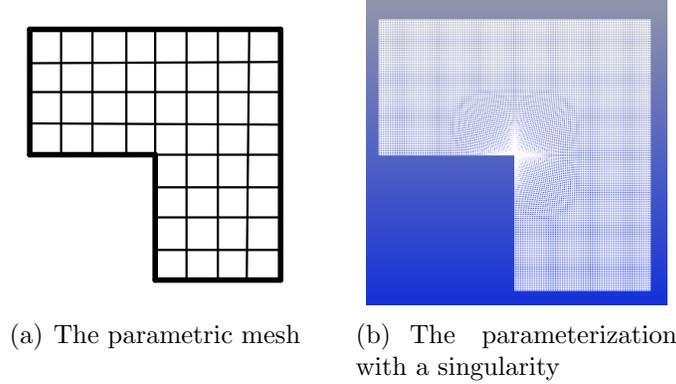


Figure 5:

h (the size of cell)	The error ($\ e_h\ _{L^2}$)	The error order
1	0.00481796	
1/2	0.001955	1.3013
1/4	0.000334744	2.5460
1/8	4.08307e-05	3.0353
1/16	4.59187e-06	3.1525
1/32	4.97948e-07	3.2052

We will solve the elliptic boundary value problem (1). The exact solution is $(x^2 + y^2)^{-1/4}xy(1 - x^2)(1 - y^2) \in \mathbf{H}^2(\Omega)$.

h (the size of cell)	The error ($\ e_h\ _{L^2}$)	The error order
1	0.000878348	
1/2	0.000114956	2.9337
1/4	7.81197e-06	3.8793
1/8	6.27692e-07	3.6376
1/16	4.90175e-08	3.6787
1/32	3.61415e-09	3.7616
1/64	2.47436e-10	3.8685

4 Conclusion and the future work

In this paper, the impact of singular parameterizations on L^2 convergence rates for solving PDEs in Isogeometric analysis has been discussed by deriving a L^2 priori estimate. By locating singularities of a parameterization at proper positions, the L^2 convergence rates given within the framework of Isogeometric analysis is better than the one within FEM framework. In the future, we will continue to work on the error estimates in H^s norm in order to avoid the regularity problems brought by singular parameterizations and make full use of singularities of parameterizations properly.

Appendix

Lemma 2.2 Let $H_1(s, t), H_2(s, t), \dots, H_n(s, t)$ be homogenous bicubic Hermite splines over Σ . Then

$$H_i \circ \mathcal{P}_\delta^{-1} \in \mathbf{H}^1(\Omega), \quad \forall i = 1, 2, \dots, n.$$

Proof. Denote $\mathcal{P}_\delta(s, t) = (x(s, t), y(s, t)) = (s^\delta \cos t, t^\delta \sin t)$, where $(s, t) \in [0, 2] \times [0, 2]$. Its Jacobian $J = s^{2\delta-1}$. In the following, we will check that, for each $H_i(s, t)$, the L^2 -norm and H^1 semi-norm are bounded. On the one hand,

$$\|H_i \circ \mathcal{P}_\delta^{-1}\|_{L^2}^2 = \int_{\Omega} (H_i \circ \mathcal{P}_\delta^{-1})^2 dx dy = \int_{\Sigma} (H_i(s, t))^2 s^{2\delta-1} ds dt.$$

Then, $\|H_i \circ \mathcal{P}_\delta^{-1}\|_{L^2}$ is bounded. On the other hand,

$$\begin{aligned} |H_i \circ \mathcal{P}_\delta^{-1}|_{H^1}^2 &= \int_{\Omega} \left(\left(\frac{\partial H_i \circ \mathcal{P}_\delta^{-1}}{\partial x} \right)^2 + \left(\frac{\partial H_i \circ \mathcal{P}_\delta^{-1}}{\partial y} \right)^2 \right) dx dy \\ &= \int_{[0,2] \times [0,2]} \frac{\left(\frac{\partial H_i}{\partial s}, \frac{\partial H_i}{\partial t} \right) \hat{J} \hat{J}^T \left(\frac{\partial H_i}{\partial s}, \frac{\partial H_i}{\partial t} \right)^T}{J} ds dt \\ &= \int_{[0,2] \times [0,2]} \frac{s^2 \frac{\partial H_i}{\partial s} + \delta^2 \frac{\partial H_i}{\partial t}}{s} ds dt, \end{aligned}$$

where $\hat{J} = \begin{pmatrix} s^\delta \cos t & s^\delta \sin t \\ -\delta s^{\delta-1} \sin t & \delta s^{\delta-1} \cos t \end{pmatrix}$.

Moreover, $s |H_i(s, t)|$ because $H_i(s, t)$ is homogenous. So, $s \left| \frac{\partial H_i}{\partial t} \right|$. Thus, $|H_i \circ \mathcal{P}_\delta^{-1}|_{H^1}$ is bounded.

In conclusion, $H_i \circ \mathcal{P}_\delta^{-1} \in \mathbf{H}^1(\Omega)$. □

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