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# On one class of permutation polynomials over finite fields of characteristic two \*

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**Abstract.** Polynomials of type  $x^{q^3+q^2+q+2}+bx$  over the field  $\mathbb{F}_{q^4}$ , where  $q = 2^m$ ,  $m \geq 2$ , are considered. All cases when these polynomials are permutation polynomials are classified.

## 1 Introduction

Let  $q = p^m$  be a prime power and let  $\mathbb{F}_q$  denote the finite field of order  $q$ . A polynomial  $f(x) \in \mathbb{F}_q[x]$  is called a permutation polynomial (PP) of  $\mathbb{F}_q$  if  $f(x)$  induces a one-to-one mapping of  $\mathbb{F}_q$  onto itself. It is well known that permutation polynomials have applications in a variety of areas, including such areas as cryptography for the secure transmission of information, and in combinatorics for the construction of various kinds of combinatorial designs (see [4, 6] and references there). Recently, permutation polynomials have been studied extensively in the literature (see [3, 7, 9, 12], for example).

A polynomial  $f(x) \in \mathbb{F}_q[x]$  is called a complete permutation polynomial (CPP), if both  $f(x)$  and  $f(x) + x$  are PP over  $\mathbb{F}_q$ . Although there are some results on CPP over  $\mathbb{F}_q$  [3, 4, 6, 8, 10, 12], still very few classes of them are known, even for monomial functions. For a positive integer  $d$  and  $a \in \mathbb{F}_q^*$ , a monomial function  $ax^d$  is a CPP over  $\mathbb{F}_q$  if and only if  $\gcd(d, q-1) = 1$  and  $ax^d + x$  is a PP over  $\mathbb{F}_q$ .

In [1, 2] we gave a complete description of PP of type

$$f(x) = x^{1+\frac{q^2-1}{q-1}} + bx$$

over  $\mathbb{F}_{q^2}$  (or, monomial CPP of type  $f(x) = ax^d$ ,  $d = (q^2 - 1)/(q - 1) + 1$ ) and

$$f(x) = x^{1+\frac{q^3-1}{q-1}} + bx$$

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over  $\mathbb{F}_{q^3}$  (or, monomial CPP of type  $f(x) = ax^d$ ,  $d = (q^3 - 1)/(q - 1) + 1$ ) for the fields of any characteristic. Here we extend such description to polynomials of type

$$f(x) = x^{1+\frac{q^4-1}{q-1}} + bx, \quad (1)$$

over  $\mathbb{F}_{q^4}$  (or, monomial CPP of type  $f(x) = ax^d$ ,  $d = (q^4 - 1)/(q - 1) + 1$ ) but only for the fields of even characteristic ( $q = 2^m$ ).

For odd  $m$ , such polynomials were considered [12]. In particular, it was proved (on base of Dickson polynomials) the following result.

**Theorem 1** [12] *Let  $q = 2^m$  where  $m \geq 3$  is odd. A polynomial of the type (1) is a PP over  $\mathbb{F}_{q^4}$  if the element  $b$  looks as follows:*

$$\left. \begin{array}{l} 1) b = u(1 + \beta + \beta^2) + v\beta^3; \\ 2) b = u(1 + \beta + \beta^3) + v(\beta + \beta^2) \\ \text{or } b = u(1 + \beta^3) + v(1 + \beta + \beta^2); \\ 3) b = u(\beta + \beta^3) + v(\beta^2 + \beta^3), \end{array} \right\} \quad (2)$$

where  $u, v$  run through  $\mathbb{F}_q$ ,  $(u, v) \neq (0, 0)$ , and where  $\beta$  is a root of  $x^4 + x + 1$ .

However, it was not proved that there do not exist other elements  $b$  for which polynomials  $x^{1+\frac{q^4-1}{q-1}} + bx$  are also permutation polynomials. Here we fill this gap and prove, firstly, these sufficient conditions from [12] are also necessary; secondly, these sufficient and necessary conditions are fulfilled only for  $b$  from [12]; and, thirdly, permutation polynomials of the type (1) do not exist for the even  $m \geq 4$  (for  $m = 2$  such polynomials exist).

Our approach as well as in [1, 2] is based on the following lemma from [8], which in our special case of  $f(x) \in \mathbb{F}_{q^4}[x]$  can be reformulated as follows:

**Lemma 1.** *The polynomial*

$$f(x) = x^{1+\frac{q^4-1}{q-1}} + bx$$

over  $\mathbb{F}_{q^4}$  is a permutation polynomial if and only if  $b \in \mathbb{F}_q^4 \setminus \mathbb{F}_q$  and the following inequality:

$$x(b+x)^{q^3+q^2+q+1} \neq y(b+y)^{q^3+q^2+q+1} \quad (3)$$

holds for all  $x, y \in \mathbb{F}_q$ , such that  $x \neq 0, y \neq 0, x \neq y$ .

## 2 Polynomials $x^{1+\frac{q^4-1}{q-1}} + bx$ , $q = 2^m$ , $m \geq 2$

Following our approach in [1, 2], first, we reduce the inequality (3) to the equation over  $x$  and  $y$ . Using that

$$\left. \begin{array}{l} x(b+x)^{q^3+q^2+q+1} + y(b+y)^{q^3+q^2+q+1} = \\ = z(x^4 + B_2x^2 + (z^3 + B_2z)x + z^4 + B_1z^3 + B_2z^2 + B_3z + B_4), \end{array} \right\} \quad (4)$$

where

$$\left. \begin{aligned} B_1 &= b + b^q + b^{q^2} + b^{q^3}, \\ B_2 &= b^{1+q} + b^{1+q^2} + b^{1+q^3} + b^{q+q^2} + b^{q+q^3} + b^{q^2+q^3}, \\ B_3 &= b^{1+q+q^2} + b^{1+q+q^3} + b^{1+q^2+q^3} + b^{q+q^2+q^3}, \\ B_4 &= b^{1+q+q^2+q^3}. \end{aligned} \right\} \quad (5)$$

and  $z = x + y$ , we can rewrite the lemma 1 as follows:

**Lemma 2.** *Let  $q = 2^m$ . The polynomial*

$$f(x) = x^{1+\frac{q^4-1}{q-1}} + bx,$$

over  $\mathbb{F}_{q^4}$  is a permutation polynomial if and only if  $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$  and the equation

$$x^4 + B_2x^2 + (z^3 + B_2z)x + z^4 + B_1z^3 + B_2z^2 + B_3z + B_4 = 0 \quad (6)$$

has no solutions  $x, z \in \mathbb{F}_q$  such that  $z \neq 0$  (for  $x = 0$  and  $x = z$ , i.e.  $y = 0$ , this equation has no solutions).

Using a new variable  $w = x + B_1$ , we obtain the equation

$$w^4 + B_2w^2 + (z^3 + zB_2)w + z^4 + B_2z^2 + Dz + E = 0, \quad (7)$$

where we denote

$$D = B_1B_2 + B_3 \quad \text{and} \quad E = B_1^4 + B_1^2B_2 + B_4. \quad (8)$$

Introducing again a new variable  $\gamma = w/z$  we arrive to the equivalent equation (recall that  $z \neq 0$ ):

$$\gamma^4 + \gamma + \frac{B_2}{z^2} \cdot (\gamma^2 + \gamma + 1) + 1 + \frac{D}{z^3} + \frac{E}{z^4} = 0. \quad (9)$$

Thus, we reduced the problem of solution of equation (6) to the problem of solution of equation (9) with  $z \neq 0$ . For convenience we denote, for an integer  $s \geq 1$  and  $q = 2^m$ , the trace function

$$\text{Tr}_{q^s}(a) = a + a^2 + a^4 + \cdots + a^{2^{sm-1}}, \quad a \in \mathbb{F}_{q^s}$$

and the relative trace function for integers  $s, r \geq 1$

$$\text{Tr}_{q^{sr} \rightarrow q^s}(a) = a + a^{q^s} + a^{q^{2s}} + \cdots + a^{q^{s(r-1)}}, \quad a \in \mathbb{F}_{q^{sr}}$$

The cases  $B_2 \neq 0$  and  $B_2 = 0$  we consider separately.

## 2.1 The case $B_2 \neq 0$ .

Because

$$\begin{aligned} & \gamma^4 + \gamma + \frac{B_2}{z^2} \cdot (\gamma^2 + \gamma + 1) + 1 + \frac{D}{z^3} + \frac{E}{z^4} \\ &= (\gamma^2 + \gamma + 1)^2 + (\gamma^2 + \gamma + 1)\left(1 + \frac{B_2}{z^2}\right) + 1 + \frac{D}{z^3} + \frac{E}{z^4}, \end{aligned} \quad (10)$$

in this case the problem of existence of a solution of the equation (9) is reduced to the existence of solutions of the two following equations:

$$\xi^2 + \xi \left(1 + \frac{B_2}{z^2}\right) + 1 + \frac{D}{z^3} + \frac{E}{z^4} = 0. \quad (11)$$

and

$$\gamma^2 + \gamma + 1 = \xi. \quad (12)$$

When  $z \neq \sqrt{B_2}$  the equation (11) has a solution if and only if

$$\text{Tr}_q \left( \frac{1 + \frac{D}{z^3} + \frac{E}{z^4}}{\left(1 + \frac{B_2}{z^2}\right)^2} \right) = \text{Tr}_q(1 + Cv + Dv^3) = 0.$$

where

$$C^4 = D\sqrt{B_2} + E + B_2^2 \quad \text{and} \quad v = \frac{1}{z + \sqrt{B_2}}.$$

**The subcase**  $(C, D) = (0, 0)$ .

In this subcase the equation (11) looks as follows:

$$\xi^2 + \xi \left(1 + \frac{B_2}{z^2}\right) + \left(1 + \frac{B_2}{z^2}\right)^2 = 0. \quad (13)$$

Here we have

**Proposition 2** *Let  $q = 2^m$  and  $m \geq 3$  be odd. Let  $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ ,  $B_2 \neq 0$ , and the following two conditions be satisfied:*

$$B_3 = B_1 B_2, \quad (14)$$

$$B_4 = B_1^4 + B_1^2 B_2 + B_2^2. \quad (15)$$

*Then the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is a permutation polynomial over  $\mathbb{F}_{q^4}$ .*

If  $D = 0$  and  $E = B_2^2$ , then for even  $m$  the equation (11) has solutions for all  $z$ ,  $z \neq 0$ , including  $z = \sqrt{B_2}$  and, obviously, in this case there exist many solutions of the equation (12). Therefore, we have the following

**Proposition 3** *Let  $q = 2^m$  and  $m \geq 2$  be even. Let  $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ ,  $B_2 \neq 0$ , and the conditions (14) and (15) be satisfied. Then the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is not a permutation polynomial over  $\mathbb{F}_{q^4}$ .*

**The subcase**  $(C, D) \neq (0, 0)$ .

For this subcase we are going to prove that for all  $m \geq 6$  solutions for the both equations (11) and (12) always exist. For this purpose it is enough to prove the existence of  $v \neq 0$  and  $v \neq \frac{1}{\sqrt{B_2}}$ , such that

$$\mathrm{Tr}_q(1 + Cv + Dv^3) = 0 \quad (16)$$

and

$$\mathrm{Tr}_q\left(1 + \frac{B_2}{z^2}\right) = 1. \quad (17)$$

Hence we obtain

**Proposition 4** *Let  $q = 2^m$ ,  $B_2 \neq 0$  and  $(C, D) \neq (0, 0)$ . Then the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is not a permutation polynomial over  $\mathbb{F}_{q^4}$  for all  $m \geq 6$ .*

## 2.2 The case $B_2 = 0$

**The subcase**  $D \neq 0$ .

Set  $u = \frac{1}{z}$ . The number  $N$  of  $\mathbb{F}_q$ -rational points of the plane absolutely irreducible non-singular curve  $P$  over  $\mathbb{F}_q$ ,

$$P = \{(\gamma, u) : \gamma^4 + \gamma + 1 + Du^3 + Eu^4 = 0\},$$

satisfies the following known (see [11]) inequality:  $N \geq q - 6\sqrt{q}$ . Because the number of points with  $u = 0$  does not exceed 4, when  $q - 6\sqrt{q} > 4$  there exists a solution  $\gamma, z \in \mathbb{F}_q$ ,  $z \neq 0$ , of the equation (9). For this case, the following proposition holds.

**Proposition 5** *Let  $q = 2^m$ ,  $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$  and*

$$B_2 = 0, \quad B_3 \neq 0.$$

*Then the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is not a permutation polynomial over  $\mathbb{F}_{q^4}$  for  $m \geq 6$ .*

**The subcase**  $D = 0$ .

Here we are obliged to consider separately odd and even  $m$ . For odd  $m$  the analysis is simple: if  $E \neq 0$ , the equation (9) has a solution for any  $\gamma$  because  $z^4$  is an automorphism of the field  $F_{2^m}$  and  $\gamma^4 + \gamma + 1 \neq 0$  for odd  $m$  and  $\gamma \in \mathbb{F}_{2^m}$ . If  $E = 0$ , then a solution of (9) does not exist. But the conditions  $B_2 = 0, D = 0, E = 0$  represent a special case of the conditions (14) and (15) and so the following proposition holds.

**Proposition 6** *Let  $q = 2^m$  and  $m \geq 3$  be odd. Let  $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$  and*

$$B_2 = 0, \quad B_3 = 0.$$

*Then the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is a permutation polynomial over  $\mathbb{F}_{q^4}$  if and only if  $B_4 = B_1^4$ .*

For even  $m$  the situation is more complicated since the equation (9), for the case  $E = 0$ , has no solutions in  $\mathbb{F}_{2^m}$  when  $m \equiv 2 \pmod{4}$ . Hence for  $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ , such that  $B_2 = 0$ ,  $B_3 = 0$ , and  $B_4 = B_1^4$ , the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is a permutation polynomial over  $\mathbb{F}_{q^4}$  for  $m \equiv 2 \pmod{4}$ . But it turned out to be that such  $b$  does not exist for even  $m$  (for odd  $m$  such  $b$  exists).

**Lemma 3.** *Let  $b \in \mathbb{F}_{q^4}$  where  $q = 2^m$  and let  $B_2$  and  $B_3$  be the expressions (5) obtained from  $b$ . Then, for even  $m$ ,*

$$B_2 = B_3 = 0$$

*if and only if  $b$  is an element of  $\mathbb{F}_q$ .*

### 3 The main results

Now we give the results for the cases  $m = 2, 3, 4, 5$ .

The direct calculations show that for  $m = 4$  there are no permutation polynomials of the type  $x^{1+\frac{q^4-1}{q-1}} + bx$  over  $\mathbb{F}_{q^4}$ ,  $q = 2^4$ . But for  $m = 2$  there are 48 such polynomials (for  $b = \alpha^i$ ,  $i = 3, 11, 37, 61, 63, 91$ , and their cyclotomic classes  $L_i = \{i, 2i, 2^2i, \dots, 2^8i\}$  modulo 255 where  $\alpha$  is a primitive element of  $\mathbb{F}_{4^4}$ ).

From here, Propositions 3, 4, 5, and Lemma 3 the following theorem is valid.

**Theorem 7** *Let  $q = 2^m$  and  $m \geq 4$  be even. The polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  over  $\mathbb{F}_{q^4}$  is not a permutation polynomial for any  $b \in \mathbb{F}_{q^4}^*$ .*

For  $m = 3$  and  $m = 5$  the direct calculations show that the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is a permutation polynomial over  $\mathbb{F}_{q^4}$  if and only if the conditions of Proposition 3 are satisfied. From here and Propositions 2, 4, 5, and 6 the following theorem is valid.

**Theorem 8** *Let  $q = 2^m$  and  $m \geq 3$  be odd. The polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  over  $\mathbb{F}_{q^4}$  is a permutation polynomial if and only if the following conditions are satisfied:*

$$b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q, \quad B_3 + B_1 B_2 = 0, \quad \text{and} \quad B_4 + B_1^4 + B_1^2 B_2 + B_2^2 = 0.$$

As we mentioned already, in [12] it was proved in a different way that conditions from Theorem 8 are sufficient conditions.

Now, for odd  $m$ , we show that we find all solutions for  $b$  when the polynomial  $x^{1+\frac{q^4-1}{q-1}} + bx$  is a permutation polynomial. Present any element  $x$  of  $\mathbb{F}_{q^4}$  as a polynomial of degree 3 over  $\mathbb{F}_q$ :

$$x = x_0 + x_1\beta + x_2\beta^2 + x_3\beta^3, \quad x_i \in \mathbb{F}_q,$$

where  $\beta$  is a primitive element of  $\mathbb{F}_{2^4}$ , i.e. it is a root of the equation  $1 + \beta + \beta^4 = 0$ . Now express  $B_1, B_2, B_3, B_4$  in terms of  $x_i$  (since they are elements of  $\mathbb{F}_q$ ):

$$\begin{aligned} B_1 &= x_3, \\ B_2 &= x_0x_3 + x_1x_2 + x_3^2, \\ B_3 &= x_0^2x_3 + x_1^3 + x_1x_2x_3 + x_1x_3^2 + x_2^3 + x_2^2x_3 + x_3^3, \\ B_4 &= x_0^4 + x_0^3x_3 + x_0^2x_1x_2 + x_0^2x_3^2 + x_0x_1^3 + x_0x_1x_2x_3 + x_0x_1x_3^2 + x_0x_2^3 + \\ &\quad + x_0x_2^2x_3 + x_0x_3^3 + x_1^4 + x_1^2x_2x_3 + x_1x_2^3 + x_1x_3^3 + x_2^4 + x_2x_3^3 + x_3^4. \end{aligned}$$

Using these expressions we obtain:  
the condition  $B_1B_2 = B_3$  is equivalent to the condition:

$$x_0x_3(x_0 + x_3) + x_1(x_1 + x_3)^2 + x_2^2(x_2 + x_3) = 0 \quad (18)$$

and the condition  $B_4 = B_1^4 + B_1^2B_2 + B_2^2$  is equivalent to the condition

$$\left. \begin{aligned} &x_0^4 + x_0^3x_3 + x_0^2x_1x_2 + x_0x_1^3 + x_0x_2^3 + x_0x_2^2x_3 + x_0x_1x_3^2 + \\ &+ x_0x_1x_2x_3 + x_1^4 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2x_3^2 + x_1x_3^3 + x_2^4 + x_2x_3^3 = 0 \end{aligned} \right\} \quad (19)$$

**Theorem 9** *All solutions of the system of two equations (18) and (19) for odd  $m$  are:*

$$\begin{aligned} (x_0 = x_2, \quad x_1 = x_2, \quad x_2, x_3), \\ (x_0 = x_2 + x_3, x_1 = x_2, \quad x_2, x_3), \\ (x_0 = 0, \quad x_1 = x_2 + x_3, x_2, x_3), \\ (x_0 = x_3, \quad x_1 = x_2 + x_3, x_2, x_3), \end{aligned}$$

where  $x_2, x_3$  run over  $\mathbb{F}_q$  and  $(x_2, x_3) \neq (0, 0)$ .

Just these solutions were given in [12], but it was not proved there that the other solutions do not exist. Thus, here we filled this gap. In the same paper [12] the authors counted also the number of their different solution:  $2(2q + 1)(q - 1)$ , that naturally coincides with the sum of the number of solutions for  $x_3 \neq 0$  equal  $4q(q - 1)$ , and for  $x_3 = 0$  equal  $2(q - 1)$ .

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