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# Fast Computation of Shifted Popov Forms of Polynomial Matrices via Systems of Modular Polynomial Equations

Vincent Neiger  
ENS de Lyon  
Laboratoire LIP, CNRS, Inria, UCBL, U. Lyon  
vincent.neiger@ens-lyon.fr

## ABSTRACT

We give a Las Vegas algorithm which computes the shifted Popov form of an  $m \times m$  nonsingular polynomial matrix of degree  $d$  in expected  $\tilde{O}(m^\omega d)$  field operations, where  $\omega$  is the exponent of matrix multiplication and  $\tilde{O}(\cdot)$  indicates that logarithmic factors are omitted. This is the first algorithm in  $\tilde{O}(m^\omega d)$  for shifted row reduction with arbitrary shifts.

Using partial linearization, we reduce the problem to the case  $d \leq \lceil \sigma/m \rceil$  where  $\sigma$  is the generic determinant bound, with  $\sigma/m$  bounded from above by both the average row degree and the average column degree of the matrix. The cost above becomes  $\tilde{O}(m^\omega \lceil \sigma/m \rceil)$ , improving upon the cost of the fastest previously known algorithm for row reduction, which is deterministic.

Our algorithm first builds a system of modular equations whose solution set is the row space of the input matrix, and then finds the basis in shifted Popov form of this set. We give a deterministic algorithm for this second step supporting arbitrary moduli in  $\tilde{O}(m^{\omega-1}\sigma)$  field operations, where  $m$  is the number of unknowns and  $\sigma$  is the sum of the degrees of the moduli. This extends previous results with the same cost bound in the specific cases of order basis computation and M-Padé approximation, in which the moduli are products of known linear factors.

## Keywords

Shifted Popov form; polynomial matrices; row reduction; Hermite form; system of modular equations.

## 1. INTRODUCTION

In this paper, we consider two problems of linear algebra over the ring  $\mathbb{K}[X]$  of univariate polynomials, for some field  $\mathbb{K}$ : computing the shifted Popov form of a matrix, and solving systems of modular equations.

### 1.1 Shifted Popov form

A polynomial matrix  $\mathbf{P}$  is row reduced [22, Section 6.3.2] if its rows have some type of minimal degree (we give precise

definitions below). Besides, if  $\mathbf{P}$  satisfies an additional normalization property, then it is said to be in Popov form [22, Section 6.7.2]. Given a matrix  $\mathbf{A}$ , the efficient computation of a (row) reduced form of  $\mathbf{A}$  and of the Popov form of  $\mathbf{A}$  has received a lot of attention recently [14, 28, 16].

In many applications one rather considers the degrees of the rows of  $\mathbf{P}$  shifted by some integers which specify degree weights on the columns of  $\mathbf{P}$ , for example in list-decoding algorithms [2, 7], robust Private Information Retrieval [12], and more generally in polynomial versions of the Copper-Smith method [9, 10]. A well-known specific shifted Popov form is the Hermite form; there has been recent progress on its fast computation [17, 15, 35]. The case of an arbitrary shift has been studied in [6].

For a shift  $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n$ , the  $\mathbf{s}$ -degree of  $\mathbf{p} = [p_1, \dots, p_n] \in \mathbb{K}[X]^{1 \times n}$  is  $\max_{1 \leq j \leq n} (\deg(p_j) + s_j)$ ; the  $\mathbf{s}$ -row degree of  $\mathbf{P} \in \mathbb{K}[X]^{m \times n}$  is  $\text{rdeg}_{\mathbf{s}}(\mathbf{P}) = (d_1, \dots, d_m)$  with  $d_i$  the  $\mathbf{s}$ -degree of the  $i$ -th row of  $\mathbf{P}$ . Then, the  $\mathbf{s}$ -leading matrix of  $\mathbf{P} = [p_{i,j}]_{i,j}$  is the matrix  $\text{lm}_{\mathbf{s}}(\mathbf{P}) \in \mathbb{K}^{m \times n}$  whose entry  $(i, j)$  is the coefficient of degree  $d_i - s_j$  of  $p_{i,j}$ .

Now, we assume that  $m \leq n$  and  $\mathbf{P}$  has full rank. Then,  $\mathbf{P}$  is said to be  $\mathbf{s}$ -reduced [22, 6] if  $\text{lm}_{\mathbf{s}}(\mathbf{P})$  has full rank. For a full rank  $\mathbf{A} \in \mathbb{K}[X]^{m \times n}$ , an  $\mathbf{s}$ -reduced form of  $\mathbf{A}$  is an  $\mathbf{s}$ -reduced matrix  $\mathbf{P}$  whose row space is the same as that of  $\mathbf{A}$ ; by row space we mean the  $\mathbb{K}[X]$ -module generated by the rows of the matrix. Equivalently,  $\mathbf{P}$  is left-unimodularly equivalent to  $\mathbf{A}$  and the tuple  $\text{rdeg}_{\mathbf{s}}(\mathbf{P})$  sorted in nondecreasing order is lexicographically minimal among the  $\mathbf{s}$ -row degrees of all matrices left-unimodularly equivalent to  $\mathbf{A}$ .

Specific  $\mathbf{s}$ -reduced matrices are those in  $\mathbf{s}$ -Popov form [22, 5, 6], as defined below. One interesting property is that the  $\mathbf{s}$ -Popov form is canonical: there is a unique  $\mathbf{s}$ -reduced form of  $\mathbf{A}$  which is in  $\mathbf{s}$ -Popov form, called the  $\mathbf{s}$ -Popov form of  $\mathbf{A}$ .

**DEFINITION 1.1 (PIVOT).** Let  $\mathbf{p} = [p_j]_j \in \mathbb{K}[X]^{1 \times n}$  be nonzero and let  $\mathbf{s} \in \mathbb{Z}^n$ . The  $\mathbf{s}$ -pivot index of  $\mathbf{p}$  is the largest index  $j$  such that  $\text{rdeg}_{\mathbf{s}}(\mathbf{p}) = \deg(p_j) + s_j$ . Then we call  $p_j$  and  $\deg(p_j)$  the  $\mathbf{s}$ -pivot entry and the  $\mathbf{s}$ -pivot degree of  $\mathbf{p}$ .

We remark that adding a constant to the entries of  $\mathbf{s}$  does not change the notion of  $\mathbf{s}$ -pivot. For example, we will sometimes assume  $\min(\mathbf{s}) = 0$  without loss of generality.

**DEFINITION 1.2 (SHIFTED POPOV FORM).** Let  $m \leq n$ , let  $\mathbf{P} \in \mathbb{K}[X]^{m \times n}$  be full rank, and let  $\mathbf{s} \in \mathbb{Z}^n$ . Then,  $\mathbf{P}$  is said to be in  $\mathbf{s}$ -Popov form if the  $\mathbf{s}$ -pivot indices of its rows are strictly increasing, the corresponding  $\mathbf{s}$ -pivot entries are monic, and in each column of  $\mathbf{P}$  which contains a pivot the nonpivot entries have degree less than the pivot entry.

In this case, the  $\mathbf{s}$ -pivot degree of  $\mathbf{P}$  is  $\boldsymbol{\delta} = (\delta_1, \dots, \delta_m) \in \mathbb{N}^m$ , with  $\delta_i$  the  $\mathbf{s}$ -pivot degree of the  $i$ -th row of  $\mathbf{P}$ .

Here, although we will encounter Popov forms of rectangular matrices in intermediate nullspace computations, our main focus is on computing shifted Popov forms of *square nonsingular matrices*. For the general case, studied in [6], a fast solution would require further developments. A square matrix in  $\mathbf{s}$ -Popov form has its  $\mathbf{s}$ -pivot entries on the diagonal, and its  $\mathbf{s}$ -pivot degree is the tuple of degrees of its diagonal entries and coincides with its column degree.

**PROBLEM 1 (SHIFTED POPOV NORMAL FORM).**

Input: *the base field  $\mathbb{K}$ , a nonsingular matrix  $\mathbf{A} \in \mathbb{K}[X]^{m \times m}$ , a shift  $\mathbf{s} \in \mathbb{Z}^m$ .*

Output: *the  $\mathbf{s}$ -Popov form of  $\mathbf{A}$ .*

Two well-known specific cases are the Popov form [27, 22] for the *uniform* shift  $\mathbf{s} = \mathbf{0}$ , and the Hermite form [19, 22] for the shift  $\mathbf{h} = (0, \delta, 2\delta, \dots, (m-1)\delta) \in \mathbb{N}^m$  with  $\delta = m \deg(\mathbf{A})$  [6, Lemma 2.6]. For a broader perspective on shifted reduced forms, we refer the reader to [6].

For such problems involving  $m \times m$  matrices of degree  $d$ , one often wishes to obtain a cost bound similar to that of polynomial matrix multiplication in the same dimensions:  $\tilde{\mathcal{O}}(m^\omega d)$  operations in  $\mathbb{K}$ . Here,  $\omega$  is so that we can multiply  $m \times m$  matrices over a commutative ring in  $\mathcal{O}(m^\omega)$  operations in that ring, the best known bound being  $\omega < 2.38$  [11, 25]. For example, one can compute  $\mathbf{0}$ -reduced [14, 16],  $\mathbf{0}$ -Popov [28], and Hermite [15, 35] forms of  $m \times m$  nonsingular matrices of degree  $d$  in  $\tilde{\mathcal{O}}(m^\omega d)$  field operations.

Nevertheless,  $d$  may be significantly larger than the average degree of the entries of the matrix, in which case the cost  $\tilde{\mathcal{O}}(m^\omega d)$  seems unsatisfactory. Recently, for the computation of order bases [30, 34], nullspace bases [36], interpolation bases [20, 21], and matrix inversion [37], fast algorithms do take into account some types of average degrees of the matrices rather than their degree. Here, in particular, we achieve a similar improvement for the computation of shifted Popov forms of a matrix.

Given  $\mathbf{A} = [a_{i,j}]_{ij} \in \mathbb{K}[X]^{m \times m}$ , we denote by  $\sigma(\mathbf{A})$  the *generic bound for  $\deg(\det(\mathbf{A}))$*  [16, Section 6], that is,

$$\sigma(\mathbf{A}) = \max_{\pi \in S_m} \sum_{1 \leq i \leq m} \overline{\deg}(a_{i, \pi_i}) \quad (1)$$

where  $S_m$  is the set of permutations of  $\{1, \dots, m\}$ , and  $\overline{\deg}(p)$  is defined over  $\mathbb{K}[X]$  as  $\overline{\deg}(0) = 0$  and  $\overline{\deg}(p) = \deg(p)$  for  $p \neq 0$ . We have  $\deg(\det(\mathbf{A})) \leq \sigma(\mathbf{A}) \leq m \deg(\mathbf{A})$ , and  $\sigma(\mathbf{A}) \leq \min(|\text{rdeg}(\mathbf{A})|, |\text{cdeg}(\mathbf{A})|)$  with  $|\text{rdeg}(\mathbf{A})|$  and  $|\text{cdeg}(\mathbf{A})|$  the sums of the row and column degrees of  $\mathbf{A}$ . We note that  $\sigma(\mathbf{A})$  can be substantially smaller than  $|\text{rdeg}(\mathbf{A})|$  and  $|\text{cdeg}(\mathbf{A})|$ , for example if  $\mathbf{A}$  has one row and one column of uniformly large degree and other entries of low degree.

**THEOREM 1.3.** *There is a Las Vegas randomized algorithm which solves Problem 1 in expected  $\tilde{\mathcal{O}}(m^\omega [\sigma(\mathbf{A})/m]) \subseteq \tilde{\mathcal{O}}(m^\omega \deg(\mathbf{A}))$  field operations.*

The ceiling function indicates that the cost is  $\tilde{\mathcal{O}}(m^\omega)$  when  $\sigma(\mathbf{A})$  is small compared to  $m$ , in which case  $\mathbf{A}$  has mostly constant entries. Here we are mainly interested in the case  $m \in \mathcal{O}(\sigma(\mathbf{A}))$ : the cost bound may be written  $\tilde{\mathcal{O}}(m^{\omega-1} \sigma(\mathbf{A}))$  and is both in  $\tilde{\mathcal{O}}(m^{\omega-1} |\text{rdeg}(\mathbf{A})|)$  and  $\tilde{\mathcal{O}}(m^{\omega-1} |\text{cdeg}(\mathbf{A})|)$ .

Previous work on fast algorithms related to Problem 1 is summarized in Table 1. The fastest known algorithm for the

Ref.	Problem	Cost bound
[18]	Hermite form	$\tilde{\mathcal{O}}(m^4 d)$
[31]	Hermite form	$\tilde{\mathcal{O}}(m^{\omega+1} d)$
[33]	Popov & Hermite forms	$\tilde{\mathcal{O}}(m^{\omega+1} d + (md)^\omega)$
[1, 2]	weak Popov form	$\tilde{\mathcal{O}}(m^{\omega+1} d)$
[26]	Popov & Hermite forms	$\mathcal{O}(m^3 d^2)$
[14]	$\mathbf{0}$ -reduction	$\tilde{\mathcal{O}}(m^\omega d)$ *
[28]	Popov form of $\mathbf{0}$ -reduced	$\tilde{\mathcal{O}}(m^\omega d)$
[17]	Hermite form	$\tilde{\mathcal{O}}(m^\omega d)$ *
[16]	$\mathbf{0}$ -reduction	$\tilde{\mathcal{O}}(m^\omega d)$
[35]	Hermite form	$\tilde{\mathcal{O}}(m^\omega d)$
[16]+[28]	$\mathbf{s}$ -Popov form for any $\mathbf{s}$	$\tilde{\mathcal{O}}(m^\omega (d + \mu))$
Here	$\mathbf{s}$ -Popov form for any $\mathbf{s}$	$\tilde{\mathcal{O}}(m^\omega [\sigma(\mathbf{A})/m])$ *

**Table 1: Fast algorithms for shifted reduction problems** ( $d = \deg(\mathbf{A})$ ; \* = probabilistic;  $\mu = \max(\mathbf{s}) - \min(\mathbf{s})$ ).

$\mathbf{0}$ -Popov form is deterministic and has cost  $\tilde{\mathcal{O}}(m^\omega d)$  with  $d = \deg(\mathbf{A})$ ; it first computes a  $\mathbf{0}$ -reduced form of  $\mathbf{A}$  [16], and then its  $\mathbf{0}$ -Popov form via normalization [28]. Obtaining the Hermite form in  $\tilde{\mathcal{O}}(m^\omega d)$  was first achieved by a probabilistic algorithm in [15], and then deterministically in [35].

For an arbitrary  $\mathbf{s}$ , the algorithm in [6] is fraction-free and uses a number of operations that is, depending on  $\mathbf{s}$ , at least quintic in  $m$  and quadratic in  $\deg(\mathbf{A})$ .

When  $\mathbf{s}$  is not uniform there is a folklore solution based on the fact that  $\mathbf{Q}$  is in  $\mathbf{s}$ -Popov form if and only if  $\mathbf{QD}$  is in  $\mathbf{0}$ -Popov form, with  $\mathbf{D} = \text{diag}(X^{s_1}, \dots, X^{s_m})$  and assuming  $\mathbf{s} \geq 0$ . Then, this solution computes the  $\mathbf{0}$ -Popov form  $\mathbf{P}$  of  $\mathbf{AD}$  using [16, 28] and returns  $\mathbf{PD}^{-1}$ . This approach uses  $\tilde{\mathcal{O}}(m^\omega (d + \mu))$  operations where  $\mu = \max(\mathbf{s}) - \min(\mathbf{s})$ , which is not satisfactory when  $\mu$  is large. For example, its cost for computing the Hermite form is  $\tilde{\mathcal{O}}(m^{\omega+2} d)$ . This is the worst case since one can assume without loss of generality that  $\mu \in \mathcal{O}(m \deg(\det(\mathbf{A}))) \subseteq \mathcal{O}(m^2 d)$  [21, Appendix A].

Here we obtain, to the best of our knowledge, the best known cost bound  $\tilde{\mathcal{O}}(m^\omega [\sigma(\mathbf{A})/m]) \subseteq \tilde{\mathcal{O}}(m^\omega d)$  for an arbitrary shift  $\mathbf{s}$ . This removes the dependency in  $\mu$ , which means in some cases a speedup by a factor  $m^2$ . Besides, this is also an improvement for both specific cases  $\mathbf{s} = \mathbf{0}$  and  $\mathbf{s} = \mathbf{h}$  when  $\mathbf{A}$  has unbalanced degrees.

One of the main difficulties in row reduction algorithms is to control the size of the manipulated matrices, that is, the number of coefficients from  $\mathbb{K}$  needed for their dense representation. A major issue when dealing with arbitrary shifts is that the size of an  $\mathbf{s}$ -reduced form of  $\mathbf{A}$  may be beyond our target cost. This is a further motivation for focusing on the computation of the  $\mathbf{s}$ -Popov form of  $\mathbf{A}$ : by definition, the sum of its column degrees is  $\deg(\det(\mathbf{A}))$ , and therefore its size is at most  $m^2 + m \deg(\det(\mathbf{A}))$ , independently of  $\mathbf{s}$ .

Consider for example  $\mathbf{A} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}$  for any  $\mathbf{0}$ -reduced  $\mathbf{B}_1$  and  $\mathbf{B}_2$  in  $\mathbb{K}[X]^{m \times m}$ . Then, taking  $\mathbf{s} = (0, \dots, 0, d, \dots, d)$  with  $d > 0$ ,  $\begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}$  is an  $\mathbf{s}$ -reduced form of  $\mathbf{A}$  for any  $\mathbf{C} \in \mathbb{K}[X]^{m \times m}$  with  $\deg(\mathbf{C}) \leq d$ ; for some  $\mathbf{C}$  it has size  $\Theta(m^2 d)$ , with  $d$  arbitrary large independently of  $\deg(\mathbf{A})$ .

Furthermore, the size of the unimodular transformation leading from  $\mathbf{A}$  to  $\mathbf{P}$  may be beyond the target cost, which is why fast algorithms for  $\mathbf{0}$ -reduction and Hermite form do not directly perform unimodular transformations on  $\mathbf{A}$  to reduce the degrees of its entries. Instead, they proceed in two steps: first, they work on  $\mathbf{A}$  to find some equations which describe its row space, and then they find a basis of solutions to these equations in  $\mathbf{0}$ -reduced form or Hermite form. We will follow a similar two-step strategy for an arbitrary shift.

It seems that some new ingredient is needed, since for both  $\mathbf{s} = \mathbf{0}$  and  $\mathbf{s} = \mathbf{h}$  the fastest algorithms use shift-specific properties at some point of the process: namely, the facts that a  $\mathbf{0}$ -reduced form of  $\mathbf{A}$  has degree at most  $\deg(\mathbf{A})$  and that the Hermite form of  $\mathbf{A}$  is triangular.

As in [17], we first compute the Smith form  $\mathbf{S}$  of  $\mathbf{A}$  and partial information on a right unimodular transformation  $\mathbf{V}$ ; this is where the probabilistic aspect comes from. This gives a description of the row space of  $\mathbf{A}$  as the set of row vectors  $\mathbf{p} \in \mathbb{K}[X]^{1 \times m}$  such that  $\mathbf{pV} = \mathbf{qS}$  for some  $\mathbf{q} \in \mathbb{K}[X]^{1 \times m}$ . Since  $\mathbf{S}$  is diagonal, this can be seen as a system of modular equations: the second step is the fast computation of a basis of solutions in  $\mathbf{s}$ -Popov form, which is our new ingredient.

## 1.2 Systems of modular equations

Hereafter,  $\mathbb{K}[X]_{\neq 0}$  denotes the set of nonzero polynomials. We fix some moduli  $\mathfrak{M} = (\mathbf{m}_1, \dots, \mathbf{m}_n) \in \mathbb{K}[X]_{\neq 0}^n$ , and for  $\mathbf{A}, \mathbf{B} \in \mathbb{K}[X]^{m \times n}$  we write  $\mathbf{A} = \mathbf{B} \bmod \mathfrak{M}$  if there exists  $\mathbf{Q} \in \mathbb{K}[X]^{m \times n}$  such that  $\mathbf{A} = \mathbf{B} + \mathbf{Q} \text{diag}(\mathfrak{M})$ . Given  $\mathbf{F} \in \mathbb{K}[X]^{m \times n}$  specifying the equations, we call *solution for*  $(\mathfrak{M}, \mathbf{F})$  any  $\mathbf{p} \in \mathbb{K}[X]^{1 \times m}$  such that  $\mathbf{pF} = 0 \bmod \mathfrak{M}$ .

The set of all such  $\mathbf{p}$  is a  $\mathbb{K}[X]$ -submodule of  $\mathbb{K}[X]^{1 \times m}$  which contains  $\text{lcm}(\mathbf{m}_1, \dots, \mathbf{m}_n) \mathbb{K}[X]^{1 \times m}$ , and is thus free of rank  $m$  [24, p. 146]. Then, we represent any basis of this module as the rows of a matrix  $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ , called a *solution basis for*  $(\mathfrak{M}, \mathbf{F})$ . Here, for example for the application to Problem 1, we are interested in such bases that are  $\mathbf{s}$ -reduced, in which case  $\mathbf{P}$  is said to be an *s-minimal solution basis for*  $(\mathfrak{M}, \mathbf{F})$ . The unique such basis which is in  $\mathbf{s}$ -Popov form is called the *s-Popov solution basis for*  $(\mathfrak{M}, \mathbf{F})$ .

PROBLEM 2 (MINIMAL SOLUTION BASIS).

Input: *the base field  $\mathbb{K}$ , moduli  $\mathfrak{M} = (\mathbf{m}_1, \dots, \mathbf{m}_n) \in \mathbb{K}[X]_{\neq 0}^n$ , a matrix  $\mathbf{F} \in \mathbb{K}[X]^{m \times n}$  such that  $\deg(\mathbf{F}_{*,j}) < \deg(\mathbf{m}_j)$ , a shift  $\mathbf{s} \in \mathbb{Z}^m$ .*

Output: *an s-minimal solution basis for  $(\mathfrak{M}, \mathbf{F})$ .*

Well-known specific cases of this problem are *Hermite-Padé approximation* with a single equation modulo some power of  $X$ , and *M-Padé approximation* [3, 32] with moduli that are products of known linear factors. Moreover, an *s-order basis for  $\mathbf{F}$*  and  $(\sigma_1, \dots, \sigma_n)$  [34] is an  $\mathbf{s}$ -minimal solution basis for  $(\mathfrak{M}, \mathbf{F})$  with  $\mathfrak{M} = (X^{\sigma_1}, \dots, X^{\sigma_n})$ .

An overview of fast algorithms for Problem 2 is given in Table 2. For M-Padé approximation, and thus in particular for order basis computation, there is an algorithm to compute the  $\mathbf{s}$ -Popov solution basis using  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  operations, with  $\sigma = \deg(\mathbf{m}_1) + \dots + \deg(\mathbf{m}_n)$  [21]. Here, for  $n \in \mathcal{O}(m)$ , we extend this result to arbitrary moduli.

THEOREM 1.4. *Assuming  $n \in \mathcal{O}(m)$ , there is a deterministic algorithm which solves Problem 2 using  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  field operations, with  $\sigma = \deg(\mathbf{m}_1) + \dots + \deg(\mathbf{m}_n)$ , and returns the s-Popov solution basis for  $(\mathfrak{M}, \mathbf{F})$ .*

We note that Problem 2 is a minimal interpolation basis problem [5, 20] when the so-called *multiplication matrix*  $\mathbf{M}$  is block diagonal with companion blocks. Indeed,  $\mathbf{p}$  is a solution for  $(\mathfrak{M}, \mathbf{F})$  if and only if  $\mathbf{p}$  is an *interpolant for*  $(\mathbf{E}, \mathbf{M})$  [20, Definition 1.1], where  $\mathbf{E} \in \mathbb{K}^{m \times \sigma}$  is the concatenation of the coefficient vectors of the columns of  $\mathbf{F}$  and  $\mathbf{M} \in \mathbb{K}^{\sigma \times \sigma}$  is  $\text{diag}(\mathbf{M}_1, \dots, \mathbf{M}_n)$  with  $\mathbf{M}_j$  the companion

matrix associated with  $\mathbf{m}_j$ . In this context, the multiplication  $\mathbf{p} \cdot \mathbf{E}$  defined by  $\mathbf{M}$  as in [5, 20] precisely corresponds to  $\mathbf{pF} \bmod \mathfrak{M}$ .

In particular, Theorem 1.4 follows from [20, Theorem 1.4] when  $\sigma \in \mathcal{O}(m)$ . If some of the moduli have small degree, we use this result for base cases of our recursive algorithm.

Ref.	Cost bound	Moduli	Particularities
[3, 32]	$\mathcal{O}(m^2\sigma^2)$	split	
[4]	$\mathcal{O}(m\sigma^2)$	$\mathbf{m}_j = X^{\sigma/n}$	partial basis
[4]	$\tilde{\mathcal{O}}(m^\omega\sigma)$	$\mathbf{m}_j = X^{\sigma/n}$	
[14]	$\tilde{\mathcal{O}}(m^\omega\sigma/n)$	$\mathbf{m}_j = X^{\sigma/n}$	
[30]	$\tilde{\mathcal{O}}(m^\omega \lceil \sigma/m \rceil)$	$\mathbf{m}_j = X^{\sigma/n}$	partial basis, $ \mathbf{s}  \leq \sigma$
[34]	$\tilde{\mathcal{O}}(m^\omega \lceil \sigma/m \rceil)$	$\mathbf{m}_j = X^{\sigma/n}$	$ \mathbf{s}  \leq \sigma$
[8]	$\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$ , probabilistic	any	returns a single small degree solution
[20]	$\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$	split	$ \mathbf{s}  \leq \sigma$
[20]	$\tilde{\mathcal{O}}(m\sigma^{\omega-1})$	any	$\mathbf{s}$ -Popov, $\sigma \in \mathcal{O}(m)$
[21]	$\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$	split	$\mathbf{s}$ -Popov
Here	$\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$	any	$\mathbf{s}$ -Popov

Table 2: Fast algorithms for Problem 2 ( $n \in \mathcal{O}(m)$ ); *partial basis* = returns small degree rows of an  $\mathbf{s}$ -minimal solution basis; *split* = product of known linear factors).

In the case of M-Padé approximation, knowing the moduli as products of linear factors leads to rewriting the problem as a minimal interpolation basis computation with  $\mathbf{M}$  in Jordan form [5, 20]. Since  $\mathbf{M}$  is upper triangular, one can then rely on recurrence relations to solve the problem iteratively [3, 32, 4, 5]. The fast algorithms in [4, 14, 34, 20, 21], beyond the techniques used to achieve efficiency, are essentially divide-and-conquer versions of this iterative solution and are thus based on the same recurrence relations.

However, for arbitrary moduli the matrix  $\mathbf{M}$  is not triangular and there is no such recurrence in general. Then, a natural idea is to relate solution bases to nullspace bases: Problem 2 asks to find  $\mathbf{P}$  such that there is some quotient  $\mathbf{Q}$  with  $[\mathbf{P}|\mathbf{Q}]\mathbf{N} = \mathbf{0}$  for  $\mathbf{N} = [\mathbf{F}^\top | -\text{diag}(\mathfrak{M})]^\top$ . More precisely,  $[\mathbf{P}|\mathbf{Q}]$  can be obtained as a  $\mathbf{u}$ -minimal nullspace basis of  $\mathbf{N}$  for the shift  $\mathbf{u} = (\mathbf{s} - \min(\mathbf{s}), \mathbf{0}) \in \mathbb{N}^{m+n}$ .

Using recent ingredients from [17, 21] outlined in the next paragraphs, the main remaining difficulty is to deal with this nullspace problem when  $n = 1$ . Here, we give a  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  algorithm to solve it using its specific properties:  $\mathbf{N}$  is the column  $[\mathbf{F}^\top | \mathbf{m}_1]^\top$  with  $\deg(\mathbf{F}) < \deg(\mathbf{m}_1) = \sigma$ , and the last entry of  $\mathbf{u}$  is  $\min(\mathbf{u})$ . First, when  $\max(\mathbf{u}) \in \mathcal{O}(\sigma)$  we show that  $[\mathbf{P}|\mathbf{Q}]$  can be efficiently obtained as a submatrix of the  $\mathbf{u}$ -Popov order basis for  $\mathbf{N}$  and order  $\mathcal{O}(\sigma)$ . Then, when  $\max(\mathbf{u})$  is large compared to  $\sigma$  and assuming  $\mathbf{u}$  is sorted non-decreasingly,  $\mathbf{P}$  has a lower block triangular shape. We show how this shape can be revealed, along with the  $\mathbf{s}$ -pivot degree of  $\mathbf{P}$ , using a divide-and-conquer approach which splits  $\mathbf{u}$  into two shifts of amplitude about  $\max(\mathbf{u})/2$ .

Then, for  $n \geq 1$  we use a divide-and-conquer approach on  $n$  which is classical in such contexts: two solution bases  $\mathbf{P}^{(1)}$  and  $\mathbf{P}^{(2)}$  are computed recursively in shifted Popov form and are multiplied together to obtain the  $\mathbf{s}$ -minimal solution basis  $\mathbf{P}^{(2)}\mathbf{P}^{(1)}$  for  $(\mathfrak{M}, \mathbf{F})$ . However this product is usually not in  $\mathbf{s}$ -Popov form and may have size beyond our target cost. Thus, as in [21], instead of computing  $\mathbf{P}^{(2)}\mathbf{P}^{(1)}$ , we use  $\mathbf{P}^{(2)}$  and  $\mathbf{P}^{(1)}$  to deduce the  $\mathbf{s}$ -pivot degree of  $\mathbf{P}$ .

In both recursions above, we focus on finding the  $\mathbf{s}$ -pivot degree of  $\mathbf{P}$ . Using ideas and results from [17, 21], we show that this knowledge about the degrees in  $\mathbf{P}$  allows us to complete the computation of  $\mathbf{P}$  within the target cost.



## 2. FAST COMPUTATION OF THE SHIFTED POPOV SOLUTION BASIS

Hereafter, we call *s-minimal degree* of  $(\mathfrak{M}, \mathbf{F})$  the *s-pivot degree*  $\delta$  of the *s-Popov solution basis* for  $(\mathfrak{M}, \mathbf{F})$ ;  $\delta$  coincides with the column degree of this basis. A central result for the cost analysis is that  $|\delta| = \delta_1 + \dots + \delta_m$  is at most  $\sigma = \deg(\mathbf{m}_1) + \dots + \deg(\mathbf{m}_n)$ . This is classical for M-Padé approximation [32, Theorem 4.1] and holds for minimal interpolation bases in general (see for example [20, Lemma 7.17]).

### 2.1 Solution bases from nullspace bases and fast algorithm for known minimal degree

This subsection summarizes and slightly extends results from [17, Section 3]. We first show that the *s-Popov solution basis* for  $(\mathfrak{M}, \mathbf{F})$  is the principal  $m \times m$  submatrix of the *u-Popov nullspace basis* of  $[\mathbf{F}^\top | \text{diag}(\mathfrak{M})]^\top$  for some  $\mathbf{u} \in \mathbb{Z}^{m+n}$ .

LEMMA 2.1. *Let  $\mathfrak{M} = (\mathbf{m}_1, \dots, \mathbf{m}_n) \in \mathbb{K}[X]_{\neq 0}^n$ ,  $\mathbf{s} \in \mathbb{Z}^m$ ,  $\mathbf{F} \in \mathbb{K}[X]^{m \times n}$  with  $\deg(\mathbf{F}_{*,j}) < \deg(\mathbf{m}_j)$ ,  $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ , and  $\mathbf{w} \in \mathbb{Z}^n$  be such that  $\max(\mathbf{w}) \leq \min(\mathbf{s})$ . Then,  $\mathbf{P}$  is the *s-Popov solution basis* for  $(\mathfrak{M}, \mathbf{F})$  if and only if  $[\mathbf{P}|\mathbf{Q}]$  is the *u-Popov nullspace basis* of  $[\mathbf{F}^\top | \text{diag}(\mathfrak{M})]^\top$  for some  $\mathbf{Q} \in \mathbb{K}[X]^{m \times n}$  and  $\mathbf{u} = (\mathbf{s}, \mathbf{w}) \in \mathbb{Z}^{m+n}$ . In this case,  $\deg(\mathbf{Q}) < \deg(\mathbf{P})$  and  $[\mathbf{P}|\mathbf{Q}]$  has *s-pivot index*  $(1, 2, \dots, m)$ .*

PROOF. Let  $\mathbf{N} = [\mathbf{F}^\top | \text{diag}(\mathfrak{M})]^\top$ . It is easily verified that  $\mathbf{P}$  is a solution basis for  $(\mathfrak{M}, \mathbf{F})$  if and only if there is some  $\mathbf{Q} \in \mathbb{K}[X]^{m \times n}$  such that  $[\mathbf{P}|\mathbf{Q}]$  is a nullspace basis of  $\mathbf{N}$ .

Now, having  $\deg(\mathbf{F}_{*,j}) < \deg(\mathbf{m}_j)$  implies that any  $[\mathbf{p}|\mathbf{q}] \in \mathbb{K}[X]^{1 \times (m+n)}$  in the nullspace of  $\mathbf{N}$  satisfies  $\deg(\mathbf{q}) < \deg(\mathbf{p})$ , and since  $\max(\mathbf{w}) \leq \min(\mathbf{s})$  we get  $\text{rdeg}_{\mathbf{w}}(\mathbf{q}) < \text{rdeg}_{\mathbf{s}}(\mathbf{p})$ . In particular, for any matrix  $[\mathbf{P}|\mathbf{Q}] \in \mathbb{K}[X]^{m \times (m+n)}$  such that  $[\mathbf{P}|\mathbf{Q}]\mathbf{N} = 0$ , we have  $\text{lm}_{\mathbf{u}}([\mathbf{P}|\mathbf{Q}]) = [\text{lm}_{\mathbf{s}}(\mathbf{P}) | \mathbf{0}]$ . This implies that  $\mathbf{P}$  is in *s-Popov form* if and only if  $[\mathbf{P}|\mathbf{Q}]$  is in *u-Popov form* with *s-pivot index*  $(1, \dots, m)$ .  $\square$

We now show that, when we have *a priori* knowledge about the *s-pivot entries* of a *s-Popov nullspace basis*, it can be computed efficiently via an *s-Popov order basis*.

LEMMA 2.2. *Let  $\mathbf{s} \in \mathbb{Z}^{m+n}$  and let  $\mathbf{N} \in \mathbb{K}[X]^{(m+n) \times n}$  be of full rank. Let  $\mathbf{B} \in \mathbb{K}[X]^{m \times (m+n)}$  be the *s-Popov nullspace basis* for  $\mathbf{N}$ ,  $(\pi_1, \dots, \pi_m)$  be its *s-pivot index*,  $(\delta_1, \dots, \delta_m)$  be its *s-pivot degree*, and  $\delta \geq \deg(\mathbf{B})$  be a degree bound. Then, let  $\mathbf{u} = (u_1, \dots, u_{m+n}) \in \mathbb{Z}_{\leq 0}^{m+n}$  with*

$$u_j = \begin{cases} -\delta - 1 & \text{if } j \notin \{\pi_1, \dots, \pi_m\}, \\ -\delta_i & \text{if } j = \pi_i. \end{cases}$$

Writing  $(\sigma_1, \dots, \sigma_n)$  for the column degree of  $\mathbf{N}$ , let  $\tau_j = \sigma_j + \delta + 1$  for  $1 \leq j \leq n$  and let  $\mathbf{A}$  be the *u-Popov order basis* for  $\mathbf{N}$  and  $(\tau_1, \dots, \tau_n)$ . Then,  $\mathbf{B}$  is the submatrix of  $\mathbf{A}$  formed by its rows at indices  $\{\pi_1, \dots, \pi_m\}$ .

PROOF. First,  $\mathbf{B}$  is in *u-Popov form* with  $\text{rdeg}_{\mathbf{u}}(\mathbf{B}) = \mathbf{0}$ . Define  $\mathbf{C} \in \mathbb{K}[X]^{(m+n) \times (m+n)}$  whose *i*-th row is  $\mathbf{B}_{j,*}$  if  $i = \pi_j$  and  $\mathbf{A}_{i,*}$  if  $i \notin \{\pi_1, \dots, \pi_m\}$ : we want to prove  $\mathbf{C} = \mathbf{A}$ .

Let  $\mathbf{p} = [p_j]_j \in \mathbb{K}[X]^{1 \times (m+n)}$  be a row of  $\mathbf{A}$ , and assume  $\text{rdeg}_{\mathbf{u}}(\mathbf{p}) < 0$ . This means  $\deg(p_j) < -u_j$  for all  $j$ , so that  $\deg(\mathbf{p}) < \max(-\mathbf{u}) = \delta + 1$ . Then, for all  $1 \leq j \leq n$  we have  $\deg(\mathbf{p}\mathbf{N}_{*,j}) < \sigma_j + \delta + 1 = \tau_j$ , and from  $\mathbf{p}\mathbf{N}_{*,j} = 0 \pmod{X^{\tau_j}}$  we obtain  $\mathbf{p}\mathbf{N}_{*,j} = 0$ , which is absurd by minimality of  $\mathbf{B}$ . As a result,  $\text{rdeg}_{\mathbf{u}}(\mathbf{A}) \geq \mathbf{0} = \text{rdeg}_{\mathbf{u}}(\mathbf{B})$  componentwise.

Besides,  $\mathbf{C}\mathbf{F} = \mathbf{0} \pmod{(X^{\tau_1}, \dots, X^{\tau_n})}$  and since  $\mathbf{C}$  has its *u-pivot entries* on the diagonal, it is *u-reduced*: by minimality of  $\mathbf{A}$ , we obtain  $\text{rdeg}_{\mathbf{u}}(\mathbf{A}) = \text{rdeg}_{\mathbf{u}}(\mathbf{C})$ . Then, it is easily verified that  $\mathbf{C}$  is in *u-Popov form*, hence  $\mathbf{C} = \mathbf{A}$ .  $\square$

In particular, computing the *s-Popov nullspace basis*  $\mathbf{B}$ , when its *s-pivot index*, its *s-pivot degree*, and  $\delta \geq \deg(\mathbf{B})$  are known, can be done in  $\tilde{\mathcal{O}}(m^{\omega-1}(\sigma + n\delta))$  with  $\sigma = \sigma_1 + \dots + \sigma_n$  using the order basis algorithm in [21].

As for Problem 2, with Lemma 2.1 this gives an algorithm for computing  $\mathbf{P}$  and the quotients  $\mathbf{Q} = -\mathbf{P}\mathbf{F}/\text{diag}(\mathfrak{M})$  when we know *a priori* the *s-minimal degree*  $\delta$  of  $(\mathfrak{M}, \mathbf{F})$ . Here, we would choose  $\delta = \max(\delta) \geq \deg([\mathbf{P}|\mathbf{Q}])$ : in some cases  $\delta = \Theta(\sigma)$  and this has cost bound  $\tilde{\mathcal{O}}(m^{\omega-1}(\sigma + n\sigma))$ , which exceeds our target  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$ . An issue is that  $\mathbf{Q}$  has size  $\mathcal{O}(mn\sigma)$  when  $\mathbf{P}$  has columns of large degree; yet here we are not interested in  $\mathbf{Q}$ . This can be solved using partial linearization to expand the columns of large degree in  $\mathbf{P}$  into more columns of smaller degree as in the next result, which holds in general for interpolation bases [21, Lemma 4.2].

LEMMA 2.3. *Let  $\mathfrak{M} \in \mathbb{K}[X]_{\neq 0}^n$  with entries having degrees  $(\sigma_1, \dots, \sigma_n)$ . Let  $\mathbf{F} \in \mathbb{K}[X]^{m \times n}$  and  $\mathbf{s} \in \mathbb{Z}^m$ . Furthermore, let  $\delta = (\delta_1, \dots, \delta_m)$  denote the *s-minimal degree* of  $(\mathfrak{M}, \mathbf{F})$ .*

Writing  $\sigma = \sigma_1 + \dots + \sigma_n$ , let  $\delta = \lceil \sigma/m \rceil \geq 1$ , and for  $i \in \{1, \dots, m\}$  write  $\delta_i = (\alpha_i - 1)\delta + \beta_i$  with  $\alpha_i \geq 1$  and  $0 \leq \beta_i < \delta$ , and let  $\tilde{m} = \alpha_1 + \dots + \alpha_m$ . Define  $\tilde{\delta} \in \mathbb{N}^{\tilde{m}}$  as

$$\tilde{\delta} = (\underbrace{\delta, \dots, \delta}_{\alpha_1}, \beta_1, \dots, \underbrace{\delta, \dots, \delta}_{\alpha_m}, \beta_m) \quad (2)$$

and the expansion-compression matrix  $\mathcal{E} \in \mathbb{K}[X]^{\tilde{m} \times m}$  as

$$\mathcal{E} = \begin{bmatrix} 1 \\ X^\delta \\ \vdots \\ X^{(\alpha_1-1)\delta} & \ddots & 1 \\ & & X^\delta \\ & & \vdots \\ & & X^{(\alpha_m-1)\delta} \end{bmatrix}. \quad (3)$$

Let  $\mathbf{d} = -\tilde{\delta} \in \mathbb{Z}^{\tilde{m}}$  and  $\mathbf{P} \in \mathbb{K}[X]^{\tilde{m} \times \tilde{m}}$  be the *d-Popov solution basis* for  $(\mathfrak{M}, \mathcal{E}\mathbf{F} \pmod{\mathfrak{M}})$ . Then,  $\mathbf{P}$  has *d-pivot degree*  $\tilde{\delta}$  and the *s-Popov solution basis* for  $(\mathfrak{M}, \mathbf{F})$  is the submatrix of  $\mathbf{P}\mathcal{E}$  formed by its rows at indices  $\{\alpha_1 + \dots + \alpha_i, 1 \leq i \leq m\}$ .

This leads to Algorithm 1, which solves Problem 2 efficiently when the *s-minimal degree*  $\delta$  is known *a priori*.

#### ALGORITHM 1 (KNOWNDEGPOLMODSYS).

*Input:* polynomials  $\mathfrak{M} = (\mathbf{m}_1, \dots, \mathbf{m}_n) \in \mathbb{K}[X]_{\neq 0}^n$ , a matrix  $\mathbf{F} \in \mathbb{K}[X]^{m \times n}$  with  $\deg(\mathbf{F}_{*,j}) < \deg(\mathbf{m}_j)$ , a shift  $\mathbf{s} \in \mathbb{Z}^m$ ,  $\delta = (\delta_1, \dots, \delta_m)$  the *s-minimal degree* of  $(\mathfrak{M}, \mathbf{F})$ .

*Output:* the *s-Popov solution basis* for  $(\mathfrak{M}, \mathbf{F})$ .

1.  $\delta \leftarrow \lceil (\deg(\mathbf{m}_1) + \dots + \deg(\mathbf{m}_n))/m \rceil$ ,  
 $\alpha_i \leftarrow \lfloor \delta_i/\delta \rfloor + 1$  for  $1 \leq i \leq m$ ,  $\tilde{m} \leftarrow \alpha_1 + \dots + \alpha_m$ ,  
 $\tilde{\delta}$  as in (2),  $\mathcal{E}$  as in (3),  $\tilde{\mathbf{F}} \leftarrow \mathcal{E}\mathbf{F} \pmod{\mathfrak{M}}$
2.  $\mathbf{u} \leftarrow (-\tilde{\delta}, -\delta - 1, \dots, -\delta - 1) \in \mathbb{Z}^{\tilde{m}+n}$   
 $\tau \leftarrow (\deg(\mathbf{m}_j) + \delta + 1)_{1 \leq j \leq n}$
3.  $\tilde{\mathbf{P}} \leftarrow$  the *u-Popov order basis* for  $[\tilde{\mathbf{F}}^\top | \text{diag}(\mathfrak{M})]^\top$  and  $\tau$   
 $\mathbf{P} \leftarrow$  the principal  $\tilde{m} \times \tilde{m}$  submatrix of  $\tilde{\mathbf{P}}$
4. Return the submatrix of  $\mathbf{P}\mathcal{E}$  formed by the rows at indices  $\alpha_1 + \dots + \alpha_i$  for  $1 \leq i \leq m$

PROPOSITION 2.4. *Algorithm KNOWNDEGPOLMODSYS is correct. Writing  $\sigma = \deg(\mathbf{m}_1) + \dots + \deg(\mathbf{m}_n)$  and assuming  $\sigma \geq m \geq n$ , it uses  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  operations in  $\mathbb{K}$ .*

PROOF. By Lemmas 2.3 and 2.1, since  $\min(-\tilde{\delta}) > -\delta - 1$  and  $\mathbf{u} = (-\tilde{\delta}, -\delta - 1, \dots, -\delta - 1)$ , the  $-\delta$ -Popov solution

basis for  $(\mathfrak{M}, \tilde{\mathbf{F}})$  is the principal  $\tilde{m} \times \tilde{m}$  submatrix of the  $\mathbf{u}$ -Popov nullspace basis  $\mathbf{B}$  for  $[\tilde{\mathbf{F}}^\top \text{diag}(\mathfrak{M})]^\top$ , and  $\mathbf{B}$  has  $\mathbf{u}$ -pivot index  $\{1, \dots, \tilde{m}\}$ ,  $\mathbf{u}$ -pivot degree  $\tilde{\delta}$ , and  $\deg(\mathbf{B}) \leq \delta$ . Then, by Lemma 2.2,  $\mathbf{B}$  is formed by the first  $\tilde{m}$  rows of  $\tilde{\mathbf{P}}$  at Step 3, hence  $\mathbf{P}$  is the  $\mathbf{d}$ -Popov solution basis for  $(\mathfrak{M}, \mathbf{F})$ . The correctness then follows from Lemma 2.3.

Since  $|\delta| \leq \sigma$ ,  $\mathcal{E}$  has  $\tilde{m} \leq 2m$  rows and  $\mathcal{E}\mathbf{F} \bmod \mathfrak{M}$  can be computed in  $\tilde{\mathcal{O}}(m\sigma)$  operations using fast polynomial division [13]. The cost bound of Step 3 follows from [21, Theorem 1.4] since  $\tau_1 + \dots + \tau_n = \sigma + n(1 + \lceil \sigma/m \rceil) \in \mathcal{O}(\sigma)$ .  $\square$

## 2.2 The case of one equation

We now present our main new ingredients, focusing on the case  $n = 1$ . First, we show that when the shift  $\mathbf{s}$  has a small *amplitude*  $\text{amp}(\mathbf{s}) = \max(\mathbf{s}) - \min(\mathbf{s})$ , one can solve Problem 2 via an order basis computation at small order.

LEMMA 2.5. *Let  $\mathbf{m} \in \mathbb{K}[X]_{\neq 0}$ ,  $\mathbf{s} \in \mathbb{Z}^m$ , and  $\mathbf{F} \in \mathbb{K}[X]^{m \times 1}$  with  $\deg(\mathbf{F}) < \deg(\mathbf{m}) = \sigma$ . Then, for any  $\tau \geq \text{amp}(\mathbf{s}) + 2\sigma$ , the  $\mathbf{s}$ -Popov solution basis for  $(\mathbf{m}, \mathbf{F})$  is the principal  $m \times m$  submatrix of the  $\mathbf{u}$ -Popov order basis for  $[\mathbf{F}^\top | \mathbf{m}]^\top$  and  $\tau$ , with  $\mathbf{u} = (\mathbf{s}, \min(\mathbf{s})) \in \mathbb{Z}^{m+1}$ .*

PROOF. Let  $\mathbf{A} = \begin{bmatrix} \mathbf{p} & \mathbf{q} \\ \mathbf{p} & \mathbf{q} \end{bmatrix}$  denote the  $\mathbf{u}$ -Popov order basis for  $[\mathbf{F}^\top | \mathbf{m}]^\top$  and  $\tau$ , where  $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$  and  $q \in \mathbb{K}[X]$ . Consider  $\mathbf{B} = [\tilde{\mathbf{P}} | \tilde{\mathbf{q}}]$  the  $\mathbf{u}$ -Popov nullspace basis for  $[\mathbf{F}^\top | \mathbf{m}]^\top$ : thanks to Lemma 2.1, it is enough to prove that  $\mathbf{B} = [\mathbf{P} | \mathbf{q}]$ .

First, we have  $\text{rdeg}(\mathbf{p}) \leq \deg(q)$  by choice of  $\mathbf{u}$ , so that  $q\mathbf{m} \neq 0$  implies  $\deg(\mathbf{p}\mathbf{F} + q\mathbf{m}) = \deg(q) + \sigma$ . Since  $\mathbf{p}\mathbf{F} + q\mathbf{m} = 0 \bmod X^\tau$ , this gives  $\deg(q) + \sigma \geq \tau$ . This also shows that the  $\mathbf{u}$ -pivot entries of  $\mathbf{B}$  are located in  $\tilde{\mathbf{P}}$ .

Then, since the sum of the  $\mathbf{u}$ -pivot degrees of  $\mathbf{A}$  is at most  $\tau$ , the sum of the  $\mathbf{s}$ -pivot degrees of  $\mathbf{P}$  is at most  $\sigma$ ; with  $[\mathbf{P} | \mathbf{q}]$  in  $\mathbf{u}$ -Popov form, this gives  $\deg(\mathbf{q}) < \sigma + \text{amp}(\mathbf{s}) \leq \tau - \sigma$ . We obtain  $\deg(\mathbf{P}\mathbf{F} + \mathbf{q}\mathbf{m}) < \tau$ , so that  $\mathbf{P}\mathbf{F} + \mathbf{q}\mathbf{m} = 0$ . Thus, the minimality of  $\mathbf{B}$  and  $\mathbf{A}$  gives the conclusion.  $\square$

When  $\text{amp}(\mathbf{s}) \in \mathcal{O}(\sigma)$ , this gives a fast solution to our problem. In what follows, we present a divide-and-conquer approach on  $\text{amp}(\mathbf{s})$ , with base case  $\text{amp}(\mathbf{s}) \in \mathcal{O}(\sigma)$ .

We first give an overview, assuming  $\mathbf{s}$  is non-decreasing. A key ingredient is that when  $\text{amp}(\mathbf{s})$  is large compared to  $\sigma$ , then  $\mathbf{P}$  has a lower block triangular shape, since it is in  $\mathbf{s}$ -Popov form with sum of  $\mathbf{s}$ -pivot degrees  $|\delta| \leq \sigma$ . Typically, if  $s_{i+1} - s_i \geq \sigma$  for some  $i$  then  $\mathbf{P} = \begin{bmatrix} \mathbf{P}^{(1)} & \mathbf{0} \\ * & \mathbf{P}^{(2)} \end{bmatrix}$  with  $\mathbf{P}^{(1)} \in \mathbb{K}[X]^{i \times i}$ . Even though the block sizes are unknown in general, we show that they can be revealed efficiently along with  $\delta$  by a divide-and-conquer algorithm, as follows.

First, we use a recursive call with the first  $j$  entries  $\mathbf{s}^{(0)}$  of  $\mathbf{s}$  and  $\mathbf{F}^{(0)}$  of  $\mathbf{F}$ , where  $j$  is such that  $\text{amp}(\mathbf{s}^{(0)})$  is about half of  $\text{amp}(\mathbf{s})$ . This reveals the first  $i \leq j$  entries  $\delta^{(1)}$  of  $\delta$  and the first  $i$  rows  $[\mathbf{P}^{(1)} | \mathbf{0}]$  of  $\mathbf{P}$ , with  $\mathbf{P}^{(1)} \in \mathbb{K}[X]^{i \times i}$ . A central point is that  $\text{amp}(\mathbf{s}^{(2)})$  is about half of  $\text{amp}(\mathbf{s})$  as well, where  $\mathbf{s}^{(2)}$  is the tail of  $\mathbf{s}$  starting at the entry  $i + 1$ .

Then, knowing the degrees  $\delta^{(1)}$  allows us to set up an order basis computation that yields a *residual*, that is, a column  $\mathbf{G} \in \mathbb{K}[X]^{(m-i) \times 1}$  and a modulus  $\mathfrak{n}$  such that we can continue the computation of  $\mathbf{P}$  using a second recursive call, which consists in computing the  $\mathbf{s}^{(2)}$ -Popov solution basis for  $(\mathfrak{n}, \mathbf{G})$ . From these two calls we obtain  $\delta$ , and then we recover  $\mathbf{P}$  using Algorithm 1.

Now we present the details. We fix  $\mathbf{F} \in \mathbb{K}[X]^{m \times 1}$ ,  $\mathbf{m} \in \mathbb{K}[X]_{\neq 0}$  with  $\sigma = \deg(\mathbf{m}) > \deg(\mathbf{F})$ ,  $\mathbf{s} \in \mathbb{Z}^m$ ,  $\mathbf{P}$  the  $\mathbf{s}$ -Popov

solution basis for  $(\mathbf{m}, \mathbf{F})$ , and  $\delta$  its  $\mathbf{s}$ -pivot degree. In what follows,  $\boldsymbol{\pi}^{\mathbf{s}} = (\pi_1, \dots, \pi_m)$  is any permutation of  $\{1, \dots, m\}$  such that  $(s_{\pi_1}, \dots, s_{\pi_m})$  is non-decreasing.

Then, for  $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{Z}^m$  we write  $\mathbf{t}_{[i:j]}$  for the subtuple of  $\mathbf{t}$  formed by its entries at indices  $\{\pi_i, \dots, \pi_j\}$ , and for a matrix  $\mathbf{M} \in \mathbb{K}[X]^{m \times m}$  we write  $\mathbf{M}_{[i:j, k:l]}$  for the submatrix of  $\mathbf{M}$  formed by its rows at indices  $\{\pi_i, \pi_{i+1}, \dots, \pi_j\}$  and columns at indices  $\{\pi_k, \pi_{k+1}, \dots, \pi_l\}$ . The main ideas in this subsection can be understood by focusing on the case of a non-decreasing  $\mathbf{s}$ , taking  $\pi_i = i$  for all  $i$ : then we have  $\mathbf{t}_{[i:j]} = (t_i, t_{i+1}, \dots, t_j)$  and  $\mathbf{M}_{[i:j, k:l]} = (\mathbf{M}_{u,v})_{i \leq u \leq j, k \leq v \leq l}$ .

We now introduce the notion of splitting index, which will help us to locate zero blocks in  $\mathbf{P}$ .

DEFINITION 2.6 (SPLITTING INDEX). *Let  $\mathbf{d} \in \mathbb{N}^m$ ,  $\mathbf{t} \in \mathbb{Z}^m$  and  $\boldsymbol{\pi}^{\mathbf{t}} = (\mu_i)_i$ . Then,  $i \in \{1, \dots, m-1\}$  is a splitting index for  $(\mathbf{d}, \mathbf{t})$  if  $d_{\mu_j} + t_{\mu_j} - t_{\mu_{j+1}} < 0$  for all  $j \in \{1, \dots, i\}$ .*

In particular, if  $i$  is a splitting index for  $(\delta, \mathbf{s})$ , then we have  $[\mathbf{P}_{[i:i]} | \mathbf{P}_{[i:i+1:i]}] = [\mathbf{P}_{[i:i]} | \mathbf{0}]$ . Our algorithm first looks for such a splitting index, and then uses  $\mathbf{P}_{[i:i+1:i]} = \mathbf{0}$  to split the problem into two subproblems of dimensions  $i$  and  $m-i$ .

To find a splitting index, we rely on the following property: if  $(\mathbf{d}, \mathbf{t})$  does not admit a splitting index, then  $|\mathbf{d}| > \text{amp}(\mathbf{t})$ . This allows us to partition  $\mathbf{s}$  into  $\ell$  subtuples which all contain a splitting index, as follows.

Given  $\alpha \in \mathbb{Z}_{>0}$  we let  $\ell = 1 + \lceil \text{amp}(\mathbf{s})/\alpha \rceil$  and we consider the subtuples  $\mathbf{s}_1, \dots, \mathbf{s}_\ell$  of  $\mathbf{s}$  where  $\mathbf{s}_k$  consists of the entries of  $\mathbf{s}$  in  $\{\min(\mathbf{s}) + (k-1)\alpha, \dots, \min(\mathbf{s}) + k\alpha - 1\}$ ; this gives a subroutine  $\text{PARTITION}(\mathbf{s}, \alpha) = (\mathbf{s}_1, \dots, \mathbf{s}_\ell)$ . Now we take  $\alpha \geq 2\sigma$  and we assume  $s_{\pi_{i+1}} - s_{\pi_i} \leq \sigma$  for  $1 \leq i < m$  without loss of generality [21, Appendix A]. Then, for  $1 \leq k < \ell$ , since  $|\delta| \leq \sigma$  and  $\text{amp}(\mathbf{t}) \geq \sigma$  with  $\mathbf{t} = (\mathbf{s}_k, \min(\mathbf{s}_{k+1}))$ , by the above remark  $\mathbf{s}_k$  contains a splitting index for  $(\delta, \mathbf{s})$ .

Still, we do not know in advance which entries of  $\mathbf{s}_k$  correspond to splitting indices for  $(\delta, \mathbf{s})$ . Thus we recursively compute the  $\mathbf{s}$ -Popov solution basis  $\mathbf{P}^{(0)}$  for  $\mathbf{s}_1, \dots, \mathbf{s}_{\ell/2}$ , and we are now going to prove that this gives us a splitting index which divides the computation into two subproblems, the first of which has been already solved by computing  $\mathbf{P}^{(0)}$ .

LEMMA 2.7. *Let  $j \in \{2, \dots, m\}$ ,  $\mathbf{s}^{(0)} = \mathbf{s}_{[j:]}$ ,  $\mathbf{P}^{(0)}$  be the  $\mathbf{s}^{(0)}$ -Popov solution basis for  $(\mathbf{m}, \mathbf{F}_{[j:]})$ , and  $\delta^{(0)}$  be its  $\mathbf{s}^{(0)}$ -pivot degree. Suppose that there is a splitting index  $i \leq j$  for  $(\delta^{(0)}, \mathbf{s}^{(0)})$ . Let  $\mathbf{P}^{(1)} \in \mathbb{K}[X]^{i \times i}$  be the  $\mathbf{s}^{(1)}$ -Popov solution basis for  $(\mathbf{m}, \mathbf{F}_{[i:]})$  with  $\mathbf{s}^{(1)} = \mathbf{s}_{[i:]}$ , and let  $\delta^{(1)}$  be its  $\mathbf{s}^{(1)}$ -pivot degree. Then  $i$  is a splitting index for  $(\delta, \mathbf{s})$  and  $\mathbf{P}_{[i:i]} = \mathbf{P}^{(1)} = \mathbf{P}^{(0)}$ , hence  $\delta_{[i:i]} = \delta^{(1)} = \delta_{[i:i]}^{(0)}$  (where  $\mathbf{P}^{(0)}$  and  $\delta^{(0)}$  are indexed by  $\{\pi_1, \dots, \pi_j\}$  sorted increasingly).*

PROOF. Since  $i$  is a splitting index for  $(\delta^{(0)}, \mathbf{s}^{(0)})$  we have  $[\mathbf{P}_{[i:i]}^{(0)} | \mathbf{P}_{[i:i+1:i]}^{(0)}] = [\mathbf{Q} | \mathbf{0}]$  for some  $\mathbf{Q} \in \mathbb{K}[X]^{i \times i}$ . Now, for any  $\mathbf{B} \in \mathbb{K}[X]^{m \times m}$  with  $[\mathbf{B}_{[i:i, i]} | \mathbf{B}_{[i:i+1:i]}] = [\mathbf{P}^{(1)} | \mathbf{0}]$ ,  $\mathbf{B}_{[i:i]}$  is in  $\mathbf{s}$ -Popov form with its rows being solutions for  $(\mathfrak{M}, \mathbf{F})$ . Then, by minimality of  $\mathbf{P}$ ,  $\mathbf{P}_{[i:i]}$  has  $\mathbf{s}$ -pivot degree at most  $\delta^{(1)}$  componentwise, so that  $i$  is also a splitting index for  $(\delta, \mathbf{s})$ , and in particular  $[\mathbf{P}_{[i:i]} | \mathbf{P}_{[i:i+1:i]}] = [\mathbf{R} | \mathbf{0}]$  for some  $\mathbf{R} \in \mathbb{K}[X]^{i \times i}$ . It remains to prove that  $\mathbf{Q} = \mathbf{R} = \mathbf{P}^{(1)}$ .

Since  $\mathbf{R}\mathbf{F}_{[i]} = 0 \bmod \mathfrak{m}$  and  $\mathbf{R} = \mathbf{P}_{[i:i]}$  is in  $\mathbf{s}^{(1)}$ -Popov form, proving that all solutions  $\mathbf{p} \in \mathbb{K}[X]^{1 \times i}$  for  $(\mathfrak{m}, \mathbf{F}_{[i]})$  are in the row space of  $\mathbf{R}$  is enough to obtain  $\mathbf{R} = \mathbf{P}^{(1)}$ . Since  $\mathbf{q} \in \mathbb{K}[X]^{1 \times m}$  defined by  $[\mathbf{q}_{[i]} | \mathbf{q}_{[i+1:i]}] = [\mathbf{p} | \mathbf{0}]$  is a solution for  $(\mathbf{m}, \mathbf{F})$ ,  $\mathbf{q} = \boldsymbol{\lambda}\mathbf{P}$  for some  $\boldsymbol{\lambda} \in \mathbb{K}[X]^{1 \times m}$ . Now  $\mathbf{P}$  is nonsingular, thus  $\mathbf{P}_{[i:i+1:i]} = \mathbf{0}$  implies that  $[\boldsymbol{\lambda}_{[i]} | \boldsymbol{\lambda}_{[i+1:i]}] =$

$[\boldsymbol{\mu}|\mathbf{0}]$  with  $\boldsymbol{\mu} \in \mathbb{K}[X]^{1 \times i}$ , hence  $\mathbf{p} = \mathbf{q}_{[i]} = \boldsymbol{\lambda}_{[i]} \mathbf{P}_{[i,i]} + \boldsymbol{\lambda}_{[i+1:i]} \mathbf{P}_{[i+1:i,i]} = \boldsymbol{\mu} \mathbf{Q}$ . Similar arguments give  $\mathbf{Q} = \mathbf{P}^{(1)}$ .  $\square$

The next two lemmas show that knowing  $\boldsymbol{\delta}^{(1)}$ , which is  $\boldsymbol{\delta}_{[i]}$ , allows us to compute a so-called *residual*  $(\mathbf{n}, \mathbf{G})$  from which we can complete the computation of  $\boldsymbol{\delta}$  and  $\mathbf{P}$ .

LEMMA 2.8. *Let  $\mathbf{s}^{(2)} = \mathbf{s}_{[i+1:i]}$ ,  $\mathbf{d} = -\boldsymbol{\delta}^{(1)} + \min(\mathbf{s}^{(2)}) - 2\sigma \in \mathbb{Z}^i$ ,  $\mathbf{v} \in \mathbb{Z}^m$  be such that  $[\mathbf{v}_{[i]}|\mathbf{v}_{[i+1:i]}] = [\mathbf{d}|\mathbf{s}^{(2)}]$ , and  $\mathbf{u} = (\mathbf{v}, \min(\mathbf{d})) \in \mathbb{Z}^{m+1}$ . Let  $\begin{bmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{bmatrix}$  be the  $\mathbf{u}$ -Popov order basis for  $[\mathbf{F}^\top|\mathbf{m}]^\top$  and  $2\sigma$ , where  $\mathbf{A} \in \mathbb{K}[X]^{m \times m}$  and  $q \in \mathbb{K}[X]$ . Then we have  $\deg(q) \geq \sigma$ ,  $\mathbf{A}_{[i,i,i+1]} = \mathbf{0}$ ,  $\mathbf{p}_{[i+1:i]} = \mathbf{0}$ , and  $[\mathbf{A}_{[i,i,i]}|\mathbf{q}_{[i]}] = [\mathbf{P}^{(1)}|\mathbf{q}^{(1)}]$  with  $\mathbf{q}^{(1)} = -\mathbf{P}^{(1)}\mathbf{F}_{[i]}/m$ .*

PROOF. Since  $\mathbf{u} = (\mathbf{v}, \min(\mathbf{v}))$  we have  $\deg(\mathbf{p}) \leq \deg(q)$ , and since  $\deg(\mathbf{F}) < \deg(m)$  the degree of  $\mathbf{p}\mathbf{F} + \mathbf{q}m$  is  $\deg(q) + \sigma$ ; then  $\mathbf{p}\mathbf{F} + \mathbf{q}m = 0 \pmod{X^{2\sigma}}$  implies  $\deg(q) + \sigma \geq 2\sigma$ . Now, since  $\mathbf{A}$  is in  $\mathbf{v}$ -Popov form and  $\deg(\mathbf{A}) \leq 2\sigma - \deg(q) < 2\sigma$ , from  $\min(\mathbf{s}^{(2)}) \geq \max(\mathbf{d}) + 2\sigma$  we get  $\mathbf{A}_{[i,i,i+1]} = \mathbf{0}$ . Besides,  $\mathbf{p}_{[i+1:i]} = \mathbf{0}$  since either  $\deg(q) < 2\sigma$  and then  $\min(\mathbf{s}^{(2)}) > \min(\mathbf{d}) + \deg(q)$ , or  $\mathbf{A}$  is the identity matrix and then  $\mathbf{p} = \mathbf{0}$ .

Furthermore, by Lemma 2.1  $[\mathbf{P}^{(1)}|\mathbf{q}^{(1)}]$  is the  $(\mathbf{d}, \min(\mathbf{d}))$ -Popov nullspace basis for  $[\mathbf{F}^\top|\mathbf{m}]^\top$ , with  $(\mathbf{d}, \min(\mathbf{d}))$ -pivot index  $\{1, \dots, i\}$ ,  $(\mathbf{d}, \min(\mathbf{d}))$ -pivot degree  $\boldsymbol{\delta}^{(1)}$  and degree at most  $\max(\boldsymbol{\delta}^{(1)})$ . Then, as in the proof of Lemma 2.2, one can show that  $[\mathbf{A}_{[i,i,i]}|\mathbf{q}_{[i]}] = [\mathbf{P}^{(1)}|\mathbf{q}^{(1)}]$ .  $\square$

Thus, up to row and column permutations this order basis is  $\begin{bmatrix} \mathbf{P}^{(1)} & \mathbf{0} & \mathbf{q}^{(1)} \\ * & \mathbf{P}^{(2)} & * \\ * & * & q \end{bmatrix}$  with  $\mathbf{P}^{(2)} = \mathbf{A}_{[i+1:i+1]} \in \mathbb{K}[X]^{(m-i) \times (m-i)}$  in  $\mathbf{s}^{(2)}$ -Popov form; let  $\boldsymbol{\delta}^{(2)}$  denote its  $\mathbf{s}^{(2)}$ -pivot degree.

LEMMA 2.9. *Let  $\mathbf{n} = X^{-2\sigma}(\mathbf{p}_{[i+1:i]}\mathbf{F}_{[i+1:i]} + \mathbf{q}m) \in \mathbb{K}[X]$  and  $\mathbf{G} = X^{-2\sigma}(\mathbf{A}_{[i+1:i,i]} \mathbf{F} + \mathbf{q}_{[i+1:i]}\mathbf{m}) \in \mathbb{K}[X]^{(m-i) \times 1}$ . Then,  $\deg(\mathbf{G}) < \deg(\mathbf{n}) \leq \sigma - |\boldsymbol{\delta}^{(1)}| - |\boldsymbol{\delta}^{(2)}|$ . Let  $\mathbf{P}^{(3)}$  be the  $\mathbf{t}$ -Popov solution basis for  $(\mathbf{n}, \mathbf{G})$  with  $\mathbf{t} = \text{rdeg}_{\mathbf{s}^{(2)}}(\mathbf{P}^{(2)})$  and  $\boldsymbol{\delta}^{(3)}$  be its  $\mathbf{t}$ -pivot degree. Then,  $(\boldsymbol{\delta}_{[i]}, \boldsymbol{\delta}_{[i+1:i]}) = (\boldsymbol{\delta}^{(1)}, \boldsymbol{\delta}^{(2)} + \boldsymbol{\delta}^{(3)})$ .*

PROOF. The sum  $|\boldsymbol{\delta}^{(1)}| + |\boldsymbol{\delta}^{(2)}| + \deg(q)$  of the  $\mathbf{u}$ -pivot degrees of  $\begin{bmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{bmatrix}$  is at most the order  $2\sigma$ . Thus, we have  $\deg(\mathbf{n}) = \deg(q) - \sigma \leq \sigma - |\boldsymbol{\delta}^{(1)}| - |\boldsymbol{\delta}^{(2)}|$ ,  $\deg(\mathbf{A}_{[i+1:i,i]}) < |\boldsymbol{\delta}^{(1)}| \leq \sigma$ ,  $\deg(\mathbf{A}_{[i+1:i,i+1]}) \leq |\boldsymbol{\delta}^{(2)}| \leq \sigma$ , and  $\deg(\mathbf{q}_{[i+1:i]}) < \deg(q)$ . This implies  $\deg(\mathbf{G}) < \deg(q) - \sigma = \deg(\mathbf{n})$ .

Let  $\mathbf{q}^{(3)} = -\mathbf{P}^{(3)}\mathbf{G}/\mathbf{n}$  and  $t = \text{rdeg}_{\mathbf{u}}([\mathbf{p}|q]) = \deg(q) + \min(\mathbf{d}) \leq \min(\mathbf{s}^{(2)}) \leq \min(\mathbf{t})$ . By Lemma 2.1,  $[\mathbf{P}^{(3)}|\mathbf{q}^{(3)}]$  is the  $(\mathbf{t}, t)$ -Popov nullspace basis for  $[\mathbf{G}^\top|\mathbf{n}]^\top$ . Defining  $\mathbf{B} \in \mathbb{K}[X]^{m \times m}$  and  $\mathbf{c} \in \mathbb{K}[X]^{m \times 1}$  by  $\begin{bmatrix} \mathbf{B}_{[i,i,i]} & \mathbf{B}_{[i,i,i+1]} & \mathbf{c}_{[i]} \\ \mathbf{B}_{[i+1:i,i]} & \mathbf{B}_{[i+1:i,i+1]} & \mathbf{c}_{[i+1:i]} \end{bmatrix} =$

$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{(3)} & \mathbf{q}^{(3)} \end{bmatrix}$ , then  $[\mathbf{B} \ \mathbf{c}] \begin{bmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{bmatrix}$  is a  $\mathbf{u}$ -minimal nullspace basis of  $[\mathbf{F}^\top|\mathbf{m}]^\top$  [36, Theorem 3.9]. Thus Lemma 2.1 implies that  $\tilde{\mathbf{P}} = [\mathbf{B} \ \mathbf{c}] \begin{bmatrix} \mathbf{A} \\ \mathbf{p} \end{bmatrix}$  is a  $\mathbf{v}$ -minimal solution basis for  $(\mathbf{m}, \mathbf{F})$ .

It is easily checked that  $\tilde{\mathbf{P}}$  is in  $\mathbf{v}$ -Popov form, so that the  $\mathbf{v}$ -Popov form of  $\tilde{\mathbf{P}}$  is  $\mathbf{P}$  and its  $\mathbf{v}$ -pivot degree is  $\boldsymbol{\delta}$ . Besides  $\begin{bmatrix} \mathbf{P}_{[i,i,i]} & \mathbf{P}_{[i,i,i+1]} \\ \mathbf{P}_{[i+1:i,i]} & \mathbf{P}_{[i+1:i,i+1]} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{(3)} & \mathbf{P}^{(1)} \\ \mathbf{A}_{2,1} + \mathbf{q}^{(3)}\mathbf{A}_{3,1} & \mathbf{P}^{(3)}\mathbf{P}^{(2)} \end{bmatrix}$ , so that  $(\boldsymbol{\delta}_{[i]}, \boldsymbol{\delta}_{[i+1:i]}) = (\boldsymbol{\delta}^{(1)}, \boldsymbol{\delta}^{(2)} + \boldsymbol{\delta}^{(3)})$  [21, Section 3].  $\square$

This results in Algorithm 2. It takes as input  $\alpha$  which dictates the amplitude of the subtuples that partition  $\mathbf{s}$ ; as mentioned above, the initial call can be made with  $\alpha = 2\sigma$ .

PROPOSITION 2.10. *Algorithm POLMODSYSONE is correct and uses  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  operations in  $\mathbb{K}$ .*

PROOF. The correctness follows from the results in this subsection. By [21, Theorem 1.4], each leaf of the recursion at Step 1.a in dimension  $m$  uses  $\tilde{\mathcal{O}}(m^{\omega-1}\alpha)$  operations.

ALGORITHM 2 (POLMODSYSONE).

*Input:* a polynomial  $\mathbf{m} \in \mathbb{K}[X]_{\neq 0}$  of degree  $\sigma$ , a column  $\mathbf{F} \in \mathbb{K}[X]^{m \times 1}$  with  $\deg(\mathbf{F}) < \deg(m)$ , a shift  $\mathbf{s} \in \mathbb{Z}^m$ , a parameter  $\alpha \in \mathbb{Z}_{>0}$  with  $\alpha \geq 2\sigma$ .

*Output:* the  $\mathbf{s}$ -Popov solution basis for  $(\mathbf{m}, \mathbf{F})$  and the  $\mathbf{s}$ -minimal degree  $\boldsymbol{\delta}$  of  $(\mathbf{m}, \mathbf{F})$ .

1. If  $\text{amp}(\mathbf{s}) \leq 2\alpha$ :
  - a.  $\mathbf{A} \leftarrow$  the  $(\mathbf{s}, \min(\mathbf{s}))$ -Popov order basis for  $[\mathbf{F}^\top|\mathbf{m}]^\top$  and  $2\alpha + 2\sigma$ ; return the principal  $m \times m$  submatrix of  $\mathbf{A}$  and the degrees of its diagonal entries
2. Else: /\*  $\ell = 1 + \lfloor \text{amp}(\mathbf{s})/\alpha \rfloor \geq 3$  \*/
  - a.  $(\mathbf{s}_1, \dots, \mathbf{s}_\ell) \leftarrow \text{PARTITION}(\mathbf{s}, \alpha)$ ,  
 $j \leftarrow$  sum of the lengths of  $\mathbf{s}_1, \dots, \mathbf{s}_{\lfloor \ell/2 \rfloor}$ ,  $\mathbf{s}^{(0)} \leftarrow \mathbf{s}_{[j]}$ ,  
 $(\mathbf{P}^{(0)}, \boldsymbol{\delta}^{(0)}) \leftarrow \text{POLMODSYSONE}(\mathbf{m}, \mathbf{F}_{[j]}, \mathbf{s}^{(0)}, \alpha)$
  - b.  $i \leftarrow$  the largest splitting index for  $(\boldsymbol{\delta}^{(0)}, \mathbf{s}^{(0)})$ ,  $\boldsymbol{\delta}^{(1)} \leftarrow \boldsymbol{\delta}_{[i]}^{(0)}$ ,  $\mathbf{s}^{(2)} \leftarrow \mathbf{s}_{[i+1:i]}$ ,  $\mathbf{d} = -\boldsymbol{\delta}^{(1)} + \min(\mathbf{s}^{(2)}) - 2\sigma$ ,  $\mathbf{v} \in \mathbb{Z}^m$  with  $[\mathbf{v}_{[i]}|\mathbf{v}_{[i+1:i]}] \leftarrow [\mathbf{d}|\mathbf{s}^{(2)}]$ ,  $\mathbf{u} = (\mathbf{v}, \min(\mathbf{d}))$
  - c.  $\begin{bmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{bmatrix} \leftarrow$   $\mathbf{u}$ -Popov order basis for  $[\mathbf{F}^\top|\mathbf{m}]^\top$  and  $2\sigma$ ,  
 $\boldsymbol{\delta}^{(2)} \leftarrow$  the  $\mathbf{s}^{(2)}$ -pivot degree of  $\mathbf{A}_{[i+1:i,i+1]}$ ,  
 $\mathbf{G} \leftarrow X^{-2\sigma}(\mathbf{A}_{[i+1:i,i]} \mathbf{F} + \mathbf{q}_{[i+1:i]}\mathbf{m})$ ,  
 $\mathbf{n} \leftarrow X^{-2\sigma}(\mathbf{p}_{[i+1:i]} \mathbf{F}_{[i+1:i]} + \mathbf{q}m)$ .
  - d.  $\mathbf{t} \leftarrow \mathbf{s}^{(2)} + \boldsymbol{\delta}^{(2)} = \text{rdeg}_{\mathbf{s}^{(2)}}(\mathbf{A}_{[i+1:i,i+1]})$ ,  
 $(\mathbf{P}^{(3)}, \boldsymbol{\delta}^{(3)}) \leftarrow \text{POLMODSYSONE}(\mathbf{n}, \mathbf{G}, \mathbf{t}, \alpha)$
  - e.  $\boldsymbol{\delta} \in \mathbb{N}^m$  with  $(\boldsymbol{\delta}_{[i]}, \boldsymbol{\delta}_{[i+1:i]}) \leftarrow (\boldsymbol{\delta}^{(1)}, \boldsymbol{\delta}^{(2)} + \boldsymbol{\delta}^{(3)})$ ,  
 $\mathbf{P} \leftarrow \text{KNOWNDEGPOLMODSYS}(\mathbf{m}, \mathbf{F}, \mathbf{s}, \boldsymbol{\delta})$
  - f. Return  $(\mathbf{P}, \boldsymbol{\delta})$

Running the algorithm with initial input  $\alpha = 2\sigma$ , the recursive tree has depth  $\mathcal{O}(\log(\ell)) = \mathcal{O}(\log(1 + \text{amp}(\mathbf{s})/2\sigma))$ , with  $\text{amp}(\mathbf{s})/2\sigma \in \mathcal{O}(m^2)$  [21, Appendix A]. All recursive calls are for a modulus of degree  $\sigma < \alpha$ . The order basis computation at Step 2.c uses  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  operations; the computation of  $\mathbf{G}$  and  $\mathbf{n}$  at Step 2.c can be done in time  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  using partial linearization as in Lemma 2.11 below; Step 2.e uses  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  operations by Proposition 2.4.

On a given level of the tree, the sum of the dimensions of the column vector in input of each sub-problem is in  $\mathcal{O}(m)$ . Since  $a^{\omega-1} + b^{\omega-1} \leq (a+b)^{\omega-1}$  for all  $a, b > 0$ , each level of the tree uses a total of  $\tilde{\mathcal{O}}(m^{\omega-1}\alpha)$  operations.  $\square$

## 2.3 Fast divide-and-conquer algorithm

Now that we have an efficient algorithm for  $n = 1$ , our main algorithm uses a divide-and-conquer approach on  $n$ . Similarly to [21, Algorithm 1], from the two bases obtained recursively we first deduce the  $\mathbf{s}$ -minimal degree  $\boldsymbol{\delta}$ , and then we use this knowledge to compute  $\mathbf{P}$  with Algorithm 1. When  $\sigma = \deg(\mathbf{m}_1) + \dots + \deg(\mathbf{m}_n) \in \mathcal{O}(m)$ , we rely on the algorithm LINEARIZATIONMIB in [20, Algorithm 9].

The computation of the so-called *residual* at Step 3.c can be done efficiently using partial linearization, as follows.

LEMMA 2.11. *Let  $\mathfrak{M} = (\mathbf{m}_j)_j \in \mathbb{K}[X]_{\neq 0}^n$ ,  $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ ,  $\mathbf{F} \in \mathbb{K}[X]^{m \times n}$  with  $m \geq n$  and  $\deg(\mathbf{F}_{*,j}) < \sigma_j = \deg(\mathbf{m}_j)$ , and let  $\sigma \geq m$  such that  $\sigma \geq \sigma_1 + \dots + \sigma_n$  and  $|\text{cdeg}(\mathbf{P})| \leq \sigma$ . Then  $\mathbf{P}\mathbf{F} \pmod{\mathfrak{M}}$  can be computed in  $\tilde{\mathcal{O}}(m^{\omega-1}\sigma)$  operations.*

PROOF. Using notation from Lemma 2.3, we let  $\tilde{\mathbf{P}} \in \mathbb{K}[X]^{m \times \tilde{m}}$  such that  $\mathbf{P} = \tilde{\mathbf{P}}\mathcal{E}$  and  $\deg(\tilde{\mathbf{P}}) < \lceil |\text{cdeg}(\mathbf{P})|/m \rceil$ . As above,  $\tilde{\mathbf{F}} = \mathcal{E}\mathbf{F} \pmod{\mathfrak{M}}$  can be computed in time  $\tilde{\mathcal{O}}(m\sigma)$ . Here we want to compute  $\mathbf{P}\mathbf{F} \pmod{\mathfrak{M}} = \tilde{\mathbf{P}}\tilde{\mathbf{F}} \pmod{\mathfrak{M}}$ .







- for  $1 \leq i \leq m$  and  $2 \leq j \leq \alpha_i - 1$ , the row  $m + (\alpha_1 - 1) + \dots + (\alpha_{i-1} - 1) + j$  of  $\mathcal{L}_\delta^c(\mathbf{A})$  is  $[0, \dots, 0, -X^\delta, 1, 0, \dots, 0]$  where 1 is on the diagonal.

Defining the row partial linearization  $\mathcal{L}_\delta^r(\mathbf{A})$  of  $\mathbf{A}$  similarly, both linearizations are related by  $\mathcal{L}_\delta^r(\mathbf{A}) = \mathcal{L}_\delta^c(\mathbf{A}^\top)^\top$ .

Now we show that for a well-chosen  $\mathbf{u}$ , one can directly read the  $\mathbf{s}$ -Popov form of  $\mathbf{A}$  as a submatrix of the  $\mathbf{u}$ -Popov form of  $\mathcal{L}_\delta^r(\mathbf{A})$  (resp.  $\mathcal{L}_\delta^c(\mathbf{A})$ ).

LEMMA 3.4. Let  $\mathbf{A} \in \mathbb{K}[X]^{m \times m}$  be nonsingular,  $\mathbf{s} \in \mathbb{Z}^m$ ,  $\mathbf{P}$  be the  $\mathbf{s}$ -Popov form of  $\mathbf{A}$ , and  $\delta \in \mathbb{N}^m$ . We have that:

- (i) if  $\tilde{m}$  is the dimension of  $\mathcal{L}_\delta^r(\mathbf{A})$  and  $\mathbf{u} = (\mathbf{s}, t, \dots, t)$  is in  $\mathbb{Z}^{\tilde{m}}$  with  $t \geq \max(\mathbf{s}) + \deg(\mathbf{P})$ , then the  $\mathbf{u}$ -Popov form of  $\mathcal{L}_\delta^r(\mathbf{A})$  is  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ * & \mathbf{I} \end{bmatrix}$ ;
- (ii) if  $\tilde{m}$  is the dimension of  $\mathcal{L}_\delta^c(\mathbf{A})$ ,  $\mathbf{E}$  is as in (4), and  $\mathbf{u} = (\mathbf{s}, \mathbf{t}) \in \mathbb{Z}^{\tilde{m}}$  for any  $\mathbf{t} \in \mathbb{Z}^{\tilde{m}-m}$ , then the  $\mathbf{u}$ -Popov form of  $\mathcal{L}_\delta^c(\mathbf{A})$  is  $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E} & \mathbf{I} \end{bmatrix}$  is  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ ;
- (iii) if  $\tilde{m}$  is the dimension of  $\mathcal{L}_\delta^c(\mathbf{A})$  and  $\mathbf{u} = (\mathbf{s}, t, \dots, t)$  is in  $\mathbb{Z}^{\tilde{m}}$  with  $t \geq \max(\mathbf{s}) + \deg(\mathbf{P})$ , then the  $\mathbf{u}$ -Popov form of  $\mathcal{L}_\delta^c(\mathbf{A})$  is  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ * & \mathbf{I} \end{bmatrix}$ .

PROOF. (i)  $\mathcal{L}_\delta^r(\mathbf{A})$  is left-unimodularly equivalent to  $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{bmatrix}$  for some  $\mathbf{B} \in \mathbb{K}[X]^{(\tilde{m}-m) \times m}$  [16, Theorem 10 (i)]. Then, let  $\mathbf{R}$  be the remainder of  $\mathbf{B}$  modulo  $\mathbf{P}$ , that is, the unique matrix in  $\mathbb{K}[X]^{(\tilde{m}-m) \times m}$  which has column degree bounded by the column degree of  $\mathbf{P}$  componentwise and such that  $\mathbf{R} = \mathbf{B} + \mathbf{Q}\mathbf{P}$  for some matrix  $\mathbf{Q}$  (see for example [22, Theorem 6.3-15], noting that  $\mathbf{P}$  is  $\mathbf{0}$ -column reduced).

Let  $\mathbf{W}$  denote the unimodular matrix such that  $\mathbf{P} = \mathbf{W}\mathbf{A}$ . Then,  $\begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{Q}\mathbf{W} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{bmatrix}$  is left-unimodularly equivalent to  $\mathcal{L}_\delta^r(\mathbf{A})$ . Besides, since  $\deg(\mathbf{R}) < \deg(\mathbf{P})$ , we have that  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{bmatrix}$  is in  $\mathbf{u}$ -Popov form by choice of  $t$ .

(ii) The matrix  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$  is obviously in  $\mathbf{u}$ -Popov form: it remains to prove that it is left-unimodularly equivalent to  $\mathcal{L}_\delta^c(\mathbf{A}) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E} & \mathbf{I} \end{bmatrix}$ . Let  $\mathbf{T}$  denote the trailing principal submatrix  $\mathbf{T} = \mathcal{L}_\delta^c(\mathbf{A})_{m+1 \dots \tilde{m}, m+1 \dots \tilde{m}}$ , and let  $\mathbf{W}$  be the unimodular matrix such that  $\mathbf{W}\mathbf{P} = \mathbf{A}$ . Then,  $\mathbf{T}$  is unit lower triangular, thus unimodular, and by construction of  $\mathcal{L}_\delta^c(\mathbf{A})$ , for some matrix  $\mathbf{B}$  we have  $\mathcal{L}_\delta^c(\mathbf{A}) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ .

(iii) From (ii),  $\mathcal{L}_\delta^c(\mathbf{A})$  is left-unimodularly equivalent to  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{E} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ -\mathbf{E} & \mathbf{I} \end{bmatrix}$ . Using arguments in the proof of (i) above, by choice of  $t$  the  $\mathbf{u}$ -Popov form of  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ -\mathbf{E} & \mathbf{I} \end{bmatrix}$  is  $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{bmatrix}$  with  $\mathbf{R}$  the remainder of  $-\mathbf{E}$  modulo  $\mathbf{P}$ .  $\square$

In the usual case where  $\deg(\mathbf{P})$  is not known a priori, one may choose  $t$  using the inequality  $\deg(\mathbf{P}) \leq \deg(\det(\mathbf{P})) = \deg(\det(\mathbf{A})) \leq m \deg(\mathbf{A})$ .

This result implies Proposition 3.2 thanks to the following remark from [16]. Let  $\pi_1, \pi_2$  be permutation matrices such that  $\mathbf{B} = \pi_1 \mathbf{A} \pi_2 = [b_{i,j}]_{i,j}$  satisfies  $\deg(b_{i,i}) \geq \deg(b_{j,k})$  for all  $j, k \geq i$  and  $1 \leq i \leq m$ . Defining  $\mathbf{d} = (d_i)_i \in \mathbb{N}^m$  by  $d_i = \overline{\deg(b_{i,i})} = \begin{cases} \deg(b_{i,i}) & \text{if } b_{i,i} \neq 0 \\ 0 & \text{otherwise} \end{cases}$ , we have  $d_1 + \dots + d_m \leq \sigma(\mathbf{A})$  by definition of  $\sigma(\mathbf{A})$  in (1). Let  $\delta = \pi_1^{-1} \mathbf{d}$ , where  $\mathbf{d}$  is seen as a column vector, and  $\gamma = \text{cdeg}(\mathcal{L}_\delta^r(\mathbf{A}))$ . Then the matrix  $\tilde{\mathbf{A}} = \mathcal{L}_\gamma^r(\mathcal{L}_\delta^r(\mathbf{A}))$  is  $\tilde{m} \times \tilde{m}$  with  $\tilde{m} < 3m$ , and we have  $\deg(\tilde{\mathbf{A}}) \leq \lceil \sigma(\mathbf{A})/m \rceil$  [16, Corollary 3]. Lemma 3.4 further shows that the  $\mathbf{s}$ -Popov form of  $\mathbf{A}$  is the principal  $m \times m$  submatrix of the  $\mathbf{u}$ -Popov form of  $\tilde{\mathbf{A}}$ , for the shift  $\mathbf{u} = (\mathbf{s}, t, \dots, t) \in \mathbb{Z}^{\tilde{m}}$  with  $t = \max(\mathbf{s}) + m \deg(\mathbf{A})$ .

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