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Hydrodynamic Limit for the Velocity-Flip Model

Marielle Simon*

Abstract

We study the diffusive scaling limit for a chain of N coupled oscillators. In order to provide the system with good ergodic properties, we perturb the Hamiltonian dynamics with random flips of velocities, so that the energy is locally conserved. We derive the hydrodynamic equations by estimating the relative entropy with respect to the local equilibrium state, modified by a correction term.

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Introduction

This paper aims at proving the hydrodynamic limit for a Hamiltonian system of N coupled oscillators. The ergodic properties of Hamiltonian dynamics are poorly understood, especially when the size of the system goes to infinity. That is why we perturb it by an additional conservative mixing noise, as it has been proposed for the first time by Olla, Varadhan and Yau [16] in the context of gas dynamics, and then in [11] in the context of Hamiltonian lattice dynamics (see e.g. [1], [2], [6], [7], [3], [5], [10], [15] for more recent related works).

We are interested in the macroscopic behavior of this system as N goes to infinity, after rescaling space and time with the diffusive scaling. The system is considered under periodic boundary conditions – more precisely we work on the one-dimensional discrete torus $\mathbb{T}_N := \{0, ..., N-1\}$. The configuration space is denoted by $\Omega_N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$. A typical configuration is given by $\omega = (p_x, r_x)_{x \in \mathbb{T}_N}$ where p_x stands for the velocity of the oscillator at site x, and r_x represents the distance between oscillator x and oscillator $x+1$. The deterministic dynamics is described by the harmonic Hamiltonian

$$\mathcal{H}_N = \sum_{x=0}^{N-1} \left[\frac{p_x^2}{2} + \frac{r_x^2}{2} \right] . \tag{0.1}$$

The stochastic perturbation is added only to the velocities, in such a way that the energy of particles is still conserved. Nevertheless, the momentum conservation is no longer valid. The added noise can be easily described: each particle independently waits an exponentially distributed time interval and then flips the sign of velocity. The strength of the noise is regulated by the parameter $\gamma > 0$. The total deformation $\sum r_x$ and the total energy $\sum (p_x^2 + r_x^2)/2$ are the only two conserved quantities.

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Thus, the Gibbs states are parametrized by two potentials, temperature and tension: for $\beta > 0$ and $\lambda \in \mathbb{R}$, the equilibrium Gibbs measures $\mu^N_{\beta,\lambda}$ on the configuration space $\Omega^N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$ are given by the product measures

$$d\mu^N_{\beta,\lambda} = \prod_{x \in \mathbb{T}_N} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta,\lambda)} dr_x dp_x \ , \tag{0.2}$$

where $e_x := (p_x^2 + r_x^2)/2$ is the energy of the particle at site x, and $Z(\beta,\lambda)$ is the normalization constant. The temperature is equal to β^{-1} and the tension is given by λ/β.

The goal is to prove that the two empirical profiles associated to the conserved quantities converge in the thermodynamic limit $N \to \infty$ to the macroscopic profiles $\mathbf{r}(t,\cdot)$ and $\mathbf{e}(t,\cdot)$ which satisfy an autonomous system of coupled parabolic equations. Let $\mathbf{r}_0 : \mathbb{T} \to \mathbb{R}$ and $\mathbf{e}_0 : \mathbb{T} \to \mathbb{R}$ be respectively the initial macroscopic deformation profile and the initial macroscopic energy profile defined on the one-dimensional torus $\mathbb{T} = [0,1]$. We want to show that the functions $\mathbf{r}(t,q)$ and $\mathbf{e}(t,q)$ defined on $\mathbb{R}_+ \times \mathbb{T}$ are solutions of

$$\begin{cases} \partial_t \mathbf{r} = \dfrac{1}{\gamma} \partial_q^2 \mathbf{r} \ , \\ \partial_t \mathbf{e} = \dfrac{1}{2\gamma} \partial_q^2 \left(\mathbf{e} + \dfrac{\mathbf{r}^2}{2} \right) \ , \end{cases} \qquad q \in \mathbb{T}, \ t \in \mathbb{R} \ , \tag{0.3}$$

with the initial conditions $\mathbf{r}(0,\cdot) = \mathbf{r}_0(\cdot)$ and $\mathbf{e}(0,\cdot) = \mathbf{e}_0(\cdot)$.

We approach this problem by using the relative entropy method, introduced for the first time by H. T. Yau [19] for a gradient [1] diffusive Ginzburg-Landau dynamics. For non-gradient models, Varadhan [18] has proposed an effective approach. Funaki et al. followed his ideas in [12] to extend the relative entropy method to some non-gradient processes and introduced the concept of local equilibrium state of second order approximation.

The usual relative entropy method works with two time-dependent probability measures. Let us denote by μ^N_0 the Gibbs local equilibrium associated to a deformation profile $\mathbf{r}_0$ and an energy profile $\mathbf{e}_0$ (see (1.8) for the explicit formula). As we work in the diffusive scaling, we look at the state of the process at time tN^2. We denote it by μ^N_t and we suppose that it starts from μ^N_0. Let $\mu^N_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}$ be the Gibbs local equilibrium associated to the profiles $\mathbf{r}(t,\cdot)$ and $\mathbf{e}(t,\cdot)$ which satisfy (0.3) [2]. If we denote by f^N_t and ϕ^N_t, respectively, the densities[3] of μ^N_t and $\mu^N_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}$ with respect to a reference equilibrium measure $\mu^N_* := \mu^N_{1,0}$, we guess that ϕ^N_t is a good approximation of the unknown density f^N_t. We measure the distance between these two densities by their relative entropy

$$H_N(t) := \int_{\Omega^N} f^N_t(\omega) \log \frac{f^N_t(\omega)}{\phi^N_t(\omega)} \, d\mu^N_*(\omega) \ . \tag{0.4}$$

Then, the strategy consists in proving that

$$\lim_{N \to \infty} \frac{H_N(t)}{N} = 0 \ , \tag{0.5}$$

and deducing that the hydrodynamic limit holds (for this last step, see [13], [16] or [4]). In the context of diffusive systems, the relative entropy method works if the following conditions are satisfied.

[1] A conservative system is called gradient if the currents corresponding to the conserved quantities are gradients.

[2] For the sake of readibility, in the following sections we will denote it by $\mu^N_{\beta_t(\cdot),\lambda_t(\cdot)}$, where $\beta_t(\cdot)$ and $\lambda_t(\cdot)$ are the two potential profiles associated to $\mathbf{r}(t,\cdot)$ and $\mathbf{e}(t,\cdot)$ (see (1.5) and (1.8)).

[3] The existence of these two densities is justified in Section 2.1.

- First, the dynamics has to be *ergodic*: the only time and space invariant measures for the infinite system, with finite local entropy, are given by mixtures of the Gibbs measures in infinite volume $\mu_{\beta,\lambda}$ (see (1.15)). From [11], we know that the velocity-flip model is ergodic in the sense above (see Theorem 1.3 for a precise statement).

- Next, we need to establish the so-called *fluctuation-dissipation equations* in the mathematics literature. Such equations express the microscopic current of energy (which here is not a discrete gradient) as the sum of a discrete gradient and a fluctuating term. More precisely, the microscopic current of energy, denoted by $j_{x,x+1}$, is defined by the local energy conservation law
$$\mathcal{L}e_x = \nabla j_{x-1,x} \tag{0.6}$$
where $\mathcal{L}$ is the generator of the infinite dynamics. The standard approach consists in proving that there exist functions f_x and h_x such that the following decomposition holds
$$j_{x,x+1} = \nabla f_x + \mathcal{L}h_x \ . \tag{0.7}$$
Equation (0.7) is called a microscopic fluctuation-dissipation equation. The term $\mathcal{L}h_x$, when integrated in time, is a martingale. Roughly speaking, $\mathcal{L}h_x$ represents rapid fluctuation, whereas ∇f_x represents dissipation. Gradient models are systems for which $h_x = 0$ with the previous notations. In general, these equations are not explicit but we are able to compute them in our model (see (A.16) and (A.17)).

- Finally, since we observe the system on a diffusive scale and the system is non-gradient, we need second order approximations. If we want to obtain the entropy estimate (0.5) of order $o(N)$, we can not work with the measure $\mu^N_{\mathbf{e}(t,\cdot),\mathbf{r}(t,\cdot)}$: we have to correct the Gibbs local equilibrium state with a small term. This idea was first introduced in [12] and then used in [17] for interacting Ornstein-Uhlenbeck processes, and in [14] for the asymmetric exclusion process in the diffusive scaling. However, as far as we know, it is the first time that this is applied for a system with several conservation laws.

Recently, Even et al. [10] used the relative entropy method for a stochastically perturbed Hamiltonian dynamics which is quite close to the dynamics of this paper: the time evolution is governed by the same Hamiltonian of anharmonic oscillators but the process is perturbed by a different noise – velocities are exchanged and not flipped. Besides, the boundary conditions are mechanical instead of periodic. Contrary to this paper, the model is studied in the hyperbolic scale, so that the authors do not need to modify the local equilibrium state.

Up to present, the derivation of hydrodynamic equations for the harmonic oscillators perturbed by the velocity-flip noise is not rigorously achieved (see e.g. [7]), mainly because the control of large energies has not been considered so far. Indeed, to perform the relative entropy method, we need to control the moments
$$\int \left[\frac{1}{N} \sum_{x \in \mathbb{T}_N} |p_x|^k \right] d\mu^N_t \ , \tag{0.8}$$
for all $k \geqslant 1$, uniformly in time and with respect to N. In fact, the only first several moments are necessary to cut-off large energies (as it is explained in Section 2.2) and we need all the others to obtain the Taylor expansion which appears in the relative entropy method (see Proposition 2.1 and

Lemma A.2). Usually, the entropy inequality (2.24) reduces the control of (0.8) to the estimate of the following equilibrium exponential moments

$$\int \exp(\delta |p_x|^k)\ d\mu_{1,0}^N$$

with $\delta > 0$ small. Unfortunately, in our model, these integrals are infinite for all $k \geqslant 3$ and all $\delta > 0$.

To avoid this problem, we could cut-off large velocities by taking a relativistic kinetic energy (as done in [16]). Nevertheless, we should change the physics of the problem by modifying the Liouville operator, and consequently the fluctuation-dissipation equations would not be available any more. Similar difficulties have already appeared in other models: in [8], Bertini et al. do not have these precious exponential moments to derive rigorously their results. In an other context, Bonetto et al. [9] study the heat conduction in anharmonic crystals with self-consistent reservoirs, and need energy bounds to complete their results. Bernardin [2] deals with a harmonic chain perturbed by a stochastic noise which is different from ours but has the same motivation: energy is conserved, momentum is not. He derives the hydrodynamic limit for a particular value of the intensity of the noise. In this case the hydrodynamic equations are simply given by two decoupled heat equations. The author highlights that good energy bounds are necessary to extend his work to other values of the noise intensity. In fact, only the following weak form is proved in his paper:

$$\lim_{N\to+\infty} \int \left[\frac{1}{N^2} \sum_{x\in\mathbb{T}_N} p_x^4 \right] d\mu_t^N = 0\ . \tag{0.9}$$

In this work, we get uniform control of (0.8) (Theorem 1.2), thanks to a remarkable property of our model: the set of convex combinations of Gaussian measures is preserved by the dynamics. This is one of the main technical novelties in our work.

The next section contains a more precise description of the results outlined here, along with the plan of the paper.

1 The Model and the Main Results

1.1 Velocity-flip Model

We consider the unpinned harmonic chain perturbed by the momentum-flip noise. Each particle has the same mass that we set equal to 1. The configuration space is denoted by $\Omega^N := (\mathbb{R}\times\mathbb{R})^{\mathbb{T}_N}$.

A typical configuration is $\omega = (\mathbf{r}, \mathbf{p}) \in \Omega^N$, where $\mathbf{r} = (r_x)_{x\in\mathbb{T}_N}$ and $\mathbf{p} = (p_x)_{x\in\mathbb{T}_N}$.

The generator of the dynamics is given by $\mathcal{L}_N := \mathcal{A}_N + \gamma\mathcal{S}_N$, where, for any continuously differentiable function $f : \Omega^N \to \mathbb{R}$,

$$\mathcal{A}_N(f) := \sum_{x\in\mathbb{T}_N} \left[(p_{x+1} - p_x)\ \partial_{r_x} f + (r_x - r_{x-1})\ \partial_{p_x} f \right] \tag{1.1}$$

and

$$\mathcal{S}_N(f)(\mathbf{r}, \mathbf{p}) := \frac{1}{2} \sum_{x\in\mathbb{T}_N} \left[f(\mathbf{r}, \mathbf{p}^x) - f(\mathbf{r}, \mathbf{p}) \right]\ . \tag{1.2}$$

Here $\mathbf{p}^x$ is the configuration obtained from $\mathbf{p}$ by the flip of p_x into $-p_x$. The parameter $\gamma > 0$ regulates the strength of the random flip of momenta.

The operator $\mathcal{A}_N$ is the Liouville operator of a chain of interacting harmonic oscillators, and $\mathcal{S}_N$ is the generator of the stochastic part of the dynamics that flips at random time the velocity of one particle. The dynamics conserves two quantities: the total deformation of the lattice $\mathcal{R} = \sum_{x\in\mathbb{T}_N} r_x$ and the total energy $\mathcal{E} = \sum_{x\in\mathbb{T}_N} e_x$, where $e_x = (p_x^2 + r_x^2)/2$. Observe that the total momentum is no longer conserved.

The deformation and the energy define a family of invariant measures depending on two parameters. For $\beta > 0$ and $\lambda \in \mathbb{R}$, we denote by $\mu^N_{\beta,\lambda}$ the Gaussian product measure on Ω^N given by

$$\mu^N_{\beta,\lambda}(d\mathbf{r}, d\mathbf{p}) = \prod_{x\in\mathbb{T}_N} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta,\lambda)} dr_x dp_x \ . \tag{1.3}$$

An easy computation gives that the partition function satisfies

$$Z(\beta,\lambda) = \frac{2\pi}{\beta} \exp\left(\frac{\lambda^2}{2\beta}\right) \ . \tag{1.4}$$

In the following, we shall denote by $\mu[\cdot]$ the expectation with respect to the measure μ. We introduce $\mathbb{L}^2(\mu^N_{\beta,\lambda})$, the space of functions f defined on Ω^N such that $\mu^N_{\beta,\lambda}[f^2] < +\infty$. This is a Hilbert space, on which $\mathcal{A}_N$ is antisymmetric and $\mathcal{S}_N$ is symmetric.

The thermodynamic relations between the averages of the conserved quantities $\bar{\mathbf{r}} \in \mathbb{R}$ and $\bar{\mathbf{e}} \in (0, +\infty)$, and the potentials $\beta \in (0, +\infty)$ and $\lambda \in \mathbb{R}$ are given by

$$\begin{cases} \bar{\mathbf{e}}(\beta,\lambda) := \mu^N_{\beta,\lambda}[e_x] = \dfrac{1}{\beta} + \dfrac{\lambda^2}{2\beta^2} \ , \\ \bar{\mathbf{r}}(\beta,\lambda) := \mu^N_{\beta,\lambda}[r_x] = -\dfrac{\lambda}{\beta} \ . \end{cases} \tag{1.5}$$

Let us notice that

$$\forall\ \beta \in (0,+\infty), \forall\ \lambda \in \mathbb{R},\ \bar{\mathbf{e}}(\beta,\lambda) > \frac{\bar{\mathbf{r}}^2(\beta,\lambda)}{2} \ . \tag{1.6}$$

We assume that the system is initially close to a local equilibrium (defined as the following).

Definition 1. *A sequence $(\mu^N)_N$ of probability measures on Ω^N is a* local equilibrium *associated to a deformation profile $\mathbf{r}_0 : \mathbb{T} \to \mathbb{R}$ and an energy profile $\mathbf{e}_0 : \mathbb{T} \to (0, +\infty)$ if for every continuous function $G : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$, we have*

$$\begin{cases} \lim\limits_{N\to\infty} \mu^N \left[\left| \dfrac{1}{N} \displaystyle\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right) r_x - \int_{\mathbb{T}} G(q)\mathbf{r}_0(q)dq \right| > \delta \right] = 0 \ , \\ \lim\limits_{N\to\infty} \mu^N \left[\left| \dfrac{1}{N} \displaystyle\sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right) e_x - \int_{\mathbb{T}} G(q)\mathbf{e}_0(q)dq \right| > \delta \right] = 0 \ . \end{cases} \tag{1.7}$$

Example 1.1. *For any integer N we define the probability measures*

$$\mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}(d\mathbf{r}, d\mathbf{p}) = \prod_{x\in\mathbb{T}_N} \frac{\exp(-\beta_0(x/N)e_x - \lambda_0(x/N)r_x)}{Z(\beta_0(\cdot),\lambda_0(\cdot))} dr_x dp_x \ , \tag{1.8}$$

where $\beta_0(\cdot)$ and $\lambda_0(\cdot)$ are related to $\mathbf{e}_0(\cdot)$ and $\mathbf{r}_0(\cdot)$ by (1.5)

$$\begin{cases} \mathbf{e_0}(\cdot) = \bar{\mathbf{e}}(\beta_0(\cdot),\lambda_0(\cdot)) \ , \\ \mathbf{r_0}(\cdot) = \bar{\mathbf{r}}(\beta_0(\cdot),\lambda_0(\cdot)) \ . \end{cases}$$

Then, the sequence $\left(\mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}\right)_N$ is a local equilibrium, and it is called the Gibbs local equilibrium state *associated to the macroscopic profiles $\mathbf{e}_0$ and $\mathbf{r}_0$. Both profiles are assumed to be continuous.*

To establish the hydrodynamic limit corresponding to the two conservation laws, we look at the process with generator $N^2\mathcal{L}_N$, namely in the diffusive scale. The configuration at time tN^2 is denoted by ω^N_t, and the law of the process $(\omega^N_t)_{t\geqslant 0}$ is denoted by μ^N_t.

1.2 The Thermodynamic Entropy

The function

$$S(\mathbf{e},\mathbf{r}) = \inf_{\beta>0,\lambda\in\mathbb{R}} \{\lambda\mathbf{r} + \beta\mathbf{e} + \log Z(\beta,\lambda)\} \tag{1.9}$$

is called the *thermodynamic entropy*. An easy computation, coming from the explicit expression of the partition function, gives

$$S(\mathbf{e},\mathbf{r}) = 1 + \log(2\pi) + \log\left(\mathbf{e} - \frac{\mathbf{r}^2}{2}\right), \quad \text{when } \mathbf{e} - \frac{\mathbf{r}^2}{2} > 0 \ . \tag{1.10}$$

The relations (1.5) can be inverted according to

$$\lambda(\mathbf{e},\mathbf{r}) = \frac{\partial S(\mathbf{e},\mathbf{r})}{\partial \mathbf{r}} \ , \qquad \beta(\mathbf{e},\mathbf{r}) = \frac{\partial S(\mathbf{e},\mathbf{r})}{\partial \mathbf{e}} \ . \tag{1.11}$$

Remark. These two equalities, together with (1.5), show that there exists a bijection between the two sets $\left\{(\beta,\lambda)\in\mathbb{R}^2 \ ; \ \beta>0\right\}$ and $\left\{(\mathbf{e},\mathbf{r})\in\mathbb{R}^2 \ ; \ \mathbf{e}>\mathbf{r}^2/2\right\}$. From the equations above, the inverted relations can be written as

$$\lambda(\mathbf{e},\mathbf{r}) = -\frac{\mathbf{r}}{\mathbf{e}-\mathbf{r}^2/2} \ , \qquad \beta(\mathbf{e},\mathbf{r}) = \frac{1}{\mathbf{e}-\mathbf{r}^2/2} \ . \tag{1.12}$$

We denote by Ψ the function

$$\begin{array}{rccc} \Psi: & \left\{(\mathbf{e},\mathbf{r})\in\mathbb{R}^2 \ ; \ \mathbf{e}>\mathbf{r}^2/2\right\} & \to & \left\{(\beta,\lambda)\in\mathbb{R}^2 \ ; \ \beta>0\right\} \\ & (\mathbf{e},\mathbf{r}) & \mapsto & \left(\dfrac{1}{\mathbf{e}-\mathbf{r}^2/2}, -\dfrac{\mathbf{r}}{\mathbf{e}-\mathbf{r}^2/2}\right) \ . \end{array}$$

If $\eta = (\mathbf{e},\mathbf{r})$ and $\chi = (\beta,\lambda)$ satisfy the relations (1.5), then η and χ are said *in duality* and we have

$$-S(\mathbf{e},\mathbf{r}) + \log Z(\beta,\lambda) = -\eta\cdot\chi \ . \tag{1.13}$$

Here, the notation $a\cdot b$ stands for the usual scalar product between a and b.

1.3 Hydrodynamic Equations

Let μ and ν be two probability measures on the same measurable space $(X, \mathcal{F})$. We define the relative entropy $H(\mu|\nu)$ of the probability measure μ with respect to the probability measure ν by

$$H(\mu|\nu) = \sup_f \left\{ \int_X f \ d\mu - \log \left(\int_X e^f \ d\nu \right) \right\} \ , \tag{1.14}$$

where the supremum is carried over all bounded measurable functions f on X.

The Gibbs states in infinite volume are the probability measures $\mu_{\beta,\lambda}$ on $\Omega = (\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$ given by

$$\mu_{\beta,\lambda}(d\mathbf{r}, d\mathbf{p}) = \prod_{x \in \mathbb{Z}} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta, \lambda)} dr_x dp_x \ . \tag{1.15}$$

We denote by $\tau_x \varphi$ the shift of φ: $(\tau_x \varphi)(\omega) = \varphi(\tau_x \omega) = \varphi(\omega(x + \cdot \ ))$. In this article the following theorem is proved.

Theorem 1.1. *Let $(\mu_0^N)_N$ be a sequence of probability measures on Ω^N which is a local equilibrium associated to a deformation profile $\mathbf{r}_0$ and an energy profile $\mathbf{e}_0$ such that $\mathbf{e}_0 > \mathbf{r}_0^2/2$ (see (1.8)). We denote by β_0 and λ_0 the potential profiles associated to $\mathbf{r}_0$ and $\mathbf{e}_0$:*

$$(\beta_0, \lambda_0) := \Psi(\mathbf{e}_0, \mathbf{r}_0) \ .$$

We assume that

$$H \left(\mu_0^N | \mu^N_{\beta_0(\cdot), \lambda_0(\cdot)} \right) = o(N) \tag{1.16}$$

and that the initial profiles are continuous.

We also assume that the energy moments are bounded: let us suppose that there exists a positive constant C which does not depend on N and t, such that

$$\forall \ k \geqslant 1, \ \mu_t^N \left[\sum_{x \in \mathbb{T}_N} e_x^k \right] \leqslant (Ck)^k \ N \ . \tag{1.17}$$

Let G be a continuous function on the torus $\mathbb{T}$ and φ be a local function which satisfies the following property: there exists a finite subset $\Lambda \subset \mathbb{Z}$ and a constant $C > 0$ such that, for all $\omega \in \Omega^N$, $\varphi(\omega) \leqslant C \left(1 + \sum_{i \in \Lambda} e_i(\omega)\right)$. Then,

$$\mu_t^N \left[\left| \frac{1}{N} \sum_x G\left(\frac{x}{N}\right) \tau_x \varphi - \int_{\mathbb{T}} G(y) \ \tilde{\varphi}(\mathbf{e}(t,q), \mathbf{r}(t,q)) dq \right| \right] \xrightarrow[N \to \infty]{} 0 \tag{1.18}$$

where $\tilde{\varphi}$ is the grand-canonical expectation of φ: in other words, for any $(\mathbf{e}, \mathbf{r}) \in \mathbb{R}^2$ such that $\mathbf{e} > \mathbf{r}^2/2$, let $(\beta, \lambda) = \Psi(\mathbf{e}, \mathbf{r})$ then

$$\tilde{\varphi}(\mathbf{e}, \mathbf{r}) = \mu_{\beta,\lambda}[\varphi] = \int_{(\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}} \varphi(\omega) \ d\mu_{\beta,\lambda}(\omega) \ . \tag{1.19}$$

Besides, $\mathbf{e}$ *and* $\mathbf{r}$ *are defined on* $\mathbb{R}_+ \times \mathbb{T}$ *and are solutions of*

$$\begin{cases} \partial_t \mathbf{r} = \dfrac{1}{\gamma}\partial_q^2 \mathbf{r} \ , \\ \partial_t \mathbf{e} = \dfrac{1}{2\gamma}\partial_q^2 \left(\mathbf{e} + \dfrac{\mathbf{r}^2}{2}\right) \ , \end{cases} \qquad q \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.20}$$

with the initial conditions $\mathbf{r}(\cdot, 0) = \mathbf{r}_0(\cdot)$ *and* $\mathbf{e}(\cdot, 0) = \mathbf{e}_0(\cdot)$.

Remarks.

1. In order to prove the theorem, we shall show afterwards that $H\left(\mu_t^N | \nu^N_{\chi_t(\cdot)}\right) = o(N)$.

   Here $\nu^N_{\chi_t(\cdot)}$ is a probability measure which is close to the Gibbs local equilibrium $\mu^N_{\beta(t,\cdot),\lambda(t,\cdot)}$ (1.8). The functions $(\beta(t,\cdot), \lambda(t,\cdot))$ are still related to $\mathbf{e}(t,\cdot)$ and $\mathbf{r}(t,\cdot)$ by (1.5).

   This fact allows to establish the hydrodynamic limit in the sense given in the theorem. For a proof, we refer the reader to [16], Corollary 2.2, [13] or [4].

2. Let us notice that the functions $\mathbf{e}, \mathbf{r}, \beta$ and λ are smooth when $t > 0$, since the system of partial differential equations is parabolic. Moreover, the function $\beta^{-1} = \mathbf{e} - \mathbf{r}^2/2$ satisfies

   $$\partial_t \left(\frac{1}{\beta}\right) = \frac{1}{2\gamma}\ \partial_q^2 \left(\frac{1}{\beta}\right) + \frac{1}{\gamma}\ |\partial_q r|^2 \geqslant \frac{1}{2\gamma}\partial_q^2 \left(\frac{1}{\beta}\right) \ . \tag{1.21}$$

   The supersolutions of the heat equation follow the minimum principle. Consequently, since there exists $c > 0$ such that the initial profile β_0 has the following property

   $$\forall\ q \in \mathbb{T}_N,\ \beta_0(q) \geqslant c > 0 \ ,$$

   then we know that the function β satisfies:

   $$\forall\ q \in \mathbb{T}_N,\ \forall\ t \in [0, T],\ \beta_t(q) \geqslant c > 0 \ . \tag{1.22}$$

3. After some integrations by parts, a simple computation shows that

   $$\partial_t \left\{ \int_{\mathbb{T}} S(\mathbf{r}(t,q), \mathbf{e}(t,q))\ dq \right\} = \frac{1}{2\gamma} \int_{\mathbb{T}} \left[\frac{\partial_q \beta(t,q)}{\beta(t,q)}\right]^2 + 2\beta(t,q)\ [\partial_q r(t,q)]^2\ dq \geqslant 0 \tag{1.23}$$

   when $\mathbf{r}$ and $\mathbf{e}$ are the solutions of the hydrodynamic equations (1.20). This fact is in agreement with the second thermodynamic principle.

In Section 3, we will show that the hypothesis on moments bounds (1.17) holds for a wide class of initial local equilibrium states. Before stating the theorem, let us give some definitions.

We denote by $\mathfrak{S}_N(\mathbb{R})$ the set of real symmetric matrices of size N. The correlation matrix $C \in \mathfrak{S}_{2N}(\mathbb{R})$ of a probability measure ν on Ω^N is the symmetric matrix $C = (C_{i,j})_{1 \leqslant i,j \leqslant 2N}$ defined by

$$C_{i,j} := \begin{cases} \nu[r_i r_j] & i,j \in \{1, ..., N\} \ , \\ \nu[r_i p_j] & i \in \{1, ..., N\},\ j \in \{N+1, ..., 2N\} \ , \\ \nu[p_i r_j] & i \in \{N+1, ..., 2N\},\ j \in \{1, ..., N\} \ , \\ \nu[p_i p_j] & i,j \in \{N+1, ..., 2N\} \ . \end{cases} \tag{1.24}$$

Let us denote by Σ_N the subset of $\mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$ defined by the following condition:

$$(m, C) \in \Sigma_N \Leftrightarrow \begin{cases} m_k = 0 & \text{for all } k = N+1 \dots 2N \ , \\ C_{i,j} = 0 & \text{for all } i \neq j \ , \\ C_{i,i} > 0 & \text{for all } i = 1 \dots 2N \ , \\ C_{i,i} - m_i^2 = C_{i+N,i+N} & \text{for all } i = 1 \dots N \ . \end{cases} \tag{1.25}$$

Precisely, it means that m is of the form $m = (m_1, \dots, m_N, 0, \dots, 0)$, and C is a diagonal matrix whose diagonal components can be written as $(m_1^2 + \alpha_1, \dots, m_N^2 + \alpha_N, \alpha_1, \dots, \alpha_N)$, where $\alpha_i > 0$ for all $i = 1 \dots N$.

For $(m, C) \in \Sigma_N$, we denote by $G_{m,C}(\cdot)$ the Gaussian measure with mean m and correlations given by the matrix C. The covariance matrix of $G_{m,C}(\cdot)$ is thus $C - m^t m$.

Lemma 1.1. *Let $\lambda(\cdot)$ and $\beta(\cdot)$ be two functions of class $\mathcal{C}^1$ defined on $\mathbb{T}$, and $\mu^N_{\beta(\cdot),\lambda(\cdot)}$ be the Gibbs local equilibrium defined by* (1.8)*. If we denote by $m_{\beta(\cdot),\lambda(\cdot)}$ and $C_{\beta(\cdot),\lambda(\cdot)}$ respectively the mean vector and the correlation matrix of $\mu^N_{\beta(\cdot),\lambda(\cdot)}$, then we have*

$$(m_{\beta(\cdot),\lambda(\cdot)}, C_{\beta(\cdot),\lambda(\cdot)}) \in \Sigma_N \quad \textit{and} \quad \mu^N_{\beta(\cdot),\lambda(\cdot)} = G_{m_{\beta(\cdot),\lambda(\cdot)}, C_{\beta(\cdot),\lambda(\cdot)}}.$$

Proof. This result comes from the explicit formula of $\mu^N_{\beta(\cdot),\lambda(\cdot)}$ given in (1.8). First, notice that each momentum p_x is centered under $\mu^N_{\beta(\cdot),\lambda(\cdot)}$ and

$$\mu^N_{\beta(\cdot),\lambda(\cdot)}[r_x] = -\frac{\lambda}{\beta}\left(\frac{x}{N}\right) \ . \tag{1.26}$$

Second, we easily obtain the following expressions:

$$m_{\beta(\cdot),\lambda(\cdot)} = \left(-\frac{\lambda}{\beta}\left(\frac{0}{N}\right), \cdots, -\frac{\lambda}{\beta}\left(\frac{N-1}{N}\right), \underbrace{0, \cdots, 0}_{N} \right) \ , \tag{1.27}$$

$$C_{\beta(\cdot),\lambda(\cdot)} = \begin{pmatrix} D & 0_N \\ 0_N & D' \end{pmatrix} \qquad \text{where} \begin{cases} D = \operatorname{diag}\left(\cdots, \dfrac{1}{\beta(x/N)} + \dfrac{\lambda^2(x/N)}{\beta^2(x/N)}, \cdots \right) \ , \\ D' = \operatorname{diag}\left(\cdots, \dfrac{1}{\beta(x/N)}, \cdots \right). \end{cases} \tag{1.28}$$

□

Now we state our second main theorem, which will be proved in Section 3.

Theorem 1.2. *We assume that the initial probability measure μ_0^N is a convex combination of Gibbs local equilibrium states. More precisely, let σ be a probability measure whose support is included in Σ_N. We assume that σ satisfies:*

$$\textit{for all } k \geqslant 1, \ \int [K(m, C)]^k \ d\sigma(m, C) < \infty \ , \tag{1.29}$$

where $K(m,C) := \sup_{i=1\dots N} \{C_{i,i}\}$. *We define the initial probability measure* μ_0^N *by*

$$\mu_0^N(\cdot) = \int G_{m,C}(\cdot)\ d\sigma(m,C)\ . \tag{1.30}$$

Then, (1.17) *holds, and the conclusions of Theorem 1.1 are valid.*

Remark. As in [7], we could consider a more general model, with a pinning potential. Instead of the deformation r_x, we now introduce the position q_x of the particle x. The new pinning Hamiltonian is given by

$$\mathcal{H}_N^p = \sum_{x\in\mathbb{T}_N} \frac{p_x^2}{2} + \nu^2 \sum_{x\in\mathbb{T}_N} \frac{q_x^2}{2} + \sum_{\substack{|x-y|=1,\\ x,y\in\mathbb{T}_N}} \frac{(q_x-q_y)^2}{4}\ . \tag{1.31}$$

The strength of the pinning potential is regulated by the parameter $\nu > 0$. The energy of site x is now given by

$$e_x = \frac{p_x^2}{2} + \nu^2 \frac{q_x^2}{2} + \frac{1}{4} \sum_{y;|x-y|=1} (q_x - q_y)^2\ . \tag{1.32}$$

The stochastic operator $\mathcal{S}_N^p$ remains equal to $\mathcal{S}_N$, and the Liouville operator $\mathcal{A}_N^p$ can be written as follows:

$$\mathcal{A}_N^p = \sum_{x\in\mathbb{T}_N} \left\{ p_x\ \partial_{q_x} - [(\nu^2-\Delta)q]_x\ \partial_{p_x} \right\}\ , \tag{1.33}$$

where Δ is the discrete Laplacian: $(\Delta u)_x = u_{x+1} + u_{x-1} - 2u_x$.

Because of the presence of the pinning, the bulk dynamics conserves only one quantity: the total energy $\sum_x e_x$. It follows that the Gibbs equilibrium measures μ_β^N are fully characterized by the temperature β^{-1}. Under μ_β^N, the variables p_x are independent of the q_x and are independent identically Gaussian variables of variance β^{-1}. The q_x are distributed according to a centered Gaussian process with covariances given by

$$\mu_\beta^N(q_x q_y) = \Gamma(x-y),\ \text{such that}\ \left[(\nu^2-\Delta)\right]\Gamma(z) = \frac{1}{\beta}\mathbb{1}_{z=0}\ . \tag{1.34}$$

Observe that there exists $C := C(\nu)$ independent of N such that $\left|\mu_\beta^N(q_x q_y)\right| \leqslant C^{-1} e^{-C|x-y|}$ for any $N \geqslant 1$.

These correlations make computations more technical, but the hydrodynamic limit can be established by following the proof here (in [7], Section 3.2, a heuristic argument is given). Assume that the system is initially distributed according to a Gibbs local equilibrium associated to the energy profile $\mathbf{e}_0(\cdot)$, and define $\mathbf{e}(t,\cdot)$ as the evolved profile in the diffusive scale. Then, if the energy moments are bounded like (1.17), $\mathbf{e}$ is the solution of the following heat equation

$$\begin{cases} \partial_t \mathbf{e} = \partial_q (D(\mathbf{e})\partial_q \mathbf{e})\ , \\ \mathbf{e}(0,\cdot) = \mathbf{e}_0(\cdot)\ , \end{cases} \qquad q\in\mathbb{T},\ t\in\mathbb{R}, \tag{1.35}$$

where $D(\mathbf{e})$ is the *diffusivity* given by

$$D := D(\mathbf{e}) = \frac{1/\gamma}{2+\nu^2+\sqrt{\nu^2(\nu^2+4)}}\ . \tag{1.36}$$

In our model, where the state space is not compact, what matters is the existence of moments bounds. We will see in Section 3 that this existence can be easily justified by following the same ideas which work for the unpinned model.

For the sake of simplicity, we will denote by $\mathbf{e}_t(\cdot)$, $\mathbf{r}_t(\cdot)$, $\lambda_t(\cdot)$ and $\beta_t(\cdot)$, respectively, the functions $q \to \mathbf{e}(t,q)$, $q \to \mathbf{r}(t,q)$, $q \to \lambda(t,q)$, and $q \to \beta(t,q)$ defined on $\mathbb{T}$.

1.4 Ergodicity of the Infinite Velocity-flip Model

We conclude this part by giving the theorem of ergodicity, which is proved in [4], Sections 2.2 and 2.4.2, by following the ideas of [11]. Let us define, for all finite subsets $\Lambda \subset \mathbb{Z}$, and for two probability measures ν and μ on $\Omega := (\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$, the restricted relative entropy

$$H_\Lambda(\nu|\mu) := H(\nu_\Lambda|\mu_\Lambda) \tag{1.37}$$

where ν_Λ and μ_Λ are the marginal distributions of ν and μ on Ω.

The Gibbs states in infinite volume are the probability measures $\mu_{\beta,\lambda}$ on Ω given by

$$d\mu_{\beta,\lambda} := \prod_{x\in\mathbb{Z}} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta,\lambda)} dr_x dp_x \ . \tag{1.38}$$

The formal generator of the infinite dynamics is denoted by $\mathcal{L}$.

Theorem 1.3. *Let ν be a probability measure on the configuration space Ω such that*

1. *ν has finite density entropy: there exists $C > 0$, such that for all finite subsets $\Lambda \subset \mathbb{Z}$,*

$$H_\Lambda(\nu|\mu_*) \leqslant C|\Lambda| \ , \tag{1.39}$$

   with $\mu_* := \mu_{1,0}$ a reference Gibbs measure on $(\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$,

2. *ν is translation invariant,*

3. *ν is stationary, i.e. for any compactly supported and differentiable function $F(\mathbf{r},\mathbf{p})$,*

$$\int \mathcal{A}(F)\ d\nu = 0 \ . \tag{1.40}$$

4. *the conditional probability distribution of $\mathbf{p}$ given the probability distribution of $\mathbf{r}$, denoted by $\nu(\mathbf{p}|\mathbf{r})$, is invariant by any flip $\mathbf{p} \to \mathbf{p}^x$, with $x \in \mathbb{Z}$.*

Then, ν is a mixture of infinite Gibbs states.

Corollary 1.1. *If ν is a probability measure on Ω satisfying 1, 2 and if ν is stationary in the sense that: for any compactly supported and differentiable function $F(\mathbf{r},\mathbf{p})$,*

$$\int \mathcal{L}(F)\ d\nu = 0 \ , \tag{1.41}$$

then ν is a mixture of infinite Gibbs states.

The outline of the rest of the paper is as follows. In the next section we expose the strategy of the proof. We introduce the relative entropy $H_N(t)$ of μ_t^N with respect to a corrected local equilibrium, and we prove a Gronwall estimate of the entropy production of the form

$$\partial_t H_N(t) \leqslant C\ H_N(t) + o(N)\ , \tag{1.42}$$

where $C > 0$ does not depend on N. In Section 3 we prove Theorem 1.2.

We suppose that t belongs to a compact set $[0, T]$, T fixed. All estimates are uniform in $t \in [0, T]$.

2 Entropy Production

2.1 Introduction to the Method

For the sake of simplificity, we denote all couples of the form $(\beta(\cdot), \lambda(\cdot))$ by $\chi(\cdot)$.

The corrected Gibbs local equilibrium state $\nu^N_{\chi_t(\cdot)}$ is defined by

$$\nu^N_{\chi_t(\cdot)} := \frac{1}{Z(\chi_t(\cdot))} \prod_{x \in \mathbb{T}_N} \exp\left(-\beta_t\left(\frac{x}{N}\right) e_x - \lambda_t\left(\frac{x}{N}\right) r_x + \frac{1}{N} F\left(t, \frac{x}{N}\right) \cdot \tau_x h(\mathbf{r}, \mathbf{p})\right) dr_x dp_x \tag{2.1}$$

where $Z(\chi_t(\cdot))$ is the partition function and F, h are functions which will be precised later on. The notation $a \cdot b$ still stands for the usual scalar product between a and b. An estimate of the partition function $Z(\chi_t(\cdot))$ is performed in Appendix A.

We are going to use the relative entropy method, with the corrected local Gibbs state $\nu^N_{\chi_t(\cdot)}$ instead of the usual one $\mu^N_{\chi_t(\cdot)}$. We define

$$H_N(t) := H\left(\mu_t^N | \nu^N_{\chi_t(\cdot)}\right) = \int_{\Omega^N} f_t^N(\omega) \log \frac{f_t^N(\omega)}{\phi_t^N(\omega)} d\mu^N_{1,0}(\omega)\ , \tag{2.2}$$

where f_t^N is the density of μ_t^N with respect to the reference measure $\mu^N_{1,0}$. This is a solution, in the sense of the distributions, of the Fokker-Planck equation

$$\partial_t f_t = N^2\ \mathcal{L}_N^* f_t \tag{2.3}$$

where $\mathcal{L}_N^* = -\mathcal{A}_N + \gamma S_N$ is the adjoint of $\mathcal{L}_N$ in $\mathbb{L}^2(\mu^N_{1,0})$. In the same way, ϕ_t^N is the density of $\nu^N_{\chi_t(\cdot)}$ with respect to $\mu^N_{1,0}$ (which here is easily computable).

Thus, our purpose is now to prove (1.42). We begin with the following lemma.

Lemma 2.1.

$$\partial_t H_N(t) \leqslant \int \frac{1}{\phi_t^N}\left(N^2 \mathcal{L}_N^* \phi_t^N - \partial_t \phi_t^N\right) f_t^N d\mu_{1,0} = \mu_t^N\left[\frac{1}{\phi_t^N}\left(N^2 \mathcal{L}_N^* \phi_t^N - \partial_t \phi_t^N\right)\right]\ . \tag{2.4}$$

Proof. The case where f_t^N is smooth is proved in [13], Chapter 6, Lemma 1.4. Here, we do not know that f_t^N is smooth, so that we refer the reader to the proof in [5], Section 3.2, which can be easily followed. □

Now, we choose the correction term. We consider

$$\begin{cases} F\left(t,\frac{x}{N}\right) := \left(-\beta_t'\left(\frac{x}{N}\right), -\lambda_t'\left(\frac{x}{N}\right)\right) \ , \\ \tau_x h(\mathbf{r},\mathbf{p}) := \left(\frac{r_x}{2\gamma}\left(p_{x+1}+p_x+\frac{\gamma}{2}r_x\right), \frac{p_{x+1}}{\gamma}\right) \ . \end{cases} \tag{2.5}$$

Thus,

$$\phi_t^N(\mathbf{r},\mathbf{p}) = \frac{(Z(1,0))^n}{Z(\chi_t(\cdot))} \prod_{x\in\mathbb{T}_N} \exp\left(e_x\left(-\beta_t\left(\frac{x}{N}\right)+1\right) - \lambda_t\left(\frac{x}{N}\right)r_x + \frac{1}{N}F\left(t,\frac{x}{N}\right)\cdot\tau_x h(\mathbf{r},\mathbf{p})\right) \ . \tag{2.6}$$

We define $\xi_x := (e_x, r_x)$ and $\eta(t,q) := (\mathbf{e}(t,q), \mathbf{r}(t,q))$. If f is a vectorial function, we denote its differential by Df.

In Appendix A, the following technical result is proved.

Proposition 2.1. *The term $(\phi_t^N)^{-1}\left(N^2\mathcal{L}_N^*\phi_t^N - \partial_t\phi_t^N\right)$ is given by the sum of five terms in which a microscopic expansion up to the first order appears. In other words,*

$$\frac{1}{\phi_t^N}\left(N^2\mathcal{L}_N^*\phi_t^N - \partial_t\phi_t^N\right) =$$
$$= \sum_{k=1}^{5}\sum_{x\in\mathbb{T}_N} v_k\left(t,\frac{x}{N}\right)\left[J_x^k - H_k\left(\eta\left(t,\frac{x}{N}\right)\right) - (DH_k)\left(\eta\left(t,\frac{x}{N}\right)\right)\cdot\left(\xi_x - \eta\left(t,\frac{x}{N}\right)\right)\right] + o(N) \tag{2.7}$$

where

k	J_x^k	$H_k(\mathbf{e},\mathbf{r})$	$v_k(t,q)$
1	$p_x^2 + r_x r_{x-1} + 2\gamma p_x r_{x-1}$	$\mathbf{e} + \frac{\mathbf{r}^2}{2}$	$\frac{-1}{2\gamma}\partial_q^2\beta(t,q)$
2	$r_x + \gamma p_x$	$\mathbf{r}$	$-\frac{1}{\gamma}\partial_q^2\lambda(t,q)$
3	$p_x^2\ (r_x + r_{x-1})^2$	$(2\mathbf{e}-\mathbf{r}^2)\left(\mathbf{e}+\frac{3}{2}\mathbf{r}^2\right)$	$\frac{1}{4\gamma}[\partial_q\beta(t,q)]^2$
4	$p_x^2\ (r_x + r_{x-1})$	$\mathbf{r}\ (2\mathbf{e}-\mathbf{r}^2)$	$\frac{1}{\gamma}\partial_q\beta(t,q)\ \partial_q\lambda(t,q)$
5	p_x^2	$\mathbf{e} - \frac{\mathbf{r}^2}{2}$	$\frac{1}{\gamma}[\partial_q\lambda(t,q)]^2$

A priori the first term on the right-hand side of (2.7) is of order N, but we want to take advantage of these microscopic Taylor expansions. First, we need to cut-off large energies in order to work with bounded variables only. Second, the strategy consists in performing a one-block estimate: we replace the empirical truncated current, which is averaged over a microscopic box centered at x, by its mean with respect to a Gibbs measure with the parameters corresponding to the microscopic averaged profiles.

A one-block estimate will be performed for each term of the form

$$\sum_{x\in\mathbb{T}_N} v_k\left(t,\frac{x}{N}\right)\left[J_x^k - H_k\left(\eta\left(t,\frac{x}{N}\right)\right) - (DH_k)\left(\eta\left(t,\frac{x}{N}\right)\right)\cdot\left(\xi_x - \eta\left(t,\frac{x}{N}\right)\right)\right] \ . \tag{2.8}$$

In the following the index k is omitted, whenever this does not cause confusion. We follow the lines of the proof given in [4], Section 3.3 and inspired from [16]. A sketch of the proof for the one-block estimate is given in Appendix B.

2.2 Cut-off of Large Energies

For $x \in \mathbb{T}_N$, we define $A_{x,M} := \{e_x + e_{x-1} \leqslant M\}$, $J_{x,M} := J_x \, \mathbb{1}_{A_{x,M}}$, and $\xi_{x,M} := \xi_x \, \mathbb{1}_{e_x \leqslant M}$.

Then, $J_{x,M}$ and $\xi_{x,M}$ are bounded by $C(M) > 0$.

We use twice the Cauchy-Schwartz inequality to write

$$\begin{aligned}\mu_t^N \left[\sum_{x \in \mathbb{T}_N} v\left(t, \frac{x}{N}\right) \, J_x \, \mathbb{1}_{A^c_{x,M}} \right] &\leqslant \mu_t^N \left[\left(\sum_{x \in \mathbb{T}_N} v^2\left(t, \frac{x}{N}\right) J_x^2 \right)^{1/2} \left(\sum_{x \in \mathbb{T}_N} \mathbb{1}_{A^c_{x,M}} \right)^{1/2} \right] \\ &\leqslant \left(\mu_t^N \left[\sum_{x \in \mathbb{T}_N} v^2\left(t, \frac{x}{N}\right) J_x^2 \right] \right)^{1/2} \left(\mu_t^N \left[\sum_{x \in \mathbb{T}_N} \mathbb{1}_{A^c_{x,M}} \right] \right)^{1/2} . \end{aligned} \tag{2.9}$$

First, $v^2(t, x/N)$ is bounded by a constant which does not depend on N. Second, the term J_x^2 can be bounded above by the squared energy e_x^2. The hypothesis (1.17) shows that there exists C_0 which does not depend on N such that

$$\left(\mu_t^N \left[\sum_{x \in \mathbb{T}_N} v^2\left(t, \frac{x}{N}\right) J_x^2 \right] \right)^{1/2} \leqslant C_0 \, N^{1/2} . \tag{2.10}$$

Moreover, Markov inequality proves that

$$\mu_t^N \left[\sum_{x \in \mathbb{T}_N} \mathbb{1}_{A^c_{x,M}} \right] \leqslant \sum_{x \in \mathbb{T}_N} \mu_t^N \left[\mathbb{1}_{e_x > M/2} \right] + \mu_t^N \left[\mathbb{1}_{e_{x-1} > M/2} \right] \leqslant \frac{4}{M} \sum_{x \in \mathbb{T}_N} \mu_t^N [e_x] \leqslant \frac{C_1}{M} \, N . \tag{2.11}$$

Finally, we obtain a constant C independent of N such that

$$\mu_t^N \left[\sum_{x \in \mathbb{T}_N} v\left(t, \frac{x}{N}\right) \, J_x \, \mathbb{1}_{A^c_{x,M}} \right] \leqslant CN \, \varepsilon(M) . \tag{2.12}$$

Observe that this estimate is in agreement with the Gronwall inequality we want to prove, since we are going to divide by N. Thus, the error term is of order $1/M$ that goes to 0 as $M \to \infty$.

Consequently, J_x can be replaced by $J_{x,M}$ in (2.8), and similarly, ξ_x can be replaced by $\xi_{x,M}$.

2.3 One-block Estimate

Now we prove that

$$\frac{1}{N}\mu_t^N\left[\sum_{x\in\mathbb{T}_N} v\left(t,\frac{x}{N}\right)\left[J_{x,M}-H\left(\eta\left(t,\frac{x}{N}\right)\right)-(DH)\left(\eta\left(t,\frac{x}{N}\right)\right)\cdot\left(\xi_{x,M}-\eta\left(t,\frac{x}{N}\right)\right)\right]\right]\leqslant$$
$$\leqslant C\,\frac{H_N(t)}{N}+\varepsilon(N,M) \quad (2.13)$$

with $\varepsilon(N,M)\xrightarrow[M\to\infty,N\to\infty]{}0$.

We denote by $\Lambda_\ell(y)$ the box of length ℓ centered around y. We introduce the microscopic average profiles

$$\eta_{\ell,M}(y):=\frac{1}{\ell}\sum_{j\in\Lambda_\ell(y)}\xi_{j,M}\ . \quad (2.14)$$

We split $\mathbb{T}_N$ into $p=N/\ell$ boxes $\Lambda_\ell(x_j)$ centered at x_j. Here ℓ is assumed to divide N for simplicity. We will first let $N\to\infty$, then $\ell\to\infty$ and then $M\to\infty$.

First of all, we want to replace

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N} v\left(t,\frac{x}{N}\right)J_{x,M} \quad (2.15)$$

by

$$\frac{1}{p}\sum_{j=1}^{p} v\left(t,\frac{x_k}{N}\right)\left[\frac{1}{\ell}\sum_{i\in\Lambda_\ell(x_j)}J_{i,M}\right]\ . \quad (2.16)$$

The error term produced during this step can be written as

$$|R_N|=\frac{1}{N}\left|\sum_{j=1}^{p}\sum_{i\in\Lambda_\ell(x_j)}\left[v\left(t,\frac{i}{N}\right)-v\left(t,\frac{x_j}{N}\right)\right]J_{i,M}\right|\leqslant C_1(M)\frac{\ell}{N}\ . \quad (2.17)$$

The last inequality comes from the smoothness of v, more precisely

$$\left|v\left(t,\frac{i}{N}\right)-v\left(t,\frac{x_j}{N}\right)\right|\leqslant C_0\frac{\ell}{N}\ . \quad (2.18)$$

Similarly, we perform the same estimates for the other terms and it remains to prove that

$$\mu_t^N\left[\frac{1}{p}\sum_{j=1}^{p} v\left(t,\frac{x_j}{N}\right)\left\{\frac{1}{\ell}\sum_{i\in\Lambda_\ell(x_j)}J_{i,M}-H\left(\eta\left(t,\frac{x_j}{N}\right)\right)\right.\right.$$
$$\left.\left.-(DH)\left(\eta\left(t,\frac{x_j}{N}\right)\right)\cdot\left(\eta_{\ell,M}(x_j)-\eta\left(t,\frac{x_j}{N}\right)\right)\right\}\right] \quad (2.19)$$

vanishes as $M,N,\ell\to\infty$, the limit in N taken first, then the limit in ℓ and finally the limit in M. The additive term which appears after performing this replacement can be bounded above by a term

$\varepsilon_{N,M,\ell}$ which depends on N, M and ℓ, but which is independent of the particular splitting of $\mathbb{T}_N$ into p boxes. This term is of order $o(N)$ in the Gronwall inequality we want to prove, in the sense that

$$\lim_{M\to\infty}\lim_{\ell\to\infty}\lim_{N\to\infty} N^{-1}\ \mu_t^N[\varepsilon_{N,M,\ell}] = 0\ . \tag{2.20}$$

Now we want to perform a one-block estimate. The main idea consists in replacing $\ell^{-1}\sum_{i\in\Lambda_\ell(x_j)} J_{i,M}$ by $H(\eta_{\ell,M}(x_j))$. This is achieved thanks to the ergodicity of the dynamics (see Theorem 1.3). In order to use this ergodicity property, we have to work with a space translation invariant measure. To obtain such a probability measure, we introduce a second average over the $x_j,\ 1\leqslant j\leqslant p$. For each $k\in\{0,...,\ell-1\}$, we can split $\mathbb{T}_N$ into p disjoint boxes of length ℓ by writing

$$\forall\ k=0,...,\ell-1,\ \mathbb{T}_N=\bigcup_{j=1}^{p}\Lambda_\ell(x_j+k)\ . \tag{2.21}$$

Then, we average the different splittings mentioned above. More precisely, in Appendix B we recall how to prove

$$\limsup_{M\to\infty}\limsup_{\ell\to\infty}\limsup_{N\to\infty}\frac{1}{\ell}\sum_{k=0}^{\ell-1}\mu_t^N\left[\frac{1}{p\ell}\sum_{j=1}^{p}\left|v\left(t,\frac{[x_j+k]}{N}\right)\sum_{i\in\Lambda_\ell(x_j+k)}J_{i,M}-H(\eta_{\ell,M}(x_j+k))\right|\right]=0. \tag{2.22}$$

2.4 Large Deviations

The previous estimates are valid for any splitting of $\mathbb{T}_N$ into p boxes of length ℓ. Thus, it would be sufficient to prove (2.19) with every x_i replaced by x_i+k for arbitrary $k\in\{1,...,\ell-1\}$. Consequently, it is sufficient to prove (2.19) in an averaged form. Then, from the one-block estimate, we have to deal with

$$\frac{1}{\ell}\sum_{k=0}^{\ell-1}\mu_t^N\left[\frac{1}{N}\sum_{j=1}^{p}v\left(t,\frac{[x_j+k]}{N}\right)\Omega\left(\eta_{\ell,M}(x_j+k),\eta\left(t,\frac{[x_j+k]}{N}\right)\right)\right], \tag{2.23}$$

where $\Omega(\mathbf{w},\mathbf{u}):=H(\mathbf{w})-H(\mathbf{u})-DH(\mathbf{u})\cdot(\mathbf{w}-\mathbf{u})$.

By definition of the entropy, for any $\alpha>0$ and any positive measurable function f we have

$$\int f\ d\mu\leqslant\frac{1}{\alpha}\left\{\log\left(\int e^{\alpha f}\ d\nu\right)+H(\mu|\nu)\right\}\ . \tag{2.24}$$

This inequality, known as the *entropy inequality*, allows to show that: for any $\alpha>0$, (2.23) is less than or equal to

$$\frac{H_N(t)}{\alpha\ N}+\frac{1}{\ell}\sum_{k=0}^{\ell-1}\frac{1}{\alpha N}\log\nu^N_{\chi_t(\cdot)}\left[e^{\alpha\ell\sum_{j=1}^{p}v\left(t,\frac{[x_j+k]}{N}\right)\Omega\left(\eta_{\ell,M}(x_j+k),\eta\left(t,\frac{[x_j+k]}{N}\right)\right)}\right]\ . \tag{2.25}$$

Notice that the last integral converges because all quantities are bounded.

The first term is in agreement with the Gronwall inequality we want to obtain. We look at the second term. Since we have arranged the sum over p disjoint blocks which are independently distributed by $\nu^N_{\chi_t(\cdot)}$, the second term is equal to

$$\frac{1}{\ell}\sum_{k=0}^{\ell-1}\frac{1}{\alpha N}\sum_{j=1}^{p}\log \nu^N_{\chi_t(\cdot)}\left[e^{\alpha\ell v\left(t,\frac{[x_j+k]}{N}\right)\Omega\left(\eta_{\ell,M}(x_j+k),\eta\left(t,\frac{[x_j+k]}{N}\right)\right)}\right]. \tag{2.26}$$

We are going to show that this expression vanishes as $M,N,\ell\to\infty$ by using the large deviation properties of the measure $\nu^N_{\chi_t(\cdot)}$, that locally is almost homogeneous. In fact, by using the smoothness for the various involved functions, we can substitute the inhomogeneous product measure $\nu^N_{\chi_t(\cdot)}$ restricted to $\Lambda_\ell(x_j+k)$ with the homogeneous product measure $\mu^N_{\chi_t([x_j+k]/N)}$, in each expectation of the previous expression. More precisely, we have the following lemma.

Lemma 2.2.

$$M_1(N,\ell,k,M):=\frac{1}{\alpha N}\sum_{j=1}^{p}\log \nu^N_{\chi_t(\cdot)}\left[e^{\alpha\ell\left|v\left(t,\frac{[x_j+k]}{N}\right)\Omega\left(\eta_{\ell,M}(x_j+k),\eta\left(t,\frac{[x_j+k]}{N}\right)\right)\right|}\right] \tag{2.27}$$

can be replaced by

$$M_2(N,\ell,k,M):=\frac{1}{\alpha N}\sum_{j=1}^{p}\log \mu^N_{\chi_t([x_j+k]/N)}\left[e^{\alpha\ell\left|v\left(t,\frac{[x_j+k]}{N}\right)\Omega\left(\eta_{\ell,M}(x_j+k),\eta\left(t,\frac{[x_j+k]}{N}\right)\right)\right|}\right]. \tag{2.28}$$

The difference between these two terms is less than or equal to a small term which depends on ℓ (but not on k) and vanishes in the N limit: there exists a constant $C(\ell,M,N)$ which does not depend on k such that

$$M_1(N,\ell,k,M)-M_2(N,\ell,k,M)\leqslant C(\ell,M,N)\quad \textit{and}\quad C(\ell,M,N)\xrightarrow[N\to\infty]{}0\ . \tag{2.29}$$

Remark. In the following, we will prove that

$$\limsup_{M\to\infty}\limsup_{\ell\to\infty}\limsup_{N\to\infty}M_2(N,\ell,k,M)=0\ . \tag{2.30}$$

In addition to this lemma, this implies that

$$\limsup_{M\to\infty}\limsup_{\ell\to\infty}\limsup_{N\to\infty}M_1(N,\ell,k,M)=0\ , \tag{2.31}$$

since $M_1(N,\ell,k,M)$ is always nonnegative, and we know that, for all sequences a_n and b_n,

$$\limsup a_n\leqslant \limsup(a_n-b_n)+\limsup b_n\ . \tag{2.32}$$

Proof. For each $j\in\{1,...,p\}$, the function

$$F_j:=\exp\left\{\alpha\ell\left|v\left(t,\frac{[x_j+k]}{N}\right)\ \Omega\left(\eta_{\ell,M}(x_j+k),\eta\left(t,\frac{[x_j+k]}{N}\right)\right)\right|\right\} \tag{2.33}$$

is bounded above by $e^{C\ell}$, $C > 0$ (since $\eta_{\ell,M}$ is bounded and t belongs to a compact set), and depends on the configuration only through the coordinates in $\Lambda_\ell(x_j+k)$. Thus, each expectation appearing in the sum can be taken w.r.t the restriction to $\Lambda_\ell(x_j+k)$ of $\nu^N_{\chi_t(\cdot)}$. These restrictions are inhomogeneous product measures but with slowly varying parameters and hence, each term $\log \nu^N_{\chi_t(\cdot)}[F_j]$ can be replaced by $\log \mu^N_{\chi_t([x_j+k]/N)}[F_j]$ with a small error.

Indeed, the difference between these two terms is equal to

$$\log \mu^N_{\chi_t([x_j+k]/N)}\left[1+\frac{F_j\ (h_j-1)}{\mu^N_{\chi_t([x_j+k]/N)}[F_j]}\right] = \log\left(1+\frac{\mu^N_{\chi_t([x_j+k]/N)}[F_j\ (h_j-1)]}{\mu^N_{\chi_t([x_j+k]/N)}[F_j]}\right) \tag{2.34}$$

with

$$h_j := \exp\left\{\sum_{i\in\Lambda_\ell(x_j+k)} \xi_{i,M}\cdot\left[\chi_t\left(\frac{i}{N}\right)-\chi_t\left(\frac{x_j+k}{N}\right)\right]+\right.$$
$$\left.-\frac{1}{N}F\left(t,\frac{i}{N}\right)\cdot\tau_i h+\left[\log Z\left(\chi_t\left(\frac{i}{N}\right)\right)-\log Z\left(\chi_t\left(\frac{x_j+k}{N}\right)\right)\right]\right\}. \tag{2.35}$$

The inequality $\log(1+x)\leqslant|x|$ (true for any real x) and the fact that $\mu^N_{\chi_t([x_j+k]/N)}[F_j]\geqslant 1$ reduces us to estimate

$$\mu^N_{\chi_t([x_j+k]/N)}[|F_j(h_j-1)|]\ . \tag{2.36}$$

By using the smoothness of χ_t and the inequality $|e^x-1|\leqslant|x|e^{|x|}$, one easily shows that there exist positive constants C_0, $C(\ell)$, and $\bar\beta$ which do not depend on j such that

$$|F_j(h_j-1)|\leqslant\frac{C(\ell)\ell}{N}\left(\sum_{i\in\Lambda_\ell(x_j+k)}[e_{i,M}+e_{i+1,M}+1]\right)\exp\left\{\frac{C_0\ell}{N}\sum_{i\in\Lambda_\ell(x_j+k)}[e_{i,M}+e_{i+1,M}+1]\right\}\ ,$$
$$\text{and}\quad \left.\frac{d\mu^N_{\chi_t([x_j+k]/N)}}{d\mu^N_{\bar\beta,0}}\right|_{\Lambda_\ell(x_j+k)}\leqslant C(\ell)\ . \tag{2.37}$$

Hence, the total error performing by these replacements is bounded above:

$$M_1(N,\ell,k,M)-M_2(N,\ell,k,M)\leqslant\frac{1}{\alpha N}C_1(\ell,M)\ \mu^N_{\bar\beta,0}\left[\exp\left\{\frac{C_0}{p}\sum_{i\in\Lambda_\ell(0)}[e_{i,M}+e_{i+1,M}+1]\right\}\right]$$

for some positive constant $C_1(\ell,M)$.

It trivially goes to 0 as N goes to infinity for each given fixed ℓ. □

Lastly, we have to show that the limit

$$\limsup_{M\to\infty}\limsup_{\ell\to\infty}\limsup_{N\to\infty}\frac{1}{\ell}\sum_{k=0}^{\ell-1}\frac{1}{\alpha N}\sum_{j=1}^{p}\log\mu^N_{\chi_t([x_j+k]/N)}\left[e^{\alpha\ell v\left(t,\frac{[x_j+k]}{N}\right)\ \Omega\left(\eta_{\ell,M},\eta\left(t,\frac{[x_j+k]}{N}\right)\right)}\right] \tag{2.38}$$

vanishes. Here, $\eta_{\ell,M}:=\eta_{\ell,M}(0)=\ell^{-1}\sum_{i\in\Lambda_\ell(0)}\xi_{i,M}$.

The limit in p results in an integral over $\mathbb{T}$ because we have a Riemann sum. Moreover, the integral does not depend on k so that the averaging over k disappears in the p limit. Hence, the point is to estimate

$$\limsup_{M\to\infty}\limsup_{\ell\to\infty}\frac{1}{\alpha\ell}\int_{\mathbb{T}}\log\,\mu^N_{\chi_t(q)}\left[e^{\alpha\ell v(t,q)\;\Omega\left(\eta_{\ell,M},\eta(t,q)\right)}\right]\,dq\;. \tag{2.39}$$

According to Laplace-Varadhan theorem applied to these product measures $\mu^N_{\chi_t(q)}$, and according to the dominated convergence theorem, the previous limit is equal to

$$\limsup_{M\to\infty}\frac{1}{\alpha}\int_{\mathbb{T}}\sup_{\mathbf{z}\in\mathbb{R}^2}\left\{\alpha v(t,q)\;\Omega(\mathbf{z},\eta(t,q))-I_M(\mathbf{z},\eta(t,q))\right\}dq\;, \tag{2.40}$$

where $I_M(\mathbf{z},\eta(t,q))$ is the rate function of the sequence $\left\{k^{-1}\sum_{i=1}^k\xi_{i,M}\right\}_k$ as $(r_x,p_x)_{x\in\mathbb{T}_N}$ are distributed according to the homogeneous product measure $\mu^N_{\chi_t(q)}$.

The function I_M is the Legendre transform of the cumulant-generating function of $\xi_{0,M}$:

$$I_M(\mathbf{z},\eta(t,q))=\sup_{\mathbf{y}\in\mathbb{R}^2}\left\{\mathbf{y}\cdot\mathbf{z}-\log\mu^N_{\chi_t(q)}[e^{\mathbf{y}\cdot\xi_{0,M}}]\right\}\;. \tag{2.41}$$

Hence

$$\liminf_{M\to\infty}I_M(\mathbf{z},\eta(t,q))\geqslant\sup_{\mathbf{y}\in\mathbb{R}^2}\left\{\mathbf{y}\cdot\mathbf{z}-\log\mu^N_{\chi_t(q)}[e^{\mathbf{y}\cdot\xi_0}]\right\}=I(\mathbf{z},\eta(t,q))\;, \tag{2.42}$$

where $I(\mathbf{z},\eta(t,q))$ is the rate function of $\left\{k^{-1}\sum_{i=1}^k\xi_i\right\}_k$ as $(r_y,p_y)_y$ are distributed according to the homogeneous product measure $\mu^N_{\chi_t(q)}$.

It follows, by Fatou's lemma, that (2.40) is smaller than or equal to

$$\frac{1}{\alpha}\int_{\mathbb{T}}\sup_{\mathbf{z}}\left\{\alpha v(t,q)\;\Omega(\mathbf{z},\eta(t,q))-I(\mathbf{z},\eta(t,q))\right\}dq\;. \tag{2.43}$$

From now on we omit the dependance in (t,q) of the involved functions v and η. Recall that χ and η are in duality (see (1.13)). An easy computation gives that

$$\begin{aligned}I(\mathbf{z},\eta)&=\sup_{\mathbf{y}}\left\{\mathbf{y}\cdot\mathbf{z}-\log\left(\int_{\mathbb{R}^2}e^{\mathbf{y}\cdot\xi}e^{\chi\cdot\xi-\log Z(\chi)}d\mathbf{r}d\mathbf{p}\right)\right\}\\&=\sup_{\mathbf{y}}\{\mathbf{y}\cdot\mathbf{z}-\log Z(\chi+\mathbf{y})+\log Z(\chi)\}\\&=\log Z(\chi)\;+\;\mathbf{z}\cdot\chi-S(\mathbf{z})\;,\end{aligned} \tag{2.44}$$

where the last equality follows from the equality between the Fenchel-Legendre transform of $\log Z$ and the function $-S$. We observe that $I(\eta,\eta)=0$ and $D_{\mathbf{z}}I(\mathbf{z},\eta)=0$. Furthermore, I is strictly convex in $\mathbf{z}$:

$$(D^2_{\mathbf{z}}I)(\mathbf{z},\eta)=(D^2_{\mathbf{z}}\{-S\})(\mathbf{z})>0\;. \tag{2.45}$$

Since $\Omega(\eta,\eta)=0$ and $(D_{\mathbf{z}}\Omega)(\mathbf{z},\eta)=(DH)(\mathbf{z})-DH(\eta)$, we also get: $(D_{\mathbf{z}}\Omega)(\eta,\eta)=0$.

Lemma 2.3. *For $\alpha>0$ sufficiently small,*

$$\forall\;\mathbf{z}\in\mathbb{R}^2,\;\forall\;q\in\mathbb{T},\;\alpha v(t,q)\Omega(\mathbf{z},\eta(t,q))\leqslant I(\mathbf{z},\eta(t,q))\;. \tag{2.46}$$

Proof. An easy computation provides an explicit expression for the rate function: if $\mathbf{z} = (z_1, z_2)$ and $\eta = (e, r)$ with $e - r^2/2 > 0$ then

$$I(\mathbf{z}, \eta) = \frac{1}{e - r^2/2} \left(\frac{r^2}{2} - z_2 r + z_1 \right) - \log \left(\frac{z_1 - z_2^2/2}{e - r^2/2} \right) - 1 \ . \tag{2.47}$$

From the inequality $-\log x \geqslant -x + 1$ (satisfied for any $x > 0$), we get

$$I(\mathbf{z}, \eta) \geqslant \frac{1}{2(e - r^2/2)} (r - z_2)^2 \ . \tag{2.48}$$

Thus, for a given η, the rate function $\mathbf{z} \to I(\mathbf{z}, \eta)$ is such that $I(\mathbf{z}, \eta) \geqslant c_\eta |\mathbf{z} - \eta|^2$, where c_η is a positive constant. Moreover, according to (1.22),

$$\forall \ t \in [0, T], \ \forall \ q \in \mathbb{T}, \ c_{\eta(t,q)} \geqslant c > 0 \tag{2.49}$$

Let us fix $\mathbf{z} \in \mathbb{R}^2$. From the Taylor-Lagrange theorem, there exists a positive constant C such that

$$\Omega(\mathbf{z}, \eta(t, q)) \leqslant C |\mathbf{z} - \eta(t, q)|^2 \leqslant I(\mathbf{z}, \eta(t, q)) \ . \tag{2.50}$$

More precisely, C is equal to

$$\sup_{(t,q) \in [0,T] \times \mathbb{T}} \| D^2 H(\eta(t, q)) \|^2 \ . \tag{2.51}$$

Since v is uniformly bounded, the result is proved. □

Consequently, for α small enough,

$$\sup_{\mathbf{z}} \{ \alpha v(t, q) \ \Omega(\mathbf{z}, \eta(t, q)) - I(\mathbf{z}, \eta(t, q)) \} = 0 \ , \tag{2.52}$$

and we have finally proved that

$$\partial_t H_N(t) \leqslant C \ H_N(t) + R_{N,\ell,M}(t) \tag{2.53}$$

with

$$\lim_{M \to \infty} \lim_{\ell \to \infty} \lim_{N \to \infty} \frac{1}{N} \int_0^t R_{N,\ell,M}(s) \ ds = 0 \ . \tag{2.54}$$

By Gronwall's inequality we obtain: $H_N(t)/N \xrightarrow[N \to \infty]{} 0$ and Theorem 1.1 is proved.

3 Proof of Theorem 1.2: Moments Bounds

In the following, we prove the two conditions on the moments bounds for a class of local equilibrium states. First, we assume that the initial law μ_0^N is exactly the Gibbs local equilibrium measure $\mu^N_{\beta_0(\cdot), \lambda_0(\cdot)}$. Second, we extend the proof to the case where μ_0^N is a convex combination of Gibbs local equilibrium measures.

We need to control the moments $\mu_t^N \left[\sum_x e_x^k \right]$ for all $k \geqslant 1$. The first two bounds ($k = 1, 2$) would be sufficient to justify the cut-off of the currents, but here we need more because of Lemma A.2 (which is necessary to prove Proposition 2.1). Since the chain is harmonic, Gibbs states are Gaussian. We

recall that all Gaussian moments can be expressed in terms of variances and covariances. In the following, we first give an other representation of the dynamics of the process, and then we prove the bounds and precise their dependence on k.

Let us highlight that, from now on, we consider the process with generator $\mathcal{L}_N$: it is not accelerated any more. The law of this new process $(\tilde{\omega}_t)_{t\geqslant 0}$ is denoted by $\tilde{\mu}_t^N$. At the end of this part, Theorem 1.2 will be easily deduced since all estimates will not depend on t, and the following equality still holds:

$$\mu_t^N = \tilde{\mu}_{tN^2}^N \ . \tag{3.1}$$

Remarks.

1. In the following, we always respect the decomposition of the space $\Omega^N = \mathbb{R}^N \times \mathbb{R}^N$. Let us recall that the first N components stand for $\mathbf{r}$ and the last N components stand for $\mathbf{p}$. All vectors and matrices are written according to this decomposition. Let ν be a measure on Ω^N. We denote by $m \in \mathbb{R}^{2N}$ its mean vector and by $C \in \mathfrak{M}_{2N}(\mathbb{R})$ its correlation matrix (see (1.24)). We can write m and C as

$$m = (\rho, \pi) \in \mathbb{R}^{2N} \quad \text{and} \quad C = \begin{pmatrix} U & Z^* \\ Z & V \end{pmatrix} \in \mathfrak{S}_{2N}(\mathbb{R}) \ , \tag{3.2}$$

   where $\rho := \nu[\mathbf{r}] \in \mathbb{R}^N$, $\pi := \nu[\mathbf{p}] \in \mathbb{R}^N$ and $U, V, Z \in \mathfrak{M}_N(\mathbb{R})$.

2. Thanks to the convexity inequality $(a+b)^k \leqslant 2^{k-1}\ (a^k + b^k)$, for $a, b > 0$, we can write

$$e_x^k \leqslant \frac{1}{2}\ \left(p_x^{2k} + r_x^{2k}\right) \ . \tag{3.3}$$

   Thus, instead of proving (1.17) we will show

$$\mu_t^N \left[\sum_{x\in\mathbb{T}_N} p_x^{2k} \right] \leqslant (Ck)^k\ N \quad \text{and} \quad \mu_t^N \left[\sum_{x\in\mathbb{T}_N} r_x^{2k} \right] \leqslant (Ck)^k\ N \ . \tag{3.4}$$

3.1 Poisson Process and Gaussian Measures

We are going to use a graphical representation of the process $(\tilde{\omega}_t)_{t\geqslant 0}$.

Let us define

$$A := \begin{pmatrix} 0 & \cdots & \cdots & 0 & -1 & 1 & & (0) \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \\ \vdots & & & \vdots & 0 & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & & 0 & \vdots & & & \vdots \\ & \ddots & \ddots & 0 & \vdots & & & \vdots \\ (0) & & -1 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathfrak{M}_{2N}(\mathbb{R}) \ . \tag{3.5}$$

We now consider $(m_t, C_t)_{t\geqslant 0}$, a Markov process on $\mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$ whose generator is denoted by $\mathcal{G}$ and defined as follows.

Take $m := (\rho, \pi) \in \mathbb{R}^{2N}$ and $C := \begin{pmatrix} U & Z^* \\ Z & V \end{pmatrix} \in \mathfrak{S}_{2N}(\mathbb{R})$, where ρ, π are two vectors in $\mathbb{R}^N$, U, V are two symmetric matrices in $\mathfrak{S}_N(\mathbb{R})$ and Z is a matrix in $\mathfrak{M}_N(\mathbb{R})$. Hereafter, we denote by Z^* the transpose of the matrix Z.

The generator $\mathcal{G}_N$ is given by

$$(\mathcal{G}_N v)(m, C) := (\mathcal{K}_N v)(m, C) + \gamma\ (\mathcal{H}_N v)(m, C)\ , \tag{3.6}$$

where

$$\mathcal{K}_N := \sum_{i,j\in\mathbb{T}_N} (-AC + CA)_{i,j}\ \partial_{C_{i,j}} + \sum_{i\in\mathbb{T}_N} \{(\pi_{i+1} - \pi_i)\partial_{\rho_i} + (\rho_i - \rho_{i-1})\partial_{\pi_i}\}\ , \tag{3.7}$$

and

$$(\mathcal{H}_N v)(m, C) := \frac{1}{2} \sum_{k\in\mathbb{T}_N} [v(m^k, C^k) - v(m, C)]\ . \tag{3.8}$$

Here,

$$m^k = (\rho, \pi^k) \ \text{ and } \ C^k = \Sigma_k^* \cdot C \cdot \Sigma_k = \begin{pmatrix} U & Z^{k*} \\ Z^k & V^k \end{pmatrix}\ . \tag{3.9}$$

In these last two formulas, π^k is the vector obtained from π by the flip of π_k into $-\pi_k$, and Σ_k is defined as

$$\Sigma_k = \begin{pmatrix} I_n & 0_n \\ 0_n & I_n - 2E_{k,k} \end{pmatrix}. \tag{3.10}$$

More precisely,

$$Z^k_{i,j} = (-1)^{\delta_{k,i}} Z_{i,j} \ \text{ and } \ V^k_{i,j} = (-1)^{(\delta_{k,i}+\delta_{k,j})} V_{i,j}\ . \tag{3.11}$$

We denote by $\mathbb{P}_{m_0,C_0}$ the law of the process $(m_t, C_t)_{t\geqslant 0}$ starting from (m_0, C_0), and by $\mathbb{E}_{m_0,C_0}[\cdot]$ the expectation with respect to $\mathbb{P}_{m_0,C_0}$.

For $t \geqslant 0$ fixed, let $\theta^t_{m_0,C_0}(\cdot,\cdot)$ be the law of the random variable $(m_t, C_t) \in \mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$, knowing that the process starts from (m_0, C_0).

Recall that we denote by $G_{m,C}(\cdot)$ the Gaussian measure on Ω^N with mean $m \in \mathbb{R}^{2N}$ and correlation matrix $C \in \mathfrak{S}_{2N}(\mathbb{R})$.

Lemma 3.1. *Let $\mu_0^N := \mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}$ be the Gibbs equilibrium state defined by* (1.8)*, where $\lambda_0(\cdot)$ and $\beta_0(\cdot)$ are the two macroscopic potential profiles.*

Then,

$$\tilde{\mu}_t^N = \int G_{m,C}(\cdot)\ d\theta^t_{m_0,C_0}(m, C) \tag{3.12}$$

where

$$m_0 := \left(-\frac{\lambda_0}{\beta_0}\left(\frac{0}{N}\right), \cdots, -\frac{\lambda_0}{\beta_0}\left(\frac{N-1}{N}\right), \underbrace{0, \cdots, 0}_{N} \right) \tag{3.13}$$

and

$$C_0 := \begin{pmatrix} D & 0_N \\ 0_N & D' \end{pmatrix} \qquad \textit{with} \begin{cases} D = \mathrm{diag}\left(\cdots, \dfrac{1}{\beta_0(x/N)} + \dfrac{\lambda_0^2(x/N)}{\beta_0^2(x/N)}, \cdots\right), \\ D' = \mathrm{diag}\left(\cdots, \dfrac{1}{\beta_0(x/N)}, \cdots\right). \end{cases} \tag{3.14}$$

Proof. We begin with the graphical representation of the process $(\tilde{\omega}_t)_{t\geqslant 0}$, which is based on the Harris description. Let $(N_i)_{i\in\mathbb{T}_N}$ be a sequence of independent standard Poisson processes of intensity γ. In other words, we put on each site $i \in \mathbb{T}_N$ an exponential clock of mean $1/\gamma$. At time 0 the process has an initial state ω_0. Let $T_1 = \inf_{t\geqslant 0}\{\exists\ i \in \mathbb{T}_N,\ N_i(t) = 1\}$ and i_1 the site where the infimum is achieved.

During the interval $[0, T_1)$, the process follows the deterministic evolution given by the generator $\mathcal{A}_N$. More precisely, let $F : (\mathbf{r}, \mathbf{p}) \in \mathbb{T}_N^2 \to A \cdot (\mathbf{r}, \mathbf{p}) \in \mathbb{T}_N^2$ where A is given by (3.5). Then, for any continuously differentiable function $f : \Omega^N \to \mathbb{R}$,

$$\mathcal{A}_N f(\omega) = A \cdot Df(\omega)\ , \tag{3.15}$$

and during the time interval $[0, T_1)$, $\tilde{\omega}_t$ follows the evolution given by the system: $dy/dt = F(y)$. At time T_1, the momentum p_{i_1} is flipped, and gives a new configuration. Then, the system starts again with the deterministic evolution up to the time of the next flip, and so on. Let $\xi := (i_1, T_1), \ldots, (i_k, T_k), \ldots$ be the sequence of sites and ordered times for which we have a flip, and let us denote its law by $\mathbb{P}$. Conditionally to ξ, the evolution is deterministic, and the state of the process $\tilde{\omega}_t^\xi$ is given by

$$\forall\ t \in [T_k, T_{k+1}),\ \tilde{\omega}_t^\xi = e^{(t-T_k)A} \circ F_{i_k} \circ e^{(T_k - T_{k-1})A} \circ F_{i_{k-1}} \circ \cdots \circ e^{T_1 A}\omega_0\ , \tag{3.16}$$

where F_i is the map $\omega = (\mathbf{r}, \mathbf{p}) \to (\mathbf{r}, \mathbf{p}^i)$.

If initially the process starts from ω_0 which is distributed according to a Gaussian measure μ_0^N, then $\tilde{\omega}_t^\xi$ is distributed according to a Gaussian measure $\tilde{\mu}_t^\xi$. Then, the density $\tilde{\mu}_t^N$ is given by

$$\tilde{\mu}_t^N(\cdot) = \int \tilde{\mu}_t^\xi(\cdot)\ d\mathbb{P}(\xi)\ . \tag{3.17}$$

More precisely, the mean vector m_t^ξ and the correlation matrix C_t^ξ of $\tilde{\mu}_t^\xi$ can be related to the mean vector m_0 and the correlation matrix C_0 of μ_0^N:

$$m_t^\xi = e^{(t-T_k)A} \cdot \Sigma_{i_k} \cdot e^{(T_k - T_{k-1})A} \cdot \Sigma_{i_{k-1}} \cdots e^{T_1 A} \cdot m_0\ , \tag{3.18}$$

and

$$C_t^\xi = e^{(t-T_k)A} \cdot \Sigma_{i_k} \cdot e^{(T_k - T_{k-1})A} \cdots \Sigma_{i_1} \cdot e^{T_1 A} \cdot C_0 \cdot e^{-T_1 A} \cdot \Sigma_{i_1}^* \cdots e^{-(T_k - T_{k-1})A} \cdot \Sigma_{i_k}^* e^{-(t-T_k)A}\ . \tag{3.19}$$

Equations (3.18) and (3.19) also give a graphical representation of the process $(m_t, C_t)_{t\geqslant 0}$: during the interval $[0, T_1)$, m_t follows the evolution given by the (vectorial) system

$$\frac{dy}{dt} = F(y) \tag{3.20}$$

(where F has been previously introduced for the process $\tilde{\omega}_t$). At time T_1, the component m_{i_1+N} (which corresponds to the mean of p_{i_1}) is flipped, and gives a new mean vector. Then, the deterministic evolution goes on up to the time of the next flip, and so on.

In the same way, during the interval $[0, T_1)$, C_t follows the evolution given by the (matrix) system:

$$\frac{dM}{dt} = -AM + MA \tag{3.21}$$

(where A has been previously defined). At time T_1, all the components $C_{i_1,j}$ and C_{i,i_1} when $j \neq i_1$ and $i \neq i_1$ are flipped and the matrix C_{T_1} becomes $\Sigma_{i_1} \cdot C_{T_1} \cdot \Sigma_{i_1}^*$. The generator of this Markov process $(m_t, C_t)_{t \geqslant 0}$ is exactly the one defined by (3.6). Consequently, for $t \geqslant 0$, the law of the random variable (m_t, C_t) is $\theta^t_{m_0,C_0}$, where

$$m_0 = \left(-\frac{\lambda_0}{\beta_0}\left(\frac{0}{N}\right), \cdots, -\frac{\lambda_0}{\beta_0}\left(\frac{N-1}{N}\right), \underbrace{0, \cdots, 0}_{N} \right) \tag{3.22}$$

and

$$C_0 = \begin{pmatrix} D & 0_N \\ 0_N & D' \end{pmatrix} \quad \text{where} \quad \begin{cases} D = \text{diag}\left(\cdots, \dfrac{1}{\beta_0(x/N)} + \dfrac{\lambda_0^2(x/N)}{\beta_0^2(x/N)}, \cdots \right) , \\ D' = \text{diag}\left(\cdots, \dfrac{1}{\beta_0(x/N)}, \cdots \right) , \end{cases} \tag{3.23}$$

as it can be deduced from Lemma 1.1. Recall that in this section, μ_0^N is given by

$$\mu_0^N(d\mathbf{r}, d\mathbf{p}) = \prod_{x \in \mathbb{T}_N} \frac{\exp\left(-\beta_0\left(x/N\right) e_x - \lambda_0(x/N) r_x\right)}{Z(\beta_0(\cdot), \lambda_0(\cdot))} dr_x dp_x \ . \tag{3.24}$$

It follows that the density $\tilde{\mu}_t^N$ is equal to

$$\tilde{\mu}_t^N(\cdot) = \int \tilde{\mu}_t^\xi(\cdot)\ d\mathbb{P}(\xi) = \int G_{m,C}(\cdot)\ d\theta^t_{m_0,C_0}(m, C) \ . \tag{3.25}$$

□

Remark. Observe that

$$\tilde{\mu}_t^N[p_x] = \int G_{m,C}(p_x)\ d\theta^t_{m_0,C_0}(m, C) = \int \pi_x\ d\theta^t_{m_0,C_0}(m, C) = \mathbb{E}_{m_0,C_0}[\pi_x(t)] \ , \tag{3.26}$$

$$\tilde{\mu}_t^N[r_x] = \int G_{m,C}(r_x)\ d\theta^t_{m_0,C_0}(m, C) = \int \rho_x\ d\theta^t_{m_0,C_0}(m, C) = \mathbb{E}_{m_0,C_0}[\rho_x(t)] \ . \tag{3.27}$$

Lemma 3.2. *Let $(m_t, C_t)_{t \geqslant 0}$ be the Markov process defined above. As previously done, we introduce $\rho(t), \pi(t) \in \mathbb{R}^N$ and $U(t), V(t), Z(t) \in \mathfrak{M}_N(\mathbb{R})$ such that*

$$m_t = (\rho(t), \pi(t)) \quad \textit{and} \quad C_t = \begin{pmatrix} U(t) & Z^*(t) \\ Z(t) & V(t) \end{pmatrix} \tag{3.28}$$

Then,

$$\mathbb{P}_{m_0,C_0} \text{ - a. s. } , \ \forall\ t \geqslant 0, \ \begin{cases} \pi_y^2(t) \leqslant V_{y,y}(t) \ , \\ \rho_y^2(t) \leqslant U_{y,y}(t) \ . \end{cases} \tag{3.29}$$

Proof. First of all, let us notice that the quantities $V_{y,y}(t)-\pi_y^2(t)$ and $U_{y,y}(t)-\rho_y^2(t)$ are the diagonal components of the symmetric matrix $S_t := m_t \cdot {}^t m_t - C_t$. From Lemma 3.1, we have

$$S_t = \int S_t^\xi \; d\mathbb{P}(\xi) \; . \tag{3.30}$$

For any sequence of sites and ordered times $\xi = (i_1, T_1), \ldots, (i_k, T_k), \ldots$, the symmetric matrix S_t^ξ is positive because this is the matrix of covariances of $\tilde{\omega}_t^\xi$. It follows that S_t is positive, and its diagonal components are all positive. □

Remark. In the case of the pinned chain, the matrix A is slightly different, but all the notations and conclusions are still valid. The initial correlation matrix for the pinned model is not more diagonal, but has non-trivial values on the upper and lower diagonals. The initial mean vector is equal to $0_{\mathbb{R}^{2N}}$.

3.2 The Evolution of $(m_t, C_t)_{t\geqslant 0}$

Thanks to the regularity of β_0 and λ_0, we know that there exists a constant K which does not depend on N such that

$$\begin{cases} \dfrac{1}{N}\displaystyle\sum_{i,j}\left[(U_{i,j})^2(0) + (V_{i,j})^2(0) + 2(Z_{i,j})^2(0)\right] \leqslant K \; , \\ \dfrac{1}{N}\displaystyle\sum_{i}\left[U_{i,i}(0) + V_{i,i}(0)\right] \leqslant K \; , \\ \dfrac{1}{N}\displaystyle\sum_{i}\Big[(U_{i,i})^k(0) + (V_{i,i})^k(0)\Big] \leqslant K^k \; , \text{ for all } k \geqslant 1 \; . \end{cases} \tag{3.31}$$

Moreover, one can easily show that

$$\mathcal{G}\left(\sum_{i,j}(U_{i,j})^2 + (V_{i,j})^2 + 2(Z_{i,j})^2\right) = 0 \quad \text{and} \quad \mathcal{G}\left(\sum_i U_{i,i} + V_{i,i}\right) = 0 \; . \tag{3.32}$$

It results that the two first inequalities of (3.31) are actually uniform in t, in the sense that

$$\begin{cases} \dfrac{1}{N}\mathbb{E}_{m_0,C_0}\left[\displaystyle\sum_{i,j}\left[(U_{i,j}(t))^2 + (V_{i,j}(t))^2 + 2(Z_{i,j}(t))^2\right]\right] \leqslant K \; , \\ \dfrac{1}{N}\mathbb{E}_{m_0,C_0}\left[\displaystyle\sum_{i}\left[U_{i,i}(t) + V_{i,i}(t)\right]\right] \leqslant K \; . \end{cases} \tag{3.33}$$

We are going to see how this last inequality can be used in order to show (1.17). We denote by $u_k(t)$ and $v_k(t)$ the two quantities

$$\begin{cases} u_k(t) = \mathbb{E}_{m_0,C_0}\left[\displaystyle\sum_{i\in\mathbb{T}_N} U_{i,i}^k(t)\right] \; , \\ v_k(t) = \mathbb{E}_{m_0,C_0}\left[\displaystyle\sum_{i\in\mathbb{T}_N} V_{i,i}^k(t)\right] \; . \end{cases} \tag{3.34}$$

Let us make the link with (1.17). In view of (3.25), we can write

$$\tilde{\mu}_t^N \left[p_y^{2k} \right] = \int G_{m,C} \left[p_y^{2k} \right] \, d\theta^t_{m_0,C_0}(m,C) \ , \tag{3.35}$$

$$\tilde{\mu}_t^N \left[r_y^{2k} \right] = \int G_{m,C} \left[r_y^{2k} \right] \, d\theta^t_{m_0,C_0}(m,C) \ . \tag{3.36}$$

We use the convexity inequality $(a+b)^{2k} \leqslant 2^{2k-1} \ (a^{2k} + b^{2k})$ - which is true for all $a, b \in \mathbb{R}$ - to get

$$\begin{aligned} \tilde{\mu}_t^N \left[p_y^{2k} \right] &= \int G_{m,C} \left[(p_y - \pi_y + \pi_y)^{2k} \right] \, d\theta^t_{m_0,C_0}(m,C) \\ &\leqslant 2^{2k-1} \int G_{m,C} \left[(p_y - \pi_y)^{2k} \right] \, d\theta^t_{m_0,C_0}(m,C) + 2^{2k-1} \int \pi_y^{2k} \, d\theta^t_{m_0,C_0}(m,C) \ . \end{aligned} \tag{3.37}$$

We deal with the two terms of the sum, separately. First, observe that Gaussian centered moments are easily computable:

$$G_{m,C} \left[(p_y - \pi_y)^{2k} \right] = \left(V_{y,y} - \pi_y^2 \right)^k \frac{(2k)!}{k! \ 2^k} \ . \tag{3.38}$$

Then,

$$\sum_{y \in \mathbb{T}_N} \int \left(V_{y,y} - \pi_y^2 \right)^k \frac{(2k)!}{k! \ 2^k} \, d\theta^t_{m_0,C_0}(m,C) \leqslant \frac{(2k)!}{k! \ 2^k} \left(v_k(t) + \mathbb{E}_{m_0,C_0} \left[\sum_{y \in \mathbb{T}_N} \pi_y^{2k}(t) \right] \right) \ . \tag{3.39}$$

In the same way,

$$\sum_{y \in \mathbb{T}_N} \int \left(U_{y,y} - \rho_y^2 \right)^k \frac{(2k)!}{k! \ 2^k} \, d\theta^t_{m_0,C_0}(m,C) \leqslant \frac{(2k)!}{k! \ 2^k} \left(u_k(t) + \mathbb{E}_{m_0,C_0} \left[\sum_{y \in \mathbb{T}_N} \rho_y^{2k}(t) \right] \right) \ . \tag{3.40}$$

Lemma 3.2 shows that

$$\begin{aligned} \mathbb{E}_{m_0,C_0} \left[\sum_{y \in \mathbb{T}_N} \pi_y^{2k}(t) \right] &\leqslant \mathbb{E}_{m_0,C_0} \left[\sum_{y \in \mathbb{T}_N} V_{y,y}^k(t) \right] = v_k(t) \ , \\ \mathbb{E}_{m_0,C_0} \left[\sum_{y \in \mathbb{T}_N} \rho_y^{2k}(t) \right] &\leqslant \mathbb{E}_{m_0,C_0} \left[\sum_{y \in \mathbb{T}_N} U_{y,y}^k(t) \right] = u_k(t) \ . \end{aligned} \tag{3.41}$$

As a result,

$$\sum_y \tilde{\mu}_t^N \left[p_y^{2k} \right] \leqslant \frac{(2k)!}{k!} \ v_k(t) \sim 2 \left(\frac{4}{e} \right)^k k^k \ v_k(t) \ , \tag{3.42}$$

$$\sum_y \tilde{\mu}_t^N \left[r_y^{2k} \right] \leqslant \frac{(2k)!}{k!} \ u_k(t) \sim 2 \left(\frac{4}{e} \right)^k k^k \ u_k(t) \ . \tag{3.43}$$

In a few words, to get (1.17), we need to estimate the two quantities $u_k(t)$ and $v_k(t)$, which are related to C_t. That is what we do in the next section.

Remark. In the case of the pinned model, the p_x and q_x remain centered during the evolution: for all $t > 0$, $m_t = 0_{\mathbb{R}^{2N}}$. This simplifies the study since we do not need to center the variables. The result is the same: we need to estimate $u_k(t)$ and $v_k(t)$.

3.3 The Correlation Matrix

Lemma 3.3. *For any integer k not equal to 0, there exists a positive constant K which does not depend on N and t such that*

$$\begin{cases} v_k(t) \leqslant K^k \ N \ , \\ u_k(t) \leqslant K^k \ N \ . \end{cases} \tag{3.44}$$

Proof. First of all, (3.33) shows that, uniformly in t,

$$\begin{cases} u_1(t) \leqslant KN \\ u_2(t) \leqslant KN \end{cases} \text{ and } \begin{cases} v_1(t) \leqslant KN \\ v_2(t) \leqslant KN \ . \end{cases} \tag{3.45}$$

We observe that

$$u_k(t) + v_k(t) = \mathbb{E}_{m_0,C_0}\left[\sum_{i\in\mathbb{T}_N} C^k_{i,i}(t)\right] = \int \sum_{i\in\mathbb{T}_N} (C^\xi_{i,i})^k(t) \ d\mathbb{P}(\xi) \ . \tag{3.46}$$

Thanks to the dynamics description, we know the expression of the correlation matrix: conditionally to ξ, for all $t \in [T_k, T_{k+1})$,

$$C^\xi(t) = e^{(t-T_k)A}\cdot\Sigma_{i_k}\cdot e^{(T_k-T_{k-1})A}\cdots\Sigma_{i_1}\cdot e^{T_1A}\cdot C_0\cdot e^{-T_1A}\cdot\Sigma^*_{i_1}\cdots e^{-(T_k-T_{k-1})A}\cdot\Sigma^*_{i_k}e^{-(t-T_k)A} \ , \tag{3.47}$$

Consequently, since C_0 and $C^\xi(t)$ are similar, we have:

$$\forall \ k \in \mathbb{N}, \ \mathrm{Tr}([C^\xi(t)]^k) = \mathrm{Tr}(C_0^k) = O(N) \ . \tag{3.48}$$

More precisely,

$$\mathrm{Tr}(C_0^k) = \sum_{i\in\mathbb{T}_N} U^k_{i,i}(0) + V^k_{i,i}(0) = \sum_{i\in\mathbb{T}_N} \frac{1}{\beta_0^k(i/N)} + \left(\frac{1}{\beta_0(i/N)} + \frac{\lambda_0^2(i/N)}{\beta_0^2(i/N)}\right)^k \ . \tag{3.49}$$

From (3.31) we get $\mathrm{Tr}(C_0^k) \leqslant NK^k$, where K does not depend on N, ξ and t:

$$K := \sup_{u\in[0,1]}\left\{\frac{1}{\beta_0(u)} + \frac{\lambda_0^2(u)}{\beta_0^2(u)}\right\} \ . \tag{3.50}$$

Now we show that the same inequality holds for $\sum_i [C^\xi_{i,i}]^k(t)$. The matrix $C^\xi(t)$ is symmetric, hence diagonalizable, and after denoting its eigenvalues by $\lambda_1, ..., \lambda_{2N}$, we can write

$$\mathrm{Tr}([C^\xi(t)]^k) = \sum_{i=1}^{2N} \lambda_i^k \ . \tag{3.51}$$

We have now to compare $\sum_{i=1}^{2N}\lambda_i^k$ with $\sum_{i=1}^{2N}[C^\xi_{i,i}]^k(t)$. But, if we denote by P the orthogonal matrix of the eigenvectors of $C^\xi(t)$, then we get $C^\xi(t) = (P^\xi_t)^*\cdot D\cdot P^\xi_t$, where D is the diagonal matrix with

the eigenvalues $\lambda_1, ..., \lambda_{2N}$. For the sake of simplicity, we denote by $(P_{i,j})_{i,j}$ the components of P_t^ξ. Then,

$$[C_{i,i}^\xi]^k(t) = \left(\sum_{j,l} P_{i,j}^* D_{j,l} P_{l,i}\right)^k = \left(\sum_j P_{i,j}^* \lambda_j P_{j,i}\right)^k = \left(\sum_j P_{i,j}^* P_{j,i} \cdot \lambda_j\right)^k . \tag{3.52}$$

But, $\sum_j P_{i,j}^* P_{j,i} = 1$, since D is an orthogonal matrix. Consequently, we can use the convexity inequality, and we obtain

$$\sum_i [C_{i,i}^\xi]^k(t) \leqslant \sum_i \sum_j P_{i,j}^* P_{j,i} \lambda_j^k \leqslant \sum_j \lambda_j^k = \mathrm{Tr}([C^\xi(t)]^k) \leqslant NK^k . \tag{3.53}$$

Hence,

$$u_k(t) + v_k(t) \leqslant \int NK^k \, d\mathbb{P}(\xi) \leqslant NK^k . \tag{3.54}$$

□

Remark. We notice that the same proof works for the pinned case. The only difference is about the initial matrix C_0, but the smoothness of the profile β_0 is still true, and the estimate $\mathrm{Tr}(C_0^k) = O(N)$ is valid.

3.4 When μ_0^N is a Convex Combination of Gibbs Measures

As in Theorem 1.2, we now suppose that the initial probability measure μ_0^N is a convex combination of Gibbs states defined by

$$\mu_0^N(\cdot) = \int G_{m_0,C_0}(\cdot) \, d\sigma(m_0, C_0) . \tag{3.55}$$

If initially the process starts from ω_0 which is distributed according to a Gaussian measure G_{m_0,C_0}, we know from Lemma 3.1 that $\tilde{\omega}_t$ is distributed according to a convex combination of Gaussian measures written as

$$\int G_{m,C}(\cdot) \, d\theta_{m_0,C_0}^t(m, C) . \tag{3.56}$$

Consequently, in the case where μ_0^N is given by (3.55), the law of the process $\tilde{\omega}_t$ is given by

$$\tilde{\mu}_t^N(\cdot) = \int \left\{ \int G_{m,C}(\cdot) \, d\theta_{m_0,C_0}^t(m, C) \right\} d\sigma(m_0, C_0) . \tag{3.57}$$

Let us recall that we want to control, for $k \geqslant 1$, $\tilde{\mu}_t^N \left[\sum_{x \in \mathbb{T}_N} p_x^{2k}\right]$ and $\tilde{\mu}_t^N \left[\sum_{x \in \mathbb{T}_N} r_x^{2k}\right]$. Following the lines of the previous section, we notice that it is sufficient to control two quantities:

$$\begin{cases} \displaystyle\int \mathbb{E}_{m_0,C_0}\left[\sum_{i \in \mathbb{T}_N} U_{i,i}^k(t)\right] d\sigma(m_0, C_0) , \\ \displaystyle\int \mathbb{E}_{m_0,C_0}\left[\sum_{i \in \mathbb{T}_N} V_{i,i}^k(t)\right] d\sigma(m_0, C_0) . \end{cases} \tag{3.58}$$

Lemma 3.3 gives a constant $C(\lambda_0, \beta_0)$ which does not depend on N and t such that

$$\begin{cases} \mathbb{E}_{m_0,C_0}\left[\sum_{i\in\mathbb{T}_N} U_{i,i}^k(t)\right] \leqslant [C(\lambda_0,\beta_0)]^k\ N\ , \\ \mathbb{E}_{m_0,C_0}\left[\sum_{i\in\mathbb{T}_N} V_{i,i}^k(t)\right] \leqslant [C(\lambda_0,\beta_0)]^k\ N\ . \end{cases} \tag{3.59}$$

More precisely,

$$C(\lambda_0,\beta_0) = \sup_{u\in[0,1]} \left\{ \frac{1}{\beta_0(u)} + \frac{\lambda_0^2(u)}{\beta_0^2(u)} \right\}\ . \tag{3.60}$$

In order to keep the same control, we have to suppose that, for all $k \geqslant 1$,

$$\int [K(m,C)]^k\ d\sigma(m,C) < \infty, \quad \text{where } K(m,C) := \sup_{i\in\mathbb{T}_N} C_{i,i}\ . \tag{3.61}$$

Finally, let us observe that all estimates are given for $\tilde{\mu}_t^N$ but are still true for the accelerated law μ_t^N. Indeed, the constants that appear do not depend on N and t.

A Proof of the Taylor Expansions

Now we prove Proposition 2.1. For the sake of simplicity, we define

$$\begin{cases} g_x(\mathbf{r},\mathbf{p}) := -\frac{r_x}{2\gamma}(p_{x+1} + p_x + \frac{\gamma}{2} r_x)\ , \\ f_x(\mathbf{r},\mathbf{p}) := -\frac{p_{x+1}}{\gamma}\ , \\ \delta_x(\mathbf{r},\mathbf{p}) := \beta_t'\left(\frac{x}{N}\right)\ g_x + \lambda_t'\left(\frac{x}{N}\right)\ f_x = F\left(t, \frac{x}{N}\right) \cdot \tau_x h(\mathbf{r},\mathbf{p})\ . \end{cases} \tag{A.1}$$

First we will compute the first part that appears in the integral $N^2\left(\phi_t^N\right)^{-1}\mathcal{L}_N^*\phi_t^N$, then we will compute the second part $-\partial_t\phi_t^N/\phi_t^N \times f_t^N$.

A.1 First Term: the Adjoint Operator

Lemma A.1.

$$\begin{aligned} \mathcal{A}\phi_t^N =& \frac{\phi_t^N}{N^2}\sum_{x\in\mathbb{T}_N} \beta_t''\left(\frac{x}{N}\right)\left[p_{x+1}r_x + \frac{p_x^2 + r_x r_{x-1}}{2\gamma}\right] - \lambda_t''\left(\frac{x}{N}\right)\left[p_{x+1} + \frac{r_{x+1}}{\gamma}\right] \\ &+ \frac{\phi_t^N}{N^2}\sum_{x\in\mathbb{T}_N}\left[\mathcal{L}^*(\delta_x) + \mathcal{A}(\delta_x)\right] + o\left(\frac{1}{N}\right). \end{aligned} \tag{A.2}$$

Proof. First, remind that the expression of ϕ_t^N is given by

$$\phi_t^N(\mathbf{r},\mathbf{p}) = \frac{(Z(1,0))^n}{Z(\chi_t(\cdot))} \prod_{x\in\mathbb{T}_N} \exp\left(e_x\left(-\beta_t\left(\frac{x}{N}\right)+1\right) - \lambda_t\left(\frac{x}{N}\right) r_x + \frac{1}{N}F\left(t,\frac{x}{N}\right)\cdot \tau_x h(\mathbf{r},\mathbf{p})\right). \tag{A.3}$$

By definition,

$$\mathcal{A}\phi_t^N = \phi_t^N \sum_{x\in\mathbb{T}_N}\left[\left(1-\beta_t\left(\frac{x}{N}\right)\right)\mathcal{A}(e_x) - \lambda_t\left(\frac{x}{N}\right)\mathcal{A}(r_x)\right] + \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\mathcal{A}(\delta_x)\ . \tag{A.4}$$

We write down the two conservation laws:

$$\mathcal{A}(e_x) = j^e_{x+1} - j^e_x \quad \text{where } j^e_x := p_x r_{x-1}\ , \tag{A.5}$$
$$\mathcal{A}(r_x) = j^r_{x+1} - j^r_x \quad \text{where } j^r_x := p_x\ . \tag{A.6}$$

Hence,

$$\mathcal{A}\phi_t^N = \phi_t^N \sum_{x\in\mathbb{T}_N}\left[\left(1-\beta_t\left(\frac{x}{N}\right)\right)\nabla(j^e_x)_x - \lambda_t\left(\frac{x}{N}\right)\nabla(j^r_x)_x\right] + \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\mathcal{A}(\delta_x)\ . \tag{A.7}$$

where $\nabla(f)_x = f_{x+1} - f_x$.

We are interesting in the first two terms in the sum, and we compute a discrete summation by part. Indeed,

$$\sum_{y\in\mathbb{T}_N} f_y \nabla(g)_y = -\sum_{y\in\mathbb{T}_N} g_{y+1}\nabla(f)_y\ . \tag{A.8}$$

We obtain the following terms:

$$\beta_t\left(\frac{x+1}{N}\right) - \beta_t\left(\frac{x}{N}\right) = \beta_t'\left(\frac{x}{N}\right)\frac{1}{N} + \beta_t''\left(\frac{x}{N}\right)\frac{1}{N^2} + O\left(\frac{1}{N^3}\right), \tag{A.9}$$
$$\lambda_t\left(\frac{x+1}{N}\right) - \lambda_t\left(\frac{x}{N}\right) = \lambda_t'\left(\frac{x}{N}\right)\frac{1}{N} + \lambda_t''\left(\frac{x}{N}\right)\frac{1}{N^2} + O\left(\frac{1}{N^3}\right). \tag{A.10}$$

First of all, we look at the term obtained in the sum with $O\left(N^{-3}\right)$. We want to prove

$$N^2\int\sum_{x\in\mathbb{T}_N} p_{x+1}r_x\ O\left(\frac{1}{N^3}\right) f_t^N d\mu^N_{1,0} \leqslant C\ H_N(t) + o(N)\ . \tag{A.11}$$

We use the entropy inequality. Let $\varepsilon : \mathbb{N}\to\mathbb{R}$ be a bounded function. We get

$$\frac{1}{N}\int\sum_{x\in\mathbb{T}_N} p_{x+1}r_x\ \varepsilon(N) f_t^N d\mu^N_{1,0} \leqslant \frac{H_N(t)}{\alpha} + \frac{1}{\alpha}\log\int\exp\left(\frac{\alpha}{N}\sum_x p_{x+1}r_x\ \varepsilon(N)\right)\phi_t^N d\mu^N_{1,0}\ . \tag{A.12}$$

But, let us recall the inequality $p_{x+1}r_x \leqslant (p^2_{x+1}+r^2_x)/2$ and for N large enough, we have

$$\nu^N_{\chi_t(\cdot)}\left[\exp\left(\frac{\alpha}{N}p_x^2\varepsilon(N)\right)\right] \sim_{N\to\infty} \sqrt{\frac{2\pi\ 2N}{N\beta - 2\alpha\varepsilon(N)}} \times \sqrt{\frac{\beta}{2\pi}} = O(1)\ . \tag{A.13}$$

We obtain a similar estimate for $\nu^N_{\chi_t(\cdot)}\left[\exp\left(\alpha N^{-1} r_x \varepsilon(N)\right)\right]$.

Therefore, we have showed

$$\frac{1}{N}\int \sum_{x\in\mathbb{T}_N} p_{x+1} r_x^2\ \varepsilon(N) f_t^N d\mu_{1,0} \leqslant \frac{H_N(t)}{\alpha} + O(1)\ . \tag{A.14}$$

Hence,

$$\begin{aligned}\mathcal{A}\phi_t^N =& \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\left[\beta_t'\left(\frac{x}{N}\right)p_{x+1}r_x + \lambda_t'\left(\frac{x}{N}\right)p_{x+1}\right] + \frac{\phi_t^N}{N^2}\sum_{x\in\mathbb{T}_N}\left[\beta_t''\left(\frac{x}{N}\right)p_{x+1}r_x + \lambda_t''\left(\frac{x}{N}\right)p_{x+1}\right]\\ &+ \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\mathcal{A}(\delta_x) + o\left(\frac{1}{N}\right)\ .\end{aligned} \tag{A.15}$$

Moreover, we can compute two equations which are called "fluctuation-dissipation equations". In other words, we decompose the current of energy and the current of deformation as the sum of a discrete gradient and a dissipative term:

$$p_{x+1} = \nabla\left(\frac{-r_x}{\gamma}\right)_x + \mathcal{L}^*(f_x)\ , \tag{A.16}$$

$$p_{x+1}r_x = \nabla\left(-\frac{p_x^2 + r_x r_{x-1}}{2\gamma}\right)_x + \mathcal{L}^*(g_x)\ . \tag{A.17}$$

We use the two equations (A.16) and (A.17), and we obtain

$$\begin{aligned}\mathcal{A}\phi_t^N =& \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\left\{\beta_t'\left(\frac{x}{N}\right)\left[\nabla\left(-\frac{p_x^2 + r_x r_{x-1}}{2\gamma}\right)_x + \mathcal{L}^*(g_x)\right] + \lambda_t'\left(\frac{x}{N}\right)\left[\nabla\left(\frac{-r_x}{\gamma}\right)_x + \mathcal{L}^*(f_x)\right]\right\}\\ &+ \frac{\phi_t^N}{N^2}\sum_{x\in\mathbb{T}_N}\left[\beta_t''\left(\frac{x}{N}\right)p_{x+1}r_x + \lambda_t''\left(\frac{x}{N}\right)p_{x+1}\right] + \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\mathcal{A}(\delta_x) + o\left(\frac{1}{N}\right).\end{aligned} \tag{A.18}$$

We sum again by part, on the two terms with a gradient, and we obtain as before

$$\begin{aligned}\mathcal{A}\phi_t^N =& \frac{\phi_t^N}{N^2}\sum_{x\in\mathbb{T}_N}\left\{\beta_t''\left(\frac{x}{N}\right)\left[\frac{p_{x+1}^2 + r_x r_{x+1}}{2\gamma} + p_{x+1}r_x\right] + \lambda_t''\left(\frac{x}{N}\right)\left[\frac{r_{x+1}}{\gamma} + p_{x+1}\right]\right\}\\ &+ \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\left\{\mathcal{A}(\delta_x) + \mathcal{L}^*(\delta_x)\right\} + o\left(\frac{1}{N}\right).\end{aligned} \tag{A.19}$$

We get the result. □

Lemma A.2.

$$\mathcal{S}\phi_t^N = \frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\mathcal{S}(\delta_x) + \frac{\phi_t^N}{4N^2}\sum_{y\in\mathbb{T}_N}\left(\sum_{x\in\mathbb{T}_N}\delta_x(\mathbf{p}^y) - \delta_x(\mathbf{p})\right)^2 + \phi_t^N\ \varepsilon(N)\ , \tag{A.20}$$

where $\mu_t^N\left[N^2\varepsilon(N)\right] = o(N)$.

Proof. Thanks to the exponential term, we have

$$\mathcal{S}\phi_t^N = \frac{\phi_t^N}{2} \sum_{y\in\mathbb{T}_N} \left\{ \exp\left[\frac{1}{N} \sum_{x\in\mathbb{T}_N} \delta_x(\mathbf{p}^y) - \delta_x(\mathbf{p}) \right] - 1 \right\} . \tag{A.21}$$

The main idea consists in noting that $e^x - 1 = x + x^2/2 + o(x^2)$. We are going to give a rigorous proof of this estimate in our context thanks to the hypothesis on the energy moments. More precisely, in view of (1.42) and Lemma 2.1, we want to prove that

$$N^2 \mu_t^N \left[\sum_{y\in\mathbb{T}_N} \sum_{k\geqslant 3} \frac{F_y^k}{k!\ N^k} \right] = o(N), \text{ where } F_y = \sum_{x\in\mathbb{T}_N} \left(\delta_x(\mathbf{p}^y) - \delta_x(\mathbf{p})\right) . \tag{A.22}$$

Let us compute F_y. We notice that in the following expression,

$$\sum_{x\in\mathbb{T}_N} -\beta_t'\left(\frac{x}{N}\right) \frac{r_x}{2\gamma}(p_{x+1} + p_x + \frac{\gamma}{2} r_x) - \lambda_t'\left(\frac{x}{N}\right) \frac{p_{x+1}}{\gamma} , \tag{A.23}$$

the only terms which are changing when we flip $\mathbf{p}$ into $\mathbf{p}^y$ are

- the term when $x = y$, and the difference is

$$\frac{r_y p_y}{\gamma} \beta_t'\left(\frac{y}{N}\right) , \tag{A.24}$$

- the term when $x = y - 1$, and the difference is

$$\frac{r_{y-1} p_y}{\gamma} \beta_t'\left(\frac{y-1}{N}\right) + \lambda_t'\left(\frac{y-1}{N}\right) \frac{2p_y}{\gamma} . \tag{A.25}$$

In other words, we have to show that

$$N\ \mu_t^N \left[\sum_{y\in\mathbb{T}_N} \sum_{k\geqslant 3} \frac{|F_y|^k}{k!\ N^k} \right] \xrightarrow[N\to\infty]{} 0 . \tag{A.26}$$

with

$$\begin{aligned} |F_y(t)| &= \left| \frac{r_y p_y}{\gamma} \beta_t'\left(\frac{y}{N}\right) + \frac{r_{y-1} p_y}{\gamma} \beta_t'\left(\frac{y-1}{N}\right) + \lambda_t'\left(\frac{y-1}{N}\right) \frac{2p_y}{\gamma} \right| \\ &\leqslant C_0\ |r_y p_y| + C_1\ |r_{y-1} p_y| + C_2\ |p_y| \\ &\leqslant C_0\ \frac{r_y^2 + p_y^2}{2} + C_1\ \frac{r_{y-1}^2 + p_y^2}{2} + C_2\ (1 + p_y^2) \\ &\leqslant K\ (1 + e_y + e_{y-1}) , \end{aligned}$$

where K is a constant which does not depend on N and t.

First of all, we introduce the space $A_y = \{e_y \leqslant 1,\ e_{y-1} \leqslant 1\}$.

$$N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{(e_y + e_{y-1} + 1)^k\ K^k\ \mathbb{1}_{\{e_y\leqslant 1,\ e_{y-1}\leqslant 1\}}}{k!\ N^k} \right] \leqslant N \sum_{y\in\mathbb{T}_N} \sum_{k\geqslant 3} \frac{(3K)^k}{k!\ N^k}$$
$$= N^2 \sum_{k\geqslant 3} \frac{(3K)^k}{k!\ N^k} \xrightarrow[N\to\infty]{} 0 \tag{A.27}$$

Since we have $(e_y + e_{y-1})^k \mathbb{1}_{A_y^C} \leqslant (2e_y + e_{y-1})^k$, we deduce $(e_y + e_{y-1})^k \mathbb{1}_{A_y^C} \leqslant C_0^k\ e_y^k + C_1^k e_{y-1}^k$. Consequently,

$$N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{|F_y|^k\ K^k\ \mathbb{1}_{A_y^C}}{k!\ N^k} \right] \leqslant N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{e_y^k\ K'^k}{k!\ N^k} \right] + N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{e_{y-1}^k\ K'^k}{k!\ N^k} \right] . \tag{A.28}$$

Now we deal with $N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} e_y^k/(k! N^k) \right]$. Remind that $e_y^k \leqslant 2\ (p_y^{2k} + r_y^{2k})$.
We are reduced to prove that

$$N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{p_y^{2k}}{k!\ N^k} \right] \xrightarrow[N\to\infty]{} 0 \quad \text{and} \quad N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{r_y^{2k}}{k!\ N^k} \right] \xrightarrow[N\to\infty]{} 0 \ . \tag{A.29}$$

We can flip the summations thanks to Fubini theorem. From the hypothesis on the moments bounds we get

$$N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{p_y^{2k}}{k!\ N^k} \right] \leqslant N^2 \sum_{k\geqslant 3} \frac{(C\ k)^k}{k!\ N^k} \xrightarrow[N\to\infty]{} 0 \ . \tag{A.30}$$

This last limit is deduced from the property of the series $S(x) := \sum_{k\geqslant 3} k^k\ x^{k-2}/(k!)$. It is a power series which has a strictly positive radius and is continuous at 0. Then,

$$N^2 \sum_{k\geqslant 3} \frac{(C\ k)^k}{k!\ N^k} = C^2 S\left(\frac{C}{N}\right) \xrightarrow[N\to\infty]{} 0 \ . \tag{A.31}$$

The same happens for the second sum. It follows that

$$N \sum_{y\in\mathbb{T}_N} \mu_t^N \left[\sum_{k\geqslant 3} \frac{F_y^k}{k!\ N^k} \right] \xrightarrow[N\to\infty]{} 0 \ . \tag{A.32}$$

□

After adding the two terms and get some simplifications, we obtain this following final result.

Proposition A.1.

$$\frac{1}{\phi_t^N} N^2 \mathcal{L}_N^* \phi_t^N = \sum_{x\in\mathbb{T}_N} \left\{ -\partial_q^2 \beta\left(t, \frac{x}{N}\right) \left[\frac{p_{x+1}^2 + r_{x+1} r_x}{2\gamma} + p_{x+1} r_x \right] - \partial_q^2 \lambda\left(t, \frac{x}{N}\right) \left[\frac{r_{x+1}}{\gamma} + p_{x+1} \right] \right\}$$
$$+ \frac{1}{4\gamma} \sum_{x\in\mathbb{T}_N} p_x^2 \left[r_x \partial_q \beta\left(t, \frac{x}{N}\right) + r_{x-1} \partial_q \beta\left(t, \frac{x-1}{N}\right) + 2\partial_q \lambda\left(t, \frac{x-1}{N}\right) \right]^2 + o(N) \ . \tag{A.33}$$

Proof. There are simplifications when we write $(-\mathcal{A}+\gamma\mathcal{S})(\phi_t^N)$. Actually,

$$\frac{\phi_t^N}{N}\sum_{x\in\mathbb{T}_N}\{-\mathcal{A}(\delta_x)+\gamma\mathcal{S}(\delta_x)-\mathcal{L}^*(\delta_x)\}=0\ . \tag{A.34}$$

The result follows. □

A.2 Second Term: Logarithmic Derivative

First, we notice that $\partial_t\phi_t^N/\phi_t^N=\partial_t\{\log(\phi_t^N)\}$. Moreover,

$$\begin{aligned}\log(\phi_t^N)=C+\sum_{x\in\mathbb{T}_N}e_x\left(-\beta_t\left(\frac{x}{N}\right)+1\right)-\lambda_t\left(\frac{x}{N}\right)r_x-\beta_t'\left(\frac{x}{N}\right)\frac{r_x}{2\gamma N}(p_{x+1}+p_x+\frac{\gamma}{2}r_x)\\ +\lambda_t'\left(\frac{x}{N}\right)\frac{p_x}{\gamma N}-\log\left[Z\left(\beta_t\left(\cdot\right),\lambda_t\left(\cdot\right)\right)\right]\ .\end{aligned} \tag{A.35}$$

We need to estimate the partition function $Z(\beta_t(\cdot),\lambda_t(\cdot))$. More precisely, we compare this new partition function to the exact partition function

$$\tilde{Z}(\beta_t(\cdot),\lambda_t(\cdot))=\prod_{x\in\mathbb{T}_N}\frac{2\pi}{\beta_t(x/N)}\exp\left(\frac{\lambda_t^2(x/N)}{2\beta_t(x/N)}\right). \tag{A.36}$$

We prove the following lemma.

Lemma A.3.

$$\left|\partial_t\log Z(\beta_t(\cdot),\lambda_t(\cdot))-\partial_t\log\tilde{Z}(\beta_t(\cdot),\lambda_t(\cdot))\right|=O(1)\quad \textit{when } N\to\infty. \tag{A.37}$$

Proof. First of all, remind that the exact expression of $Z_t:=Z(\beta_t(\cdot),\lambda_t(\cdot))$ can be written as

$$\begin{aligned}Z_t=\int_{\mathbb{R}^{2N}}\Bigg[\prod_{x\in\mathbb{T}_N}\exp\Bigg\{-\beta_t\left(\frac{x}{N}\right)e_x-\lambda_t\left(\frac{x}{N}\right)r_x\\ -\frac{1}{N}\beta_t'\left(\frac{x}{N}\right)\frac{r_x}{2\gamma}\left(p_{x+1}+p_x+\frac{\gamma}{2}r_x\right)-\frac{1}{N}\lambda_t'\left(\frac{x}{N}\right)\frac{p_{x+1}}{\gamma}\Bigg\}\Bigg]d\mathbf{p}d\mathbf{r}\\ =\exp\left\{\frac{1}{2}\|b_t\|^2\right\}\int_{\mathbb{R}^{2N}}\exp\left\{-\frac{1}{2}\langle X-b_t,C_t(X-b_t)\rangle\right\}dX=\exp\left\{\frac{1}{2}\|b_t\|^2\right\}(2\pi)^N|\det(C_t)|^{1/2}.\end{aligned} \tag{A.38}$$

where b_t is a vector and C_t is a symmetric positive matrix.

More precisely, one can see that

$$\|b_t\|^2=\sum_{x\in\mathbb{T}_N}\frac{\lambda_t^2}{\beta_t}\left(\frac{x}{N}\right)+\frac{1}{N}\sum_{x\in\mathbb{T}_N}h_t\left(\frac{x}{N}\right) \tag{A.39}$$

where h_t is a function that can be easily expressed with $\lambda_t,\beta_t,\lambda_t'$ and β_t'. Then, h_t is smooth.

Moreover, C_t can be written as $C_t = D_t + N^{-1}H_t$ with D_t a diagonal matrix and H_t a symmetric matrix which has at most three non-zero components on each row and each column. More precisely,

$$D_t = \begin{pmatrix} \ddots & & (0) \\ & \beta_t(x/N) & \\ (0) & & \ddots \end{pmatrix}, \tag{A.40}$$

$$H_t = \begin{pmatrix} \begin{pmatrix} \ddots & & (0) \\ & -(1/4)\beta_t'(x/N) & \\ (0) & & \ddots \end{pmatrix} & \begin{pmatrix} \ddots & -(2\gamma)^{-1}\beta_t'(x/N) & (0) \\ & -(2\gamma)^{-1}\beta_t'(x/N) & \ddots \\ (0) & & \ddots \end{pmatrix} \\ \begin{pmatrix} \ddots & & (0) \\ \ddots & -(2\gamma)^{-1}\beta_t'(x/N) & \\ (0) & -(2\gamma)^{-1}\beta_t'(x/N) & \ddots \end{pmatrix} & (0) \end{pmatrix} \tag{A.41}$$

Now we write

$$\partial_t \log Z_t = \frac{1}{2}\sum_x \partial_t \left(\frac{\lambda_t^2}{\beta_t}\left(\frac{x}{N}\right)\right) + \frac{1}{2}\partial_t \log\det(C_t) + \frac{1}{N}\sum_x \partial_t h_t\left(\frac{x}{N}\right), \tag{A.42}$$

$$\partial_t \log \tilde{Z}_t = \frac{1}{2}\sum_x \partial_t \left(\frac{\lambda_t^2}{\beta_t}\left(\frac{x}{N}\right)\right) + \frac{1}{2}\partial_t \log\det(D_t)\ . \tag{A.43}$$

But, $\left|N^{-2}\sum_x \partial_t h_t\left(x/N\right)\right| = O(1)$ since h_t is smooth.

It remains to show that the following quantity is bounded above by a constant that does not depend on N:

$$\left|\partial_t\left(\log\frac{\det C_t}{\det D_t}\right)\right| = \left|\partial_t\left[\log\det\left(I + \frac{1}{N}D_t^{-1}H_t\right)\right]\right| = \left|\frac{\partial_t\{\det(I + D_t^{-1}H_t/N)\}}{\det(I + D_t^{-1}H_t/N)}\right|\ . \tag{A.44}$$

We denote by K_t the matrix $D_t^{-1}H_t$, which also has at most three non-zero components on each row and each column, and by K_t' the derivative of K_t with respect to t. We notice that for N large enough, the matrix $I + K_t/N$ is invertible, and we have

$$\left|\partial_t\left(\log\frac{\det C_t}{\det D_t}\right)\right| = \left|\frac{\mathrm{Tr}({}^t\mathrm{com}(I + K_t/N)\cdot(I + K_t'/N))}{\det(I + K_t/N)}\right| = \left|\mathrm{Tr}\left[\left(I + \frac{1}{N}K_t\right)^{-1}\left(I + K_t'\right)\right]\right|\ , \tag{A.45}$$

where $\mathrm{com}(A)$ is the comatrix of A.

Now we deal with $(I + K_t/N)^{-1}$:

$$\left(I + \frac{1}{N}K_t\right)^{-1} = I - K_t + \sum_{k\geqslant 2}\frac{(-1)^k}{N^k}K_t^k\ . \tag{A.46}$$

But, the component (i,j) of K_t^k can be written as $\sum_{i_1,\dots,i_k} a_{i,i_1}\ a_{i_1,i_2}\cdots\ a_{i_k,j}$ where $a_{i,j}$ are the components of K_t. We know that there are at most three non-zero components on each row and

each column, and that they are all bounded by a constant C that does not depend on N (since β_t and λ_t are smooth). Then, it implies that $|\mathrm{Tr}(K_t^k)| \leqslant N3^kC$.

It follows that

$$\left|\mathrm{Tr}\left[\left(I+\frac{1}{N}K_t\right)^{-1}\right]\right| = \left|\mathrm{Tr}\left(I-K_t+\sum_{k\geqslant 2}\frac{(-1)^k}{N^k}K_t^k\right)\right| \leqslant 1+|\mathrm{Tr}(K_t)|+C\sum_{k\geqslant 2}\frac{3^k}{N^{k-1}} = O(1)\ , \tag{A.47}$$

because $\mathrm{Tr}(K_t) = O(1)$ (we can compute it and again use the smoothness of the profiles).

In the same way, we show that

$$\left|\mathrm{Tr}\left[K_t'\left(I+\frac{1}{N}K_t\right)^{-1}\right]\right| = O(1)\ . \tag{A.48}$$

It ends the proof. □

We deduce from the previous result that

$$\partial_t \log\left[Z\left(\beta_t\left(\cdot\right),\lambda_t\left(\cdot\right)\right)\right] = \sum_{x\in\mathbb{T}_N} -\frac{\partial_t\beta_t(x/N)}{\beta_t(x/N)} + \partial_t\lambda_t(x/N)\frac{\lambda_t(x/N)}{\beta_t(x/N)} - \frac{\partial_t\beta_t(x/N)}{2}\frac{\lambda_t^2(x/N)}{\beta_t^2(x/N)} + O(1)\ . \tag{A.49}$$

Consequently, we get the following statement.

Proposition A.2.

$$\begin{aligned}\partial_t\{\log(\phi_t^N)\} = \sum_{x\in\mathbb{T}_N} &- e_x\partial_t\beta\left(t,\frac{x}{N}\right) - r_x\partial_t\lambda\left(t,\frac{x}{N}\right) - \frac{r_x}{2\gamma N}\partial_t\partial_q\beta\left(t,\frac{x}{N}\right)\left(p_{x+1}+p_x+\frac{\gamma}{2}r_x\right)\\ &- \frac{p_x}{\gamma N}\partial_t\partial_q\lambda\left(t,\frac{x}{N}\right) + \frac{\partial_t\beta(t,x/N)}{\beta(t,x/N)}\\ &- \partial_t\lambda(t,x/N)\frac{\lambda(t,x/N)}{\beta(t,x/N)} + \frac{\partial_t\beta(t,x/N)}{2}\frac{\lambda^2(t,x/N)}{\beta^2(t,x/N)},\end{aligned} \tag{A.50}$$

$$\partial_t\{\log(\phi_t^N)\} = \sum_{x\in\mathbb{T}_N} -\left[e_x-\mathbf{e}\left(t,\frac{x}{N}\right)\right]\partial_t\beta\left(t,\frac{x}{N}\right) + \left[r_x-\mathbf{r}\left(t,\frac{x}{N}\right)\right]\partial_t\lambda\left(t,\frac{x}{N}\right) + O(1)\ . \tag{A.51}$$

A.3 Ending Proof

We are now able to prove the Taylor expansion. According to the results of the two previous parts, we have

$$\begin{aligned}\frac{1}{\phi_t^N}N^2\mathcal{L}_N^*\phi_t^N - \partial_t\{\log(\phi_t^N)\} = \sum_{x\in\mathbb{T}_N}\Bigg\{&-\partial_q^2\beta\left(t,\frac{x}{N}\right)\left[\frac{p_x^2+r_{x-1}r_x}{2\gamma}+p_xr_{x-1}\right] - \partial_q^2\lambda\left(t,\frac{x}{N}\right)\left[\frac{r_x}{\gamma}+p_x\right]\\ &+\frac{p_x^2}{4\gamma}\left[(r_x+r_{x-1})\partial_q\beta\left(t,\frac{x}{N}\right)+2\partial_q\lambda\left(t,\frac{x}{N}\right)\right]^2\\ &+\left[e_x-\mathbf{e}\left(t,\frac{x}{N}\right)\right]\partial_t\beta\left(t,\frac{x}{N}\right)+\left[r_x-\mathbf{r}\left(t,\frac{x}{N}\right)\right]\partial_t\lambda\left(t,\frac{x}{N}\right)\Bigg\} + o(N)\ .\end{aligned} \tag{A.52}$$

Using the notations introduced in Section 2, it becomes:

$$\frac{1}{\phi_t^N} N^2 \mathcal{L}_N^* \phi_t^N - \partial_t \{\log(\phi_t^N)\} = \sum_{x \in \mathbb{T}_N} \Big\{ -\frac{1}{2\gamma} \partial_q^2 \beta \left(t, \frac{x}{N}\right) J_x^1 - \frac{1}{\gamma} \partial_q^2 \lambda \left(t, \frac{x}{N}\right) J_x^2$$
$$+ \frac{1}{4\gamma} \left[\partial_q \beta \left(t, \frac{x}{N}\right)\right]^2 J_x^3 + \frac{1}{\gamma} \partial_q \beta \left(t, \frac{x}{N}\right) \partial_q \lambda \left(t, \frac{x}{N}\right) J_x^4$$
$$+ \frac{1}{\gamma} \left[\partial_q \lambda \left(t, \frac{x}{N}\right)\right]^2 J_x^5$$
$$+ \left[e_x - \mathbf{e}\left(t, \frac{x}{N}\right)\right] \partial_t \beta \left(t, \frac{x}{N}\right) + \left[r_x - \mathbf{r}\left(t, \frac{x}{N}\right)\right] \partial_t \lambda \left(t, \frac{x}{N}\right) \Big\} + o(N) \ . \tag{A.53}$$

We denote by H_k the function defined as follows:

$$H_k \left(\eta \left(t, \frac{x}{N}\right)\right) = \mu^N_{\chi_t(x/N)} \left[J_0^k\right] . \tag{A.54}$$

The explicit formulations for H_k are given by Proposition 2.1. The sum

$$\sum_{x \in \mathbb{T}_N} \Big\{ -\frac{1}{2\gamma} \partial_q^2 \beta \left(t, \frac{x}{N}\right) H_1 \left(\eta \left(t, \frac{x}{N}\right)\right) - \frac{1}{\gamma} \partial_q^2 \lambda \left(t, \frac{x}{N}\right) H_2 \left(\eta \left(t, \frac{x}{N}\right)\right)$$
$$+ \frac{1}{4\gamma} \left[\partial_q \beta \left(t, \frac{x}{N}\right)\right]^2 H_3 \left(\eta \left(t, \frac{x}{N}\right)\right) + \frac{1}{\gamma} \partial_q \beta \left(t, \frac{x}{N}\right) \partial_q \lambda \left(t, \frac{x}{N}\right) H_4 \left(\eta \left(t, \frac{x}{N}\right)\right)$$
$$+ \frac{1}{\gamma} \left[\partial_q \lambda \left(t, \frac{x}{N}\right)\right]^2 H_5 \left(\eta \left(t, \frac{x}{N}\right)\right) \Big\} \tag{A.55}$$

is of order $o(N)$ (thanks to the regularity of the functions $\mathbf{e}, \mathbf{r}, \beta, \lambda$), so that we can introduce it in the right member of the equality (A.52).

Then, we obtain after computations

$$-\frac{\partial_q^2 \beta}{2\gamma} \partial_{\mathbf{e}} H_1 - \frac{\partial_q^2 \lambda}{\gamma} \partial_{\mathbf{e}} H_2 + \frac{[\partial_q \beta]^2}{4\gamma} \partial_{\mathbf{e}} H_3 + \frac{\partial_q \beta \partial_q \lambda}{\gamma} \partial_{\mathbf{e}} H_4 + \frac{[\partial_q \lambda]^2}{\gamma} d_{\mathbf{e}} H_5 = -\partial_t \beta \ , \tag{A.56}$$

and

$$-\frac{\partial_q^2 \beta}{2\gamma} \partial_{\mathbf{r}} H_1 - \frac{\partial_q^2 \lambda}{\gamma} \partial_{\mathbf{r}} H_2 + \frac{[\partial_q \beta]^2}{4\gamma} \partial_{\mathbf{r}} H_3 + \frac{\partial_q \beta \partial_q \lambda}{\gamma} \partial_{\mathbf{r}} H_4 + \frac{[\partial_q \lambda]^2}{\gamma} \partial_{\mathbf{r}} H_5 = \partial_t \lambda \ . \tag{A.57}$$

Indeed, these two quantities are respectively equal to

$$\frac{\partial_q^2 \beta}{2\gamma} - \frac{[\partial_q \beta]^2}{\gamma} \left(\mathbf{e} + \frac{\mathbf{r}^2}{2}\right) - 2\mathbf{r} \frac{\partial_q \beta \partial_q \lambda}{\gamma} - \frac{[\partial_q \lambda]^2}{\gamma} \ , \tag{A.58}$$

and

$$\frac{\partial_q^2 \beta}{2\gamma} \mathbf{r} + \frac{\partial_q^2 \lambda}{\gamma} - \frac{[\partial_q \beta]^2}{2\gamma} \mathbf{r} \left(2\mathbf{e} - 3\mathbf{r}^2\right) - \frac{\partial_q \beta \partial_q \lambda}{\gamma} \left(2\mathbf{e} - 3\mathbf{r}^2\right) + \mathbf{r} \frac{[\partial_q \lambda]^2}{\gamma} \ . \tag{A.59}$$

This concludes the proof and gives Proposition 2.1.

B Proof of the One-block Estimate

We just give a sketch of the proof, which is done in [4], Section 3.4. First, we define the space time average of distribution:

$$\bar{f}^N = \frac{1}{tN} \sum_{i=1}^{N} \int_0^t \tau_i f_s^N ds \ , \tag{B.1}$$

and $\bar{f}_k^N$ its projection on $\{(r_i, p_i) \in \mathbb{R}^{2(k+1)} \ ; \ i \in \Lambda_k := \{-[k/2]-1, \dots, [k/2]+1\}\}$.

We also denote $d\nu^N = \bar{f}^N \prod_{i \in \mathbb{T}_N} dr_i dp_i$ and $d\nu_k^N = \bar{f}_k^N \prod_{i \in \mathbb{T}_N} dr_i dp_i$ the corresponding probability measures on $\mathbb{R}^{2N}$ and $\mathbb{R}^{2(k+1)}$.

Observe first that (2.22) can be rewritten as

$$t \limsup_{M \to \infty} \limsup_{\ell \to \infty} \limsup_{N \to \infty} \int \left\{ \left| \frac{1}{\ell} \sum_{i \in \Lambda_\ell(0)} J_{i,M} - H(\eta_{\ell,M}(0)) \right| \right\} d\nu^N = 0 \ , \tag{B.2}$$

because

$$\frac{1}{\ell} \sum_{k=0}^{\ell-1} \frac{1}{p} \sum_{j=1}^{p} \tau_{x_j+k} = \frac{1}{N} \sum_{x=1}^{N} \tau_x. \tag{B.3}$$

We can prove the first following lemma.

Lemma B.1. *For each fixed k, the sequence of probability measures $(\nu_k^N)_{N \geqslant k}$ is tight.*

For any k let ν_k be a limit point of the sequence $(\nu_k^N)_{N \geqslant 1}$. The sequence of probability measures $(\nu_k)_{k \geqslant 1}$ forms a consistent family and by Kolmogorov's theorem there exists a unique probability measure ν on $(\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$ such that the restriction of ν on $\{(r_i, p_i) \in \mathbb{R}^{2(k+1)} \ ; \ i \in \Lambda_k\}$ is ν_k. One has easily that ν is invariant by translations.

Lemma B.2. *For any bounded smooth local function $F(\mathbf{r}, \mathbf{p})$, we have $\displaystyle\int \mathcal{L} F d\nu = 0$.*

Then, ν is a convex combination of grand canonical Gibbs measures $\mu_\chi = \mu_{\beta,\lambda}$: $\nu = \int d\rho(\chi) \mu_\chi$, with ρ a probability measure such that $\int d\rho(\chi) \mu_\chi[e_j] \leqslant C_0$ for any $j \in \mathbb{Z}$.

Hence, it results that

$$\begin{aligned}
&\limsup_{M \to \infty} \limsup_{\ell \to \infty} \limsup_{N \to \infty} \int \left\{ \left| \frac{1}{\ell} \sum_{i \in \Lambda_\ell(0)} J_{i,M} - H(\eta_{\ell,M}(0)) \right| \right\} d\nu^N \\
&\quad = \limsup_{M \to \infty} \limsup_{\ell \to \infty} \int d\rho(\chi) \int \left\{ \left| \frac{1}{\ell} \sum_{i \in \Lambda_\ell(0)} J_{i,M} - H(\eta_{\ell,M}(0)) \right| \right\} d\mu_\chi \\
&\quad = \limsup_{M \to \infty} \int d\rho(\chi) \left[\limsup_{\ell \to \infty} \int \left\{ \left| \frac{1}{\ell} \sum_{i \in \Lambda_\ell(0)} J_{i,M} - H(\eta_{\ell,M}(0)) \right| \right\} d\mu_\chi \right] ,
\end{aligned} \tag{B.4}$$

where the last equality is a consequence of the dominated convergence theorem. Since μ_χ is ergodic with respect to $\{\tau_x \; ; \; x \in \mathbb{Z}\}$, the last term is equal to

$$\limsup_{M\to\infty} \int d\rho(\chi) \left|\mu_\chi \left[J_{0,M}\right] - H(\mu_\chi[\eta_{0,M}])\right| . \tag{B.5}$$

As $M \to \infty$, $\mu_\chi[J_{0,M}]$ converges to $\mu_\chi[J_0] = H\left(\mu_\chi[\xi_0]\right)$ and $\mu_\chi[\xi_{0,M}]$ to $\mu_\chi[\xi_0]$.

By Fatou's lemma, the limit in M is equal to 0 and this concludes the proof of the one-block lemma.

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