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Abstract. Betweenness centrality is one of the basic concepts in the analysis of social networks. Initial definition for the betweenness of a node in a graph is based on the fraction of the number of geodesics (shortest paths) between any two nodes that given node lies on, to the total number of the shortest paths connecting these nodes. This method has quadratic complexity and does not take into account indirect paths. We propose a new concept of betweenness centrality for weighted network, beta current flow centrality, based on Kirchhoff's law for electric circuits. In comparison with the original current flow centrality and alpha current flow centrality, this new measure can be computed for larger networks. The results of numerical experiments for some examples of networks, in particular, for the popular social network VKontakte as well as the comparison with PageRank method are presented.

Keywords: beta current flow centrality, betweenness centrality, PageRank, weighted graph, social networks.

1 Introduction

The online social networks gave impulse to the development of new graph-theoretical methods for network analysis. Furthermore, social network analysis methods are applied in many other fields such as: economics, physics, biology and information technologies.

One of the basic concepts in the analysis of social networks is betweenness centrality, a measure of centrality that is based on how well a node i is situated in terms of the paths that it lies on [11]:

$$c_B(i) = \frac{1}{n_B} \sum_{s,t \in V} \frac{\sigma_{s,t}(i)}{\sigma_{s,t}}, \tag{1}$$

where $\sigma_{s,t}$ is the total number of geodesics (shortest paths) between nodes s and t, $\sigma_{s,t}(i)$ is the number of geodesics between s and t that i lies on. The

denominator n_B captures that the node i could lie on paths between as many as $n_B = (n-1)(n-2)/2$ pairs of other nodes. The complexity of the fastest algorithm to find $c_B(i)$ is $O(mn)$ where m is the number of edges and presented in [8].

One of shortcomings of the betweenness centrality is that it takes into accounts only the shortest paths, ignoring the paths that might be one or two steps longer, while the edges on such paths can be important for communication processes in the network. In order to take such paths into account, Brandes and Fleischer [9] and Newman [19] introduced the current flow betweenness centrality (CF-centrality, for short). In [9, 19] the graph is regarded as an electrical network with edges being unit resistances. The CF-centrality of an edge is the amount of current that flows through it, averaged over all source-destination pairs, when one unit of current is induced at the source, and the destination (sink) is connected to the ground.

However, the modification proposed in [9,19] comes with a cost. In comparison with the original betweenness centrality, the bottleneck in the computation of CF-centrality is the matrix inversion with complexity $O(n^3)$. To mitigate this high complexity, in [2] the authors suggested a modification of CF-centrality, where in addition to the grounded sink, every node is attached to the ground with some small conductance proportional to the node degree.

The proposal in [2] makes the underlying linear system strongly diagonally dominant and reduces the computational cost of CF-centrality significantly but still needs to apply averaging over all source-destination pairs. In the current work, we go further and suggest to ground all nodes equally, which leads to averaging only over source nodes and reduces further computational cost. We refer to our new method as beta current flow centrality (βCF-centrality, for short).

Additionally, in contrast to the works [2,9,19], we consider weighted networks. Of course, the original betweenness centrality can easily be extended to weighted networks with integer weights. Namely, transform each link of the weight k into k parallel links of weight 1. We obtain a multigraph. The shortest path between two nodes is determined the same way as in unweighted graph. But the number of geodesics becomes larger because of the multi-links. For instance, if the nodes i_1 and i_2 are connected by k links and the nodes i_2 and i_3 are connected by l links, then the nodes i_1 and i_3 are connected by $k \cdot l$ paths. Applying the formula (1) to the nodes of multigraph we derive the centrality value for weighted graph, but with a very significant increase in computation cost. In the worst case scenario of k links between any two nodes the complexity of the algorithm to find $c_B(i)$ is $O(mn^k)$. In contrast, we note that our proposed method has the same computational complexity for weighted and non-weighted graphs.

Finally, we would like to note that, due to its relatively small computational cost, the proposed βCF-centrality is very well suited to serve as a characteristic function in the Myerson vector [1, 18]. The concept of betweenness centrality via the Myerson vector was proposed in [12–14]. Considering the nodes in the network as players and the links as connections between players they formulate

a communication game. The imputation of the general payoff in this cooperative game can be used for ranking of players and, respectively, for nodes of the graph. In [17] for the communication game with special characteristic function it was proposed a fairly simple imputation procedure based on the generating function and was shown that the resulting imputation agrees with the Myerson value. The advantage of the Myerson value is in taking into account the impact of all coalitions. Using the current flow betweenness centrality as a weight of any subset of the network it is possible to determine a new characteristic function and then rank the nodes as the Myerson value. This approach extends the game-theoretic approach from non-weighted to weighted graphs.

2 Beta current flow centrality based on Kirchhoff's law

Consider a weighted graph $G = (V, E, W)$, where V is the set of nodes, E is the set of edges, and W is the matrix of weights, i.e.,

$$W(G) = \begin{pmatrix} 0 & w_{1,2} & \dots & w_{1,n} \\ w_{2,1} & 0 & \dots & w_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n,1} & w_{n,2} & \dots & 0 \end{pmatrix},$$

where $w_{i,j} \geqslant 0$ is weight of the edge connecting the nodes i and j, $n = |V|$ is the number of nodes. Note that $w_{i,j} = 0$ if nodes i and j are not adjacent. Here we assume that G is undirected graph, i.e. $w_{i,j} = w_{j,i}$. By random walk interpretation, the method can in fact be extended to directed networks.

Next we introduce the diagonal degree matrix:

$$D(G) = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix},$$

where $d_i = \sum_{j=1}^{n} w_{i,j}$ is the sum of weights of the edges which are adjacent to node i in graph G. The Laplacian matrix $L(G)$ for weighted graph G is defined as follows:

$$L(G) = D(G) - W(G) = \begin{pmatrix} d_1 & -w_{1,2} & \dots & -w_{1,n} \\ -w_{2,1} & d_2 & \dots & -w_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n,1} & -w_{n,2} & \dots & d_n \end{pmatrix}. \tag{2}$$

Let the graph G' be converted from the graph G by extension with an additional node $n + 1$ connected with all nodes of the graph G with the links of constant conductance β. Thus, we obtain the Laplacian matrix for the modified

graph G' as:

$$L(G') = D(G') - W(G') = \begin{pmatrix} d_1+\beta & -w_{1,2} & \dots & -w_{1,n} & -\beta \\ -w_{2,1} & d_2+\beta & \dots & -w_{2,n} & -\beta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_{n,1} & -w_{n,2} & \dots & d_n+\beta & -\beta \\ -\beta & -\beta & \dots & -\beta & \beta n \end{pmatrix}. \quad (3)$$

Suppose that a unit of current enters into the node $s \in V$ and the node $n+1$ is grounded. Let φ_i^s be the electric potential at node i when an electric charge is located at node s. The vector of all potentials $\varphi^s(G') = [\varphi_1^s, \dots, \varphi_n^s, \varphi_{n+1}^s]^T$ for the nodes of graph G' is determined by the following system of equations (Kirchhoff's current law):

$$L(G')\varphi^s(G') = b'_s, \quad (4)$$

where b'_s is the vector of $n+1$ components with the values:

$$b'_s(i) = \begin{cases} 1 & i = s, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

The Laplacian matrix (2) is singular. The potential values can be determined up to a constant. Hence, without loss of generality, we can assume that the potential in node $n+1$ is equal to 0 (grounded node). Then, from (3) it follows that

$$\tilde{\varphi}^s(G') = \tilde{L}(G')^{-1} b_s, \quad (6)$$

where $\tilde{\varphi}^s(G')$, $\tilde{L}(G')$ and b_s are obtained from (3) by deleting the last row and column corresponding to node $n+1$. Notice that in $\varphi^s(G')$ and b'_s zero elements are deleted. This yields

$$\tilde{\varphi}^s(G') = [D(G) - W(G) + \beta I]^{-1} b_s, \quad (7)$$

where I is a unity matrix of size n.

Thus we can consider the vector $\tilde{\varphi}^s(G')$ as the vector of potential values for the nodes of graph G, that is,

$$\tilde{\varphi}^s(G) = [L(G) + \beta I]^{-1} b_s.$$

Rewrite (7) in the following form:

$$\tilde{\varphi}^s(G) = [(D(G) + \beta I) - W(G)]^{-1} b_s =$$

$$= [I - (D(G) + \beta I)^{-1} D(G) D^{-1}(G) W(G)]^{-1} (D(G) + \beta I)^{-1} b_s.$$

The matrices $(D(G) + \beta I)^{-1}$ and $(D(G) + \beta I)^{-1} D(G)$ are diagonal with the elements $\frac{1}{d_i+\beta}$ and $\frac{d_i}{d_i+\beta}, i = 1, ..., n$, denote these matrices as D_1 and D_2, respectively. The matrix $D^{-1}(G)W(G)$ is stochastic. Denote it as P. Consequently, we have

$$\tilde{\varphi}^s(G) = [I - D_2 P]^{-1} D_1 b_s = \sum_{k=0}^{\infty} (D_2 P)^k D_1 b_s. \quad (8)$$

From (8) it follows that the potential vector can be calculated by the recursion:

$$\tilde{\varphi}^s_{k+1}(G) = D_2 P \tilde{\varphi}^s_k(G) + D_1 b_s, \quad \tilde{\varphi}^s_0(G) = 0.$$

Note that the convergence is guaranteed since the matrix D_2P is substochastic. The rate of convergence can be easily regulated by the value of β.

The current let-through the link $e = (i, j)$ according to Ohm's law is $x^s_e = |\varphi^s_i - \varphi^s_j| \cdot w_{i,j}$. Consequently, given that the electric charge is in node s, the mean value of the current passing through node i is:

$$x^s(i) = \frac{1}{2}(b_s(i) + \sum_{e:i\in e} x^s_e), \tag{9}$$

where

$$b_s(i) = \begin{cases} 1 & i = s, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define beta current flow centrality (βCF-centrality) of node i as follows:

$$CF_\beta(i) = \frac{1}{n} \sum_{s\in V} x^s(i). \tag{10}$$

We note that the above equation and the law of large numbers can be used to make a simple, light complexity, Monte Carlo type method for quick estimation of βCF-centrality. Specifically, we can take a small subset of nodes, $V_1 \subset V$, chosen independently and uniformly as source nodes in order to approximate βCF-centrality:

$$CF_\beta(i) \approx \frac{1}{|V_1|} \sum_{s\in V_1} x^s(i). \tag{11}$$

Let us now investigate the limiting cases of large and small values of β. First, assume that β is large. Then, we can derive the following asymptotics for the potential vector.

$$\tilde{\varphi}^s = [L + \beta I]^{-1} b_s = \frac{1}{\beta}[I + \frac{1}{\beta}L]^{-1} b_s = \frac{1}{\beta} b_s - \frac{1}{\beta^2} L b_s + \mathrm{o}\left(\frac{1}{\beta^2}\right)$$

From the above asymptotics, we can conclude that $x^s(s) = 1/2(1+d_s/\beta)+\mathrm{o}(1/\beta)$ and $x^s(i) = \mathrm{o}(1)$, for $i \neq s$, and consequently,

$$CF_\beta(i) = \frac{1}{2n} + \mathrm{o}(1), \quad \text{as} \quad \beta \to \infty,$$

which does not give informative ranking. Now for the other case $\beta \to 0$, we can derive the following asymptotics

$$\tilde{\varphi}^s = [L + \beta I]^{-1} b_s = \left[\frac{1}{\beta}\frac{1}{n}\underline{1}\underline{1}^T + L^\sharp + \mathrm{O}(\beta)\right] b_s = \frac{1}{\beta}\frac{1}{n}\underline{1} + L^\sharp_{\star,s} + \mathrm{O}(\beta),$$

where we have used the Laurent series expansion for inversion of singularly perturbed matrices (see e.g., [4, Chapter 2]) with $\underline{1}$ denoting vector of ones of appropriate dimension, and $L^\sharp = [L - 1/n\underline{1}\underline{1}^T]^{-1} - 1/n\underline{1}\underline{1}^T$ denoting the group inverse of the Laplacian. Thus, we have

$$x_e^s = |L^\sharp_{i,s} - L^\sharp_{j,s}|w_{i,j} + \mathrm{o}(1),$$

and hence a well-defined and non-trivial limit for βCF-centrality exists when $\beta \to 0$.

3 Illustrative examples

3.1 Weighted network of six nodes

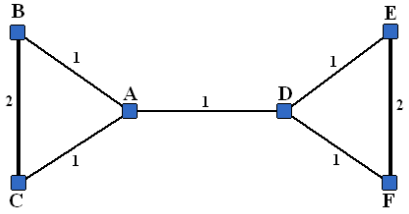

Fig. 1. Weighted network of six nodes.

Let us start with a simple six nodes network example which nicely explains the properties of the beta current flow centrality (see Fig.1). We compute all main measures of centrality for that weighted graph with six nodes. The results of computation are presented in Table 1. We see that classical betweenness centrality evaluates only the nodes A and D and gives 0 to other four nodes, even though they are obviously also important. The PageRank method ranks all nodes with equal values and thus it is indiscriminatory in this particular case. The current flow betweenness centrality and the βCF-centrality evaluate all nodes in quite similar manner. In particular, they both give rather high values to nodes A and D. As we mentioned in the introduction, the comparative advantage of the βCF-centrality in its small computational costs.

Table 1. Measures of centrality for weighted graph with six nodes

Nodes	A	B	C	D	E	F
Original betweenness centrality	6	0	0	6	0	0
PageRank centrality $\alpha = 0.85$	1/6	1/6	1/6	1/6	1/6	1/6
Current flow betweenness centrality	1.12	0.66	0.66	1.12	0.66	0.66
βCF-centrality $\beta = 1$	0.27	0.19	0.19	0.27	0.19	0.19

3.2 Star graph

Consider a graph G of n nodes in the form of a star. Let node 1 be the center of the star. The modified Laplacian matrix in this case is given by

$$\tilde{L} = L + \beta I = D(G) - W(G) + \beta I$$

$$= \begin{pmatrix} n-1+\beta & -1 & \dots & -1 \\ -1 & 1+\beta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 1+\beta \end{pmatrix}.$$

Its inverse matrix is

$$\tilde{L}^{-1} = (L + \beta I)^{-1}$$

$$= \frac{1}{\beta(1+\beta)(n+\beta)} \begin{pmatrix} (1+\beta)^2 & 1+\beta & 1+\beta & \dots & 1+\beta \\ 1+\beta & 1+\beta(n+\beta) & 1+\beta & \dots & 1 \\ 1+\beta & 1 & 1+\beta(n+\beta) & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+\beta & 1 & 1 & \dots & 1+\beta(n+\beta) \end{pmatrix}.$$

If we take as a source node $s = 1$, we find from (9) that

$$x^s(1) = \frac{1}{2}\left(1 + \frac{n-1}{n+\beta}\right),$$

and

$$x^s(i) = \frac{1}{2(n+\beta)}, \quad i = 2, ..., n.$$

And for a source node $s \neq 1$, we obtain

$$x^s(1) = \frac{2n-3+\beta}{2(1+\beta)(n+\beta)},$$

$$x^s(s) = \frac{1}{2}\left(1 + \frac{n-1+\beta}{(1+\beta)(n+\beta)}\right),$$

$$x^s(i) = \frac{1}{2(1+\beta)(n+\beta)}, \quad i \neq 1, s.$$

The latter yields that the βCF-centrality for the star graph is

$$CF_\beta(1) = \frac{1}{2n}\left(1 + \frac{n-1}{n+\beta} + (n-1)\frac{2n-3+\beta}{(1+\beta)(n+\beta)}\right) = \frac{1}{2n} + \frac{(n-1)(n-1+\beta)}{n(1+\beta)(n+\beta)},$$

$$CF_\beta(i) = \frac{1}{2n}\left(\frac{1}{n+\beta} + 1 + \frac{n-1+\beta}{(1+\beta)(n+\beta)} + (n-2)\frac{1}{(1+\beta)(n+\beta)}\right)$$

$$= \frac{1}{2n} + \frac{n-1+\beta}{n(1+\beta)(n+\beta)}, \quad i = 2, ..., n.$$

In particular, we can conclude from the above expressions that if $\beta \to \infty$ all nodes obtain the same value $1/(2n)$. And if $\beta \to 0$ and n is large, the central node obtains a value very close to one and the other nodes have nearly zero value. This is in agreement with the general asymptotics derived in the previous section.

This example also shows that the βCF-centrality can be viewed as a flexible characteristic function and thus efficiently used in the calculation of the Myerson vector.

3.3 The results of computer experiments with online social network VKontakte

In this subsection we consider the weighted graph extracted from the popular Russian social network VKontakte. The graph corresponds to the online community devoted to game theory. This community consists of 483 participants. As a weight of a link we take the number of common friends between the participants. In fact, the probability that two participants are familiar depends on the number of common friends [14]. This approach is often used in online social networks for link recommendation.

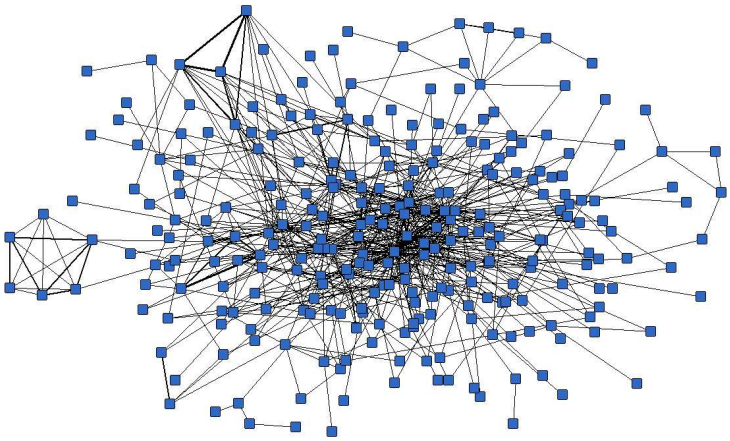

Fig. 2. Principal component of the community Game Theory in the social network VKontakte (number of nodes: 275, number of edges: 805 and mean path's length: 3.36).

In Fig. 2 we show the principal component of the community Game Theory, which consists of 275 nodes. It is difficult to see from Fig. 2 which nodes are

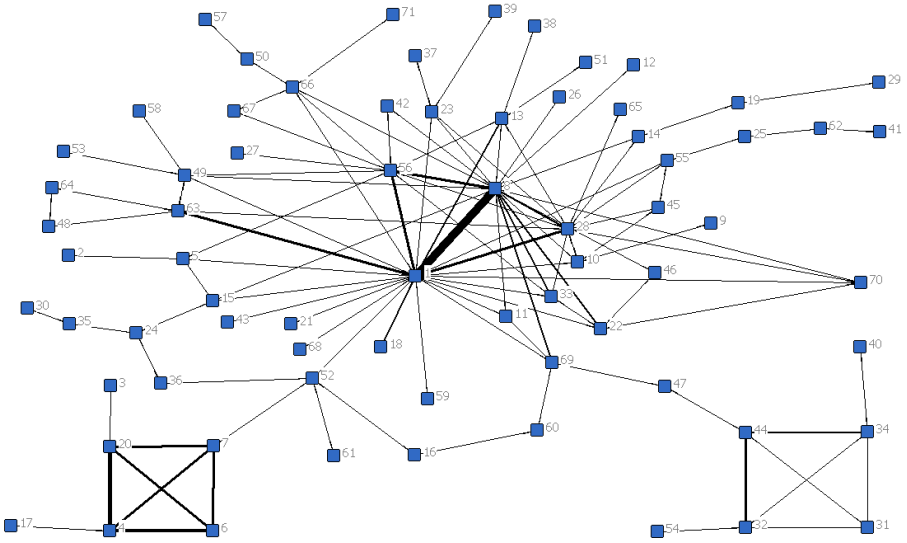

Fig. 3. Principal component of the community Game Theory in the social network VKontakte after deleting the links weighted less than 3 (number of nodes: 71, number of edges: 116 and mean path's length: 3.75).

more important with respect to the community connection structure. Then, we have converted this graph to another modified graph by deleting the links whose weights are less than three. This new weighted graph is presented in Fig. 3. The thickness of a link depends on the link weight, i.e. on the number of common friends.

The results of computing the βCF-centrality for the social network VKontakte are given in Table 2. Here we take $\beta = 0.3$. It is useful to compare these values of βCF-centrality with the results corresponding to the PageRank and classical notation of centrality using the shortest paths [20] for the parameter $\alpha = 1.5$. We present in the table only the lists of top-10 nodes for each centrality measure.

From Table 2 we find that all four methods ranked two main nodes 1 and 8 in the same order. We can already see that, as in the six node network example, βCF-centrality is more similar to CF-centrality and betweenness centrality than to PageRank.

On Figure 3 we can see that node 52 connects the subgraph $\{3, 4, 6, 7, 17, 20\}$ with the rest of the graph. Thus, we can expect that node 52 deserves high centrality rank. Similarly, we also expect that node 7 should have high centrality rank. The ranking according to βCF-centrality confirms this intuitive expectation, as they take positions 3 and 7, respectively (See Table 2). We also note that nodes 4, 20, 6, 17 and 3 took positions 22, 24, 36, 68 and 69, respectively. However, PageRank gives to nodes 52 and 7 only positions 22 and 12, respec-

tively. Furthermore, under PageRank ranking nodes 4, 20 and 6 took positions 6, 8 and 10, respectively. Namely, PageRank ranks nodes 4, 20 and 6 higher than node 52. This does not correspond at all to our intuition.

Finally, in Table 3 we present the results of the Monte Carlo approach (see equation (11)) with sampling only 10% of the nodes as sources. Nodes 1 and 8 as before lead the ranking and there are 6 correct elements in the top-10 basket of nodes [3]. Monte Carlo approach also determines correctly the ranks of the key nodes 52 and 7.

4 Conclusion

Betweenness centrality measure is an important tool in the analysis of social networks. The structure of a network is represented by a graph. The original betweenness centrality measure is based on the assumption that the information is transmitted along geodesics (shortest paths) between any two nodes. There is a criticism of this approach that it does not take into account information spread along non-shortest paths. The current flow betweenness centrality based on electric circuit interpretation tries to mitigate this shortcoming. However, this comes with the increase of computational cost. We introduce here the βCF-centrality method which depends on the parameter β. This method is versatile, has lower computational complexity and can be easily used as characteristic function in the Myerson vector.

Table 2. Measures of centrality for top nodes of social network VKontakte.

Nodes	βCF-centrality ($\beta = 0.3$)	Nodes	PageRank centrality ($\alpha = 0.85$)	Nodes	Weighted betweenness centrality "tnet" ($\alpha = 1.5$)	Nodes	CF-centrality
1	0.4168	1	0.1359	1	1846	1	0.6406
8	0.3143	8	0.1189	8	1398	8	0.4919
52	0.1463	56	0.0432	52	500	69	0.2946
69	0.1454	28	0.0366	69	494	52	0.2748
28	0.1299	44	0.0277	47	384	28	0.2095
56	0.1273	4	0.0267	44	331	56	0.1942
7	0.1002	32	0.0252	63	331	47	0.1880
15	0.0931	20	0.0244	7	325	44	0.1649
66	0.0922	63	0.0228	55	265	15	0.1645
63	0.0896	6	0.0212	15	228	7	0.1642

Table 3. The results of the Monte Carlo approach with sampling only 10% of the nodes as sources

Nodes	βCF-centrality ($\beta = 0.3$)	Nodes	Monte Carlo approach 10% of the nodes
1	0.4168	1	0.5043
8	0.3143	8	0.4134
52	0.1463	52	0.2468
69	0.1454	23	0.2307
28	0.1299	28	0.2255
56	0.1273	20	0.2003
7	0.1002	7	0.1982
15	0.0931	24	0.1871
66	0.0922	63	0.1789
63	0.0896	10	0.1786
47	0.0889	15	0.1763
24	0.0880	55	0.1756
44	0.0842	36	0.1613
55	0.0801	69	0.1565
49	0.0725	12	0.1457
23	0.0702	39	0.1438
13	0.0699	45	0.1403
10	0.0610	56	0.1397
14	0.0598	3	0.1360
25	0.0564	4	0.1234

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