Invariantization and Polynomial Systems with Symmetry

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Abstract

Assuming the variety of a polynomial set is invariant under a group action, we provide a set of invariants that define the same variety. The contribution is about infinite algebraic groups, the case of finite group being previously known. We introduce for those a new definition of the concept of algebraic invariantization [8]. It is based on the construction of rational invariants by Hubert and Kogan [7], a construction for which we provide here new simplified proofs.
Keywords: Rational invariants; Symmetric polynomial system; Section.

1 Introduction

Consider the variety $\mathcal{F}$ in $\mathbb{C}^n$ of a finite set of polynomials F in $\mathbb{C}[z_1, \ldots, z_n]$. $\mathcal{F}$ can be invariant under the action of a group $\mathcal{G}$ without the polynomials in F being themselves invariant. When $\mathcal{G}$ is a finite group acting regularly, a system of polynomial invariants $\tilde{F}$ that have the same variety as F can be determined explicitely. The construction of $\tilde{F}$ can be found for instance in the proof of [21, Proposition 2.6.4] and this construction also applies for $\mathcal{G}$-invariant semi-algebraic sets [2]. The existence of such a $\tilde{F}$ for compact group is proved in [1]. It was nonetheless an open question in [2] whether there exists a constructive approach. The present article aims to provide such a construction for rational actions of any algebraic group.

For a given set F of polynomials as above we shall determine a set of invariant functions $\tilde{F}$ such that the zero set of $\tilde{F}$ is equal to the variety of F outside of some proper closed algebraic set $\mathcal{W}$. The elements of $\tilde{F}$ are rational invariants and the restriction to an open set is therefore unavoidable for a general statement. Yet $\mathcal{W}$ is independent of F. Concomittantly the ements of $\tilde{F}$ are actually polynomials in a fixed finite set of generating rational invariants; $\mathcal{W}$ contains the varieties of their denominators.

The construction of the set $\tilde{F}$ above is based on the notion of algebraic *invariantization* first introduced in [8]. The new definition of invariantization given here allows to use an analogous argument to the finite group case to prove that the zero set of $\tilde{F}$ is equal to the variety of F. In the special case of scalings we retrieve the results obtained in [9].

Algebraic invariantization builds on the construction of the field of rational invariants based on the notion of section (a.k.a. as cross-section or quasi-section) to the orbits of the group action. This geometric construction and its algorithmic realization was first presented in [7]. The results were recasted in another terminology in [5, Section 4.10]. We give here a new set of simpler proofs that are have the geometric content that permit the interpretation of the main result of the present paper.

In next section we define the group actions to be considered as well as the notion of *section of degree e* to the orbits. We show how to compute a finite set of generating rational invariants. They are the coefficients of the reduced Gröbner basis of the *orbit-section* ideal. These generators are furthermore endowed with an algorithm that allows to rewrite any other rational invariants in terms of them. In Section 3 we give a

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new constructive definition of invariantization w.r.t. a given section to the orbits: to any polynomial f is associated e *symmetrizations*, where e is the degree of the section to the orbits. These symmetrizations are polynomials in the generating invariants. If F is a set of polynomials whose variety is invariant under the group action then the set $\tilde{F}$ of the symmetrisations of the elements of F have the same zero set. The result has to be understood within a $\mathcal{G}$-invariant open set where the reduced Gröbner basis of the orbit-section ideal specializes well.

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2 Construction of rational invariants

We shall first introduce the notions and notations to be used in the article. We then review the construction of rational invariants that appeared in [7] with a simplified set of proofs. We also base this construction on a better definition of section to the orbit. Theorem 2.5 and Theorem 2.6 can then be compared respectively with [7, Theorem 3.5] and [7, Theorem 3.7].

2.1 Rational action of an algebraic group

$\mathbb{K}$ is a field of characteristic zero, $\overline{\mathbb{K}}$ is an algebraically closed field extension of $\mathbb{K}$. We deal here with $\mathcal{Z}$ an affine space $\overline{\mathbb{K}}^n$, but we believe our constructions can be extended to an irreducible algebraic variety. The groups we consider are affine algebraic groups. They are given by an affine algebraic variety $\mathcal{G}$ endowed with a group operation $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and an inverse $\mathcal{G} \to \mathcal{G}$ given by regular maps. To be explicit, we assume that $\mathcal{G}$ is embedded in $\overline{\mathbb{K}}^l$ and $G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_l]$ is its defining ideal. The coordinate ring $\mathbb{K}[\mathcal{G}]$ can be identified with the quotient algebra $\mathbb{K}[\lambda_1, \ldots, \lambda_l]/G$.

A rational action of $\mathcal{G}$ on $\mathcal{Z}$ is defined by a homomorphism ρ from $\mathcal{G}$ to the group of birational maps of $\mathcal{Z}$. In practice it is given by a rational map $\mathcal{G} \times \mathcal{Z} \dashrightarrow \mathcal{Z}$, $(\lambda, z) \mapsto \lambda \star z = \rho(\lambda)(z)$ defined by quotients of polynomials:

$$\lambda \star z = \left(\frac{h_1(\lambda, z)}{h_0(\lambda, z)}, \ldots, \frac{h_n(\lambda, z)}{h_0(\lambda, z)} \right)$$

where $h_0, h_1, \ldots, h_n \in \mathbb{K}[\lambda, z]$. When we write $(\lambda, z) \in \mathcal{G} \times \mathcal{Z}$ we mean that (λ, z) belongs to the open set of $\mathcal{G} \times \mathcal{Z}$ where $\lambda \star z$ is well defined. The orbit $\mathcal{O}_z$ of $z \in \mathcal{Z}$ is the image of the rational map $\mathcal{G} \dashrightarrow \mathcal{Z}$, $\lambda \mapsto \lambda \star z$.

A rational action of $\mathcal{G}$ on $\mathcal{Z}$ induces an action on the field of rational functions $\mathbb{K}(\mathcal{Z})$ given by $(\lambda \star f)(z) = f(\lambda^{-1} \star z)$. The set of rational invariants $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ is the subfield of $\mathbb{K}(\mathcal{Z})$ of rational functions f s.t. $\lambda \star f = f$, for all $\lambda \in \mathcal{G}$.

Lemma 2.1 *If p/q is a rational invariant, with $p, q \in \mathbb{K}[z]$ relatively prime, then the varieties $\mathcal{V}(p)$ and $\mathcal{V}(q)$ are invariant under the action of $\mathcal{G}$.*

PROOF: By hypothesis $p(z)\, q(\lambda \star z) = q(z)\, p(\lambda \star z)$ for all $(\lambda, z) \in \mathcal{G} \times \mathcal{Z}$. Hence $p(\lambda \star z) = 0$ for all $(\lambda, z) \in \mathcal{G} \times \mathcal{V}(p) \setminus \mathcal{V}(q)$. Since p and q are relatively prime, $\mathcal{V}(p) \setminus \mathcal{V}(q)$ is dense in $\mathcal{V}(p)$. Hence $p(\lambda \star z)$ has to vanish on the whole of $\mathcal{G} \times \mathcal{V}(p)$. □

When $\mathcal{G}$ is connected and acts regularly on $\mathcal{Z}$, one can further conclude that p and q are *semi-invariants* with the same *weight*, i.e. $p(\lambda \star z) = \chi(\lambda)\, p(z)$ and $q(\lambda \star z) = \chi(\lambda)\, q(z)$ where $\chi : \mathcal{G} \to \bar{\mathbb{K}}^*$ is a group morphism [19, Theorem 3.1 and 3.3].

2.2 Sections to the orbits

Definition 2.2 *For a given rational action of $\mathcal{G}$, an irreducible variety $\mathcal{P}$ is a* section of degree e *to the orbits if there exists an open dense subset $\mathcal{U}$ of $\mathcal{Z}$ such that the orbits of $\mathcal{U}$ intersect $\mathcal{P}$ at exactly e points.*

Obviously, a section cannot be contained in a proper $\mathcal{G}$-invariant subvariety of $\mathcal{Z}$. Yet the present notion of section is not restrictive. Most irreducible subvarieties of complementary dimension to the generic orbits are sections. Without further knowledge on the geometry of the generic orbits one can always choose an affine linear space as a section [7, Theorem 3.3], or even the level set of some of coordinates [8, Theorem 1.6]. For a generic affine linear space the degree of the section it defines is the degree of the orbits. Sections of lower degree can be obtained by taking into consideration the points at infinity or the singular points of the closure of generic orbits. Section of degree one are of particular interest, as we shall point out at several places.

Proposition 2.3 *Assume the generic orbits have dimension d and take $P \subset \mathbb{K}[Z_1, \ldots, Z_n]$ as a prime ideal of codimension d. Consider*

$$A = (h_0(\lambda, z)\, Z_1 - h_1(\lambda, z),\ \ldots,\ h_0(\lambda, z)\, Z_n - h_n(\lambda, z)) \subset \mathbb{K}[z, Z, \lambda] \tag{1}$$

Then the variety $\mathcal{P}$ of P is a section if the ideal

$$I_{\mathcal{P}} = (G + A + P) : h_0^{\infty} \cap \mathbb{K}(z)[Z]. \tag{2}$$

is zero dimensional. The dimension e of the quotient algebra $\mathbb{K}(z)[Z]/I_{\mathcal{P}}$ as a $\mathbb{K}(z)$-vector space is the degree of the section.

Indeed, by specializing z to a generic point in $\mathcal{Z}$, the ideal $I_{\mathcal{P}}$ becomes the ideal of the intersection of the orbit $\mathcal{O}_z$ of z with the section $\mathcal{P}$. We shall refer to the ideal $I_{\mathcal{P}}$ as the *orbit-section* ideal.

Given the equations of an irreducible variety we can thus determine if it is a section and compute its degree by computing the Gröbner basis[1] of the elimination ideal $I_{\mathcal{P}}$. The degree e of the section $\mathcal{P}$ is the number of monomials *under the staircase* defined by the leading monomials of the Gröbner basis of $I_{\mathcal{P}}$. As we shall see in next section, the reduced Gröbner bases of $I_{\mathcal{P}}$ also delivers a generating set of invariants for the action.

Example 2.4 Scalings in the plane. *Consider the action of the multiplicative group $\mathbb{K}^*$ given by*

$$\begin{array}{rccc} \star: & \mathbb{K}^* \times \mathbb{K}^2 & \to & \mathbb{K}^2 \\ & (\lambda, (x, y)) & \mapsto & (\lambda^a x, \lambda^b y) \end{array}$$

where a and b are positive integers that we assume here relatively prime. The ideal of the orbit of $(x, y) \in \mathbb{K}^2 \setminus \{(0, 0)\}$ is then given by

$$O = (x^b Y^a - y^a X^b).$$

Note that the orgin is in the closure of all the orbits. There is therefore no non constant polynomial invariant for this action [5, Lemma 2.4.5].

A generic affine line in $\mathbb{K}^2$ is a section of degree $\max(a, b)$. But $P = (X - 1)$ defines a section of degree a since the ideal of the intersection of the orbit of $(x, y) \in \mathbb{K}^2 \setminus \{(0, y) \,|\, y \in \mathbb{K}\}$ with the variety $\mathcal{P}$ of P is

$$I = \left(X - 1, Y^a - \frac{y^a}{x^b}\right).$$

[1] The reader is invited to consult for instance [3, 4] for the basic concepts and results about Gröbner bases.

A section of degree 1 *is provided by the Bezout coefficients* $\alpha, \beta \in \mathbb{Z}$ *s.t.* $\alpha a - \beta b = 1$. *Without loss of generality one can assume that* $\alpha, \beta \in \mathbb{N}$. *If we choose* $P = \left(X^\alpha - Y^\beta\right)$ *then the ideal of the intersection of the orbit of* $(x, y) \in \mathbb{K}^2 \setminus \{(0, y) \,|\, y \in \mathbb{K}\}$ *with the variety* $\mathcal{P}$ *of* P *is*

$$I_{\mathcal{P}} = \left(X - \left(\frac{y^a}{x^b}\right)^\beta,\ Y - \left(\frac{y^a}{x^b}\right)^\alpha\right).$$

This generalizes for scalings in any dimension, i.e. diagonal linear actions of the algebraic torus $(\mathbb{K}^*)^r$*: we can compute the (binomial) equations of a section of degree one with linear algebra over the integers [9, 10].*

2.3 Generating invariants and rewriting

Observe that $\mathcal{O}_z = \mathcal{O}_{\lambda \star z}$. Since $I_{\mathcal{P}}$ is the ideal of $\mathcal{O}_z \cap \mathcal{P} = \mathcal{O}_{\lambda \star z} \cap \mathcal{P}$ for a generic $\bar{z} \in \mathcal{Z}$ a canonical representation of the ideal $I_{\mathcal{P}}$ must be defined over $\mathbb{K}(z)^{\mathcal{G}}$. If we fix a term ordering on the monomials in Z, the reduced Gröbner basis of $I_{\mathcal{P}}$ is such a canonical representative.

Theorem 2.5 *The coefficients of a reduced Gröbner basis of* $I_{\mathcal{P}}$ *belong to* $\mathbb{K}(z)^{\mathcal{G}}$.

PROOF: For a given term order, the reduced Gröbner basis of an ideal is unique. Let B be the reduced Gröbner basis for $I_{\mathcal{P}}$ for a given term order on Z. As such it consists of monic polynomials in $\mathbb{K}(z)[Z]$.

There is a closed proper subset $\mathcal{W}$ of $\mathcal{Z}$ s.t. for $z \in \mathcal{Z} \setminus \mathcal{W}$ the image of B under specialization is a (reduced) Gröbner basis for the ideal whose variety is the intersection of $\mathcal{O}_z$ with $\mathcal{P}$. Since $\mathcal{O}_z = \mathcal{O}_{\lambda \star z}$, the specializations of B to z and to $\lambda \star z$ bring the same reduced Gröbner basis, for a generic $\lambda \in \mathcal{G}$. Therefore $B \subset \mathbb{K}(z)^{\mathcal{G}}[Z]$. □

In this construction it is clear that the coefficients of the Gröbner basis of $I_{\mathcal{P}}$ *separate* generic orbits. According to [20, Theorem 2] or [19, Lemma 2.1], we can deduce that they form a generating set. The alternative proof we give next is constructive. We show how to rewrite any invariant in terms of the coefficients of the reduced Gröbner basis.

Theorem 2.6 *Consider* $\{r_1, \dots, r_m\} \in \mathbb{K}(z)^{\mathcal{G}}$ *the coefficients of a reduced Gröbner basis* B *of* $I_{\mathcal{P}}$. *Then* $\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}(r_1, \dots, r_m)$ *and we can rewrite any rational invariant* $\frac{p}{q}$, *with* $p, q \in \mathbb{K}[z]$ *relatively prime, in terms of those as follows.*

Take a new set of indeterminates $y_1, \dots, y_m$ *and consider the set* $\bar{B} \subset \mathbb{K}[y, Z]$ *obtained from* B *by substituting* r_i *by* y_i. *Let* $a(y, Z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(y) Z^\alpha$ *and* $b(y, Z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(y) Z^\alpha$ *in* $\mathbb{K}[y, Z]$ *be the normal forms*[2] *of* $p(Z)$ *and* $q(Z)$ *w.r.t.* $\bar{B}$. *There exists* $\alpha \in \mathbb{N}^m$ *s.t.* $b_\alpha(r) \neq 0$ *and for any such* α *we have* $\frac{p(z)}{q(z)} = \frac{a_\alpha(r)}{b_\alpha(r)}$.

PROOF: We first note that neither $q(Z)$, nor $p(Z)$, belong to P. Indeed $\mathcal{P} \subset \mathcal{V}(q)$ and $\mathcal{V}(q)$ invariant (Lemma 2.1) would imply $q = 0$ since the orbits of the points on $\mathcal{P}$ fill an open dense set of $\mathcal{Z}$ by hypothesis.

We obviously have $P \subset I_{\mathcal{P}} \cap \mathbb{K}[Z]$ and therefore the projection of $\mathcal{V}(I_{\mathcal{P}} \cap \mathbb{K}[z, Z])$ is included in $\mathcal{P}$. Conversely, for a generic point Z on $\mathcal{P}$, the points $(\lambda \star Z, Z)$, for $\lambda \in \mathcal{G}$ lies $\mathcal{V}(I_{\mathcal{P}} \cap \mathbb{K}[z, Z])$. The projection of this latter on the second component is thus dense in $\mathcal{P}$. Hence $I_{\mathcal{P}} \cap \mathbb{K}[Z] = P$.

Therefore neither $p(Z)$ nor $q(Z)$ belong to $I_{\mathcal{P}}$. The normal forms of $q(Z)$ and $p(Z)$ w.r.t. B are, respectively, $b(r, Z)$ and $a(r, Z)$ and they are thus both different from zero.

Since p/q is invariant, $p(z)\, q(\lambda \star z) \equiv q(z)\, p(\lambda \star z) \mod G$. Hence $p(z)q(Z) - q(z)p(Z)$ belongs to $I_{\mathcal{P}}$ so that its normal form with respect to B must be zero: $p(z)\, b(r, Z) = q(z)\, a(r, Z)$. The conclusion follows. □

Theorem 2.6 applies in particular to polynomial invariants. We immediately see that:

[2] For the reductions in $\mathbb{K}[y, Z]$ the term order on Z is extended to a block order $y \ll Z$ so that the set of leading term of $\bar{B}$ is equal to the set of leading terms of B.

Corollary 2.7 *Any polynomial invariant can be written as a polynomial in* $\{r_1, \dots, r_m\}$.

Therefore a case of special interest is when the coefficients of the reduced Gröbner basis have no denominator.

Proposition 2.8 *If the coefficients of a reduced Gröbner basis of* $I_{\mathcal{P}}$ *are polynomials, then they generate* $\mathbb{K}[z]^{\mathcal{G}}$.

Example 2.9 *Consider the linear action of* $\mathrm{SO}_2(\overline{\mathbb{K}})$ *on* $\mathcal{Z} = \overline{\mathbb{K}}^3$ *acting by rotation on the* (x, y)*-plane. Choose the section* $\mathcal{P}$ *of degree* 2 *given by* $P = (X)$. *Then* $I_{\mathcal{P}} = (X, Y^2 - (x^2 + y^2), Z - z)$. *Thus* $r = x^2 + y^2$ *and* z *form a generating set for* $\mathbb{K}(x, y, z)^{\mathcal{G}}$, *but also for* $\mathbb{K}[x, y, z]^{\mathcal{G}}$.

More generallly, when the coefficients are the quotients of invariant polynomials, they provide generators for a localisation of the invariant ring. The generators of the invariant ring can then be computed following [5, Section 4.1.2].

Section of degree one are also of special interest. For those, the reduced Gröbner basis of $I_{\mathcal{P}}$ w.r.t. any term order is of the form $\{Z_1 - r_1(z), \dots, Z_n - r_n(z)\}$, where $r_i \in \mathbb{K}(z)^{\mathcal{G}}$. The rewriting described in Theorem 2.6 is then a simple substitution: if f is a rational invariant then $f(z_1, \dots, z_n) = f(r_1(z), \dots, r_n(z))$.

Example 2.10 *Following up on Example 2.4,* $f(x, y) = f\left(r^{\beta}, r^{\alpha}\right)$ *for any invariant* f.

2.4 Section, quasi-section, cross-section, partial section

The present concept of *section of degree e* appears as *quasi-section* in [19]. In [7] we defined *cross-sections of degree e* but the two notions actually differ.

In [7, Definition 3.1] an irreducible ideal P, of complementary dimension to the generic orbits, defines a cross-section if the ideal $O + P$ is zero-dimensional and radical, where $O = (G + A) : h_0^{\infty} \cap \mathbb{K}(z)[Z]$ is the ideal of the generic orbit. The degree of the cross-section is then the dimension of the $\mathbb{K}(z)$-vector space $\mathbb{K}(z)[Z]/(O + P)$. This is the number of points of intersection of the closure of a generic orbit with the variety $\mathcal{P}$ of P. The following example shows how this can differ from the present notion of section.

Example 2.11 *Consider the scaling* $\lambda \star (x, y) = (\lambda^2 x, \lambda^3 y)$ *and* $\mathcal{P}$ *as the variety of* $P = (Y - X)$. *On one hand, the orbit-section ideal is*

$$I_{\mathcal{P}} = \left(Y - \frac{x^3}{y^2}, X - \frac{x^3}{y^2}\right)$$

so that $\mathcal{P}$ *is a section of degree one. On the other hand* $O = \left(y^2 X^3 - x^3 Y^2\right)$ *so that*

$$O + P = \left(X - Y, Y^2\left(Y - \frac{x^3}{y^2}\right)\right).$$

The closures of the generic orbits contain the origin, as does $\mathcal{P}$. $\mathcal{P}$ *fails to be a cross-section because* $O + P$ *is not radical.*

Both concepts lead to valid constructions for the fields of rational invariants. The present concept of section nonetheless appears as more favorable. For instance, sections of degree one can be obtained for any scaling, i.e. diagonal representation of tori [9, 10].

In [7] we also proved Theorem 2.5 and 2.6 for the orbit ideal $O = (G + A) : h_0^{\infty}$ and it is natural to consider intermediate cases, i.e. where $\mathcal{P}$ is of lower dimension than the codimension of the generic orbits and thus the ideal $(G + A + P) : h_0^{\infty}$ is not zero dimensional. The proofs of Theorem 2.5 and 2.6 used the following properties:

- $\mathcal{P}$ is not contained in any invariant hypersurface
- $(G + A + P) : h_0^\infty \cap \mathbb{K}[Z] = P$.

Those properties would be sufficient to define a notion of *partial section*. The ideas and the results were molded in another terminology for the textbook [5, Section 4.10].

3 Invariantisation & Symmetrization

We shall define a constructive concept of invariantization and symmetrization with respect to a section. We prove that within a $\mathcal{G}$-invariant open set, which depends on the chosen section and a term order, the symmetrization of a polynomial system produces invariants with the same variety. This $\mathcal{G}$-invariant open set is given as the locus with good specialisation properties of the reduced Gröbner basis of the orbit-section ideal.

3.1 Specializations

The variety of the orbit-section ideal $I_\mathcal{P}$ defined in previous section is the intersection of the section $\mathcal{P}$ and the generic orbit. The construction of rational invariants makes essential use of the reduced Gröbner basis B of $I_\mathcal{P}$ for a given term order on Z.

Recall that G is the ideal in $\mathbb{K}[\lambda]$ defining the group $\mathcal{G}$, that A is the set of polynomials in $\mathbb{K}[z, Z, \lambda]$ describing the action of $\mathcal{G}$ in $\mathcal{Z}$ (see Proposition 2.3) and that P is the prime ideal defining $\mathcal{P}$. If $\tilde{B}$ is a reduced Gröbner basis for $(G + A + P) : h_0^\infty$, as an ideal in $\mathbb{K}(z)[Z, \lambda]$, according to a block order that eliminates λ, then $B = \tilde{B} \cap \mathbb{K}(z)[Z]$ is the reduced Gröbner basis for $I_\mathcal{P}$. For $\bar{z} \in \mathcal{Z}$ we write $A_{\bar{z}} \subset \overline{\mathbb{K}}[Z, \lambda]$ for the specialisation of A at $\bar{z}$.

There exists a proper closed set $\mathcal{W}$ in $\mathcal{Z}$ such that for $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$ the specialisation $B_{\bar{z}}$ of B at $\bar{z}$ is precisely the reduced Gröbner basis of $(G + A_{\bar{z}} + P) : h_0^\infty \cap \overline{\mathbb{K}}[Z]$. The variety of this latter ideal is the intersection of $\mathcal{O}_{\bar{z}}$ with $\mathcal{P}$. We can determine such a set $\mathcal{W}$ with specialisation criteria as [11, Theorem 3.1] or [12, Theorem 4.3] that apply to a Gröbner basis of $(G + A + P) : h_0^\infty$ considered as an ideal of $\mathbb{K}[z][Z, \lambda]$. Yet, to compute a minimal such $\mathcal{W}$ it is necessary to resort to the computation of a comprehensive Gröbner basis [12, 22] or of a Gröbner cover [23, 14].

In the present situation, $B_{\bar{\lambda} \star \bar{z}} = B_{\bar{z}}$ for any $\bar{\lambda} \in \mathcal{G}$ so that the minimal such $\mathcal{W}$ is $\mathcal{G}$-invariant. It is left for future research to examine if this property can bring a computational advantage in determining $\mathcal{W}$, in combination with the geometric information that can be read directly on the Gröbner basis [13, 16, 17]. Let us just make one observation. Obviously the varieties of the denominators of the coefficients in B, which are $\mathcal{G}$-invariant, are contained in $\mathcal{W}$. Yet the following example shows that it is not enough to consider the product of those denominators. Among other considerations one might have to consider the denominators in $\tilde{B}$.

Example 3.1 *Consider the action of $\mathcal{G} = \overline{\mathbb{K}}^*$ on the plane given by $\lambda \star (x, y) = \left(\lambda^{-1} x, \lambda y\right)$. We have $G = (\lambda \mu - 1)$ in $\mathbb{K}[\lambda, \mu]$, $A = (X - \mu x, Y - \lambda y)$ and let us take the section defined by $P = (X - 1)$.*

The reduced Gröbner basis of $J = G + A + P$ in $\mathbb{K}(x, y)[X, Y, \lambda, \mu]$ for a block term order $\{X, Y\} \ll \{\lambda, \mu\}$ is

$$\tilde{B} = \left\{ X - 1, Y - x y, \lambda - x, \mu - \frac{1}{x} \right\}.$$

This corroborates the fact that the orbits of the points $(0, \bar{y})$ do not intersect the section defined by P. When $\bar{x} \neq 0$, the Gröbner basis for the ideal of $\mathcal{O}_{(\bar{x}, \bar{y})} \cap \mathcal{P}$ is obtained by specializing $B = \tilde{B} \cap \mathbb{K}(x, y)[X, Y] =$

$\{X-1, Y-xy\}$. *Hence for this section* $\mathcal{W} = \mathcal{V}(x)$. *This is also what appears when computing the Gröbner cover [14].*

Example 3.2 *Following on Example 2.9. The implementation of the Gröbner cover [14, 15] shows that we have to take* $\mathcal{W} = \mathcal{V}\big(x^2+y^2\big)$.

The above examples could nonetheless be deceptive in that it leads us to think that the denominators of $\tilde{B}$ provide a description of $\mathcal{W}$. Some denominators in $\tilde{B}$ are spurious in the description of $\mathcal{W}$.

3.2 Invariantization

Consider a rational action of $\mathcal{G}$ on $\mathcal{Z} = \overline{\mathbb{K}}^n$ and $\mathcal{P}$ a section of degree e. We fix a term order on Z and consider the reduced Gröbner basis B of the orbit-section ideal $I_\mathcal{P}$ in $\mathbb{K}(z)[Z]$. The coefficients of B belong to and generate $\mathbb{K}(z)^\mathcal{G}$ (Theorem 2.5 and 2.6). $\mathbb{K}(z)^\mathcal{G}$ is thus the field of definition of the orbit-section ideal $I_\mathcal{P}$ and we can consider $I_\mathcal{P}^\mathcal{G} = I_\mathcal{P} \cap \mathbb{K}(z)^\mathcal{G}[Z]$. It is a zero dimensional ideal in $\mathbb{K}(z)^\mathcal{G}[Z]$ and the dimension of $\mathbb{K}(z)^\mathcal{G}[Z]/I_\mathcal{P}^\mathcal{G}$ as a $\mathbb{K}(z)^\mathcal{G}$-vector space is e.

A polynomial $f \in \mathbb{K}(z)^\mathcal{G}[Z]$ defines an element $\bar{f}$ in the quotient $\mathbb{K}(z)^\mathcal{G}[Z]/I_\mathcal{P}$. The multiplication map

$$\begin{array}{rccc} m_f : & \mathbb{K}(z)^\mathcal{G}[Z]/I_\mathcal{P} & \to & \mathbb{K}(z)^\mathcal{G}[Z]/I_\mathcal{P} \\ & \bar{g} & \mapsto & \overline{f\,g} \end{array}$$

is a linear mapping [4, Proposition 4.1]. Note that $m_{fg} = m_f \circ m_g = m_g \circ m_f$. When f is not a zero divisor modulo $I_\mathcal{P}$ then there exists $f_1 \in \mathbb{K}(z)^\mathcal{G}[Z]$ s.t. $f\,f_1 \equiv 1 \mod I_\mathcal{P}^\mathcal{G}$. It follows that $\det m_f \neq 0$ and $m_{f_1} = (m_f)^{-1}$. We shall thus define $m_{g/f}$ as $m_{g\,f_1} = m_g (m_f)^{-1}$ when f is not a zero divisor modulo $I_\mathcal{P}^\mathcal{G}$.

$\mathbb{K}[Z]_P$ denotes the localisation of $\mathbb{K}[Z]$ at the complement of P in $K[Z]$. No element of $\mathbb{K}[Z] \setminus P$ are zero divisors modulo $I_\mathcal{P}^\mathcal{G}$. The following is thus well defined.

Definition 3.3 *For* $f \in \mathbb{K}[Z]_P$ *we define its* invariantization *with respect to the section* $\mathcal{P}$ *as the characteristic polynomial*

$$f_\mathcal{P}(z,\zeta) = \zeta^e - f_\mathcal{P}^{(1)}(z)\zeta^{e-1} + \ldots + (-1)^j f_\mathcal{P}^{(j)}(z)\,\zeta^{e-j} + \ldots + (-1)^e f_\mathcal{P}^{(e)}(z)$$

of the multiplication map m_f *by* f *in* $\mathbb{K}(z)^\mathcal{G}[Z]/I_\mathcal{P}^\mathcal{G}$. *The coefficients* $f_\mathcal{P}^{(1)}(z), \ldots, f_\mathcal{P}^{(e)}(z) \in \mathbb{K}(z)^\mathcal{G}$ *are the* symmetrizations *of* f *w.r.t.* $\mathcal{P}$.

A constructive concept of invariantization of algebraic functions was introduced in [8]. When restricted to polynomial functions, this latter is equivalent to the present concept. Thus, by [8, Theorem 3.9], $f_\mathcal{P}(z,\zeta)$ is the defining polynomial of a smooth algebraic function that is the unique local invariant that agrees with the values of f on the section $\mathcal{P}$ in the neighborhood of one of its point. The algebraic invariantization thus provides a constructive approach to the local invariantization process that is central in [6].

Given a reduced Gröbner basis of $I_\mathcal{P}$, we can identify a set of monomials that forms a basis of the $\mathbb{K}(z)^\mathcal{G}$-vector space $\mathbb{K}(z)^\mathcal{G}[Z]/I_\mathcal{P}^\mathcal{G}$ and explicitely write down the matrix of m_f in this basis. Invariantization as defined above can thus be computed algorithmically. At no additional cost, the *symmetrizations* $f_\mathcal{P}^{(1)}(z), \ldots, f_\mathcal{P}^{(e)}(z) \in \mathbb{K}(z)^\mathcal{G}$ are written in terms of the generators $\{r_1, \ldots, r_m\}$ of $\mathbb{K}(z)^\mathcal{G}$ that are read from the reduced Gröbner basis of $I_\mathcal{P}$ according to Theorem 2.6.

Proposition 3.4 *If the coefficients of the reduced Gröbner basis of* $I_\mathcal{P}$ *can be written as polynomials in* $\{r_1, \ldots, r_m\} \subset \mathbb{K}(z)^\mathcal{G}$ *then, for any* $f \in \mathbb{K}[Z]$, *we can determine polynomials* $\tilde{f}_\mathcal{P}^{(j)}$, $1 \le j \le e$, *in* $\mathbb{K}[y_1, \ldots, y_m]$ *such that the symmetrizations of* f *are*

$$f_\mathcal{P}^{(j)}(z_1, \ldots, z_n) = \tilde{f}_\mathcal{P}^{(j)}\left(r_1(z), \ldots, r_m(z)\right).$$

The case where $\mathcal{P}$ is a section of degree 1 is particularly favorable. Then $I_{\mathcal{P}} = (Z_1 - r_1(z), \ldots, Z_n - r_n(z))$ with $r_1, \ldots, r_n \in \mathbb{K}(z)^{\mathcal{G}}$ and therefore $f_{\mathcal{P}}(z, \zeta) = \zeta - f_{\mathcal{P}}^{(1)}$ with $f_{\mathcal{P}}^{(1)}(z_1, \ldots, z_n) = f(r_1(z), \ldots, r_n(z))$. Therefore $\tilde{f}_{\mathcal{P}}^{(1)} = f$.

Example 3.5 Scalings in the plane. *We follow up on Example 2.4. We choose* $P = (X^\alpha - Y^\beta)$. *It defines a section of degree one and the reduced Gröbner of* $I_{\mathcal{P}}^{\mathcal{G}}$ *is* $B = \left\{X - r^\beta,\ Y - r^\alpha\right\}$ *where* $r = \frac{y^a}{x^b}$. *Thus* $f_{\mathcal{P}}^{(1)}(x, y) = f(r^\beta, r^\alpha)$.

Example 3.6 *Following on Example 2.9 where we considered a linear action of* $\mathrm{SO}_2(\overline{\mathbb{K}})$ *on* $\mathcal{Z} = \overline{\mathbb{K}}^3$ *and chose the section defined by* $P = (X)$ *so that* $I_{\mathcal{P}} = (X, Y^2 - r, Z - z)$ *where* $r = x^2 + y^2$.

Consider the polynomials $f_1 = -x\left(x^2 + y^2 - 1\right) - y\,z$ *and* $f_2 = -y\left(x^2 + y^2 - 1\right) + x\,z$. *Neither* f_1 *nor* f_2 *is invariant.*

A basis for $\mathbb{K}(x, y)[X, Y]/I_{\mathcal{P}}$ *is given by the set of monomials* $\{1, Y\}$. *In this basis, the multiplication matrix of* $f_1(X, Y, Z)$ *and* $f_2(X, Y, Z)$ *are respectively*

$$M_1 = \begin{bmatrix} 0 & -z\,r \\ -z & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & r\,(1-r) \\ 1-r & 0 \end{bmatrix}.$$

Hence

$$f_1^{(1)} = -\mathrm{Tr}(M_1) = 0,\ f_1^{(2)} = \mathrm{Det}(M_1) = -z^2\,r,$$

and

$$f_2^{(1)} = -\mathrm{Tr}(M_2) = 0,\ f_2^{(2)} = \mathrm{Det}(M_2) = r\,(1-r)^2.$$

3.3 Geometric interpretation and connection to the finite group case

Let $\mathcal{W}$ be a proper $\mathcal{G}$-invariant closed set such that for $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$, the specialisation $B_{\bar{z}}$ of B is a Gröbner basis for an ideal whose variety is $\mathcal{O}_{\bar{z}} \cap \mathcal{P}$. We discussed how to determine such a $\mathcal{W}$ in Section 3.1. For $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$ the ideal $(B_{\bar{z}})$ has e zeros $\left\{z^{(1)}, \ldots, z^{(e)}\right\}$. They form $\mathcal{O}_{\bar{z}} \cap \mathcal{P}$, possibly with multiplicities.

Proposition 3.7 *Consider* $f \in \mathbb{K}[Z]_P$ *and* $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$. *Let* $\left\{z^{(1)}, \ldots, z^{(e)}\right\}$ *be the* e *zeros of* $B_{\bar{z}}$. *For* $1 \le j \le e$,

$$f_{\mathcal{P}}^{(j)}(\bar{z}) = S_j\left(f(z^{(1)}), \ldots, f(z^{(e)})\right)$$

where S_j *is the* j*-th symmetric polynomial in* e *variables.*

proof: The eigenvalues of m_f are the evaluations of f at the roots of $(B_{\bar{z}})$, with matching multiplicities [4, Theorem 4.5]. Hence $f_{\mathcal{P}}(z, \zeta) = \prod_{i=1}^{e}\left(\zeta - f(z^{(j)})\right)$. □

Note that $f_{\mathcal{P}}^{(1)}(z)$ is the trace of m_f and thus $f \mapsto f_{\mathcal{P}}^{(1)}$ is a linear map. But, contrary to the Reynolds operator, $(\lambda \star f)_{\mathcal{P}}^{(1)} \neq f_{\mathcal{P}}^{(1)}$ in general. For instance, in the case of Example 3.5, $(\lambda \star f)_{\mathcal{P}}^{(1)}(x, y) = f(\lambda^{-a} r^\beta, \lambda^{-b} r^\alpha)$ while $f_{\mathcal{P}}^{(1)}(x, y) = f(r^\beta, r^\alpha)$.

Corollary 3.8 *If* $f \in \mathbb{K}(z)^{\mathcal{G}}$ *then* $f_{\mathcal{P}}^{(j)} = \begin{pmatrix} j \\ e \end{pmatrix} f^j$.

proof: By Proposition 2.1, $\mathbb{K}(z)^{\mathcal{G}} \subset \mathbb{K}[z]_P$ since $\mathcal{P}$ cannot be included in any invariant hypersurface. As an invariant, f is constant on orbits: $f(\bar{z}) = f(z^{(1)}) = \ldots = f(z^{(e)})$ □

In the case where $\mathcal{G}$ is a finite group acting regularly and faithfully, we should consider $\mathcal{P} = \mathcal{Z}$ as section and its degree e is the order of group $\mathcal{G}$. For $\bar{z} \in \mathcal{Z}$, $\left\{z^{(1)}, \ldots, z^{(e)}\right\} = \{\lambda \star \bar{z} \,|\, \lambda \in \mathcal{G}\}$. Thus the invariantization of a polynomial f is

$$\zeta^e - f^{(1)}(z)\zeta^{e-1} + \ldots + (-1)^j f^{(j)}(z)\,\zeta^{e-j} + \ldots + (-1)^e f^{(e)}(z) = \prod_{\lambda\in\mathcal{G}} (\zeta - f(\lambda \star z))$$

where $f^{(j)}(z)$ is simply the i-th symmetric function on $\{f(\lambda \star z) \,|\, \lambda \in \mathcal{G}\}$. The symmetrization $f^{(j)}(z)$, for $1 \le j \le e$, are invariant polynomials. As can be read in [21, Proposition 2.6.4], if the variety $\mathcal{V}(F)$ of a finite set of polynomials F is invariant then

$$\mathcal{V}(F) = \mathcal{V}\left(f^{(i)} \,|\, f \in F,\ 1 \le i \le e\right).$$

This proves that any variety invariant under the action of a finite group is the variety of a set of invariant polynomials.

The notion of symetrization w.r.t. a section $\mathcal{P}$ we introduced for algebraic groups of positive dimension allows us to provide an analogous result for the rational action of an algebraic group of positive dimension.

3.4 Polynomial systems with symmetry

The proof of the following result is similar to the proof in the case of finite groups. A caveat is that the result is valid outside of a proper closed set $\mathcal{W}$ determined by the specialisation properties discussed in Section 3.1.

Proposition 3.9 *Let F be a set of polynomials in $\mathbb{K}[z]$ and assume that its variety $\mathcal{F}$ is $\mathcal{G}$-invariant. Consider a section $\mathcal{P}$ of degree e. Then*

$$\mathcal{V}\left(f_{\mathcal{P}}^{(i)} \,|\, f \in F,\ 1 \le i \le e\right) \setminus \mathcal{W} = \mathcal{F} \setminus \mathcal{W},$$

where $\mathcal{W}$ is the $\mathcal{G}$-invariant variety discussed in Section 3.1.

In the above proposition, $\mathcal{V}\left(f_{\mathcal{P}}^{(i)} \,|\, f \in F,\ 1 \le i \le e\right)$ stands for the variety of the numerators. The varieties of the denominators arising in $f_{\mathcal{P}}^{(j)}$ actually lies in $\mathcal{W}$ due to Proposition 3.4.

PROOF: For $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$, we note $\left\{z^{(1)}, \ldots, z^{(e)}\right\}$ the zeros of $(B_{\bar{z}})$. Each $z^{(i)}$ belongs to $\mathcal{O}_{\bar{z}} \cap \mathcal{P}$. As $\mathcal{F}$ is $\mathcal{G}$-invariant we thus have $\bar{z} \in \mathcal{F} \setminus \mathcal{W} \Leftrightarrow \left\{z^{(1)}, \ldots, z^{(e)}\right\} \subset \mathcal{F} \setminus \mathcal{W}$.

Since

$$\prod_{j=1}^{e} \left(\zeta - f(z^{(j)})\right) = \zeta^e - f_{\mathcal{P}}^{(1)}(\bar{z})\zeta^{e-1} + \ldots + (-1)^j f_{\mathcal{P}}^{(j)}(\bar{z})\,\zeta^{e-j} + \ldots + (-1)^e f_{\mathcal{P}}^{(e)}(z)$$

we have $\left(f(z^{(1)}) = 0, \ldots, f(z^{(e)}) = 0\right) \Leftrightarrow \left(f^{(j)}(\bar{z}) = 0, \forall\ 1 \le j \le e\right)$. □

Several examples of polynomial systems invariant under scaling were treated in [9]. Indeed [9, Theorem 5.3] can be seen as a special case of the above result.

Example 3.10 *Following on Example 2.9, 3.2 and 3.6, one observes that $\mathcal{V}(f_1, f_2)$ is invariant under the action of $\mathrm{SO}_2(\overline{\mathbb{K}})$ since $(f_1, f_2)^t$ is equivariant. We determined that outside $\mathcal{W} = \mathcal{V}\left(x^2 + y^2\right)$ the reduced Gröbner basis of $I_{\mathcal{P}}$ specialises to the reduced Gröbner basis of the intersection of the orbit of that point with the section $\mathcal{P}$. Applying the above construction we obtain*

$$\mathcal{V}(f_1, f_2) \setminus \mathcal{W} = \mathcal{V}\left(f_1^{(1)}, f_1^{(2)}, f_2^{(1)}, f_2^{(2)}\right) \setminus \mathcal{W} = \mathcal{V}\left(z^2, (x^2 + y^2 - 1)^2\right).$$

The above proposition combined with Proposition 3.4 thus allows us to provide the following general statement that is the main claim of this article.

Theorem 3.11 *Assume $\mathcal{P}$ is a section of degree e to the orbits of the rational action of an algebraic group $\mathcal{G}$ on $\mathcal{Z}$. We can determine a $\mathcal{G}$-invariant algebraic set $\mathcal{W}$ and $r_1, \ldots, r_m \in \mathbb{K}(z)^{\mathcal{G}}$ such that $\mathbb{K}(z)^{\mathcal{G}} = \mathbb{K}(r_1, \ldots, r_m)$ with the following properties: If the variety $\mathcal{V}(F)$ of a finite set of polynomials F in $\mathbb{K}[z]$ is $\mathcal{G}$-invariant then there exists a finite set of polynomials $\tilde{F} \subset \mathbb{K}[y_1, \ldots, y_m]$ such that for any $\bar{z} \in \mathcal{Z} \setminus \mathcal{W}$*

$$f(\bar{z}) = 0,\ \forall f \in F \quad \Leftrightarrow \quad \tilde{f}\,(r(\bar{z})) = 0,\ \forall \tilde{f} \in \tilde{F}.$$

Following the arguments in [2, Proposition 3.15], the construction we presented would work similarly for a semi-algerbaic set $\mathcal{K} = \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_m(z) \geq 0\}$ if we could ensure that the orbits in $\mathcal{K} \cap \mathcal{W}$ intersect the section $\mathcal{P}$ in e points (counting multiplicities), *i.e.* the intersection points are not complex. This is the case in the example above. It is also the case for scalings as [9, 10, Theorem 4.5] gives an explicit rational expression for the intersection of the orbit of a point with the section of degree 1 appearing in the construction presented there.

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