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Equivalent Robin Boundary Conditions for Acoustic and Elastic Media

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and Computer Science

Instituto de Matemáticas, UNAM

Mexico City, June 3 and 4, 2015



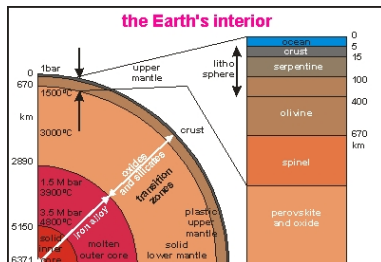
Context - Motivations

HPC-GA Project

- **Numerical Models (WP2)** : Time dependent Elasto-Acoustic Coupling
 - To reproduce **earthquakes**
 - To model the effects of an ocean on the propagation of seismic waves
- **Contributions** : Thin Layer Models for Geophysical Applications
 - Tool : Asymptotic method
 - We derive approximate models for the propagation of seismic waves and acoustic waves

Motivations

Configurations of interest

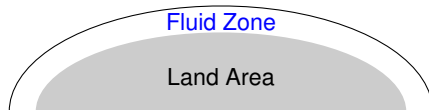


- **Configuration A** : The medium of interest is a Land Area surrounded by a [Fluid Zone](#)
- **Configuration B** : The medium of interest is a Land Area surrounded by a [Fluid Zone](#) and a part of the atmosphere

Motivations

Configuration A

- **Configuration:**



- The thickness of the **Fluid Zone** is negligible with respect to the wavelength
- **Difficulty**
Apply a FEM on a mesh with **thin cells** in the **Fluid Zone** and much larger outside
- **Sub-Configurations:**
 - 1 **Configuration 1** : A **thin layer** with a **uniform** thickness
 - 2 **Configuration 2** : A **thin layer** with a **variable** thickness

Approach : an Asymptotic Method

- 1 "Replace" the **thin layer** by an Equivalent Boundary Condition
- 2 Couple this condition with the elastic equation

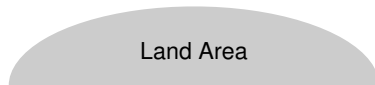


Figure : Configuration A without the **thin layer**

- 3 Apply a (DG) Finite Element Method

Bibliography

- Derivation of Equivalent Conditions for **thin layer** Problems :



ENGQUIST-NÉDÉLEC, 93



ABBOUD-AMMARI, 96



BENDALI-LEMRABET, 96



LAFITTE, 99

...



PÉRON, 2014



BUREL, PHD THESIS, 2014



DE CASTRO, DIAZ, PÉRON, 2014

Outline

- 1 Energy Estimates (Conf. 1)
- 2 Equivalent Conditions (Conf. 1)
- 3 Numerical Results (Conf. 1)
- 4 Equivalent Conditions (Conf. 2)

Outline

- 1 Energy Estimates (Conf. 1)**
- 2 Equivalent Conditions (Conf. 1)
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A Model Problem

The Problem (\mathbf{P}_ε) set in a smooth domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

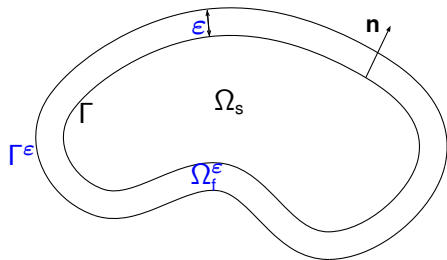
$$\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} \quad \text{in } \Omega_s$$

$$\Delta p + \left(\frac{\omega}{c_f}\right)^2 p = 0 \quad \text{in } \Omega_f^\varepsilon$$

$$\partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} \quad \text{on } \Gamma$$

$$\underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = -p \mathbf{n} \quad \text{on } \Gamma$$

$$\partial_n p - i \frac{\omega}{c_f} p = 0 \quad \text{on } \Gamma^\varepsilon$$



Issue : Uniform Estimates for solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of (\mathbf{P}_ε) as $\varepsilon \rightarrow 0$?

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

- (i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

- (i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem (\mathbf{P}_ε) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon)$, and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C\|\mathbf{f}\|_{0,\Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem (\mathbf{P}_ε) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon)$, and

$$\|\mathbf{u}_\varepsilon\|_{1, \Omega_s} + \|p_\varepsilon\|_{1, \Omega_f^\varepsilon} \leq C \|\mathbf{f}\|_{0, \Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$
Key for the proof:

To write a formulation in a fixed domain by means of the Scaling

$$S = \frac{s}{\varepsilon} \in (0, 1) \quad \text{where} \quad s \in (0, \varepsilon)$$

Outline

- 1 Energy Estimates (Conf. 1)
- 2 Equivalent Conditions (Conf. 1)**
- 3 Numerical Results (Conf. 1)
- 4 Equivalent Conditions (Conf. 2)

Methodology

- **Step 1** : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ when $\varepsilon \rightarrow 0$

$$\begin{aligned}\mathbf{u}_\varepsilon(x) &= \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \dots \\ p_\varepsilon(x) &= p_0(y_\alpha, \frac{\mathbf{s}}{\varepsilon}) + \varepsilon p_1(y_\alpha, \frac{\mathbf{s}}{\varepsilon}) + \varepsilon^2 p_2(y_\alpha, \frac{\mathbf{s}}{\varepsilon}) + \dots\end{aligned}$$

- **Step 2** : Derive Equivalent Conditions of order $k + 1 \in \mathbb{N}$.
Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \dots + \varepsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\varepsilon^{k+1})$$

- **Step 3** : Prove Stability and Convergence results for Equivalent models

Step 1 : Multiscale Expansion

First terms

1

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

2

$$p_0 = -i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}$$

3

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) - i\omega c_f \rho_f \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} = i\omega c_f \rho_f P_1(D)(\mathbf{u}_0 \cdot \mathbf{n}) \mathbf{n} & \text{on } \Gamma \end{cases}$$

where $P_1(D) = -2\mathcal{H} + i\kappa^{-1}\Delta_\Gamma$

4

$$p_1(y_\alpha, \mathbf{S}) = -i\omega c_f \rho_f (\mathbf{u}_1 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_0 \cdot \mathbf{n})) + \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{S}, \quad \mathbf{S} \in (0, 1)$$

Step 1 : Multiscale Expansion

Next terms $\mathbf{u}_2, \mathfrak{p}_2$



$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 \\ \mathbf{T}(\mathbf{u}_2) - i\omega c \rho_f \mathbf{u}_2 \cdot \mathbf{n} \mathbf{n} = i\omega c \rho_f (P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n})) \mathbf{n} \end{cases}$$



$$\mathfrak{p}_2(y_\alpha, \mathbf{S}) = a_2(y_\alpha) + b_2(y_\alpha) \mathbf{S} + c_2(y_\alpha) \mathbf{S}^2,$$

where

$$a_2 = -i\omega c \rho_f (\mathbf{u}_2 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n}))$$

$$b_2 = \omega^2 \rho_f \mathbf{u}_1 \cdot \mathbf{n}$$

$$c_2 = \omega \rho_f \left(-\omega \mathcal{H} + \frac{1}{2} i c [\Delta_\Gamma + \kappa^2 \mathbb{I}] \right) (\mathbf{u}_0 \cdot \mathbf{n}),$$

and

$$P_2(D) = -2i\kappa \mathcal{H} \mathbb{I} + ((4\mathcal{H}^2 - \mathcal{K}) \mathbb{I} - \Delta_\Gamma) \\ + (i\kappa)^{-1} [2\mathcal{H} \Delta_\Gamma - \operatorname{div}_\Gamma(\mathcal{H} \mathbb{I} - \mathcal{R}) \nabla_\Gamma] + \Delta_\Gamma (2\mathcal{H}) + (i\kappa)^{-2} \Delta_\Gamma^2.$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\epsilon^N := \mathbf{u}_\epsilon - \sum_{n=0}^N \epsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\epsilon^N := p_\epsilon - \sum_{n=0}^N \epsilon^n p_n \quad \text{in } \Omega_f^\epsilon.$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{ll} \Delta r_\epsilon^N + \kappa^2 r_\epsilon^N & = f_\epsilon \quad \text{in } \Omega_f^\epsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{r}_\epsilon^N) + \omega^2 \rho \mathbf{r}_\epsilon^N & = 0 \quad \text{in } \Omega_s \\ \partial_{\mathbf{n}} r_\epsilon^N - \rho_f \omega^2 \mathbf{r}_\epsilon^N \cdot \mathbf{n} & = g_\epsilon \quad \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\epsilon^N) + r_\epsilon^N \mathbf{n} & = 0 \quad \text{on } \Gamma \\ \partial_{\mathbf{n}} r_\epsilon^N - i\kappa r_\epsilon^N & = 0 \quad \text{on } \Gamma^\epsilon. \end{array} \right.$$

The RHS are explicit : $f_\epsilon = \mathcal{O}(\epsilon^{N-\frac{1}{2}})$ in Ω_f^ϵ , $g_\epsilon = \mathcal{O}(\epsilon^N)$ on Γ .

Optimal Estimates : $\|\mathbf{r}_\epsilon^N\|_{1,\Omega_s} + \sqrt{\epsilon} \|r_\epsilon^N\|_{1,\Omega_f^\epsilon} \leq C_N \epsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

For all $k \in \{0, 1, 2\}$, we identify a simpler problem

$$(\mathbf{P}_{\epsilon}^k) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_{\epsilon}^k) + \omega^2 \rho \mathbf{u}_{\epsilon}^k = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_{\epsilon}^k) + \mathbf{B}_{k,\epsilon}(\mathbf{u}_{\epsilon}^k \cdot \mathbf{n}) \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

with $\mathbf{B}_{k,\epsilon}$ a surfacic differential operator.

Step 2 : Equivalent Conditions

1 Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

2 Order 2 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c_f \rho_f (1 + \varepsilon (-2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma)) (\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}) \mathbf{n} = 0$$

3 Order 3 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - i\omega c_f \rho_f (\mathbb{I} + \varepsilon P_1(D) + \varepsilon^2 P_2(D)) (\mathbf{u}_\varepsilon^2 \cdot \mathbf{n}) \mathbf{n} = 0$$

where $P_1(D) = -2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma$

Step 3 : Stability and Convergence results

Functional setting : $\mathbf{u}_\varepsilon^k \in V_k := \{\mathbf{u} \in H^1(\Omega_s) \mid \mathbf{u} \cdot \mathbf{n}|_\Gamma \in H^k(\Gamma)\}$

Proposition

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{P}_\varepsilon^k)$ with data $\mathbf{f} \in L^2(\Omega_s)$ has a unique solution $\mathbf{u}_\varepsilon^k \in V_k$ and

- **Stability**

$$\|\mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C \|\mathbf{f}\|_{0,\Omega_s}$$

- **Convergence**

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C_k \varepsilon^{k+1}$$

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Numerical simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_s = D(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on Γ ($\mathbf{f} = 0$) : $p_i(\mathbf{x}) = \exp(i\omega\mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- \mathbb{P}_3 -finite elements (Lagrange) in the Library [Hou10ni](#)
- Ω_s : Isotropic elastic material with Lamé coefficients :

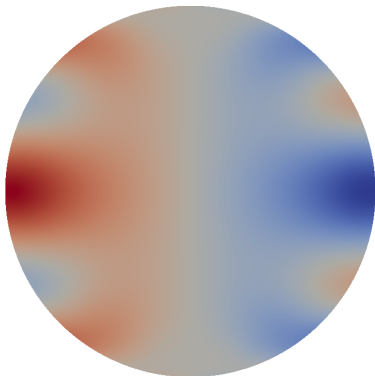
$$\mu = 26.32 \times 10^9 \quad \text{and} \quad \lambda = 51.08 \times 10^9 .$$

- $c_f = 1500 \text{ m.s}^{-1}$, $\rho_f = 1000 \text{ kg.m}^{-3}$, $\rho = 2700 \text{ kg.m}^{-3}$.

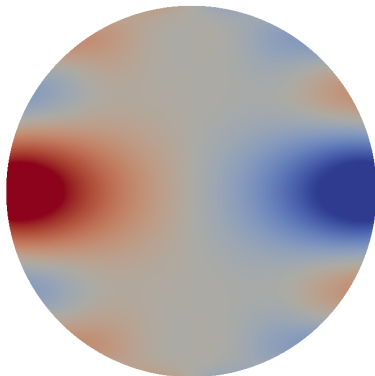
Numerical Simulations

Qualitative Comparisons

$$\varepsilon = 0.001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$
 \mathbf{u}_ε : analytical solution

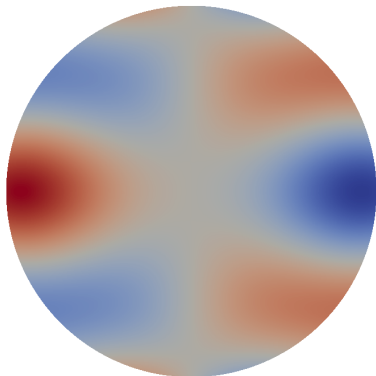


$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$
 \mathbf{u}_ε^2 (model of order 3): IPDGM

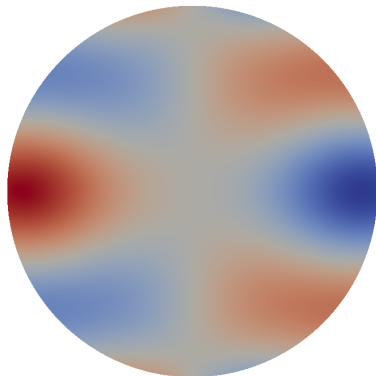
Numerical Simulations

Qualitative Comparisons

$$\varepsilon = 0.0001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$
 \mathbf{u}_ε : analytical solution



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$
 \mathbf{u}_ε^2 (model of order 3): IPDGM

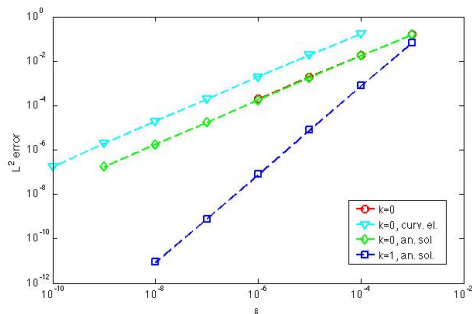
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2 -error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0, \Omega_s}$ w.r.t. ε

\mathbf{u}_ε : analytical solution

\mathbf{u}_ε^k : analytical solution or numerical solution (IPDGM)



Outline

- 1 Energy Estimates (Conf. 1)
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- 4 Equivalent Conditions (Conf. 2)**

Configuration 2

Framework in 2D

- Parameterization of the layer Ω_Γ^ϵ

$$\Omega_\Gamma^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

Configuration 2

Framework in 2D

- Parameterization of the layer Ω_f^ϵ

$$\Omega_f^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

- Euclidean metric of Ω_f^ϵ defined in the coordinates (t, s) :

$$\begin{pmatrix} (1 + s f(t) c(t))^2 + (s f'(t))^2 & s f(t) f'(t) \\ s f(t) f'(t) & f(t)^2 \end{pmatrix}$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

- Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

- Order 2 :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - i\omega c_f \rho_f \left(1 + \epsilon \left(-f(t)c(t) + i\kappa^{-1} [g(t)\partial_t + f(t)\partial_t^2] \right) \right) \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

where

$$g(t) = (2 - f^2(t)) f'(t)$$

Prospect

- 1 To implement Equivalent Conditions in 3D and in Time domain
- 2 To implement Equivalent Conditions in Configuration 2
- 3 To test these asymptotic models with real data

Thank you for your attention!

Proof of Uniform Estimates

Variational Problem (VP_ϵ)

- $V_\epsilon = H^1(\Omega_s) \times H^1(\Omega_f^\epsilon)$
- For all $\epsilon > 0$, (VP_ϵ) writes : Find $(\mathbf{u}_\epsilon, p_\epsilon) \in V_\epsilon$ such that

$$\forall (\mathbf{v}, q) \in V_\epsilon, \quad a_\epsilon((\mathbf{u}_\epsilon, p_\epsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V_\epsilon', V_\epsilon}$$

$$a_\epsilon((\mathbf{u}, p), (\mathbf{v}, q)) := \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}) \, d\mathbf{x} \\ + \int_{\Omega_f^\epsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, d\mathbf{x} + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, ds - i\kappa \int_{\Gamma^\epsilon} p \bar{q} \, ds$$

Formulation in a fixed domain

Scaled Problem (\mathfrak{P}_ε)

- $\Omega_f := \Gamma \times (0, 1)$
- $V = H^1(\Omega_s) \times H^1(\Omega_f)$
- The problem (\mathfrak{P}_ε) writes : Find $(\mathbf{u}, \mathbf{p}) \in V$ such that $\forall (\mathbf{v}, \mathbf{q}) \in V$

$$\begin{aligned} \varepsilon \int_0^1 \int_\Gamma [(\mathbb{I} + \varepsilon S\mathcal{R})^{-2} \nabla_\Gamma \mathbf{p} \nabla_\Gamma \bar{\mathbf{q}} + \varepsilon^{-2} \partial_S \mathbf{p} \partial_S \bar{\mathbf{q}} - \kappa^2 \mathbf{p} \bar{\mathbf{q}}] \det(\mathbb{I} + \varepsilon S\mathcal{R}) \, d\Omega_f \\ + a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p} \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma \\ - i\kappa \int_\Gamma \mathbf{p}(\cdot, 1) \bar{\mathbf{q}}(\cdot, 1) \det(\mathbb{I} + \varepsilon \mathcal{R}) \, d\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathbf{q}) \rangle_{V', V} \end{aligned}$$

Scaled Problem (\mathfrak{P}_ε)

Uniform estimates

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, the problem (\mathfrak{P}_ε) with data $\mathfrak{F}_\varepsilon \in V'$ has a unique solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ which satisfies

$$\sqrt{\varepsilon} \|\nabla_{\Gamma} \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon}^{-1} \|\partial_s \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'} .$$

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, any solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ of problem (\mathfrak{P}_ε) with a data $\mathfrak{F}_\varepsilon \in V'$ satisfies

$$\|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'} .$$

Sketch of the proof of the Lemma

Assume : $\exists(\mathbf{u}_m, \mathbf{p}_m) \in V^{\mathbb{N}}$ satisfying $(\mathfrak{P}_{\varepsilon_m})$ with $\varepsilon_m \rightarrow 0$ and $\mathfrak{F}_m \in V'$ s.t.

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \rightarrow 0$$

We prove successively that :

1

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C \quad \text{with} \quad W = H^1(\Omega_s) \times H^1(0, 1; L^2(\Gamma))$$

2 (up to the extraction of a subsequence)

$$\begin{aligned} (\mathbf{u}_m, \mathbf{p}_m) &\rightarrow (\mathbf{u}, \mathbf{p}) \quad \text{in } L^2(\Omega_s) \times L^2(\Omega_f) \\ (\mathbf{u}_m \cdot \mathbf{n}, \gamma_0 \mathbf{p}_m) &\rightarrow (\mathbf{u} \cdot \mathbf{n}, \gamma_0 \mathbf{p}) \quad \text{in } L^2(\Gamma) \times L^2(\Gamma) \end{aligned}$$

3

$$\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}\|_{0, \Gamma} = 1$$

4 $\mathbf{u} = 0$ and $\mathbf{p} = 0$ (using (SA)) : Contradiction

Step 3 : Proof of Convergence

(i) Derive an expansion of \mathbf{u}_ε^k and show that :

$$\begin{aligned}\mathbf{u}_\varepsilon &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k, \\ \mathbf{u}_\varepsilon^k &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k.\end{aligned}$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1, \Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k, \varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma,$$

According to the Stability result, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s} \leq C \varepsilon^{k+1}.$$

Convergence of the models

Dirichlet case

For $k \in \{0, 1, 2\}$, we plot the L^2 -error $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s}$ w.r.t. ϵ

\mathbf{u}_ϵ : analytical solution

\mathbf{u}_ϵ^k : analytical solution or numerical solution (IPDGM)

