



HAL
open science

Equivalent Robin Boundary Conditions for Acoustic and Elastic Media

Julien Diaz, Victor Péron

► **To cite this version:**

Julien Diaz, Victor Péron. Equivalent Robin Boundary Conditions for Acoustic and Elastic Media. First Inria-Mexico Workshop in Applied Mathematics and Computer Science, Jun 2015, Mexico City, Mexico. hal-01254193

HAL Id: hal-01254193

<https://inria.hal.science/hal-01254193v1>

Submitted on 11 Jan 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Equivalent Robin Boundary Conditions for Acoustic and Elastic Media

JULIEN DIAZ^{1,2}, VICTOR PÉRON^{2,1}

¹ INRIA Bordeaux Sud-Ouest, Team MAGIQUE 3D

² Université de Pau et des Pays de l'Adour

First Inria-Mexico Workshop in Applied Mathematics
and Computer Science

Instituto de Matematicas, UNAM

Mexico City, June 3 and 4, 2015



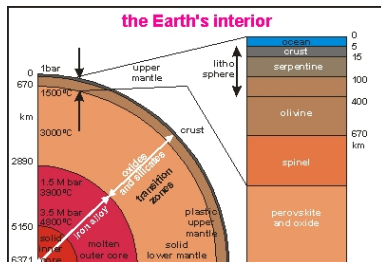
Context - Motivations

HPC-GA Project

- **Numerical Models (WP2)** : Time dependent Elasto-Acoustic Coupling
 - To reproduce **earthquakes**
 - To model the effects of an ocean on the propagation of seismic waves
- **Contributions** : Thin Layer Models for Geophysical Applications
 - Tool : Asymptotic method
 - We derive approximate models for the propagation of seismic waves and acoustic waves

Motivations

Configurations of interest

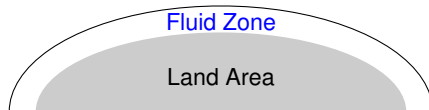


- **Configuration A** : The medium of interest is a Land Area surrounded by a [Fluid Zone](#)
- **Configuration B** : The medium of interest is a Land Area surrounded by a [Fluid Zone](#) and a part of the atmosphere

Motivations

Configuration A

- **Configuration:**



- The thickness of the **Fluid Zone** is negligible with respect to the wavelength
- **Difficulty**
Apply a FEM on a mesh with **thin cells** in the **Fluid Zone** and much larger outside
- **Sub-Configurations:**
 - 1 **Configuration 1** : A **thin layer** with a **uniform** thickness
 - 2 **Configuration 2** : A **thin layer** with a **variable** thickness

Approach : an Asymptotic Method

- 1 "Replace" the **thin layer** by an Equivalent Boundary Condition
- 2 Couple this condition with the elastic equation

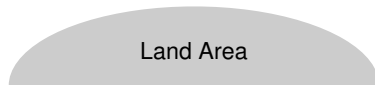


Figure : Configuration A without the **thin layer**

- 3 Apply a (DG) Finite Element Method

Bibliography

- Derivation of Equivalent Conditions for **thin layer** Problems :



ENGQUIST-NÉDÉLEC, 93



ABBOUD-AMMARI, 96



BENDALI-LEMRABET, 96



LAFITTE, 99

...



PÉRON, 2014



BUREL, PHD THESIS, 2014



DE CASTRO, DIAZ, PÉRON, 2014

Outline

- 1 Energy Estimates (Conf. 1)
- 2 Equivalent Conditions (Conf. 1)
- 3 Numerical Results (Conf. 1)
- 4 Equivalent Conditions (Conf. 2)

Outline

- 1 Energy Estimates (Conf. 1)**
- 2 Equivalent Conditions (Conf. 1)
- 3 Numerical Results (Conf. 1)
- 4 Equivalent Conditions (Conf. 2)

A Model Problem

The Problem (\mathbf{P}_ε) set in a smooth domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

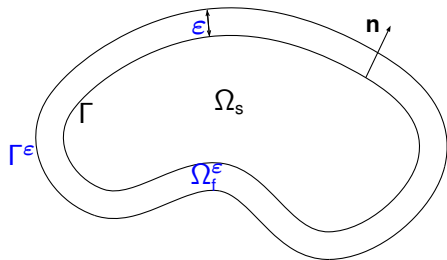
$$\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} \quad \text{in } \Omega_s$$

$$\Delta p + \left(\frac{\omega}{c_f}\right)^2 p = 0 \quad \text{in } \Omega_f^\varepsilon$$

$$\partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} \quad \text{on } \Gamma$$

$$\underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = -p \mathbf{n} \quad \text{on } \Gamma$$

$$\partial_n p - i \frac{\omega}{c_f} p = 0 \quad \text{on } \Gamma^\varepsilon$$



Issue : Uniform Estimates for solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of (\mathbf{P}_ε) as $\varepsilon \rightarrow 0$?

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

- (i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

- (i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem (\mathbf{P}_ε) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon)$, and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C\|\mathbf{f}\|_{0,\Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem (\mathbf{P}_ε) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon)$, and

$$\|\mathbf{u}_\varepsilon\|_{1, \Omega_s} + \|p_\varepsilon\|_{1, \Omega_f^\varepsilon} \leq C \|\mathbf{f}\|_{0, \Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$
Key for the proof:

To write a formulation in a fixed domain by means of the Scaling

$$S = \frac{s}{\varepsilon} \in (0, 1) \quad \text{where} \quad s \in (0, \varepsilon)$$

Outline

- 1 Energy Estimates (Conf. 1)
- 2 Equivalent Conditions (Conf. 1)**
- 3 Numerical Results (Conf. 1)
- 4 Equivalent Conditions (Conf. 2)

Methodology

- **Step 1** : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ when $\varepsilon \rightarrow 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \dots$$

$$p_\varepsilon(x) = p_0(y_\alpha, \frac{\mathbf{s}}{\varepsilon}) + \varepsilon p_1(y_\alpha, \frac{\mathbf{s}}{\varepsilon}) + \varepsilon^2 p_2(y_\alpha, \frac{\mathbf{s}}{\varepsilon}) + \dots$$

- **Step 2** : Derive Equivalent Conditions of order $k + 1 \in \mathbb{N}$.
Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \dots + \varepsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\varepsilon^{k+1})$$

- **Step 3** : Prove Stability and Convergence results for Equivalent models

Step 1 : Multiscale Expansion

First terms

1

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

2

$$p_0 = -i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}$$

3

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) - i\omega c_f \rho_f \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} = i\omega c_f \rho_f P_1(D)(\mathbf{u}_0 \cdot \mathbf{n}) \mathbf{n} & \text{on } \Gamma \end{cases}$$

where $P_1(D) = -2\mathcal{H} + i\kappa^{-1}\Delta_\Gamma$

4

$$p_1(y_\alpha, \mathbf{S}) = -i\omega c_f \rho_f (\mathbf{u}_1 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_0 \cdot \mathbf{n})) + \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{S}, \quad \mathbf{S} \in (0, 1)$$

Step 1 : Multiscale Expansion

Next terms $\mathbf{u}_2, \mathfrak{p}_2$



$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 \\ \mathbf{T}(\mathbf{u}_2) - i\omega c \rho_f \mathbf{u}_2 \cdot \mathbf{n} \mathbf{n} = i\omega c \rho_f (P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n})) \mathbf{n} \end{cases}$$



$$\mathfrak{p}_2(y_\alpha, \mathbf{S}) = a_2(y_\alpha) + b_2(y_\alpha) \mathbf{S} + c_2(y_\alpha) \mathbf{S}^2,$$

where

$$a_2 = -i\omega c \rho_f (\mathbf{u}_2 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n}))$$

$$b_2 = \omega^2 \rho_f \mathbf{u}_1 \cdot \mathbf{n}$$

$$c_2 = \omega \rho_f \left(-\omega \mathcal{H} + \frac{1}{2} i c [\Delta_\Gamma + \kappa^2 \mathbb{I}] \right) (\mathbf{u}_0 \cdot \mathbf{n}),$$

and

$$P_2(D) = -2i\kappa \mathcal{H} \mathbb{I} + ((4\mathcal{H}^2 - \mathcal{K}) \mathbb{I} - \Delta_\Gamma) \\ + (i\kappa)^{-1} [2\mathcal{H} \Delta_\Gamma - \operatorname{div}_\Gamma (\mathcal{H} \mathbb{I} - \mathcal{R}) \nabla_\Gamma + \Delta_\Gamma (2\mathcal{H})] + (i\kappa)^{-2} \Delta_\Gamma^2.$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\epsilon^N := \mathbf{u}_\epsilon - \sum_{n=0}^N \epsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\epsilon^N := p_\epsilon - \sum_{n=0}^N \epsilon^n p_n \quad \text{in } \Omega_f^\epsilon.$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{ll} \Delta r_\epsilon^N + \kappa^2 r_\epsilon^N & = f_\epsilon \quad \text{in } \Omega_f^\epsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{r}_\epsilon^N) + \omega^2 \rho \mathbf{r}_\epsilon^N & = 0 \quad \text{in } \Omega_s \\ \partial_{\mathbf{n}} r_\epsilon^N - \rho_f \omega^2 \mathbf{r}_\epsilon^N \cdot \mathbf{n} & = g_\epsilon \quad \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\epsilon^N) + r_\epsilon^N \mathbf{n} & = 0 \quad \text{on } \Gamma \\ \partial_{\mathbf{n}} r_\epsilon^N - i\kappa r_\epsilon^N & = 0 \quad \text{on } \Gamma^\epsilon. \end{array} \right.$$

The RHS are explicit : $f_\epsilon = \mathcal{O}(\epsilon^{N-\frac{1}{2}})$ in Ω_f^ϵ , $g_\epsilon = \mathcal{O}(\epsilon^N)$ on Γ .

Optimal Estimates : $\|\mathbf{r}_\epsilon^N\|_{1,\Omega_s} + \sqrt{\epsilon} \|r_\epsilon^N\|_{1,\Omega_f^\epsilon} \leq C_N \epsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

For all $k \in \{0, 1, 2\}$, we identify a simpler problem

$$(\mathbf{P}_{\epsilon}^k) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_{\epsilon}^k) + \omega^2 \rho \mathbf{u}_{\epsilon}^k = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_{\epsilon}^k) + \mathbf{B}_{k,\epsilon}(\mathbf{u}_{\epsilon}^k \cdot \mathbf{n}) \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

with $\mathbf{B}_{k,\epsilon}$ a surfacic differential operator.

Step 2 : Equivalent Conditions

1 Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

2 Order 2 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c_f \rho_f (1 + \varepsilon (-2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma)) (\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}) \mathbf{n} = 0$$

3 Order 3 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - i\omega c_f \rho_f (\mathbb{I} + \varepsilon P_1(D) + \varepsilon^2 P_2(D)) (\mathbf{u}_\varepsilon^2 \cdot \mathbf{n}) \mathbf{n} = 0$$

where $P_1(D) = -2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma$

Step 3 : Stability and Convergence results

Functional setting : $\mathbf{u}_\varepsilon^k \in V_k := \{\mathbf{u} \in H^1(\Omega_s) \mid \mathbf{u} \cdot \mathbf{n}|_\Gamma \in H^k(\Gamma)\}$

Proposition

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{P}_\varepsilon^k)$ with data $\mathbf{f} \in L^2(\Omega_s)$ has a unique solution $\mathbf{u}_\varepsilon^k \in V_k$ and

- **Stability**

$$\|\mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C \|\mathbf{f}\|_{0, \Omega_s}$$

- **Convergence**

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C_k \varepsilon^{k+1}$$

Outline

- 1 Energy Estimates (Conf. 1)
- 2 Equivalent Conditions (Conf. 1)
- 3 Numerical Results (Conf. 1)**
- 4 Equivalent Conditions (Conf. 2)

Numerical simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_s = D(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on Γ ($\mathbf{f} = 0$) : $p_i(\mathbf{x}) = \exp(i\omega\mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- \mathbb{P}_3 -finite elements (Lagrange) in the Library [Hou10ni](#)
- Ω_s : Isotropic elastic material with Lamé coefficients :

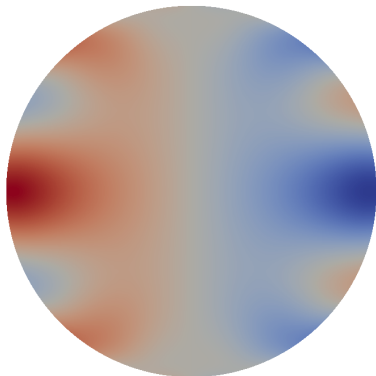
$$\mu = 26.32 \times 10^9 \quad \text{and} \quad \lambda = 51.08 \times 10^9 .$$

- $c_f = 1500 \text{ m.s}^{-1}$, $\rho_f = 1000 \text{ kg.m}^{-3}$, $\rho = 2700 \text{ kg.m}^{-3}$.

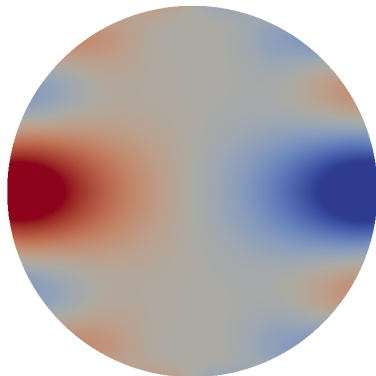
Numerical Simulations

Qualitative Comparisons

$$\varepsilon = 0.001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$
 \mathbf{u}_ε : analytical solution

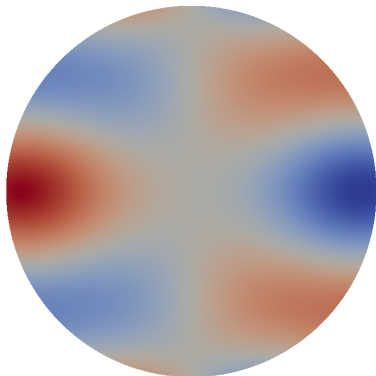


$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$
 \mathbf{u}_ε^2 (model of order 3): IPDGM

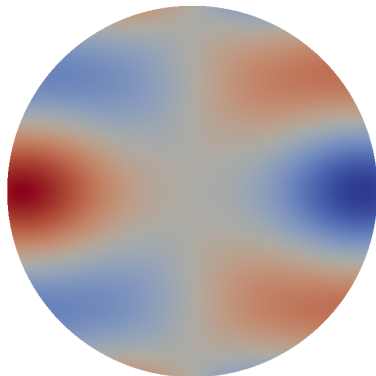
Numerical Simulations

Qualitative Comparisons

$$\varepsilon = 0.0001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$
 \mathbf{u}_ε : analytical solution



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$
 \mathbf{u}_ε^2 (model of order 3): IPDGM

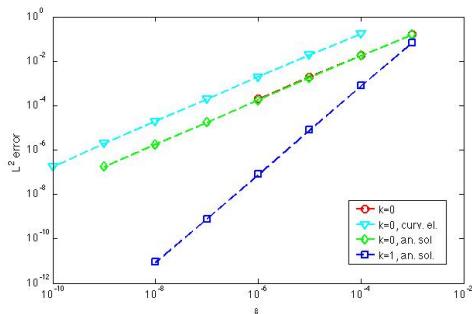
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2 -error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0, \Omega_s}$ w.r.t. ε

\mathbf{u}_ε : analytical solution

\mathbf{u}_ε^k : analytical solution or numerical solution (IPDGM)



Outline

- 1 Energy Estimates (Conf. 1)
- 2 Equivalent Conditions (Conf. 1)
- 3 Numerical Results (Conf. 1)
- 4 Equivalent Conditions (Conf. 2)**

Configuration 2

Framework in 2D

- Parameterization of the layer Ω_Γ^ϵ

$$\Omega_\Gamma^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

Configuration 2

Framework in 2D

- Parameterization of the layer Ω_f^ϵ

$$\Omega_f^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

- Euclidean metric of Ω_f^ϵ defined in the coordinates (t, s) :

$$\begin{pmatrix} (1 + s f(t) c(t))^2 + (s f'(t))^2 & s f(t) f'(t) \\ s f(t) f'(t) & f(t)^2 \end{pmatrix}$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

- Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

- Order 2 :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - i\omega c_f \rho_f \left(1 + \epsilon \left(-f(t)c(t) + i\kappa^{-1} [g(t)\partial_t + f(t)\partial_t^2] \right) \right) \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

where

$$g(t) = (2 - f^2(t)) f'(t)$$

Prospect

- 1 To implement Equivalent Conditions in 3D and in Time domain
- 2 To implement Equivalent Conditions in Configuration 2
- 3 To test these asymptotic models with real data

Thank you for your attention!

Proof of Uniform Estimates

Variational Problem (VP_ϵ)

- $V_\epsilon = H^1(\Omega_s) \times H^1(\Omega_f^\epsilon)$
- For all $\epsilon > 0$, (VP_ϵ) writes : Find $(\mathbf{u}_\epsilon, p_\epsilon) \in V_\epsilon$ such that

$$\forall (\mathbf{v}, q) \in V_\epsilon, \quad a_\epsilon((\mathbf{u}_\epsilon, p_\epsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V_\epsilon', V_\epsilon}$$

$$\begin{aligned}
 a_\epsilon((\mathbf{u}, p), (\mathbf{v}, q)) &:= \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}) \, d\mathbf{x} \\
 &+ \int_{\Omega_f^\epsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, d\mathbf{x} + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, ds - i\kappa \int_{\Gamma^\epsilon} p \bar{q} \, ds
 \end{aligned}$$

Formulation in a fixed domain

Scaled Problem (\mathfrak{P}_ε)

- $\Omega_f := \Gamma \times (0, 1)$
- $V = H^1(\Omega_s) \times H^1(\Omega_f)$
- The problem (\mathfrak{P}_ε) writes : Find $(\mathbf{u}, \mathbf{p}) \in V$ such that $\forall (\mathbf{v}, \mathbf{q}) \in V$

$$\begin{aligned} \varepsilon \int_0^1 \int_{\Gamma} [(\mathbb{I} + \varepsilon S\mathcal{R})^{-2} \nabla_{\Gamma} \mathbf{p} \nabla_{\Gamma} \bar{\mathbf{q}} + \varepsilon^{-2} \partial_S \mathbf{p} \partial_S \bar{\mathbf{q}} - \kappa^2 \mathbf{p} \bar{\mathbf{q}}] \det(\mathbb{I} + \varepsilon S\mathcal{R}) \, d\Omega_f \\ + a_s(\mathbf{u}, \mathbf{v}) + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p} \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma \\ - i\kappa \int_{\Gamma} \mathbf{p}(\cdot, 1) \bar{\mathbf{q}}(\cdot, 1) \det(\mathbb{I} + \varepsilon \mathcal{R}) \, d\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathbf{q}) \rangle_{V', V} \end{aligned}$$

Scaled Problem (\mathfrak{P}_ε)

Uniform estimates

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, the problem (\mathfrak{P}_ε) with data $\mathfrak{F}_\varepsilon \in V'$ has a unique solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ which satisfies

$$\sqrt{\varepsilon} \|\nabla_{\Gamma} \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon}^{-1} \|\partial_s \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'} .$$

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, any solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ of problem (\mathfrak{P}_ε) with a data $\mathfrak{F}_\varepsilon \in V'$ satisfies

$$\|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'} .$$

Sketch of the proof of the Lemma

Assume : $\exists(\mathbf{u}_m, \mathbf{p}_m) \in V^{\mathbb{N}}$ satisfying $(\mathfrak{P}_{\varepsilon_m})$ with $\varepsilon_m \rightarrow 0$ and $\mathfrak{F}_m \in V'$ s.t.

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \rightarrow 0$$

We prove successively that :

❶

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C \quad \text{with} \quad W = H^1(\Omega_s) \times H^1(0, 1; L^2(\Gamma))$$

❷ (up to the extraction of a subsequence)

$$\begin{aligned} (\mathbf{u}_m, \mathbf{p}_m) &\rightarrow (\mathbf{u}, \mathbf{p}) \quad \text{in } L^2(\Omega_s) \times L^2(\Omega_f) \\ (\mathbf{u}_m \cdot \mathbf{n}, \gamma_0 \mathbf{p}_m) &\rightarrow (\mathbf{u} \cdot \mathbf{n}, \gamma_0 \mathbf{p}) \quad \text{in } L^2(\Gamma) \times L^2(\Gamma) \end{aligned}$$

❸

$$\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}\|_{0, \Gamma} = 1$$

❹ $\mathbf{u} = 0$ and $\mathbf{p} = 0$ (using (SA)) : Contradiction

Step 3 : Proof of Convergence

(i) Derive an expansion of \mathbf{u}_ε^k and show that :

$$\begin{aligned}\mathbf{u}_\varepsilon &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k, \\ \mathbf{u}_\varepsilon^k &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k.\end{aligned}$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1, \Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k, \varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma,$$

According to the Stability result, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s} \leq C \varepsilon^{k+1}.$$

Convergence of the models

Dirichlet case

For $k \in \{0, 1, 2\}$, we plot the L^2 -error $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s}$ w.r.t. ϵ

\mathbf{u}_ϵ : analytical solution

\mathbf{u}_ϵ^k : analytical solution or numerical solution (IPDGM)

