



Equivalent Robin Boundary Conditions for Acoustic and Elastic Media

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Equivalent Robin Boundary Conditions for Acoustic and Elastic Media

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and Computer Science
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Mexico City, June 3 and 4, 2015



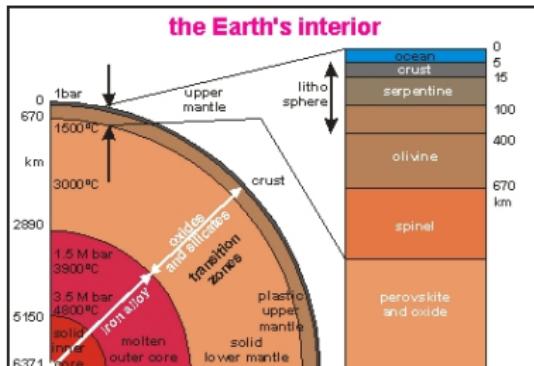
Context - Motivations

HPC-GA Project

- **Numerical Models (WP2)** : Time dependent Elasto-Acoustic Coupling
 - To reproduce **earthquakes**
 - To model the effects of an ocean on the propagation of seismic waves
- **Contributions** : Thin Layer Models for Geophysical Applications
 - Tool : Asymptotic method
 - We derive approximate models for the propagation of seismic waves and acoustic waves

Motivations

Configurations of interest



- **Configuration A :** The medium of interest is a Land Area surrounded by a **Fluid Zone**
- **Configuration B :** The medium of interest is a Land Area surrounded by a **Fluid Zone** and a part of the atmosphere

Motivations

Configuration A

- Configuration:



- The thickness of the **Fluid Zone** is negligible with respect to the wavelength
- **Difficulty**
Apply a FEM on a mesh with **thin cells** in the **Fluid Zone** and much larger outside
- **Sub-Configurations:**
 - ① **Configuration 1** : A **thin layer** with a **uniform** thickness
 - ② **Configuration 2** : A **thin layer** with a **variable** thickness

Approach : an Asymptotic Method

- ① "Replace" the **thin layer** by an Equivalent Boundary Condition
- ② Couple this condition with the elastic equation



Figure : Configuration A without the **thin layer**

- ③ Apply a (DG) Finite Element Method

Bibliography

- Derivation of Equivalent Conditions for **thin layer** Problems :
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 -  PÉRON, 2014
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Outline

1 Energy Estimates (Conf. 1)

2 Equivalent Conditions (Conf. 1)

3 Numerical Results (Conf. 1)

4 Equivalent Conditions (Conf. 2)

Outline

1 Energy Estimates (Conf. 1)

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A Model Problem

The Problem (\mathbf{P}_ε) set in a smooth domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

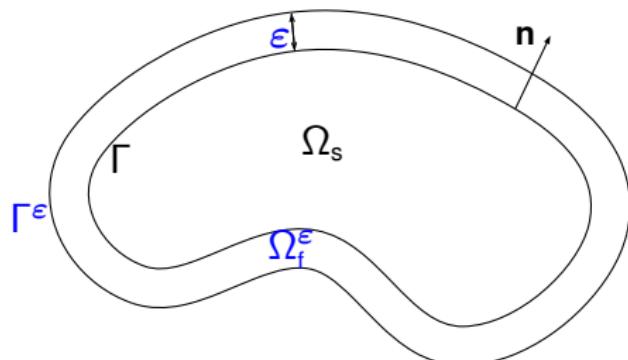
$$\nabla \cdot \underline{\sigma}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} \quad \text{in } \Omega_s$$

$$\Delta p + \left(\frac{\omega}{c_f}\right)^2 p = 0 \quad \text{in } \Omega_f^\varepsilon$$

$$\partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} \quad \text{on } \Gamma$$

$$\underline{\sigma}(\mathbf{u}) \mathbf{n} = -p \mathbf{n} \quad \text{on } \Gamma$$

$$\partial_n p - i \frac{\omega}{c_f} p = 0 \quad \text{on } \Gamma^\varepsilon$$



Issue : Uniform Estimates for solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of (\mathbf{P}_ε) as $\varepsilon \rightarrow 0$?

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

- (i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klji}$
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

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- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem (\mathbf{P}_ε) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon)$, and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C\|\mathbf{f}\|_{0,\Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$

Uniform Estimates

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$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C\|\mathbf{f}\|_{0,\Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$

Key for the proof:

To write a formulation in a fixed domain by means of the Scaling

$$S = \frac{s}{\varepsilon} \in (0, 1) \quad \text{where} \quad s \in (0, \varepsilon)$$

Outline

1 Energy Estimates (Conf. 1)

2 Equivalent Conditions (Conf. 1)

3 Numerical Results (Conf. 1)

4 Equivalent Conditions (Conf. 2)

Methodology

- **Step 1** : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ when $\varepsilon \rightarrow 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \dots$$

$$p_\varepsilon(x) = p_0(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon p_1(y_\alpha, \frac{s}{\varepsilon}) + \varepsilon^2 p_2(y_\alpha, \frac{s}{\varepsilon}) + \dots$$

- **Step 2** : Derive Equivalent Conditions of order $k+1 \in \mathbb{N}$.

Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \dots + \varepsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\varepsilon^{k+1})$$

- **Step 3** : Prove Stability and Convergence results for Equivalent models

Step 1 : Multiscale Expansion

First terms

1

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

2

$$\mathbf{p}_0 = -i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n}$$

3

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) - i\omega c_f \rho_f \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} = i\omega c_f \rho_f P_1(D)(\mathbf{u}_0 \cdot \mathbf{n}) \mathbf{n} & \text{on } \Gamma \end{cases}$$

where $P_1(D) = -2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma$

4

$$\mathbf{p}_1(y_\alpha, \mathbf{s}) = -i\omega c_f \rho_f (\mathbf{u}_1 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_0 \cdot \mathbf{n})) + \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{s}, \quad \mathbf{s} \in (0, 1)$$

Step 1 : Multiscale Expansion

Next terms $\mathbf{u}_2, \mathbf{p}_2$

-

$$\begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 \\ \mathbf{T}(\mathbf{u}_2) - i\omega c \rho_f \mathbf{u}_2 \cdot \mathbf{n} \mathbf{n} = i\omega c \rho_f (P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n})) \mathbf{n} \end{cases}$$

-

$$\mathbf{p}_2(y_\alpha, \mathcal{S}) = a_2(y_\alpha) + b_2(y_\alpha) \mathcal{S} + c_2(y_\alpha) \mathcal{S}^2 ,$$

where

$$a_2 = -i\omega c \rho_f (\mathbf{u}_2 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n}))$$

$$b_2 = \omega^2 \rho_f \mathbf{u}_1 \cdot \mathbf{n}$$

$$c_2 = \omega \rho_f \left(-\omega \mathcal{H} + \frac{1}{2} i c [\Delta_\Gamma + \kappa^2 \mathbb{I}] \right) (\mathbf{u}_0 \cdot \mathbf{n}) ,$$

and

$$\begin{aligned} P_2(D) &= -2i\kappa \mathcal{H} \mathbb{I} + ((4\mathcal{H}^2 - \mathcal{K}) \mathbb{I} - \Delta_\Gamma) \\ &+ (i\kappa)^{-1} [2\mathcal{H}\Delta_\Gamma - \operatorname{div}_\Gamma(\mathcal{H}\mathbb{I} - \mathcal{R})\nabla_\Gamma] + \Delta_\Gamma(2\mathcal{H})] + (i\kappa)^{-2} \Delta_\Gamma^2 . \end{aligned}$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\varepsilon^N := \mathbf{u}_\varepsilon - \sum_{n=0}^N \varepsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\varepsilon^N := p_\varepsilon - \sum_{n=0}^N \varepsilon^n p_n \quad \text{in } \Omega_f^\varepsilon.$$

Evaluation of the right hand sides in

$$\begin{cases} \Delta \mathbf{r}_\varepsilon^N + \kappa^2 \mathbf{r}_\varepsilon^N &= f_\varepsilon \quad \text{in } \Omega_f^\varepsilon \\ \nabla \cdot \underline{\sigma}(\mathbf{r}_\varepsilon^N) + \omega^2 \rho \mathbf{r}_\varepsilon^N &= 0 \quad \text{in } \Omega_s \\ \partial_{\mathbf{n}} \mathbf{r}_\varepsilon^N - \rho_f \omega^2 \mathbf{r}_\varepsilon^N \cdot \mathbf{n} &= g_\varepsilon \quad \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\varepsilon^N) + r_\varepsilon^N \mathbf{n} &= 0 \quad \text{on } \Gamma \\ \partial_{\mathbf{n}} r_\varepsilon^N - i\kappa r_\varepsilon^N &= 0 \quad \text{on } \Gamma^\varepsilon. \end{cases}$$

The RHS are explicit : $f_\varepsilon = \mathcal{O}(\varepsilon^{N-\frac{1}{2}})$ in Ω_f^ε , $g_\varepsilon = \mathcal{O}(\varepsilon^N)$ on Γ .

Optimal Estimates : $\|\mathbf{r}_\varepsilon^N\|_{1,\Omega_s} + \sqrt{\varepsilon} \|r_\varepsilon^N\|_{1,\Omega_f^\varepsilon} \leq C_N \varepsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

For all $k \in \{0, 1, 2\}$, we identify a simpler problem

$$(P_\varepsilon^k) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_\varepsilon^k) + \omega^2 \rho \mathbf{u}_\varepsilon^k = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k \cdot \mathbf{n}) \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

with $\mathbf{B}_{k,\varepsilon}$ a surfacic differential operator.

Step 2 : Equivalent Conditions

① Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

② Order 2 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c_f \rho_f (1 + \varepsilon (-2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma)) (\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}) \mathbf{n} = 0$$

③ Order 3 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - i\omega c_f \rho_f (\mathbb{I} + \varepsilon P_1(D) + \varepsilon^2 P_2(D)) (\mathbf{u}_\varepsilon^2 \cdot \mathbf{n}) \mathbf{n} = 0$$

where $P_1(D) = -2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma$

Step 3 : Stability and Convergence results

Functional setting : $\mathbf{u}_\varepsilon^k \in \mathbf{V}_k := \{\mathbf{u} \in \mathbf{H}^1(\Omega_s) \mid \mathbf{u} \cdot \mathbf{n}|_\Gamma \in \mathbf{H}^k(\Gamma)\}$

Proposition

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{P}_\varepsilon^k)$ with data $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$ has a unique solution $\mathbf{u}_\varepsilon^k \in \mathbf{V}_k$ and

- **Stability**

$$\|\mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C \|\mathbf{f}\|_{0,\Omega_s}$$

- **Convergence**

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C_k \varepsilon^{k+1}$$

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Numerical simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_s = D(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on Γ ($f = 0$) : $p_i(\mathbf{x}) = \exp(i\omega \mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- \mathbb{P}_3 -finite elements (Lagrange) in the Library **Hou10ni**
- Ω_s : Isotropic elastic material with Lamé coefficients :

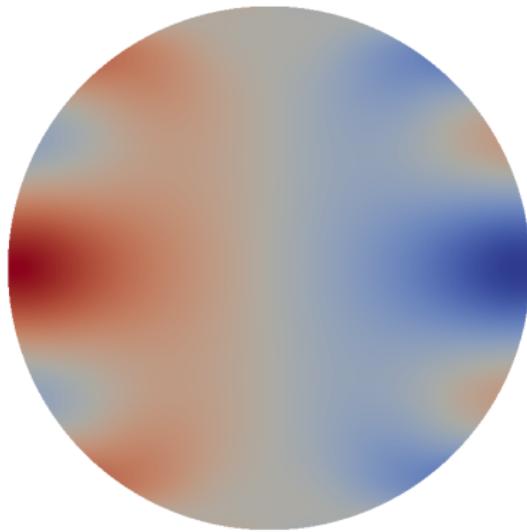
$$\mu = 26.32 \times 10^9 \quad \text{and} \quad \lambda = 51.08 \times 10^9.$$

- $c_f = 1500 \text{ m.s}^{-1}$, $\rho_f = 1000 \text{ kg.m}^{-3}$, $\rho = 2700 \text{ kg.m}^{-3}$.

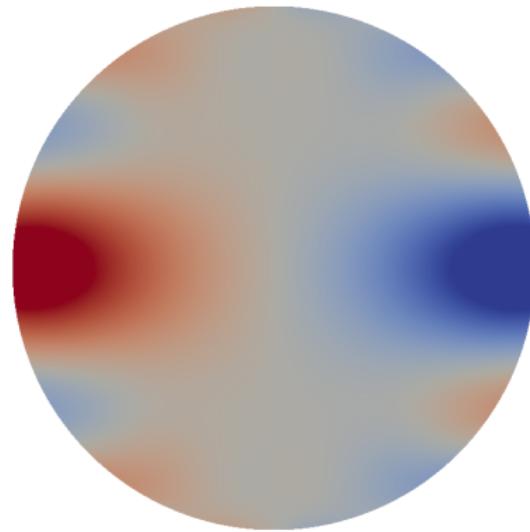
Numerical Simulations

Qualitative Comparisons

$$\varepsilon = 0.001;$$



$| \operatorname{Re} u_\varepsilon |$
 u_ε : analytical solution

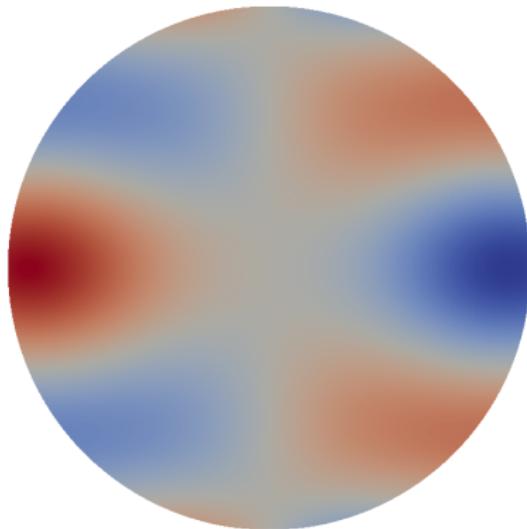


$| \operatorname{Re} u_\varepsilon^2 |$
 u_ε^2 (model of order 3) : IPDGM

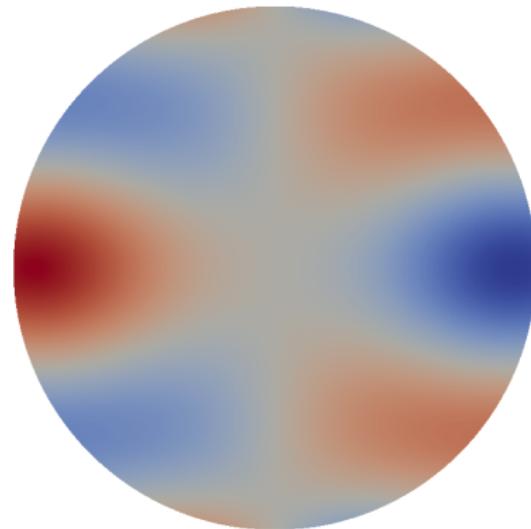
Numerical Simulations

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$$\varepsilon = 0.0001;$$



$|\operatorname{Re} u_\varepsilon|$
 u_ε : analytical solution



$|\operatorname{Re} u_\varepsilon^2|$
 u_ε^2 (model of order 3) : IPDGM

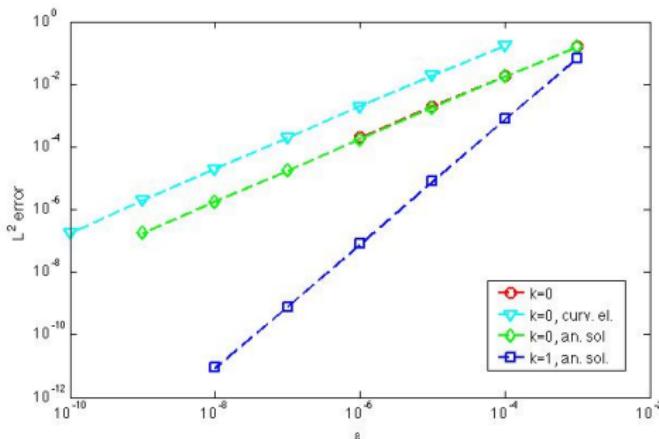
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2 -error $\|u_\varepsilon - u_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε

u_ε : analytical solution

u_ε^k : analytical solution or numerical solution (IPDGM)



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Configuration 2

Framework in 2D

- Parameterization of the layer Ω_f^ε

$$\Omega_f^\varepsilon = \{ \mathbf{x}(t) + s \mathbf{f}(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, s \in (0, \varepsilon) \}$$

t : arc-length on Γ

\mathbf{f} : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

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Framework in 2D

- Parameterization of the layer Ω_f^ε

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t : arc-length on Γ

\mathbf{f} : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

- Euclidean metric of Ω_f^ε defined in the coordinates (t, s) :

$$\begin{pmatrix} (1 + s \mathbf{f}(t) \mathbf{c}(t))^2 + (s \mathbf{f}'(t))^2 & s \mathbf{f}(t) \mathbf{f}'(t) \\ s \mathbf{f}(t) \mathbf{f}'(t) & \mathbf{f}(t)^2 \end{pmatrix}$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

- Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

- Order 2 :

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c_f \rho_f \left(1 + \varepsilon \left(-f(t)c(t) + i\kappa^{-1}[g(t)\partial_t + f(t)\partial_t^2] \right) \right) \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

where

$$g(t) = (2 - f^2(t)) f'(t)$$

Prospect

- ① To implement Equivalent Conditions in 3D and in Time domain
- ② To implement Equivalent Conditions in Configuration 2
- ③ To test these asymptotic models with real data

Thank you for your attention!

Proof of Uniform Estimates

Variational Problem (\mathbf{VP}_ε)

- $V_\varepsilon = H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon)$
- For all $\varepsilon > 0$, $(\mathbf{VP}_\varepsilon)$ writes : Find $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$ such that

$$\forall (\mathbf{v}, q) \in V_\varepsilon, \quad a_\varepsilon((\mathbf{u}_\varepsilon, p_\varepsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V'_\varepsilon, V_\varepsilon}$$

$$\begin{aligned} a_\varepsilon((\mathbf{u}, p), (\mathbf{v}, q)) &:= \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{e}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}) \, d\mathbf{x} \\ &+ \int_{\Omega_f^\varepsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, d\mathbf{x} + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{v} \cdot \mathbf{n}) \, ds - i\kappa \int_{\Gamma^\varepsilon} p \bar{q} \, ds \end{aligned}$$

Formulation in a fixed domain

Scaled Problem (\mathfrak{P}_ε)

- $\Omega_f := \Gamma \times (0, 1)$
- $V = H^1(\Omega_s) \times H^1(\Omega_f)$
- The problem (\mathfrak{P}_ε) writes : Find $(\mathbf{u}, \mathbf{p}) \in V$ such that $\forall (\mathbf{v}, \mathbf{q}) \in V$

$$\begin{aligned} & \varepsilon \int_0^1 \int_{\Gamma} [(\mathbb{I} + \varepsilon S \mathcal{R})^{-2} \nabla_{\Gamma} \mathbf{p} \nabla_{\Gamma} \bar{\mathbf{q}} + \varepsilon^{-2} \partial_s \mathbf{p} \partial_s \bar{\mathbf{q}} - \kappa^2 \mathbf{p} \bar{\mathbf{q}}] \det(\mathbb{I} + \varepsilon S \mathcal{R}) d\Omega_f \\ & \quad + a_s(\mathbf{u}, \mathbf{v}) + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p} \bar{\mathbf{v}} \cdot \mathbf{n}) d\Gamma \\ & \quad - i\kappa \int_{\Gamma} \mathbf{p}(\cdot, 1) \bar{\mathbf{q}}(\cdot, 1) \det(\mathbb{I} + \varepsilon \mathcal{R}) d\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathbf{q}) \rangle_{V', V} \end{aligned}$$

Scaled Problem (\mathfrak{P}_ε)

Uniform estimates

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, the problem (\mathfrak{P}_ε) with data $\mathfrak{F}_\varepsilon \in V'$ has a unique solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ which satisfies

$$\sqrt{\varepsilon} \|\nabla_\Gamma \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon^{-1}} \|\partial_S \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, any solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ of problem (\mathfrak{P}_ε) with a data $\mathfrak{F}_\varepsilon \in V'$ satisfies

$$\|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

Sketch of the proof of the Lemma

Assume : $\exists (\mathbf{u}_m, \mathbf{p}_m) \in V^{\mathbb{N}}$ satisfying $(\mathfrak{P}_{\varepsilon_m})$ with $\varepsilon_m \rightarrow 0$ and $\mathfrak{F}_m \in V'$ s.t.

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \rightarrow 0$$

We prove successively that :

①

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C \quad \text{with} \quad W = H^1(\Omega_s) \times H^1(0, 1; L^2(\Gamma))$$

② (up to the extraction of a subsequence)

$$(\mathbf{u}_m, \mathbf{p}_m) \rightarrow (\mathbf{u}, \mathbf{p}) \quad \text{in } L^2(\Omega_s) \times L^2(\Omega_f)$$

$$(\mathbf{u}_m \cdot \mathbf{n}, \gamma_0 \mathbf{p}_m) \rightarrow (\mathbf{u} \cdot \mathbf{n}, \gamma_0 \mathbf{p}) \quad \text{in } L^2(\Gamma) \times L^2(\Gamma)$$

③

$$\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}\|_{0, \Gamma} = 1$$

④ $\mathbf{u} = 0$ and $\mathbf{p} = 0$ (using (SA)) : Contradiction

Step 3 : Proof of Convergence

(i) Derive an expansion of \mathbf{u}_ε^k and show that :

$$\begin{aligned}\mathbf{u}_\varepsilon &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k, \\ \mathbf{u}_\varepsilon^k &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k.\end{aligned}$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma,$$

According to the Stability result, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s} \leq C \varepsilon^{k+1}.$$

