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ON PSEUDO-INVERSES OF MATRICES AND THEIR CHARACTERISTIC POLYNOMIALS IN SUPERTROPICAL ALGEBRA

ADI NIV[†]

Abstract. The only invertible matrices in tropical algebra are diagonal matrices, permutation matrices and their products. However, the pseudo-inverse A^{∇}, defined as $\frac{1}{det(A)}adj(A)$, with $det(A)$ being the tropical permanent (also called the tropical determinant) of a matrix A, inherits some classical algebraic properties and has some surprising new ones. Defining B and B' to be tropically similar if $B' = A^{\nabla}BA$, we examine the characteristic (max-)polynomials of tropically similar matrices as well as those of pseudo-inverses. Other miscellaneous results include a new proof of the identity for $det(AB)$ and a connection to stabilization of the powers of definite matrices.

Keywords: Tropical and supertropical linear algebra; characteristic polynomial; eigenvalues; Kleene star; permanent; definite matrices; pseudo-inverse.

AMSC: 15A09 (Primary), 15A15, 15A18, 15A80, 15B33

1. Introduction

The **tropical max-plus semifield** is an ordered group $\mathcal{G}$ (usually the set of real numbers $\mathbb{R}$ or the set of rational numbers $\mathbb{Q}$), together with $-\infty$, denoted as $\mathbb{T} = \mathcal{G} \bigcup \{-\infty\}$, equipped with the operations $a \oplus b = max\{a, b\}$ and $a \odot b = a+b$, denoted as $a+b$ and ab respectively (see [1] and [14]). The unit element $1_{\mathbb{T}}$ is actually the element $0 \in \mathbb{Q}$. This arithmetic enables one to simplify non-linear questions by asking them in a (pseudo)-linear setting (see [12]), which can be applied to discrete mathematics, optimization, algebraic geometry and more, as has been well reviewed in [8], [9], [11], [13], [21] and [24]. In this max-plus language, we may also use notions of linear algebra to interpret combinatorial problems, such as eigenvectors being used to solve the Longest-Distance problem (see [4]).

The intention of this paper is to use an analogous concept of the inverse matrix by passing to a wider structure called the **supertropical semiring**, equipped with the ghost ideal $G = \mathcal{G}^{\nu}$, as established and studied by Izhakian and Rowen in [16] and [17]. The use of the term "pseudo"in this paper is the same as "quasi"throughout the work of Izhakian and Rowen.

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We denote the "standard" supertropical semiring as

$$R = T \bigcup G \bigcup \{-\infty\},$$

where $T = \mathcal{G}$, which contains the so called tangible elements of the structure and where $\forall a \in T$ we have $a^\nu \in G$ are the ghost elements of the structure, as defined in [16]. So G inherits the order of $\mathcal{G}$. This enables us to distinguish between a maximal element a that is attained only once in a sum, i.e. $a \in T$ which is invertible, and a maximum that is being attained at least twice, i.e. $a + a = a^\nu \in G$, which is not invertible. Note that ν projects the standard supertropical semiring onto G, which can be identified with the usual tropical structure.

In this new supertropical sense, we use the following order relation to describe two elements that are equal up to some ghost supplement:

Definition 1.1. Let a, b be any two elements in R. We say that a **ghost surpasses** b, denoted $a \models_{gs} b$, if $a = b + ghost$, i.e. $a = b$ or $a \in G$ with $a^\nu \geq b^\nu$. We say a is ν-equivalent to b, denoted by $a \cong_\nu b$, if $a^\nu = b^\nu$.That is, when we use ν to project from R to G, identified with the tropical structure, ν-equivalence becomes equality.

For matrices $A = (a_{ij}), B = (b_{ij}) \in M_{n\times m}(R)$ (and in particular for vectors) $A \models_{gs} B$ means $a_{ij} \models_{gs} b_{ij} \quad \forall i = 1, ..., n$ and $j = 1, ..., m$.

For polynomials

$$f(x) = \sum_{i=1}^{n} a_i x^i,\ g(x) = \sum_{i=1}^{n} b_i x^i \in R[x],$$

we say that $f(x) \models_{gs} g(x)$, also denoted as $f \models_{gs} g$, when $a_i \models_{gs} b_i\ \forall i$.

Important properties of $\models_{gs}$:

1. $\models_{gs}$ is a partial order relation.
   See [19, Lemma 1.5].
2. If $a \models_{gs} b$ then $ac \models_{gs} bc$.

Considering this relation, we use the classical notion $\frac{1}{det(A)} adj(A)$ (where $det(A)$ is the permanent and then $adj(A)$ is defined as usual) to formulate results in the supertropical setting, which are inaccessible in the usual tropical setting. This notion was introduced and studied in tropical algebra by Yoeli [31] and Cuninghame-Green [7], and was further investigated by Gaubert [10], Reutenauer and Straubing [27] and Sergeev [28]. We obtain tropical theorems by considering the tangible elements. By Izhakian's results in [15], this notion satisfies that $\frac{1}{det(A)} adj(A) \cdot A$ is equal to the Id matrix on the diagonal, and $\models_{gs}$ the Id matrix off the diagonal.

In section 3 we discuss type of matrices with 0 on the diagonal and 0 determinant, defined as **definite matrices**. In section 3.1 , we establish that for the set

$$\mathcal{N} = \{A^\nabla : A \in M_n(R)\},$$

the operation ∇ is of order 2 (see Theorem 3.5), and in section 3.2 we will show that for a definite matrix A of order n

$$A^\nabla \cong_\nu A^{\nabla\nabla} \cong_\nu A^* \cong_\nu A^k,\ \forall k \geq n-1$$

(see Theorem 3.6 and Proposition 3.7). These results are extended to the supertropical setting from the results obtained over the tropical structure in [31] and [28], regarding these closure operations.

In section 4, we use the factorizability of matrices in $\mathcal{N}$ to give an alternative proof, analogous to the proof in classical linear algebra, of the property

$$det(AB) \models_{gs} det(A)det(B)$$

stated in Theorem 2.10.

In sections 5 we prove that

$$f_{A^{\triangledown}BA} \models_{gs} f_B, \text{ (see Theorem 5.4).}$$

In section 6, our considerations lead us to Conjecture 6.2:

$$|A| f_{A^{\triangledown}}(x) \models_{gs} x^n f_A(x^{-1}),$$

where $f_M(x) = det(M+xI)$ denotes the characteristic polynomial of a square matrix M. We prove 6.2 for the determinant and trace coefficients and for every 2×2 or 3×3 and triangular matrix.

A consequence would be

$$f_{A^{\triangledown\triangledown}} = f_A, \text{ whenever } f_{A^{\triangledown\triangledown}} \text{ is tangible.}$$

These properties provide a foundation for further research in representation theory and eigenspace decomposition.

2. Preliminaries

2.1. Matrices.

The work in [18], [19] and [20] shows that even though the semiring of matrices over the supertropical semiring lacks negation, it satisfies many of the classical matrix theory properties when using the ghost ideal G. Following [4] and [18] for the tropical and supertropical notations, we give some basic definitions for this theory. One may also find in [4] further combinatorial motivations for the objects discussed.

Definition 2.1. A **permutation track,** of the permutation $\pi \in S_n$, is the sequence

$$a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}$$

of n entries of the matrix $A = (a_{i,j}) \in M_n(R)$.

Definition 2.2. We define the **tropical determinant** of a matrix $A = (a_{i,j})$ to be

$$det(A) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

In the special case where $a_{i,j} \in R$, $\forall i, j$, we refer to any permutation track yielding the highest value in this sum as **a dominant permutation track**.

The tropical determinant is actually the same as *max permanent* of [4]- [7], and the above definition is a reformulation of the optimal assignment problem.

Definition 2.3. We define a matrix $A \in M_n(\mathbb{T})$ or $A \in M_n(R)$ to be **tropically singular** if there exist at least two different dominant permutation tracks. Otherwise the matrix is **tropically non-singular**.

Consequently a matrix $A \in M_n(R)$ is supertropically singular if $det(A) \in G \bigcup \{-\infty\}$ and supertropically non-singular if $det(A) \in T$. A matrix A is **strictly singular** if $det(A) = -\infty$.

Notice that over the tropical semifield we cannot determine if the matrix is tropically non-singular from the value of its determinant, which is always invertible over $\mathbb{T} \backslash \{-\infty\}$. Over the supertropical semiring however, a supertropically non-singular matrix has an invertible determinant, while a supertropically singular matrix has a non-invertible determinant.

As the definitions of singularity are identical over the tropical and supertropical structures, we will only indicate "non-singular" or "singular" and "over $\mathbb{T}$" or "over R"(which will effect the value and invertibility of the determinant).

Definition 2.4. Let $\mathbb{T}^n$ be the free module (see [19]) of rank n over the tropical semifield, and R^n be the free module of rank n over the supertropical semiring. We define the **standard base** of $\mathbb{T}^n$, and therefore of R^n, to be $e_1, ..., e_n$, where

$$e_i = \begin{cases} 1_{\mathbb{T}} = 1_R, \text{ in the } i^{th} \text{ coordinate} \\ 0_{\mathbb{T}} = 0_R, \text{ otherwise} \end{cases} .$$

Definition 2.5. The tropical **identity matrix** in the tropical matrix semiring is the $n \times n$ matrix with the standard base for its columns. We denote this matrix as

$$I_{\mathbb{T}} = I_R = I.$$

Definition 2.6. A matrix $A \in M_n(R)$ is **invertible** if there exists a matrix $B \in M_n(R)$ such that

$$AB = BA = I.$$

Definition 2.7. A square matrix $P_\pi = (a_{i,j})$ is defined to be a **permutation matrix** if there exists $\pi \in S_n$ such that $a_{i,j} = \begin{cases} 0_R, j \neq \pi(i) \\ 1_R, j = \pi(i) \end{cases}$.

Remark 2.8. A tropical matrix A is invertible if and only if it is a product of a permutation matrix P_π and a diagonal matrix D with an invertible determinant. These type of products are defined as **generalized permutation matrices**.

Definition 2.9. Following the notation in [26], we define three types of tropical elementary matrices, corresponding to the three elementary matrix operations, obtained by applying one such operation to the identity matrix.

A **transposition matrix** is obtained from the identity matrix by switching two rows (resp. columns). Multiplying a matrix A to the right of such a matrix (resp. to the left) will switch the corresponding rows (resp. columns) in A.

A **diagonal multiplier** is obtained from the identity matrix where one row (resp. column) has been multiplied by an invertible scalar. Multiplying a matrix A to the right of such a matrix (resp. to the left) will multiply the corresponding row (resp. columns) in A by the same scalar.

A **Gaussian matrix** is defined to differ from the identity matrix by having a non-zero entry r, in a non-diagonal position (i, j). Multiplying a matrix A to the right of such a matrix (resp. to the left) will add row j (resp. column), multiplied by r, to row (resp. column) i.

By Remark 2.8, a product of transposition matrices, which is a permutation matrix, is invertible. A product of diagonal multipliers, which is a diagonal matrix, is invertible. Thus a product of transposition matrices and diagonal multipliers, which is a generalized permutation matrix, is invertible. Gaussian matrices however, are not invertible, and therefore a product including a Gaussian matrix is not invertible.

Theorem 2.10. ***(The rule of determinants)*** *For $n \times n$ matrices A,B over the supertropical semiring R, we have*

$$det(AB) \models_{gs} det(A)det(B).$$

Proof. See Theorem 3.5 in [18, §3].

This theorem also follows from [10, Proposition 2.1.7]. Although S. Gaubert's proof is done in the symmetrized tropical semiring, it carries over to the supertropical case.

In this context, we note that the two transfer principles in [2, Theorem 3.3 and Theorem 3.4] allow one to obtain such results automatically in a wider class of semirings, including the supertropical semiring.

The fact that the determinant of a non-singular matrix A over R is tangible means that the matrix has one dominant permutation track. By using a permutation matrix we can relocate the corresponding permutation to the diagonal, and by using a diagonal matrix we can change the diagonal entries to 1_R, obtaining a non-singular matrix whose dominant Id-permutation track equals 1_R. That is, $A = P\bar{A}$ where P is an invertible matrix (See Remark 2.8) such that $det(P) = det(A)$ and $\bar{A}$ is a definite matrix.

Definition 2.11. $\bar{A}$ is called the **definite form of** A, P the **conductor of** A and we say that P *conducts* the dominant permutation track in A.

The process of bringing a non-singular matrix A to its definite form, can be obtained by using transposition matrices and diagonal multipliers either on the columns or on the rows of A. Respectively, we denote these conducting matrices as the **right conductor** and the **left conductor** of A, which are invertible. The **right** (resp. left) **definite form** corresponds to the right (resp. left) conductor.

Definite matrices are not the same as normal matrices, defined in [3] and [4] to have non-positive off diagonal entries. Over $\mathbb{T}$ the definite form is obtained for not strictly singular matrices, by conducting one of the dominant permutation tracks.

Looking at elementary matrices as the "atoms" of matrices, we present the following definitions:

Definition 2.12. We define a matrix to be **elementarily factorizable** if it can be factored into a product of tropical elementary matrices.

By Remark 2.8, an invertible matrix is always elementarily factorizable, while a non-invertible matrix is elementarily factorizable if and only if its definite form is factorizable.

Remark. Writing a permutation π as a product of disjoint cycles $\sigma_1, ..., \sigma_t$, the permutation track $a_{1,\pi(1)} \cdots a_{n,\pi(n)}$ can be decomposed into the **cycle tracks** $C_1, ..., C_t$, where C_i is the cycle track corresponds to the cycle σ_i.

We introduce a standard combinatorial property of definite matrices. Since this property will be in use in Section 5, we provide some details regarding the technique of its use.

Lemma 2.13. *For a definite matrix $A = (a_{i,j}) \in M_n(R)$, any sequence*

$$a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_k,i_1}, \text{ where } i_j \in \{1,...,n\} \text{ (not necessarily distinct)} \tag{2.1}$$

either describes a permutation track, or is dominated by a subsequence which describes a permutation track.

Proof. As proved in [4, Theorem 4.4], due to the determinant obtained by the 0-entries on the diagonal of a definite matrix, for every cycle track C we have that

$$C = C \cdot C_{Id} \leq P_{Id},$$

where P_{Id} is the Id-permutation track and C_{Id} is the Id-cycle track extending C to a permutation track.

We use a standard fact from graph theory, that if a path $(i_1, i_2, i_3, \cdots, i_k, i_1)$ has repeating nodes, then it can be recursively separated into (not necessarily disjoint) cycles, starting and ending at the points of repetition. Thus, (2.1) can be decomposed as the product of (not necessarily disjoint) cycle tracks satisfying

$$\prod_{j=1}^{l} C_j \leq C_i = C_i \cdot C_{Id} \quad \forall i,$$

where C_j are cycle tracks and $C_i \cdot C_{Id}$ is the extension of C_i to a permutation track.

□

Notation. For an element $a \in R$, we denote as a^ν the element $b \in G$ s.t. $a \cong_\nu b$, and as $\hat{a}$ the element $b \in T$ s.t. $a \cong_\nu b$.

For a matrix (and in particular for a vector) $A = (a_{i,j})$ we write

$$A^\nu = (a^\nu_{i,j}) \text{ and } \widehat{A} = (\widehat{a_{i,j}}).$$

For a polynomial $f(x) = \Sigma a_i x^i$ we write

$$f^\nu(x) = \Sigma a^\nu_i x^i \text{ and } \widehat{f(x)} = \Sigma \hat{a}_i x^i.$$

Definition 2.14. A **pseudo-zero** matrix Z_G is a matrix equal to 0_R on the diagonal, and whose off-diagonal entries are ghosts or 0_R.

A **pseudo-identity** matrix I_G is a nonsingular, multiplicatively idempotent matrix equal to $I + Z_G$, where Z_G is a pseudo-zero matrix. Thus, for every matrix A and a pseudo-identity I_G we have $I_G A \models_{gs} A$.

A **ghost pseudo-identity** matrix is a singular, multiplicatively idempotent matrix equal to $I^\nu + Z_G$.

Definition 2.15. The r, c-**minor** $A_{r,c}$ of a matrix $A = (a_{i,j})$ is obtained by deleting row r and column c of A. The **adjoint matrix** $adj(A)$ of A is defined as the matrix $(a'_{i,j})$, where $a'_{i,j} = det(A_{j,i})$.

The matrix A^∇ denotes $\frac{1}{det(A)} adj(A)$, when $det(A)$ is invertible, and $\left(\frac{1}{\widehat{det(A)}}\right)^\nu adj(A)$, when $det(A)$ is not invertible. Thus over R, A^∇ is defined differently for singular and

non-singular matrices. Over $\mathbb{T}$, however, A^{∇} is defined the same for every not strictly singular matrix.

Remark. Notice that $det(A_{j,i})$ may be obtained as the sum of all permutation tracks in A passing through $a_{j,i}$, with $a_{j,i}$ removed:

$$det(A_{j,i}) = \sum_{\substack{\pi \in S_n : \\ \pi(j) = i}} a_{1,\pi(1)} \cdots a_{j-1,\pi(j-1)} a_{j+1,\pi(j+1)} \cdots a_{n,\pi(n)}.$$

When writing such a permutation as the product of disjoint cycles, $det(A_{j,i})$ can be presented as:

$$det(A_{j,i}) = \sum_{\substack{\pi \in S_n : \\ \pi(j) = i}} (a_{i,\pi(i)} \cdots a_{\pi^{-1}(j),j}) C_{\pi}, \tag{2.2}$$

where $(a_{i,\pi(i)} \cdots a_{\pi^{-1}(j),j})$ is the cycle track missing $a_{j,i}$, and C_{π} is the product of the cycle tracks of π that do not include i and j.

Fact. The products AA^{∇} and $A^{\nabla}A$ are pseudo-identities when $det(A)$ is invertible, and ghost pseudo-identities otherwise.

These identities can be deduced from [10, Proposition 2.1.2], by replacing Gaubert's $\ominus a$ with a. Then $a^{\nu} \models 0_R$corresponds to $a^{\bullet} \nabla 0_R$. See [19, Theorem 2.8] for the proof in the supertropical setting.

Definition 2.16. We say that A^{∇} is **the pseudo-inverse** of A over R, denoting

$$I_A = AA^{\nabla} \text{ and } I'_A = A^{\nabla}A.$$

Theorem 2.17.
(i) $det(A \cdot adj(A)) = det(A)^n$.
(ii) $det(adj(A)) = det(A)^{n-1}$.

Proof. [18, Theorem 4.9].

Remark 2.18. For a definite matrix A, we have $A^{\nabla} = \frac{1}{det(A)} adj(A) = adj(A)$, which is also definite.

Proof. The diagonal entries in $adj(A)$ are sums of cycle tracks of A, and thus the Id summand 1_R dominates every diagonal entry. Also, by Theorem 2.17(ii), we have

$$1_R = det(A^{\nabla}),$$

as required for definite matrices. □

Proposition 2.19. $adj(AB) \models_{gs} adj(B)adj(A)$.

Proof. [18, Proposition 4.8].

Recall the classical Bruhat (LDU) decomposition, whose tropical analog in [22] is called the LDM decomposition.

Lemma 2.20.
a. If A is a not strictly singular triangular matrix over $\mathbb{T}$ (respectively non-singular over R), then A is elementarily factorizable.

b. If A is a not strictly singular matrix over $\mathbb{T}$ (respectively non-singular over R), then A^{∇} (respectively $A^{\nabla\nabla}$) is elementarily factorizable.

Proof. See the LDM decomposition in [22], or an alternative proof in [26, Lemma 6.5 and Corollary 6.6]. □

One can find a cruder factorization in [3], where sufficient conditions are established for $trop(AB) = trop(A)trop(B)$, when looking at the tropical structure as the image of a valuation over the field of Puiseux series, using the classical LDU decomposition over this field. In section 3 we will show the connection of definite matrices and factorization to the well known tropical closure operation $*$, studied in [28] and [29].

2.2. **Polynomials.**

As one can see in [16], the polynomials over the tropical structure are rather straightforward to view geometrically. We notice that the graph of a monomial $a_i x^i \in \mathbb{T}[x]$ is a line, where the power i indicates the slope. Since $\mathbb{T}$ is ordered we may present its elements on an axis, directed rightward, where if $a < b$ then a appears left to b on the $\mathbb{T}$-axis, for every pair of distinct elements $a, b \in T$. It is now easy to understand that a tropical polynomial

$$\sum_{i=0}^{n} a_i x^i \in R[x]$$

takes the value of the dominant monomial among $a_i x^i$ along the $\mathbb{T}$-axis. That having been said, it is possible that some monomials in the polynomial would not dominate for any $x \in \mathbb{T}$.

Definition 2.21. Let

$$f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$$

be a supertropical polynomial. We call monomials in $f(x)$ that dominate for some $x \in R$ **essential**, and monomials in $f(x)$ that do not dominate for any $x \in R$ **inessential**. We write

$$f^{es}(x) = \sum_{i \in I} a_i x^i \in R[x],$$

where $a_i x^i$ is an essential monomial $\forall i \in I$, called **the essential polynomial** of f.

Definition 2.22. We say that b is a $k-$**root** of a, for some $k \in \mathbb{N}$, denoted as $b = \sqrt[k]{a}$, if $b^k = a$.

Remark 2.23. If $a, b \in T$ and $a^k = b^k$ then $a^k + b^k = (a+b)^k \in G$, and therefore $a+b \in G$ and $a = b$. That is, the k-root of a tangible element is unique.

Definition 2.24. We call an element $r \in R$ a **root** of a polynomial $f(x)$ if

$$f(r) \models_{gs} 0_R.$$

We distinguish between two kinds of roots of supertropical polynomials.

Definition 2.25. We refer to roots of a polynomial being obtained as an intersection of two leading tangible monomials as **corner roots**, and to roots that are being obtained from one leading ghost monomial as **non-corner roots**.

Remark 2.26. Suppose $f(x) = \sum a_i x^i \in R[x]$. We specialize to elements $r \in R$, starting with r small and then increasing.
1. The constant term a_0 and the leading monomial $a_n x^n$ dominate first and last, respectively, due to their slopes. Furthermore, they are the only ones that are **necessarily** essential in every polynomial.
2. The intersection of an essential monomial $a_i x^i$ and the next essential monomial $a_j x^j$ where $j > i$, is the ith root of f (counting multiplicities), denoted as $\alpha_i = \sqrt[k]{\frac{a_i}{a_j}}$, and is of multiplicity $k = j - i$.
Proof: The monomial $a_i x^i$ dominates all monomials between $a_i x^i$ and $a_j x^j$. Therefore, when $r \in (\alpha_{i-1}, \alpha_i]$,

$$f(r) = a_j r^j + a_i r^i = a_j \left(r^j + \frac{a_i}{a_j} r^i \right) \in G \Rightarrow \left(r^j + \frac{a_i}{a_j} r^i \right) = r^i \left(r^{j-i} + \frac{a_i}{a_j} \right) \in G$$

which means $(r + \alpha_i)^k \in G$, and therefore α_i is a root of f with multiplicity k.

2.3. **Supertropical characteristic polynomials and eigenvalues.**

We follow the description in [19, §5].

Definition 2.27. $\forall v \in T^n$ and $A \in M_n(R)$ such that $\exists \alpha \in T \bigcup \{0_R\}$ where $Av \models_{gs} \alpha v$, we say that v is a **supertropical eigenvector** of A with a **supertropical eigenvalue** α.
The **characteristic polynomial** of A (also called the maxpolynomial) is defined to be $f_A(x) = det(xI + A)$. The tangible value of the roots of the characteristic polynomial f_A are the eigenvalues of A, as shown in [18, Theorem 7.10]. The coefficient of x^k in this polynomial is a sum of the tropical determinants of all $n - k \times n - k$ minors, obtained by deleting k chosen rows of the matrix, and their corresponding columns. These minors are defined as **principal sub-matrices**.

The combinatorial motivation for the tropical characteristic polynomial is the Best Principal Submatrix problem, and has been studied by Butkovic in [5] and [6].

Theorem 2.28. *(Supertropical Hamilton-Cayley) Any matrix A satisfies its characteristic polynomial, in the sense that $f_A(A)$ is ghost.*

Proof. See [30] or [18, Theorem 5.2] for a supertropical statement and proof. □

Proposition 2.29. *If $\alpha \in T \bigcup 0_R$ is a supertropical eigenvalue of a matrix $A \in M_n(T)$, then α^i is a supertropical eigenvalue of A^i.*

Proof. See [25, Proposition 3.2]. □

However, we notice that $\{\alpha^i : \alpha$ is an eigenvalue of a matrix A$\}$ need not be the only supertropical eigenvalues of the matrix A^i, as shown in the next example.

Example 2.30. Consider the 2×2 matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}.$$

Then $f_A(x) = x^2 + 2x + 2 \Rightarrow f_A(x) \in G \quad \text{when} \quad x = 0, 2.$

However,

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Thus $f_{A^2}(x) = x^2 + 4x + 5^\nu \Rightarrow f_{A^2}(x) \in G \quad \text{when} \quad x^\nu \leq 1^\nu \quad \text{or} \quad x = 4.$

Theorem 2.31. *Let A be in $M_n(R)$. If*

$$f_A(x) = \sum_{i=0}^{n} \alpha_i x^i \quad \textit{and} \quad f_{A^m}(x) = \sum_{i=0}^{n} \beta_i x^i$$

are the characteristic polynomials of A and its mth power, respectively, then

$$f_{A^m}(x^m) \models_{gs} f_A(x)^m.$$

Proof. See [25, Theorem 3.6]. □

Corollary 2.32.
a. If $f_{A^m} \in T[x]$ then equality holds in Theorem 2.31.
b. Every corner root of f_{A^m} is an mth power of a corner root of f_A.

Proof. See [25, Corollary 3.7 and Corollary 3.10]. □

3. The ∇ of a definite matrix

As shown in [26] and [28], reductions to definite matrices can simplify verifications of some complicated properties of matrices. On top of that, matrices of this form enjoy some interesting properties of their own.

Remark 3.1. Since ghost equivalent implies equality in the tropical setting, we will formulate our results in terms of ghost equivalence (i.e. over R), with the understanding that the corresponding tropical equalities (i.e. over $\mathbb{T}$) follow automatically.

3.1. Stabilization under the ∇ operation.

The goal in this section is to show that ∇ is a closure operation.

Lemma 3.2.
(i) $P^\nabla = P^{-1}$ whenever P is an invertible matrix.
(ii) det(PA)=det(P)det(A)=det(AP), for P an invertible matrix.
(iii) $(PA)^\nabla = A^\nabla P^\nabla$ where det(A) is invertible and P is an invertible matrix.
(iv) Let $\bar{A}$ be the left definite form of the matrix A, i.e. $A = P\bar{A}$ for some invertible matrix P. Then $A^\nabla = \bar{A}^\nabla P^{-1}$.

Proof. For (i), (iii) and (iv) see [26, Lemma 5.7].
(ii) We recall that for any permutation σ we have a corresponding permutation ρ such that $\sigma(i) = \rho(j),\ \sigma(j) = \rho(i)$ and $\sigma(k) = \rho(k)\ \forall k \neq i, j$.

As a result, if $P = E_{i,j}$ is a transposition matrix and $A = (a_{i,j})$, then

$$det(E_{i,j}A) = \sum_{\sigma} a_{i,\sigma(j)} a_{j,\sigma(i)} \prod_{k \neq j,i} a_{k,\sigma(k)} = \sum_{\rho} a_{i,\rho(i)} a_{j,\rho(j)} \prod_{k \neq j,i} a_{k,\rho(k)} = det(A) = det(E_{i,j})det(A).$$

If $P = E_{\alpha\cdot j}$ is a diagonal multiplier and $A = (a_{i,j})$, then

$$det(E_{\alpha\cdot j}A) = \sum_{\sigma} \alpha\cdot a_{j,\sigma(j)} \prod_{k\neq j} a_{k,\sigma(k)} = \alpha \sum_{\sigma} a_{j,\sigma(j)} \prod_{k\neq j} a_{k,\sigma(k)} = \alpha det(A) = det(E_{\alpha\cdot j})det(A).$$

Inductively the claim holds for every invertible matrix P.

□

Claim 3.3. *If A is a definite matrix, then $A^{\nabla}A \cong_{\nu} A^{\nabla} \cong_{\nu} AA^{\nabla}$.*

Proof. See [27] for general semirings, or [26, Claim 6.1] for the special case of the supertropical semiring. □

Corollary 3.4. *Let A be a matrix with left definite form $\bar{A}$ and right definite form $\tilde{A}$, i.e. $A = P\bar{A} = \tilde{A}Q$ for invertible matrices P and Q. Then*
a. $\bar{A}^{\nabla\nabla} \cong_{\nu} \bar{A}^{\nabla} \cong_{\nu} I'_A$, $\tilde{A}^{\nabla\nabla} \cong_{\nu} \tilde{A}^{\nabla} \cong_{\nu} I_A$.
b. $A^{\nabla\nabla} \cong_{\nu} PA^{\nabla}P$.

Proof.
a. According to [19, Corollary 4.4] we know that $A^{\nabla} \cong_{\nu} A^{\nabla}A^{\nabla\nabla}A^{\nabla}$. By applying Claim 3.3 to $\bar{A}^{\nabla}$ we can conclude

$$\bar{A}^{\nabla} \cong_{\nu} \bar{A}^{\nabla}(\bar{A}^{\nabla\nabla}\bar{A}^{\nabla}) \cong_{\nu} \bar{A}^{\nabla}\bar{A}^{\nabla\nabla} \cong_{\nu} \bar{A}^{\nabla\nabla}.$$

According to Claim 3.3

$$\bar{A}^{\nabla} \cong_{\nu} \bar{A}^{\nabla}\bar{A} = \bar{A}^{\nabla}P^{-1}P\bar{A} = A^{\nabla}A = I'_A, \text{ and } \tilde{A}^{\nabla} \cong_{\nu} \tilde{A}\tilde{A}^{\nabla} = \tilde{A}QQ^{-1}\tilde{A}^{\nabla} = AA^{\nabla} = I_A$$

b. By Lemma 3.2 we have

$$A^{\nabla\nabla} = (P\bar{A})^{\nabla\nabla} = P\bar{A}^{\nabla\nabla} \cong_{\nu} P\bar{A}^{\nabla} = PA^{\nabla}P.$$

□

Theorem 3.5. *We denote the application k times of ∇ to A as $A^{\nabla^{(k)}}$. Then*

$$A^{\nabla^{(k)}} \cong_{\nu} A^{\nabla^{(k+2)}} \ \forall k \geq 1.$$

That is, applying ∇ to A^{∇} is an operator of order 2, which means that the ∇ operation on the set $\{A^{\nabla} : A \in M_n(R)\}$ acts like the inverse operation.

Proof. In the same way as in Corollary 3.4 (b) we see that

$$(A^{\nabla})^{\nabla\nabla} = (\bar{A}^{\nabla}P^{-1})^{\nabla\nabla} = (P\bar{A}^{\nabla\nabla})^{\nabla} \cong_{\nu} (P\bar{A}^{\nabla})^{\nabla} = \bar{A}^{\nabla\nabla}P^{-1} \cong_{\nu} \bar{A}^{\nabla}P^{-1} = A^{\nabla}.$$

The general case then follows inductively. □

In summary, $\nabla\nabla$ is a sort of closure operation, which yields elementarily factorizable matrices.

Example 3.6.
(i) For every non-singular 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have that

$$A^{\nabla} = det(A)^{-1} \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

and

$$A^{\nabla\nabla} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A,$$

imply that the formula in Theorem 3.5 holds in the 2×2 case for all $k \geq 0$. We conclude that A is elementarily factorizable.

(ii) Let

$$A = \begin{pmatrix} 0 & 0 & - \\ - & 0 & 0 \\ 1 & - & 0 \end{pmatrix},$$

(where $-$ denotes $-\infty$).

Thus

$$A^{\nabla} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$A^{\nabla\nabla} = \begin{pmatrix} 0 & 0 & -1^{\nu} \\ 0^{\nu} & 0 & 0 \\ 1 & 0^{\nu} & 0 \end{pmatrix}, \text{ which is different than } A^{\nabla},$$

$$A^{\nabla^{(3)}} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = A^{\nabla} \text{ and therefore equal } A^{\nabla^{(2k-1)}} \ \forall k \in \mathbb{N},$$

and

$$A^{\nabla^{(4)}} = \begin{pmatrix} 0 & 0 & -1^{\nu} \\ 0^{\nu} & 0 & 0 \\ 1 & 0^{\nu} & 0 \end{pmatrix} = A^{\nabla\nabla} \text{ and therefore equal } A^{\nabla^{(2k)}} \ \forall k \in \mathbb{N}.$$

3.2. **The closure operation $*$ and power stabilization.**

Noticing that A^{∇} arises from supertropical algebraic considerations (as its product with A gives a pseudo-identity), we would like to make a connection to the familiar tropical concept of the Kleene star, denoted as A^*, which has been widely studied since the 60's. In the next theorem and proposition we give results relevant to [31] and [28], obtained over the tropical setting, with the understanding that tropical equality implies supertropical ghost-equivalent. Since the definition for A^* requires that $det(A)$ is bounded by 1_R, we consider here the special case of definite matrices. According to Lemma 2.20(a), we can conclude from the LDM factorization of A^* in [23] that A^* is elementarily factorizable.

Theorem 3.7. *(See* [31, Theorem 2]*) If A is an $n\times n$ definite matrix and k is a natural number, then $A^k \cong_{\nu} A^{k+1}$, $\forall k \geq n-1$.*

Proposition 3.8. *$A^{\nabla} \cong_{\nu} A^* \cong_{\nu} A^{n-1}$ when A is definite.*

Proof. For $A^{\nabla} \cong_{\nu} A^*$ see [31, Theorem 4]. As described in [28], the equivalence to A^{n-1} is immediate from the definition of A^* for a definite matrix A:

$$A^* = \sum_{i\in\mathbb{N}\bigcup 0} A^i. \tag{3.1}$$

Indeed, according to [31, Theorem 2] we have that

$$\widehat{A} \leq \widehat{A}^2 \leq ... \leq \widehat{A}^{n-1} \cong_\nu A^n \cong_\nu ...$$

(due to the diagonal, each position can only increase comparing to the corresponding position in the previous power and the value of each entry will stabilize at power $n-1$ at most). □

Combining Corollary 3.4, Theorem 3.6 and Proposition 3.7 we get

$$A^* \cong_\nu A^{n-1} \cong_\nu A^\nabla \cong_\nu A^{\nabla\nabla} \cong_\nu I_A,$$

for a definite matrix A of order n.

4. An alternative proof of Theorem 2.10: $det(AB) \models_{gs} det(A)det(B)$

The proof of Theorem 2.10 is given by means of a multilinear function. In the classical case there exists an alternative, somewhat easier proof, using the factorization of non-singular matrices into the product of elementary matrices. Due to Lemma 2.20 we can now give a tropical analog of this proof.

If A or B are strictly singular (without lost of generality B), then according to [18, Theorem 6.5] the columns of B, denoted as $C_1, ..., C_n$, are tropically dependent:

$$\sum \alpha_i C_i \in G \bigcup 0_R, \ \alpha_i \in T \forall i.$$

Denoting the rows of A as $R_1, ..., R_n$ and the columns of AB as $c_1, ..., c_n$ we get

$$\sum \alpha_i c_i = \begin{pmatrix} \sum \alpha_i R_1 C_i \\ \vdots \\ \sum \alpha_i R_n C_i \end{pmatrix} \in G_{0_R}.$$

Therefore $det(AB) \in G \bigcup 0_R$ as required.

In order to obtain the ∇ operation for (not-strictly) singular matrices we work over the tropical semifield and will prove that $det(AB)$ equals $det(A)det(B)$ plus a term that is attained twice.

Using Theorem 2.17 we have that $det(A^\nabla) = det(A)^{-1}$, and it suffices to prove the theorem for A^∇, B^∇. By Lemma 3.2(ii), if A or B are transposition matrices and diagonal multipliers then the theorem holds with equality. Due to Lemma 2.20(b) we only have to show the impact of Gaussian matrices.

Let $E = E_{row\ i+\alpha \cdot row\ j}$ be a Gaussian matrix adding row j, multiplied by α, to row i. We denote the rows of A as $R_1, ..., R_n$. Then the rows of EA are the same as A's, except for the ith row which is $R_i + \alpha R_j$, therefore

$$det(EA) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{i,\sigma(i)} \cdots a_{n,\sigma(n)} + \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots \alpha a_{j,\sigma(i)} \cdots a_{n,\sigma(n)}, \tag{4.1}$$

where the sum on the left is $det(A)$. In order to analyze the sum on the right, we note that

$$\forall i, j \ \exists! \sigma, \rho \in S_n : \ \sigma(i) = \rho(j), \sigma(j) = \rho(i), \text{ and } \sigma(k) = \rho(k) \ \forall k \neq i, j, \tag{4.2}$$

and let E fix i and j. Thus

$$det(EA) = det(A) + \tag{4.3}$$

$$\sum_{\sigma,\rho\in S_n}(a_{1,\sigma(1)}\cdots\alpha a_{j,\sigma(i)}\cdots a_{j,\sigma(j)}\cdots a_{n,\sigma(n)}+a_{1,\sigma(1)}\cdots a_{j,\rho(j)}\cdots\alpha a_{j,\rho(i)}\cdots a_{n,\sigma(n)}),$$

for σ,ρ as in (4.2). Therefore the sum is attained twice, as required. If E is multiplied on the right, then the proof is analogous using column operation instead of row operations.

As a result we have

$$|AB|=|(AB)^{\nabla}|^{-1}\models_{gs}|B^{\nabla}A^{\nabla}|^{-1}=|P_1^{-1}\bar{B}^{\nabla}\bar{A}^{\nabla}P_2^{-1}|^{-1}=|P_1||\bar{B}^{\nabla}\bar{A}^{\nabla}|^{-1}|P_2|\models_{gs}$$
$$|P_1||\bar{B}^{\nabla}|^{-1}|\bar{A}^{\nabla}|^{-1}|P_2|=|P_1^{-1}\bar{B}^{\nabla}|^{-1}|\bar{A}^{\nabla}P_2^{-1}|^{-1}=|B^{\nabla}|^{-1}|A^{\nabla}|^{-1}=|A||B|,$$

where P_1,P_2 are the right conductor of B and left conductor of A respectively, and as shown in Lemma 2.20, $\bar{A}$ and $\bar{B}$ are products of Gaussian matrices, for which the theorem holds according to (4.3).

5. Tropically conjugate and similar matrices

Although our factorization results for A^{∇} follow from those for A^*, the ∇ operation has additional applications in representation theory.

Definition 5.1. A matrix B is **tropically conjugate** to B' if there exists a non-singular matrix A such that

$$A^{\nabla}BA\models_{gs}B'. \tag{5.1}$$

If equality holds in (5.1) then we say that B is **tropically similar** to B'.

Remark 5.2. If B is tropically similar to B', then
a. $det(B')\models_{gs}det(B)$, with equality when B' is non-singular.
b. $tr(B')\models_{gs}tr(B)$.

These relations can be seen by direct computation, but are also a consequence of Theorem 5.4 below.

Following the next motivating example, we have an analog to the theorem on similar matrices in classical algebra.

Example 5.3.

1. Let

$$A=\begin{pmatrix}2&0\\1&0\end{pmatrix},\ B=\begin{pmatrix}1&2\\3&1\end{pmatrix},\ B'=\begin{pmatrix}3&1^{\nu}\\5&3\end{pmatrix}.$$

We have $A^{\nabla}BA=B'$, and therefore $f_{B'}(x)=x^2+3^{\nu}x+6^{\nu}\models_{gs}f_B(x)=x^2+1^{\nu}x+5$. As a result all supertropical eigenvalues of B are supertropical eigenvalues of $A^{\nabla}BA$.

2. By taking $B=\begin{pmatrix}0&0\\1&2\end{pmatrix}$, $B'=\begin{pmatrix}1&3\\1&2\end{pmatrix}$, $A=\begin{pmatrix}0&1^{\nu}\\-\infty&0\end{pmatrix}$ we get

$$A^{\nabla}BA=\begin{pmatrix}2^{\nu}&3^{\nu}\\1&2^{\nu}\end{pmatrix}\models_{gs}B'.$$

However, $|B|=2$ and $|B'|=4$ do not ghost-surpass each other, which shows that there need not be a ghost surpassing relation between the characteristic polynomials of tropically conjugate matrices.

Theorem 5.4. *If B is tropically similar to B', then $f_{B'} \models_{gs} f_B$.*

Proof. Let $B' = A^{\nabla}BA$, where A is a non-singular matrix.

Let $\bar{A}$ be the right definite form of A and E be its right conductor which means

$$A = \bar{A}E \text{ and } A^{\nabla} = E^{-1}\bar{A}^{\nabla}.$$

Using Lemma 3.2, we notice that $A^{\nabla}BA$ and $\bar{A}^{\nabla}B\bar{A}$ share the same characteristic polynomial, since

$$det(xI + A^{\nabla}BA) = det(xI + E^{-1}\bar{A}^{\nabla}B\bar{A}E) =$$
$$det(E)^{-1}det(xI + \bar{A}^{\nabla}B\bar{A})det(E) = det(xI + \bar{A}^{\nabla}B\bar{A}).$$

Therefore it is sufficient to prove this theorem for the definite form of A, which according to Proposition 3.8 coincides with its Kleene star and $n-1$ power.

We consider A to be a definite matrix. We denote the characteristic polynomials of B' and B as

$$f_{B'}(x) = x^n + \sum_{k=0}^{n-1} \beta'_k x^{n-k}, \quad f_B(x) = x^n + \sum_{k=0}^{n-1} \beta_k x^{n-k}$$

respectively, and the entries of the matrices as

$$A = (a_{i,j}), \quad A^{\nabla} = A^* = A^{n-1} = A^n = (a'_{i,j}) \text{ and } B = (b_{i,j}).$$

We show that $\beta'_k \models_{gs} \beta_k \ \forall k$. That is, every summand in β'_k either has an identical summand in β_k or appear twice, creating a ghost summand.

The entry of $A^{\nabla}BA$ appears in the $(i, \sigma(i))$ position is $\sum_{t,s=1}^{n} a'_{i,t}b_{t,s}a_{s,\sigma(i)}$.

The summands in β'_k (the sum of determinants of $k \times k$ principal sub-matrices) are

$$\prod_{j=1}^{k} a'_{i_j,t_j}b_{t_j,s_j}a_{s_j,\sigma(i_j)}, \tag{5.2}$$

where $\sigma \in S_k$, and $t_j, s_j \in \{1, ..., n\}$.

We can factor each permutation into disjoint cycles $(i_j \ \sigma(i_j) \ \sigma^2(i_j) \ \cdots \ \sigma^{-1}(i_j))$, and we denote i_j as j. We abbreviate $\sigma(j), \sigma^2(j), ..., \sigma^{-1}(j)$ to $\sigma, \sigma^2, ..., \sigma^{-1}$.
Therefore, each permutation track can be factored into cycle tracks:

$$\prod_{\substack{\text{cycles}\\ \text{of } \sigma}} (a'_{j,t_j}b_{t_j,s_j}a_{s_j,\sigma})(a'_{\sigma,t_\sigma}b_{t_\sigma,s_\sigma}a_{s_\sigma,\sigma^2}) \cdots (a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},s_{\sigma^{-1}}}a_{s_{\sigma^{-1}},j}), \tag{5.3}$$

obtaining a product of brackets, where the left indices are distinct, describing a permutation. Changing the location of the brackets does not change the value of this term. We use this relocation of brackets to show that identical terms are being obtained on different permutation tracks.

The term a'_{j,t_j} is a sum arising from the Kleene star:

$$a'_{j,t_j} = \sum_{r_1,...,r_{n-2}=1}^{n} (a_{j,r_1}a_{r_1,r_2} \cdots a_{r_{n-2},t_j}).$$

Thus, we write (5.3) as

$$\prod_{\substack{\text{cycles}\\ \text{of } \sigma}} (a_{j,r_{j,1}} a_{r_{j,1},r_{j,2}} \cdots a_{r_{j,n-2},t_j} b_{t_j,s_j} a_{s_j,\sigma})(a_{\sigma,r_{\sigma,1}} a_{r_{\sigma,1},r_{\sigma,2}} \cdots a_{r_{\sigma,n-2},t_\sigma} b_{t_\sigma,s_\sigma} a_{s_\sigma,\sigma^2}) \cdots$$

$$(a_{\sigma^{-1},r_{\sigma^{-1},1}} a_{r_{\sigma^{-1},1},r_{\sigma^{-1},2}} \cdots a_{r_{\sigma^{-1},n-2},t_{\sigma^{-1}}} b_{t_{\sigma^{-1}},s_{\sigma^{-1}} a_{s_{\sigma^{-1}}},j}).$$

Next, we divide our proof into disjoint cases as follows:

$$\boxed{a'_{j,t_j} b_{t_j,s_j} a_{s_j,\sigma(j)}}$$

$$\boxed{\begin{array}{c}\text{Case 1}\\ s_j = \sigma(j)\ \forall j\end{array}} \qquad \boxed{\begin{array}{c}\text{Case 2}\\ \exists j:\ s_j \neq \sigma(j)\end{array}}$$

$$\boxed{\begin{array}{c}\text{Case 1.1}\\ j = t_j\ \forall j\\ \Rightarrow (5.2) \text{ is}\\ \prod b_{j,\sigma(j)}\end{array}} \qquad \boxed{\begin{array}{c}\text{Case 1.2}\\ \exists j:\ i_j \neq t_j\\ \Rightarrow (5.2) \text{ is}\\ \prod a'_{j,t_j} b_{t_j,\sigma(j)}\end{array}}$$

In **case 1**, $s_j = \sigma(j)\ \forall j$. Thus $a_{s_j,\sigma(j)}$ is diagonal and therefore equals 0 for every j. Next, if we have the special **sub-case 1.1**, $j = t_j\ \forall j$, then a'_{j,t_j} are also diagonal entries, equal to 0. Therefore (5.2) is $\prod b_{j,\sigma(j)}$, which is a summand in f_B.

At this point we have established $\beta'_k = \beta_k + g_k$, where $g_k \in R\ \ \forall k$. It remains to prove that $g_k \in G\ \ \forall k$, that is every other summand is attained at least twice, creating a ghost summand.

In **sub-case 1.2**, stating that $\exists j:\ i_j \neq t_j$, some a'_{j,t_j} is not diagonal. Thus (5.3) is

$$\prod_{\substack{\text{cycles}\\ \text{of } \sigma}} (a_{j,r_{j,1}} a_{\underline{r_{j,1}},r_{j,2}} \cdots a_{r_{j,n-2},t_j} b_{t_j,\sigma})(a_{\sigma,r_{\sigma,1}} a_{\underline{r_{\sigma,1}},r_{\sigma,2}} \cdots a_{r_{\sigma,n-2},t_\sigma} b_{t_\sigma,\sigma^2}) \cdots$$

$$(a_{\sigma^{-1},r_{\sigma^{-1},1}} a_{\underline{r_{\sigma^{-1},1}},r_{\sigma^{-1},2}} \cdots a_{r_{\sigma^{-1},n-2},t_{\sigma^{-1}}} b_{t_{\sigma^{-1}},j}).$$

We analyze the second index of each element: $r_{j,1},\ j = 1, ..., k$.

If all these indices are distinct, then $r_{\sigma^l,1} \mapsto r_{\sigma^{l+1},1}$ describes a permutation. Its track is obtained by moving the brackets one position **forward**:

$$\prod_{\substack{\text{cycles}\\ \text{of } \sigma}} a_{j,r_{j,1}})(a_{r_{j,1},r_{j,2}} \cdots a_{r_{j,n-2},t_j} b_{t_j,\sigma} a_{\sigma,r_{\sigma,1}})(a_{r_{\sigma,1},r_{\sigma,2}} \cdots a_{r_{\sigma,n-2},t_\sigma} b_{t_\sigma,\sigma^2} \cdots$$

$$a_{\sigma^{-1},r_{\sigma^{-1},1}})(a_{r_{\sigma^{-1},1},r_{\sigma^{-1},2}} \cdots a_{r_{\sigma^{-1},n-2},t_{\sigma^{-1}}} b_{t_{\sigma^{-1}},j} =$$

$$\prod_{\substack{\text{cycles}\\ \text{of } \sigma}} (a_{r_{j,1},r_{j,1}} a_{r_{j,1},r_{j,2}} \cdots a_{r_{j,n-2},t_j} b_{t_j,\sigma} a_{\sigma,r_{\sigma,1}})(a_{r_{\sigma,1},r_{\sigma,1}} a_{r_{\sigma,1},r_{\sigma,2}} \cdots a_{r_{\sigma,n-2},t_\sigma} b_{t_\sigma,\sigma^2} a_{\sigma^2,r_{\sigma^2,1}})$$

$$\cdots (a_{r_{\sigma^{-1},1},r_{\sigma^{-1},1}} a_{r_{\sigma^{-1},1},r_{\sigma^{-1},2}} \cdots a_{r_{\sigma^{-1},n-2},t_{\sigma^{-1}}} b_{t_{\sigma^{-1}},j} a_{j,r_{j,1}}).$$

If $r_{j,1} = r_{i,1}$ for some $j \neq i$, then:

If it occurs *in the same cycle*, we obtain a new permutation by factoring this cycle into two sub-cycles, using the points of repetition (we do not change any of the other cycles):

$$[(a_{j,\underline{r_{j,1}}}a_{\underline{r_{j,1}},r_{j,2}}\cdots a_{r_{j,n-2},t_j}b_{t_j,\sigma})\cdots(a_{\sigma^l,\underline{\underline{r_{j,1}}}}a_{\underline{\underline{r_{j,1}}},r_{\sigma^l,2}}\cdots a_{r_{\sigma^l,n-2},t_{\sigma^l}}b_{t_{\sigma^l},\sigma^{l+1}})\cdots$$
$$(a_{\sigma^{-1},r_{\sigma^{-1},1}}a_{r_{\sigma^{-1},1},r_{\sigma^{-1},2}}\cdots a_{r_{\sigma^{-1},n-2},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},j})] =$$
$$[(a_{j,\underline{r_{j,1}}}a_{\underline{\underline{r_{j,1}}},r_{\sigma^l,2}}\cdots a_{r_{\sigma^l,n-2},t_{\sigma^l}}b_{t_{\sigma^l},\sigma^{l+1}})\cdots(a_{\sigma^{-1},r_{\sigma^{-1},1}}a_{r_{\sigma^{-1},1},r_{\sigma^{-1},2}}\cdots a_{r_{\sigma^{-1},n-2},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},j})]$$
$$[(a_{\sigma^l,\underline{\underline{r_{j,1}}}}a_{\underline{r_{j,1}},r_{j,2}}\cdots a_{r_{j,n-2},t_j}b_{t_j,\sigma})\cdots(a_{\sigma^{l-1},r_{\sigma^{l-1},1}}a_{r_{\sigma^{l-1},1},r_{\sigma^{l-1},2}}\cdots a_{r_{\sigma^{l-1},n-2},t_\sigma^{l-1}}b_{t_\sigma^{l-1},\sigma^l})]$$

(the squared brackets indicate cycles).

If an index $r_{j,1}$ repeats *in two different cycles*, we obtain a new permutation by joining these cycles into a single cycle, using the points of repetition (we do not change any of the other cycles):

$$[(a_{j,\underline{r_{j,1}}}a_{\underline{r_{j,1}},r_{j,2}}\cdots a_{r_{j,n-2},t_j}b_{t_j,\sigma})\cdots(a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},j})]\cdot$$
$$[(a_{i,\underline{\underline{r_{j,1}}}}a_{\underline{\underline{r_{j,1}}},r_{i,2}}\cdots a_{r_{i,n-2},t_i}b_{t_i,\sigma(i)})\cdots(a'_{\sigma^{-1}(i),t_{\sigma^{-1}(i)}}b_{t_{\sigma^{-1}(i)},i})] =$$

$$[(a_{j,\underline{r_{j,1}}}a_{\underline{\underline{r_{j,1}}},r_{i,2}}\cdots a_{r_{i,n-2},t_i}b_{t_i,\sigma(i)})\cdots(a'_{\sigma^{-1}(i),t_{\sigma^{-1}(i)}}b_{t_{\sigma^{-1}(i)},i})$$
$$(a_{i,\underline{\underline{r_{j,1}}}}a_{\underline{r_{j,1}},r_{j,2}}\cdots a_{r_{j,n-2},t_j}b_{t_j,\sigma})\cdots(a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},j})]$$

(the squared brackets indicate cycles).

In **case 2**, $\exists j:\ s_j \neq \sigma(j)$, and thus some $a_{s_j,\sigma(j)}$ is not diagonal.

We obtain the second permutation track using the algorithm of case 1.2, analyzing the indices $s_j,\ j = 1, ..., k$.
If all these indices are distinct, then $s_{\sigma^l} \mapsto s_{\sigma^{l+1}}$ describes a permutation. Even if this permutation coincides with σ, you still obtain a different summand of (5.2). Its track is obtained by moving the brackets one position **backwards**:

$$\prod_{\substack{\text{cycles}\\ \text{of }\sigma}} a'_{j,t_j}b_{t_j,s_j})(a_{s_j,\sigma}a'_{\sigma,t_\sigma}b_{t_\sigma,s_\sigma})\cdots(a_{s_\sigma^{-2},\sigma^{-1}}a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},s_{\sigma^{-1}}})(a_{s_{\sigma^{-1}},j} =$$
$$\prod_{\substack{\text{cycles}\\ \text{of }\sigma}} (a_{s_{\sigma^{-1}},j}a'_{j,t_j}b_{t_j,s_j})(a_{s_j,\sigma}a'_{\sigma,t_\sigma}b_{t_\sigma,s_\sigma})\cdots(a_{s_\sigma^{-2},\sigma^{-1}}a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},s_{\sigma^{-1}}}).$$

(This case is dual to the previous argument in the sense that 1.2 *starts* where the entries of A are diagonal, and is *being repeated* by a permutation track with no such restriction.)

In all remaining summands $s_j = s_i$, for some $j \neq i$. Then like in case 1.2, we factor and disjoint cycles, using the points of repetition, without changing the non-relevant cycles:

$$[(a'_{j,t_j}b_{t_j,\underline{s_j}}a_{\underline{s_j},\sigma})\cdots(a'_{\sigma^l,t_{\sigma^l}}b_{t_{\sigma^l},\underline{\underline{s_j}}}a_{\underline{\underline{s_j}},\sigma^{l+1}})\cdots(a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},s_{\sigma^{-1}}}a_{s_{\sigma^{-1}},j})] =$$
$$[(a'_{j,t_j}b_{t_j,\underline{s_j}}a_{\underline{\underline{s_j}},\sigma^{l+1}})\cdots(a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},s_{\sigma^{-1}}}a_{s_{\sigma^{-1}},j})]\cdot$$
$$[(a'_{\sigma^l,t_{\sigma^l}}b_{t_{\sigma^l},\underline{\underline{s_j}}}a_{\underline{s_j},\sigma})\cdots(a'_{\sigma^{l-1},t_{\sigma^{l-1}}}b_{t_{\sigma^{l-1}},s_{\sigma^{l-1}}}a_{s_{\sigma^{l-1}},\sigma^l})],$$

and

$$[(a'_{j,t_j}b_{t_j,\underline{s_j}}a_{\underline{s_j},\sigma})\cdots(a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},s_{\sigma^{-1}}}a_{s_{\sigma^{-1}},j})]\cdot$$
$$[(a'_{i,t_i}b_{t_i,\underline{\underline{s_j}}}a_{\underline{\underline{s_j}},\sigma(i)})\cdots(a'_{\sigma^{-1}(i),t_{\sigma^{-1}(i)}}b_{t_{\sigma^{-1}(i)},s_{\sigma^{-1}(i)}}a_{s_{\sigma^{-1}(i)},i})] =$$

$$[(a'_{j,t_j}b_{t_j,\underline{s_j}}a_{\underline{\underline{s_j}},\sigma(i)})\cdots(a'_{\sigma^{-1}(i),t_{\sigma^{-1}(i)}}b_{t_{\sigma^{-1}(i)},s_{\sigma^{-1}(i)}}a_{s_{\sigma^{-1}(i)},i})\cdot$$
$$(a'_{i,t_i}b_{t_i,\underline{\underline{s_j}}}a_{\underline{s_j},\sigma})\cdots(a'_{\sigma^{-1},t_{\sigma^{-1}}}b_{t_{\sigma^{-1}},s_{\sigma^{-1}}}a_{s_{\sigma^{-1}},j})].$$

The assertion then follows.

□

Corollary 5.5. *If B is tropically similar to B', then*
1. Every supertropical eigenvalue of B is a supertropical eigenvalue of B'.
2. If $f_{B'} \in T[x]$ then $f_{B'} = f_B$.
3. B satisfies $f_{B'}$ in the sense that $f_{B'}(B)$ is ghost..

Proof.
1. B is tropically similar to B'. If λ is an eigenvalue of B, then $f_{B'}(\lambda) \models_{gs} f_B(\lambda) \in G$.
2. Follows immediately from the theorem.
3. By Theorem 2.28 we get $f_{B'}(B) \models_{gs} f_B(B) \in G$.

□

6. The characteristic polynomial of the pseudo-inverse.

Current research regarding the eigenspaces of a supertropical matrix shows that these spaces may behave in an undesirable fashion, as shown in [19, Example 5.7]. The following chapter provides properties regarding the characteristic polynomial, eigenvalues, determinant and trace of A^∇, which may provide techniques for solving the dependency problem of supertropical eigenspaces.

We observe the following motivating example.

Example 6.1. Let $A = \begin{pmatrix} 1 & 0 & - \\ 3 & 4 & - \\ - & - & 1 \end{pmatrix}$ (where $-$ denotes $-\infty$). Then

$$f_A(x) = x^3 + 4x^2 + 5^\nu x + 6$$

and

$$A^\nabla = 6^{-1}\begin{pmatrix} 5 & 1 & - \\ 4 & 2 & - \\ - & - & 5 \end{pmatrix} = \begin{pmatrix} -1 & -5 & - \\ -2 & -4 & - \\ - & - & -1 \end{pmatrix}.$$

As a result we get $f_{A^\nabla}(x) = x^3 + (-1)^\nu x^2 + (-2)x - 6$. Multiplying by $det(A)$ we get

$$6x^3 + 5^\nu x^2 + 4x + 0,$$

which has the same coefficients of f_A but in opposite order. Consequently, the inverse of every supertropical eigenvalue of A is a supertropical eigenvalue of A^∇, obtained in the opposite order.

In the following conjecture we formulate how the kth coefficient of the characteristic polynomial of A^{∇} is related to the $(n-k)^{th}$ coefficient of the characteristic polynomial of A.

Conjecture 6.2. *If $A = (a_{i,j}) \in M_n(R)$ is a non-singular matrix, then*

$$det(A) f_{A^{\nabla}}(x) \models_{gs} x^n f_A(x^{-1}).$$

An equivalent formulation is

$$det(A) b_k \models_{gs} a_{n-k} \ \forall k = 0, ..., n.$$

We resolve this conjecture for some special cases described in Proposition 6.3 and Example 6.4.

Proposition 6.3. *Let $A \in M_n(R)$. We denote the characteristic polynomials of the matrix $A = (a_{i,j})$ and $A^{\nabla} = \left(\frac{a'_{i,j}}{det(A)}\right)$ as $f_A(x) = \sum_{k=0}^{n} a_k x^k$ and $f_{A^{\nabla}}(x) = \sum_{k=0}^{n} b_k x^k$ respectively, noting that $x^n f_A(x^{-1}) = \sum_{k=0}^{n} a_{n-k} x^k$. Then,*

$$det(A) b_k \models_{gs} a_{n-k}, \ \text{where } k \in \{0, n-2, n-1, n\}.$$

Proof. (i) Since $det(A) = det(A^{\nabla})^{-1}$, we have

$$det(A) b_0 = det(A) det(A^{\nabla}) = 0 = a_n$$

and

$$det(A) b_n = det(A) = a_0.$$

(ii) We have $det(A) b_{n-1} = a_1$ since

$$det(A) tr(A^{\nabla}) = det(A) b_{n-1} = det(A) \sum \frac{a'_{i,i}}{det(A)} = \sum a'_{i,i},$$

which is indeed the sum of determinants of all $(n-1) \times (n-1)$ principal sub-matrices in A.

(iii) We have $det(A) b_{n-2} \models_{gs} a_2$, where b_{n-2} is the sum of determinants of 2×2 principal sub matrices in A^{∇}, and

$$det(A) b_{n-2} = det(A) \sum_{1 \le i_1 < i_2 \le n} \sum_{\sigma \in S_2} \frac{a'_{i_1,\sigma(i_1)}}{det(A)} \frac{a'_{i_2,\sigma(i_2)}}{det(A)} . \tag{6.1}$$

For a product of entries of $A = (a_{i,j})$, we abuse the graph theory notations of $d_{out}(r)$ and $d_{in}(r)$ to denote the number of appearances of r as a left index and as a right index, respectively. We use [18, Proposition 3.17], stating that a product of kn entries of A

$$\prod_{j=1}^{kn} a_{s_j,t_j}, \quad \text{where } d_{out}(s_j) = d_{in}(t_j) = k, \ \forall s_j, t_j = 1, ..., n, \tag{6.2}$$

is a product of k permutation tracks of A.

The numerator in 6.1

$$\prod_{j=1}^{2} a'_{i_j,\sigma(i_j)} \tag{6.3}$$

is the product of two entries of the adjoin of A, located in positions $(i_1, \sigma(i_1))$ and $(i_2, \sigma(i_2))$. That is, 6.3 is on the permutation track $i_1 \mapsto \sigma(i_1),\ i_2 \mapsto \sigma(i_2)$ of the 2×2 principle sub-matrix of i_1 and i_2 in $adj(A)$. Thus, $\prod_{j=1}^{2} a'_{i_j,\sigma(i_j)} a_{\sigma(i_j),i_j}$ is the product of two permutation tracks of A:

$$[a_{1,\pi_1(1)} \cdots a_{n,\pi_1(n)}][a_{1,\pi_2(1)} \cdots a_{n,\pi_2(n)}] = \prod_{j=1}^{2} \prod_{r=1}^{n} a_{r,\pi_j(r)}, \tag{6.4}$$

where $a_{\sigma(i_j),i_j}$ appears in the track of $\pi_j \in S_n$, and

$$d_{out}(r) = d_{in}(\pi_j(r)) = 2,\ \forall r = 1, ..., n,\ j = 1, 2.$$

Product 6.4 includes

$$\prod_{j=1}^{2} a_{\sigma(i_j),i_j}, \tag{6.5}$$

which is a permutation track in a 2×2 principle sub-matrix of A, and

$$d_{out}(\sigma(i_j)) = d_{in}(i_j) = 1,\ \forall j = 1, 2.$$

Analyzing 6.3, we would like to find that when multiplied by $\frac{det(A)}{det(A)^2}$, it either describes a summand in a_2, dominated by one, or appears in another permutation track in a 2×2 principle sub-matrix of $adj(A)$. We show that by applying a switch (a change of permutation on the same set of indices) on σ. That is, we multiply 6.3 by $a_{\sigma(i_1),i_2} a_{\sigma(i_2),i_1}$, instead of $a_{\sigma(i_1),i_1} a_{\sigma(i_2),i_2}$, as we did in 6.4:

$$a'_{i_1,\sigma(i_1)} a_{\sigma(i_1),i_2} a'_{i_2,\sigma(i_2)} a_{\sigma(i_2),i_1}. \tag{6.6}$$

Since the index counting in 6.6 do not chance (from 6.4), it also describes the product of two permutation tracks of A:

$$[a_{1,\pi'_1(1)} \cdots a_{n,\pi'_1(n)}][a_{1,\pi'_2(1)} \cdots a_{n,\pi'_2(n)}] = \prod_{j=1}^{2} \prod_{r=1}^{n} a_{r,\pi'_j(r)}, \tag{6.7}$$

where $d_{out}(r) = d_{in}(\pi'_j(r)) = 2$ and $\pi'_j \in S_n,\ \forall r = 1, ..., n,\ j = 1, 2.$

We look for the location of $a_{\sigma(i_2),i_1}$ and $a_{\sigma(i_1),i_2}$ in 6.7.

<u>The summands obtaining a_2</u>:
If both appear in the same permutation track, without loss of generality π'_1, then the track of π'_2, which is equal to, or dominated by $det(A)$, appears in 6.3. That is, we can consider π'_2 to be the permutation attaining $det(A)$, otherwise, it is dominated by the summand having the track attaining $det(A)$ instead of π'_2.
Next, since $a_{\sigma(i_1),i_2} a_{\sigma(i_2),i_1}$ is a track of a permutation of S_2 in A, included in the track of π'_1, we have that $\frac{a_{1,\pi'_1(1)} \cdots a_{n,\pi'_1(n)}}{a_{\sigma(i_1),i_2} a_{\sigma(i_2),i_1}}$ is a track of a permutation of S_{n-2} in A:

$$\frac{det(A)}{det(A)^2} \frac{det(A) a_{1,\pi'_1(1)} \cdots a_{n,\pi'_1(n)}}{a_{\sigma(i_1),i_2} a_{\sigma(i_2),i_1}} = \frac{a_{1,\pi'_1(1)} \cdots a_{n,\pi'_1(n)}}{a_{\sigma(i_1),i_2} a_{\sigma(i_2),i_1}},$$

which is a summand in a_2.

The summands obtaining a ghost:
If $a_{\sigma(i_2),i_1}$ and $a_{\sigma(i_1),i_2}$ appears in different permutation tracks, without loss of generality $a_{\sigma(i_2),i_1}$ on the track of π_1' and $a_{\sigma(i_1),i_2}$ on the track of π_2', then

$$\frac{[a_{1,\pi_1'(1)}\cdots a_{n,\pi_1'(n)}][a_{1,\pi_2'(1)}\cdots a_{n,\pi_2'(n)}]}{a_{\sigma(i_2),i_1}\quad a_{\sigma(i_1),i_2}}, \tag{6.8}$$

which is identical to 6.3 (we multiplied by $a_{\sigma(i_2),i_1}a_{\sigma(i_1),i_2}$ instead of $a_{\sigma(i_1),i_1}a_{\sigma(i_2),i_2}$ in the numerator, and then canceled it in the dominator), describes the product of two adjoint-entries, located in positions $(i_1,\sigma(i_2))$ and $(i_2,\sigma(i_1))$, and described by the permutations π_1' and π_2' respectively.

As a result, products 6.3 and 6.8 describe two identical summands obtained on the permutation tracks of $\begin{cases} i_1 \mapsto \sigma(i_1) \\ i_2 \mapsto \sigma(i_2) \end{cases}$ and $\begin{cases} i_1 \mapsto \sigma(i_2) \\ i_2 \mapsto \sigma(i_1) \end{cases}$ in $adj(A)$, respectively.

□

Example 6.4.
1. If $A = (a_{i,j})$ is a triangular, non-singular matrix, then A^∇ and all principal submatrices of A and A^∇ are triangular. Denoting $A^\nabla = (b_{i,j})$, we have $b_{i,i} = a_{i,i}^{-1}$. Noting that the determinant of a triangular matrix comes from its diagonal, we get

$$det(A)f_{A^\nabla}(x) = x^n f_A(x^{-1}).$$

2. By Proposition 6.3, if $A \in M_i(R)$, where $i = 2, 3$, then

$$det(A)f_{A^\nabla}(x) \models x^i f_A(x^{-1}) \text{ (with equality when } i = 2),$$

since the coefficient of x^{i-1} is the trace, the coefficient of x^{i-2} is b_{n-2} and the constant is the determinant. The 4×4 case has been verified by a computer calculation.

Consequences of Conjecture 6.2 would be:

1. The inverse of every supertropical eigenvalue of A is a supertropical eigenvalue of A^∇, since if λ is an eigenvalue of A, then $det(A)f_{A^\nabla}(\lambda^{-1}) \models_{gs} (\lambda^{-1})^n f_A(\lambda) \in G$.

2. If $f_{A^\nabla} \in T[x]$ then $det(A)f_{A^\nabla}(x) = x^n f_A(x^{-1})$ (follows immediately).

3. If $f_{A^{\nabla\nabla}} \in T[x]$ then $f_{A^{\nabla\nabla}} = f_A$, since applying (6.2) to A and A^∇, we have

$$det(A)^{-1}f_{A^{\nabla\nabla}}(x) \models_{gs} x^n f_{A^\nabla}(x^{-1}) \text{ and } det(A)f_{A^\nabla}(x) \models_{gs} x^n f_A(x^{-1}).$$

Combining these two equations we get

$$det(A)det(A)^{-1}f_{A^{\nabla\nabla}}(x) \models_{gs} det(A)x^n f_{A^\nabla}(x^{-1}) \models_{gs} x^n x^{-n} f_A(x).$$

Then, if $f_{A^{\nabla\nabla}} \in T[x]$, we have that $f_{A^{\nabla\nabla}} = f_A$.

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