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Optimizing Lateral Boundary Conditions at Staircase-shaped Coastlines: Variational Approach

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Optimizing Lateral Boundary Conditions at Staircase-shaped Coastlines: Variational Approach.

Christine & Eugene Kazantsev

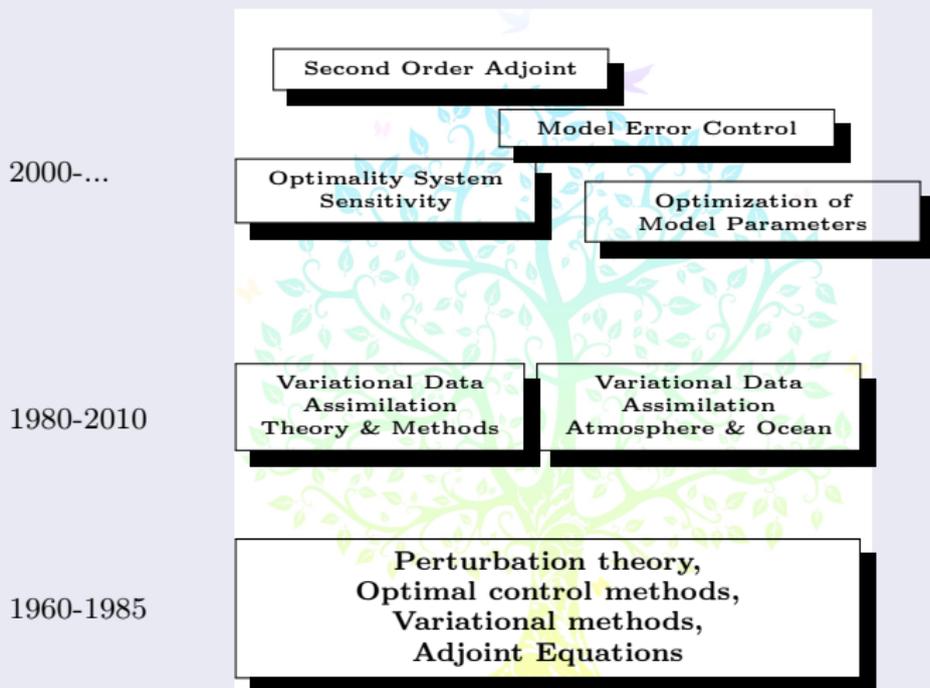
University Grenoble Alpes & INRIA
FRANCE

International conference
*Modern Problems in Numerical Mathematics and
Mathematical Modeling.*

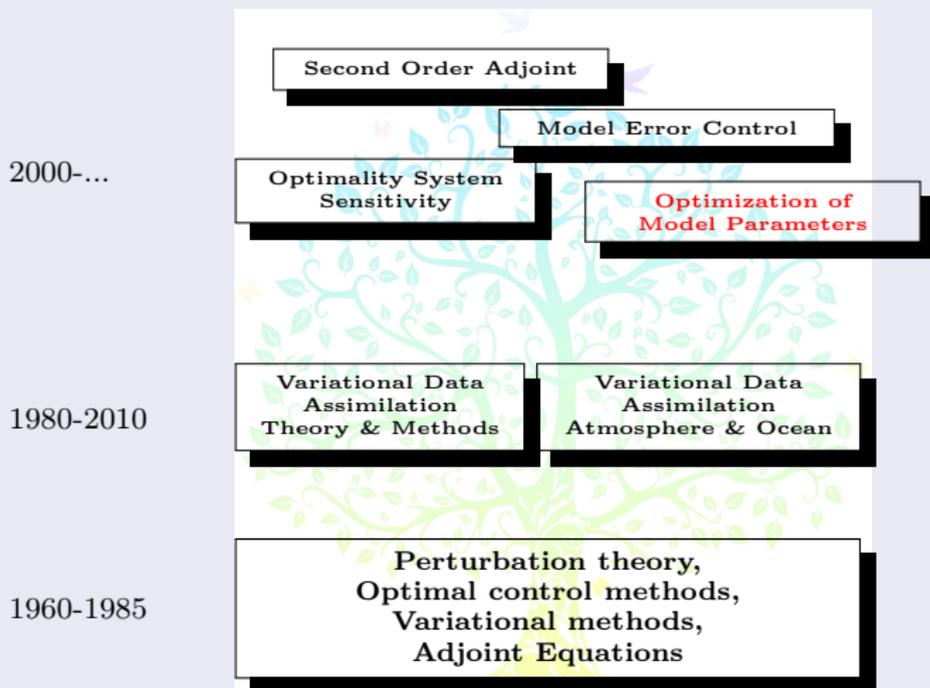
**Институт Вычислительной Математики
им. Г.И. Марчука**
Moscow, June 2015.



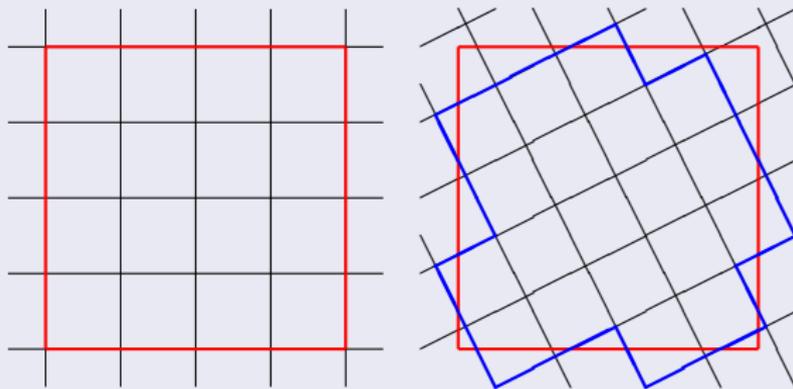
Layout:



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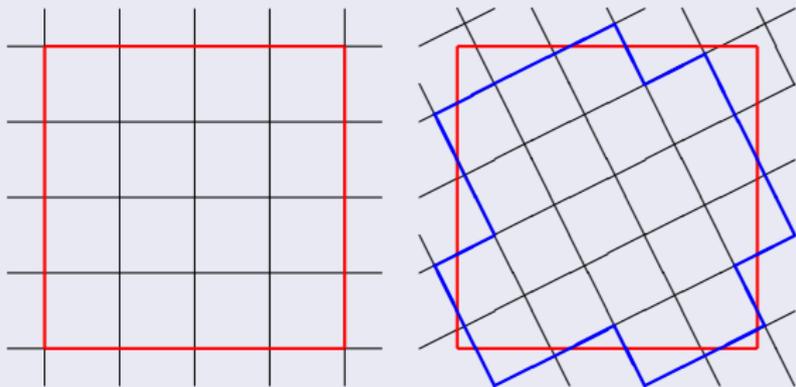


Staircase-shaped boundary



When the grid is not aligned with the model boundary, boundary conditions become difficult to prescribe.

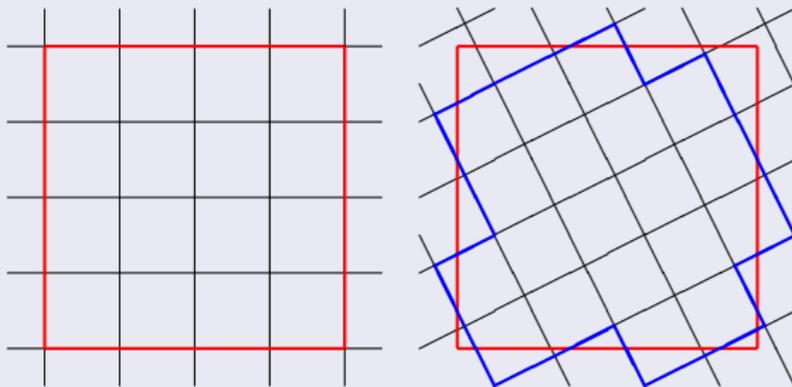
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When the grid is not aligned with the model boundary, boundary conditions become difficult to prescribe.

- Finite elements? Expensive...
-
-

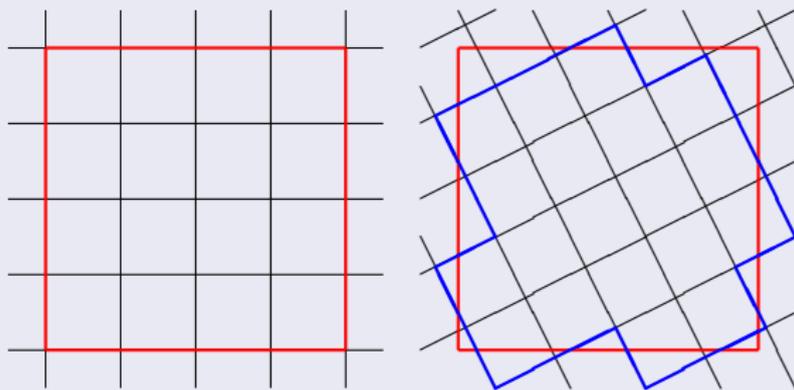
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- Better interpolation in frames of FD? May be instable...
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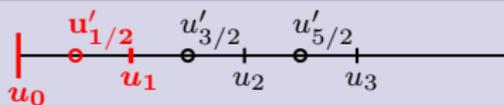
- Finite elements? Expensive...
- Better interpolation in frames of FD? May be instable...
- **Variational methods** ? We can try...

Particular discretisation for derivatives near the boundary

Boundary conditions are introduced into the model by a particular discretization of operators near the boundary.

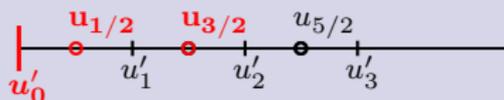
To avoid instabilities, we **control both BC and their approximation**.

Integer nodes



$$\text{if } u_0 = 0 \rightarrow \left. \frac{\partial u}{\partial x} \right|_{1/2} = \frac{u_1}{h}$$

Half-integer nodes



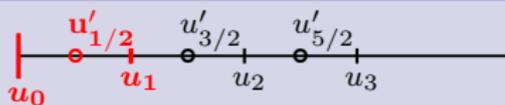
$$\text{if } u_0 = 0 \rightarrow \left. \frac{\partial u}{\partial x} \right|_0 = \frac{2u_{1/2}}{h}$$

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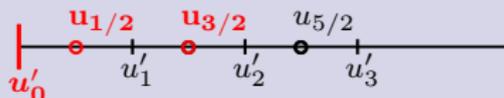
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$$\left. \frac{\partial u}{\partial x} \right|_{1/2} = \frac{\alpha_1 u_0 + \alpha_2 u_1}{h}$$

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$$\left. \frac{\partial u}{\partial x} \right|_0 = \frac{\alpha_1 u_{1/2} + \alpha_2 u_{3/2}}{h}$$

where α_1 and α_2 are the control coefficients.

Derivatives are allowed to change their properties near the boundaries in order to find the best fit with requirements of the model and data.

Nucleus for European Modelling of the Ocean (NEMO): Rectangular Box Configuration

$30^\circ \times 20^\circ$ rectangle with $\frac{1}{4}^\circ$ resolution and 5 z levels.

$120 \times 80 \times 5$ nodes in (x, y, z) coordinates, 64 time steps per day.

$$\frac{\partial u}{\partial t} = v(\omega + f) - \frac{\partial(u^2 + v^2)/2}{\partial x} - w \frac{\partial u}{\partial z} - \frac{\partial A_u^h \xi}{\partial x} + \frac{\partial A_u^h \omega}{\partial y} + \frac{\partial}{\partial z} A_u^z \frac{\partial u}{\partial z} + g \int_0^z \frac{\partial \rho(x, y, \zeta)}{\partial x} d\zeta + g \frac{\partial(\eta + T_c \phi)}{\partial x}$$

$$\xi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad \text{Divergence, Vorticity}$$

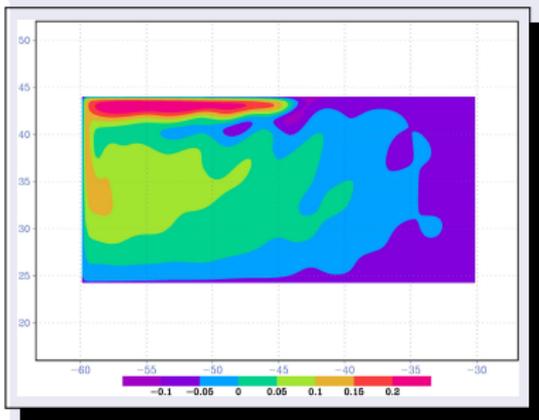
$$w = \int_H^z \xi(x, y, \zeta) d\zeta; \quad w(x, y, H) = 0 \quad \text{Vertical velocity}$$

$$A^z = 1.2 \times 10^{-4} \frac{m^2}{s}; \quad A^h = 200 \frac{m^2}{s}, \quad f = \frac{4\pi}{86400} \sin(lat)$$

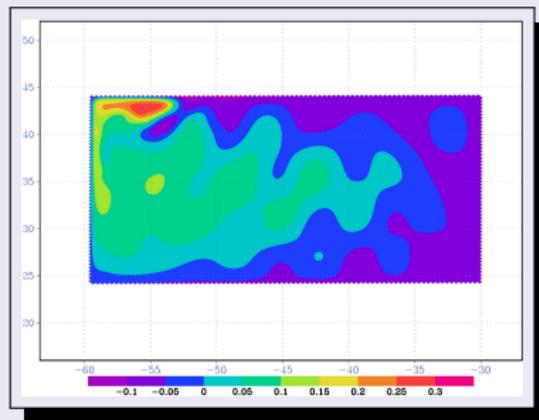
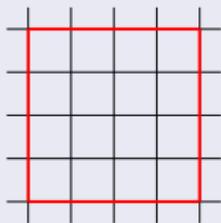
$$\left. \frac{\partial u}{\partial z} \right|_{w_0} = \frac{-0.1 \frac{N}{m^2} \cos\left(B\pi * \frac{lat - 24^\circ}{44^\circ - 24^\circ}\right)}{hz_1 \rho_0} \quad \begin{array}{l} B = 2, \quad \text{Double gyre} \\ B = 1, \quad \text{Single gyre} \end{array}$$

$(u_\perp, \omega)_{\text{Lateral Boundary}} = 0$ (Impermeability and Free-Slip conditions)

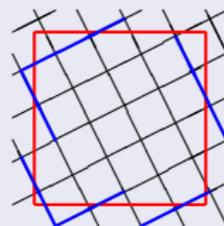
Aligned grid and rotated grid



SSH on the aligned grid



SSH on the 45° rotated grid.



Nucleus for European Modelling of the Ocean (NEMO): Rectangular Box Configuration

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$$\left. \frac{\partial v}{\partial x} \right|_{\omega_b} = \frac{\alpha_1 v_{1/2} + \alpha_2 v_{3/2}}{h}$$

$$\left. \frac{\partial u}{\partial y} \right|_{\omega_b} = \frac{\alpha_1 u_{1/2} + \alpha_2 u_{3/2}}{h}$$

$$\left. \frac{\partial v}{\partial x} \right|_{\omega_b} = \frac{\alpha_1 v_{-1/2} + \alpha_2 v_{1/2}}{h}$$

$$\left. \frac{\partial u}{\partial y} \right|_{\omega_b} = \frac{\alpha_1 u_{-1/2} + \alpha_2 u_{1/2}}{h}$$

$$\left. \frac{\partial \omega}{\partial x} \right|_v = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}$$

$$\left. \frac{\partial \omega}{\partial y} \right|_u = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}$$

$$\alpha_1 = \alpha_2 = 0$$

$$\alpha_1 = -1, \quad \alpha_2 = 1$$

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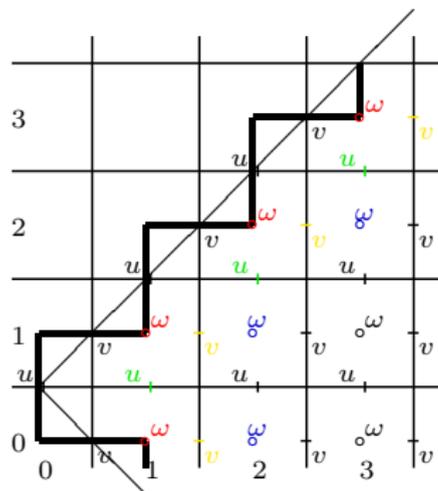


Рис.: 45° rotated grid

The model: $x(t) = \mathcal{M}_{0,t}(x(0), \alpha)$ with $x = (u, v, T, S, ssh)^T$

Cost function J

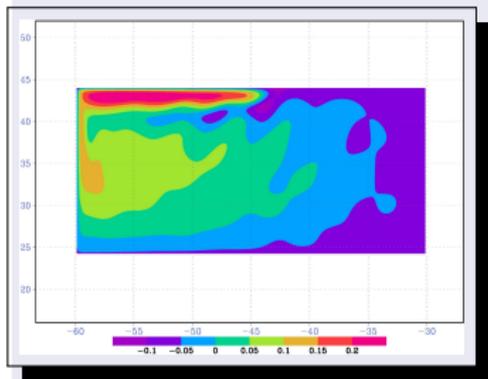
$$J = 10^{-4}(\|x(0) - x_{bgr}\|^2 + \|\alpha - \alpha_{bgr}\|^2) + \int_{t=0}^T t \iint (u - u_{\text{ref}})^2 + (v - v_{\text{ref}})^2 + (ssh - ssh_{\text{ref}})^2 dx dy dt$$

Layout:

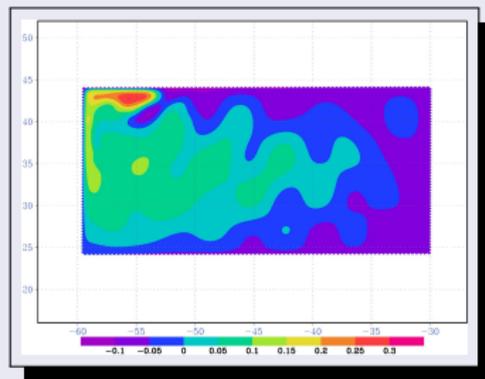
- **Joint control** of the initial point $x(0)$ (interpolation errors) and the set of α ;
- Artificially generated data by the same model on the aligned grid;
- Data Assimilation over the **50 days** window;
- Analysis of the solution on the **8 years** interval.

- Minimization is performed by M1QN3 (JC Gilbert, C.Lemarechal);
- Adjoint is generated by Tapenade (Ecuador team, INRIA).

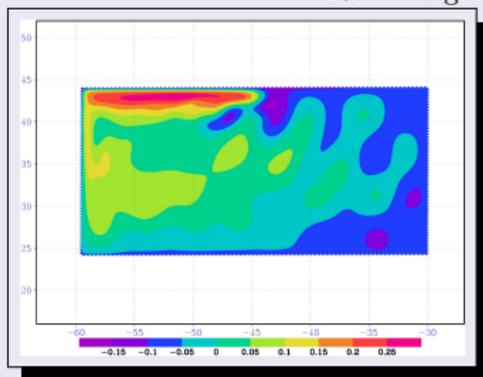
Reference, Optimal and Conventional BC 800 days later



Reference SSH

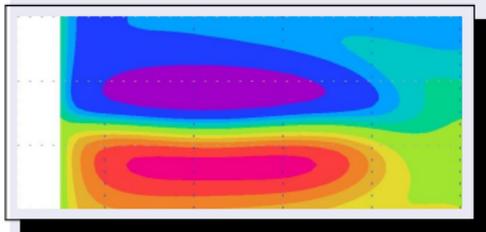


Rotated grid conventional BC SSH

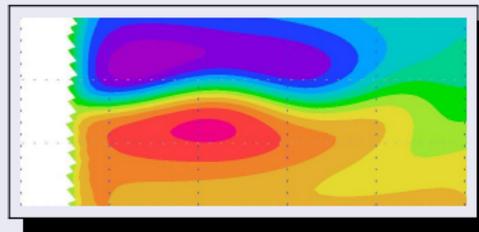


Rotated grid Optimal BC SSH

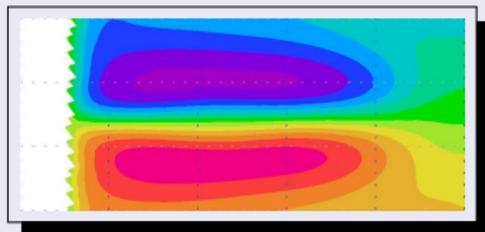
Reference, Rotated, Optimal SSH 10 years average.



Aligned Grid

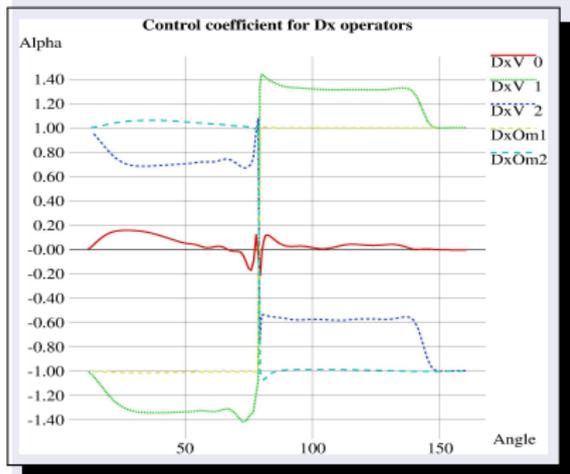


Conventional BC, Rotated (30°) Grid

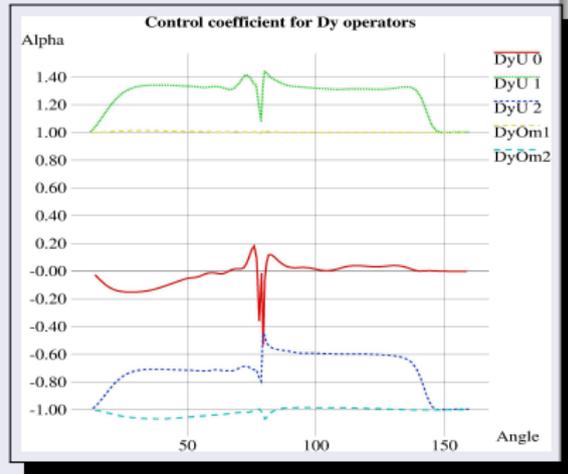


Optimized BC, Rotated (30°) Grid

Single gyre, α_1, α_2



Derivatives in x



Derivatives in y

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$$\left. \frac{\partial \omega}{\partial x} \right|_v = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}$$

$$\left. \frac{\partial \omega}{\partial y} \right|_u = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}$$

$$\alpha_1 = 0, \quad \alpha_2 = -0.1$$

$$\alpha_1 = -1.4, \quad \alpha_2 = 0.6$$

$$\alpha_1 = -1.01, \quad \alpha_2 = 1.01$$

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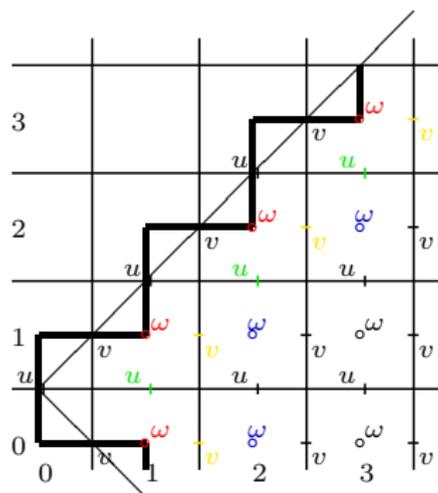


Рис.: 45° rotated grid

$$\omega_o = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 0.8 \frac{u+v}{h}$$

$$\begin{aligned} \omega_0 &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u+v}{h} \\ &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V} \cdot \vec{\tau}}{h/\sqrt{2}} \end{aligned}$$

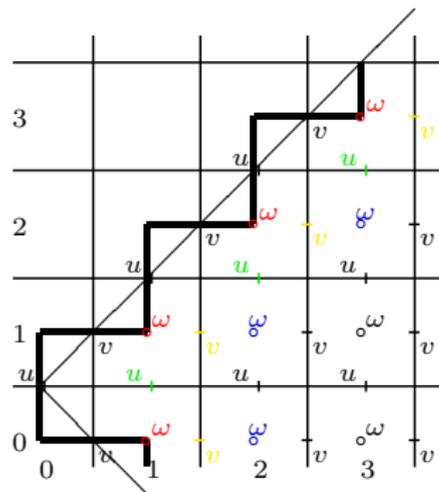


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- Free-slip condition on a **curvilinear boundary** $\omega|_{bnd} = \frac{\vec{V} \cdot \vec{\tau}}{R}$
- Optimal boundary is curvilinear with $R = -h/\sqrt{2}$

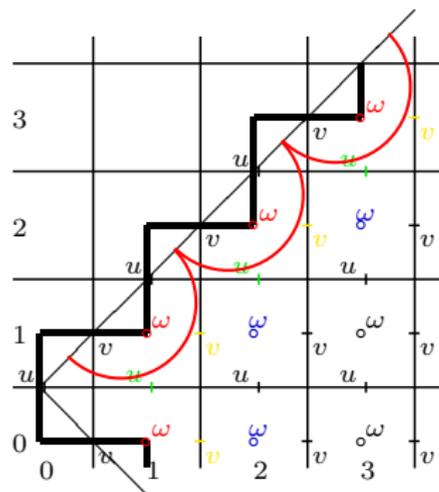
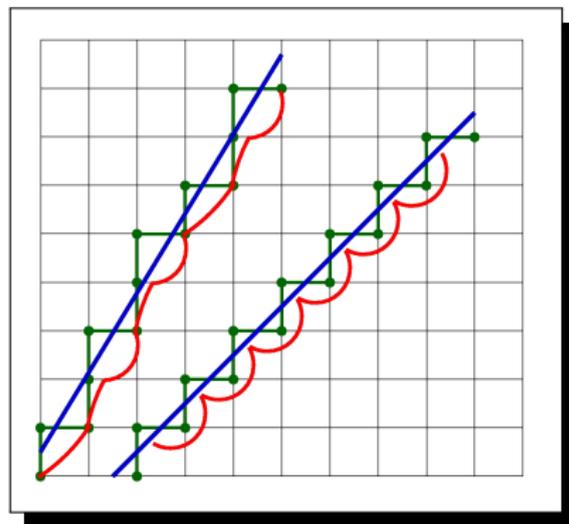


Рис.: 45° rotated grid

$$\begin{aligned}\omega_{\circ} &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u+v}{h} \\ &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V} \cdot \vec{\tau}}{h/\sqrt{2}}\end{aligned}$$

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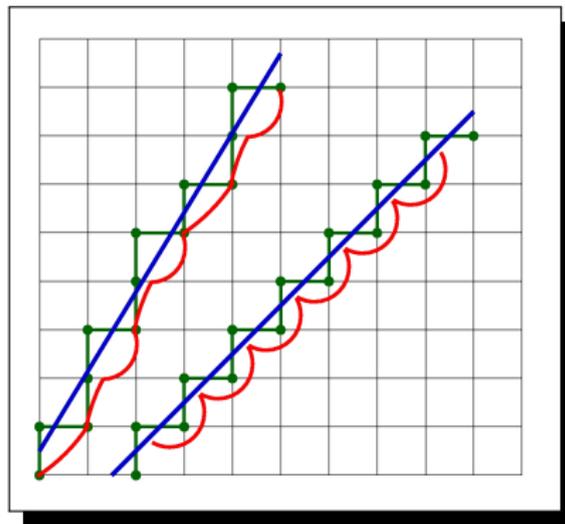
Variable $R_{30^\circ} : -h \leq R_{30^\circ} \leq 5h$.



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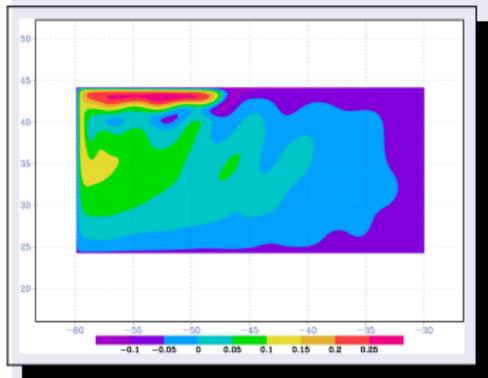
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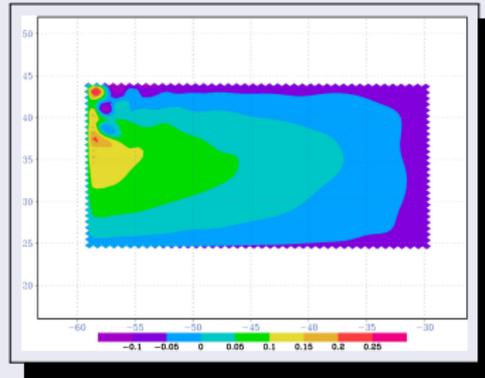
The Radius depends on the grid resolution h .

Does this curvilinear boundary **remains optimal on different resolutions?**

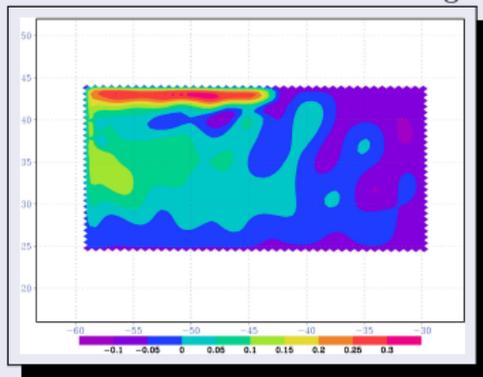
Reference, Optimal and Conventional BC 800 days later



Reference SSH

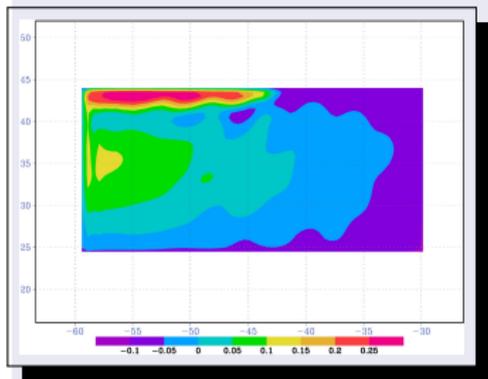


Rotated grid conventional BC SSH

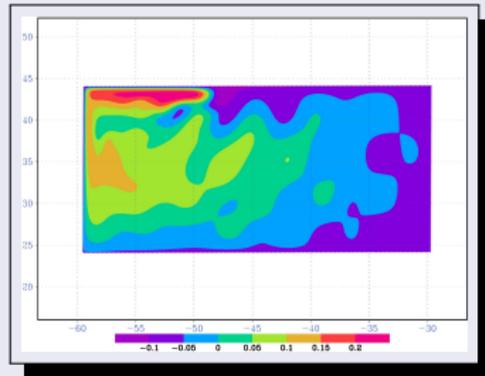


Rotated grid constant $R = -h/\sqrt{2}$ BC
SSH

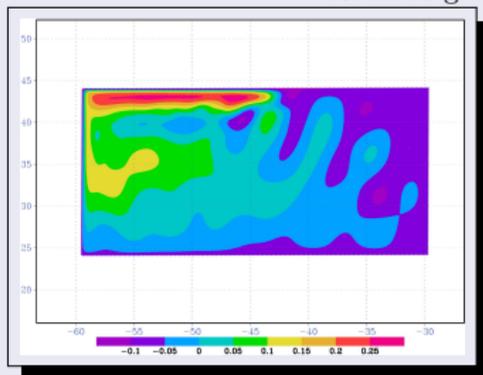
Reference, Optimal and Conventional BC 800 days later



Reference SSH



Rotated grid conventional BC SSH

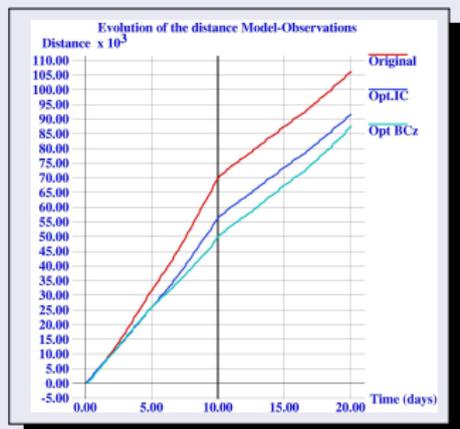


Rotated grid, constant $R = -h/\sqrt{2}$ BC
SSH

Layout:

- NEMO, Global ocean model, 2° resolution, 31 layer;
- ECMWF data issued from Jason-1 and Envisat altimetric missions and ENACT/ENSEMBLES data banque;
- Data Assimilation during 10 days interval;
- Analysis of the distance “model-observations” on 1 month interval

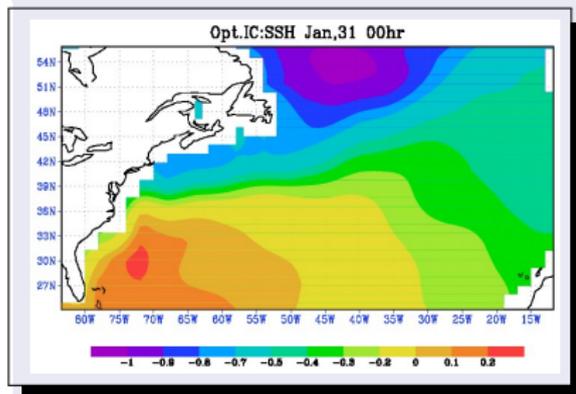
Distance “model-observations”



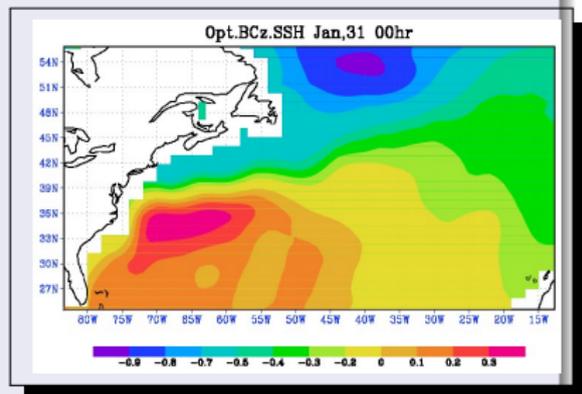
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SSH, North Atlantic, January, 31, 2006.



Optimal Initial Conditions.



Optimal BC at the bottom.

Boundary Conditions influence is **important**

- Optimal BCs allows **to correct errors** committed by the discretization
- The model **is closer** to the reference one with optimal BC
- Data assimilation allows to get an **optimal position and form** of the boundary

BUT

As well as for any adjoint parameter estimation

- The control may violate the model physics;
- The **physical meaning** of the optimal boundary is difficult to understand;
- The set of α is **not unique**;
- The problem of **identifiability** is not addressed yet;
- The problem of **stability** is not even posed.

Consequently:

It is not a parameter estimation study, but

- a way to **compensate model errors**
- showing the **most influent parameter**.

Another result of the Russian-French cooperation



2001



2015