Optimizing Lateral Boundary Conditions at Staircase-shaped Coastlines: Variational Approach
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Christine & Eugene Kazantsev

University Grenoble Alpes & INRIA FRANCE

International conference
Институт Вычислительной Математики им. Г.И. Марчука
Moscow, June 2015.
Development

1960-1985
- Perturbation theory
- Optimal control methods
- Variational methods
- Adjoint Equations

1980-2010
- Variational Data Assimilation Theory & Methods
- Variational Data Assimilation Atmosphere & Ocean
- Optimality System Sensitivity

2000-...
- Second Order Adjoint
- Model Error Control
- Optimization of Model Parameters

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Optimizing Lateral Boundary Conditions
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Optimizing Lateral Boundary Conditions
When the grid is not aligned with the model boundary, boundary conditions become difficult to prescribe.
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- Finite elements? Expensive...
Boundary conditions at rigid boundaries

Staircase-shaped boundary

When the grid is not aligned with the model boundary, boundary conditions become difficult to prescribe.

- Finite elements? Expensive...
- Better interpolation in frames of FD? May be instable...
Staircase-shaped boundary

When the grid is not aligned with the model boundary, boundary conditions become difficult to prescribe.

- Finite elements? Expensive...
- Better interpolation in frames of FD? May be instable...
- Variational methods? We can try...
Discretized derivatives

Particular discretisation for derivatives near the boundary

Boundary conditions are introduced into the model by a particular discretization of operators near the boundary. To avoid instabilities, we control both BC and their approximation.

Integer nodes

\[ u_0, u_{1/2}, u_1, u_{3/2}, u_{5/2}, u_2, u_3 \]

If \( u_0 = 0 \) → \( \frac{\partial u}{\partial x}\bigg|_{1/2} = \frac{u_1}{h} \)

Half-integer nodes

\[ u_0', u_{1/2}', u_1', u_{3/2}', u_{5/2}', u_2', u_3' \]

If \( u_0 = 0 \) → \( \frac{\partial u}{\partial x}\bigg|_{0} = \frac{2u_{1/2}}{h} \)
Discretized derivatives

Particular discretisation for derivatives near the boundary

Boundary conditions are introduced into the model by a particular discretization of operators near the boundary. To avoid instabilities, we control both BC and their approximation.

Integer nodes

\[ \begin{align*}
  &u_0, \ u_1, \ u_2, \ u_3, \ u_4, \ u_5 \\
  &\text{if } u_0 = 0 \rightarrow \left. \frac{\partial u}{\partial x} \right|_{1/2} = \frac{u_1}{h} \\
  &\left. \frac{\partial u}{\partial x} \right|_{1/2} = \frac{\alpha_1 u_0 + \alpha_2 u_1}{h}
\end{align*} \]

Half-integer nodes

\[ \begin{align*}
  &u_0', \ u_1', \ u_2', \ u_3', \ u_4', \ u_5' \\
  &\text{if } u_0 = 0 \rightarrow \left. \frac{\partial u}{\partial x} \right|_{0} = \frac{2u_{1/2}}{h} \\
  &\left. \frac{\partial u}{\partial x} \right|_{0} = \frac{\alpha_1 u_{1/2} + \alpha_2 u_{3/2}}{h}
\end{align*} \]

where \( \alpha_1 \) and \( \alpha_2 \) are the control coefficients.

Derivatives are allowed to change their properties near the boundaries in order to find the best fit with requirements of the model and data.
Nucleus for European Modelling of the Ocean (NEMO): Rectangular Box Configuration

30° × 20° rectangle with \( \frac{1°}{4} \) resolution and 5 \( z \) levels.

120 × 80 × 5 nodes in \((x, y, z)\) coordinates, 64 time steps per day.

\[
\begin{align*}
\frac{\partial u}{\partial t} &= v(\omega + f) - \frac{\partial (u^2 + v^2)/2}{\partial x} - w \frac{\partial u}{\partial z} - \frac{\partial A^h_u \xi}{\partial x} + \frac{\partial A^h_u \omega}{\partial y} + \\
&+ \frac{\partial}{\partial z} A^z_u \frac{\partial u}{\partial z} + g \int_0^z \frac{\partial \rho(x, y, \zeta)}{\partial x} d\zeta + g \frac{\partial (\eta + T_c \phi)}{\partial x} \\
\xi &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad \text{Divergence, Vorticity} \\
w &= \int_H^z \xi(x, y, \zeta) d\zeta; \quad w(x, y, H) = 0 \quad \text{Vertical velocity} \\
A^z &= 1.2 \times 10^{-4} \frac{m^2}{s}; \quad A^h = 200 \frac{m^2}{s}, \quad f = \frac{4\pi}{86400} \sin(\text{lat}) \\
\left. \frac{\partial u}{\partial z} \right|_{w_0} &= -0.1 \frac{N}{m^2} \cos \left( B\pi \star \frac{\text{lat} - 24^°}{44^° - 24^°} \right) \frac{h z_1 \rho_0}{B = 2, \quad \text{Double gyre}} \\
&= B = 1, \quad \text{Single gyre} \\
(u_\perp, \omega)_{\text{Lateral Boundary}} &= 0 \quad \text{(Impermeability and Free-Slip conditions)}
\end{align*}
\]
Single Gyre Forcing, Impermeability and Free-Slip conditions

Aligned grid and rotated grid

SSH on the aligned grid

SSH on the 45° rotated grid.
Nucleus for European Modelling of the Ocean (NEMO): Rectangular Box Configuration

30° × 20° rectangle with \( \frac{1}{4} \)° resolution and 5 z levels.

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\frac{\partial u}{\partial t} = v(\omega + f) - \frac{\partial(u^2 + v^2)/2}{\partial x} - w \frac{\partial u}{\partial z} - \frac{\partial A^h u \xi}{\partial x} + \frac{\partial A^h \omega}{\partial y} + \\
\frac{\partial}{\partial z} A^z u \frac{\partial u}{\partial z} + g \int_0^z \frac{\partial \rho(x, y, \zeta)}{\partial x} d\zeta + g \frac{\partial (\eta + T_c \phi)}{\partial x} \\
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\frac{\partial u}{\partial z} \bigg|_{w_0} = -0.1 \frac{N}{m^2} \cos \left( B \pi * \frac{lat - 24^\circ}{44^\circ - 24^\circ} \right) \frac{hz_1 \rho_0}{h} \\
\text{Double gyre} \quad B = 2, \quad \text{Single gyre} \quad B = 1 \\
(u_\perp, \omega) \text{Lateral Boundary} = 0 \quad \text{(Impermeability and Free-Slip conditions)}
Control coefficients

\[ \frac{\partial v}{\partial x} \bigg|_{\omega_b} = \frac{\alpha_1 v_{1/2} + \alpha_2 v_{3/2}}{h} \]

\[ \frac{\partial u}{\partial y} \bigg|_{\omega_b} = \frac{\alpha_1 u_{1/2} + \alpha_2 u_{3/2}}{h} \]

\[ \frac{\partial v}{\partial x} \bigg|_{\omega_b} = \frac{\alpha_1 v_{-1/2} + \alpha_2 v_{1/2}}{h} \]

\[ \frac{\partial u}{\partial y} \bigg|_{\omega_b} = \frac{\alpha_1 u_{-1/2} + \alpha_2 u_{1/2}}{h} \]

\[ \frac{\partial \omega}{\partial x} \bigg|_{v} = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h} \]

\[ \frac{\partial \omega}{\partial y} \bigg|_{u} = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h} \]

\[ \alpha_1 = \alpha_2 = 0 \]

\[ \alpha_1 = -1, \quad \alpha_2 = 1 \]

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Рис.: 45° rotated grid
The model: $x(t) = \mathcal{M}_{0,t}(x(0), \alpha)$ with $x = (u, v, T, S, ssh)^T$

**Cost function $J$**

$$J = 10^{-4} (\|x(0) - x_{bgr}\|^2 + \|\alpha - \alpha_{bgr}\|^2) +$$
$$+ \int_{t=0}^{T} \iint (u - u_{ref})^2 + (v - v_{ref})^2 + (ssh - ssh_{ref})^2 dxdydt$$

**Layout:**

- **Joint control** of the initial point $x(0)$ (interpolation errors) and the set of $\alpha$;
- Artificially generated data by the same model on the aligned grid;
- Data Assimilation over the 50 days window;
- Analysis of the solution on the 8 years interval.

- Minimization is performed by M1QN3 (JC Gilbert, C.Lemarechal);
- Adjoint is generated by Tapenade (Ecuador team, INRIA).
Reference, Optimal and Conventional BC 800 days later

Reference SSH

Rotated grid conventional BC SSH

Rotated grid Optimal BC SSH
Reference, Rotated, Optimal SSH 10 years average.

Aligned Grid

Conventional BC, Rotated (30°) Grid

Optimized BC, Rotated (30°) Grid
Optimal Boundary Conditions: 45° rotation

Single gyre, $\alpha_1, \alpha_2$

Derivatives in $x$

Derivatives in $y$
Control coefficients: $45^\circ$ rotation

$$
\frac{\partial v}{\partial x} = \frac{\alpha_1 v_{-1/2} + \alpha_2 v_{1/2}}{h}
$$

$$
\frac{\partial u}{\partial y} = \frac{\alpha_1 u_{-1/2} + \alpha_2 u_{1/2}}{h}
$$

$$
\frac{\partial v}{\partial x} = \frac{\alpha_1 v_{1/2} + \alpha_2 v_{3/2}}{h}
$$

$$
\frac{\partial u}{\partial y} = \frac{\alpha_1 u_{1/2} + \alpha_2 u_{3/2}}{h}
$$

$$
\frac{\partial \omega}{\partial x} = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}
$$

$$
\frac{\partial \omega}{\partial y} = \frac{\alpha_1 \omega_0 + \alpha_2 \omega_1}{h}
$$

$$
\alpha_1 = 0, \quad \alpha_2 = -0.1
$$

$$
\alpha_1 = -1.4, \quad \alpha_2 = 0.6
$$

$$
\alpha_1 = -1.01, \quad \alpha_2 = 1.01
$$

$$
\alpha_1 = -1.01, \quad \alpha_2 = 1.01
$$

$$
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 0.8 \frac{u + v}{h}
$$

Рис.: $45^\circ$ rotated grid
"Optimal" configuration

\[ \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u + v}{h} \]

\[ = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V} \cdot \vec{\tau}}{h/\sqrt{2}} \]

Рис.: 45° rotated grid
"Optimal" configuration

\[ \omega_0 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u + v}{h} \]
\[ = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V}.\vec{\tau}}{h/\sqrt{2}} \]

- Free-slip condition on a **curvilinear boundary** \( \omega|_{bnd} = \frac{\vec{V}.\vec{\tau}}{R} \)
- Optimal boundary is curvilinear with \( R = -h/\sqrt{2} \)

Рис.: 45° rotated grid
"Optimal" configuration

\[ \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u + v}{h} \]
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Variable \( R_{30^\circ} : -h \leq R_{30^\circ} \leq 5h \).
"Optimal" configuration

\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{u + v}{h} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\vec{V} \cdot \vec{\tau}}{h/\sqrt{2}}
\]

- Free-slip condition on a curvilinear boundary \( \omega|_{bnd} = \frac{\vec{V} \cdot \vec{\tau}}{R} \)
- Optimal boundary is curvilinear with \( R = -h/\sqrt{2} \)

Variable \( R_{30}\circ \): \(-h \leq R_{30}\circ \leq 5h\).

The Radius depends on the grid resolution \( h \).
Does this curvilinear boundary remains optimal on different resolutions?
Single Gyre Control: 45° rotation, 1°/2 resolution

Reference, Optimal and Conventional BC 800 days later

Reference SSH

Rotated grid conventional BC SSH

Rotated grid constant $R = -h/\sqrt{2}$ BC SSH
Single Gyre Control: 45° rotation, 1°/8 resolution

Reference, Optimal and Conventional BC 800 days later

Reference SSH

Rotated grid conventional BC SSH

Rotated grid, constant $R = -h/\sqrt{2}$ BC SSH
ORCA-2 Model

Layout:

- NEMO, Global ocean model, 2° resolution, 31 layer;
- ECMWF data issued from Jason-1 and Envisat altimetric missions and ENACT/ENSEMBLES data bank;
- Data Assimilation during 10 days interval;
- Analysis of the distance “model–observations" on 1 month interval

Distance “model–observations"

![Graph showing the evolution of the distance between model observations over time.](image)
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Optimal Initial Conditions.

Optimal BC at the bottom.
**Boundary Conditions influence is important**

- Optimal BCs allows **to correct errors** committed by the discretization
- The model is closer to the reference one with optimal BC
- Data assimilation allows to get an **optimal position and form** of the boundary

**BUT**

As well as for any adjoint parameter estimation

- The control may violate the model physics;
- The **physical meaning** of the optimal boundary is difficult to understand;
- The set of $\alpha$ is not unique;
- The problem of **identifiability** is not addressed yet;
- The problem of **stability** is not even posed.

**Consequently:**

It is not a parameter estimation study, but

- a way to **compensate model errors**
- showing the **most influent parameter**.
Another result of the Russian-French cooperation