

Modelling and numerical simulation of bi-temperature Euler Equations in toroidal geometry

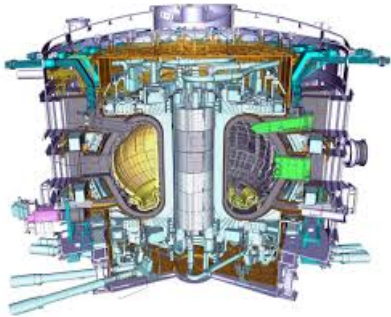
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Context of this work



src : iter.org

Hot plasma where ions and electrons may behave in independent way.

Ion and electron temperatures can be very different.

Complete model of plasma:
MHD equations.

MHD: MagnetoHydroDynamics.

Goal of this work

- Numerical method to approximate a two temperature model:
 - Two temperatures model: Bi-temperature Euler equations.
 - Numerical methods: Relaxation scheme.

- Applied to study flows in tokamaks:
 - Spatial discretization: cylindrical coordinates.
 - Adapt the finite volume scheme to this coordinate system.

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Non-conservative and conservative systems

- Non-conservative system, $E_\alpha = \varepsilon_\alpha + \frac{1}{2}\mathbf{V}^2$, $p_\alpha = p_\alpha(\rho_\alpha, \varepsilon_\alpha)$, and $c_\alpha = \rho_\alpha/\rho$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{V}) & = 0, \\ \partial_t (\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{V}) + \nabla (p_e + p_i) & = 0, \\ \partial_t (\rho_e E_e) + \nabla \cdot ((\rho_e E_e + p_e) \mathbf{V}) - u(c_i \nabla p_e - c_e \nabla p_i) & = \nu_{ei}(T_i - T_e), \\ \partial_t (\rho_i E_i) + \nabla \cdot ((\rho_i E_i + p_i) \mathbf{V}) + u(c_i \nabla p_e - c_e \nabla p_i) & = -\nu_{ei}(T_i - T_e). \end{cases}$$

- Conservative system, $S_e = p_e \rho_e^{-\gamma_e}$ et $\rho E = \rho_e E_e + \rho_i E_i$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{V}) & = 0, \\ \partial_t (\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{V}) + \nabla (p_e + p_i) & = 0, \\ \partial_t (\rho E) + \nabla \cdot ((\rho E + p_e + p_i) \mathbf{V}) & = 0, \\ \partial_t (\rho_e S_e) + \nabla \cdot (\rho_e S_e \mathbf{V}) & = (\gamma_e - 1) \rho_e^{1-\gamma_e} \nu_{ei}(T_i - T_e). \end{cases}$$

F. Coquel, and C. Marmignon, *Numerical methods for weakly ionized gas*,
 Astrophysics and Space Science, 1998.

Transformation of the system

- Rotationally invariant system.
- Can write the system in any direction \mathbf{n} , $u = \mathbf{n} \cdot \mathbf{V}$, $v = \mathbf{t} \cdot \mathbf{V}$:

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_n(\rho u) \\ \partial_t(\rho u) + \partial_n(\rho u^2 + p_e + p_i) \\ \partial_t(\rho v) + \partial_n(\rho uv) \\ \partial_t(\rho E) + \partial_n((\rho E + p_e + p_i)u) \\ \partial_t(\rho_e S_e) + \partial_n(\rho_e S_e u) \end{array} \right. \begin{array}{l} = 0, \\ = 0, \\ = 0, \\ = 0, \\ = (\gamma_e - 1)\rho_e^{1-\gamma_e} \nu_{ei}(T_i - T_e). \end{array}$$

- 1D numerical scheme.

Principle

- $\partial_t U + \partial_n F(U, \mathbf{n}) = S(U)$
- Relaxation:

$$\partial_t \begin{bmatrix} U \\ V \end{bmatrix} + \partial_n \begin{bmatrix} G_1(U, V) \\ G_2(U, V) \end{bmatrix} = \begin{bmatrix} S(U) \\ \frac{1}{\tau} Q(U, V) \end{bmatrix}.$$

If $\tau \rightarrow 0$ then $Q(U, V) \rightarrow 0$ and $G_1(U, V) \rightarrow F(U)$

- Numerically:
 - 1 Transport.
 - 2 Projection.

Relaxed system

- Relaxation variables: π_e and π_j .
- Relaxed system:

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_n(\rho u) \\ \partial_t(\rho u) + \partial_n(\rho u^2 + \pi_e + \pi_i) \\ \partial_t(\rho v) + \partial_n(\rho uv) \\ \partial_t(\rho E) + \partial_n((\rho E + \pi_e + \pi_i)u) \\ \partial_t(\rho_e S_e) + \partial_n(\rho_e S_e u) \\ \partial_t \pi_\alpha + u \partial_n \pi_\alpha + \frac{a^2 \rho_\alpha}{\rho^2} \partial_n u \end{array} \right. = \begin{array}{l} 0, \\ 0, \\ 0, \\ 0, \\ (\gamma_e - 1) \rho_e^{1-\gamma_e} \nu_{ei} (T_i - T_e), \\ \frac{1}{\tau} (\rho_\alpha - \pi_\alpha). \end{array}$$

- Stability condition on impedance: $a^2 \geq \rho^2 \max(\gamma_e \frac{\rho_e}{\rho_e}, \gamma_i \frac{\rho_i}{\rho_i})$.

D. Aregba, J. Breil, S. Brull, B. Dubroca, E. Estibals, in preparation.

Relaxation scheme

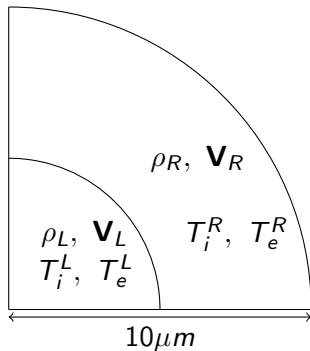
- Transport step: Godunov scheme

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_n(\rho u) \\ \partial_t(\rho u) + \partial_n(\rho u^2 + \pi_e + \pi_i) \\ \partial_t(\rho v) + \partial_n(\rho uv) \\ \partial_t(\rho E) + \partial_n((\rho E + \pi_e + \pi_i)u) \\ \partial_t(\rho_e S_e) + \partial_n(\rho_e S_e u) \\ \partial_t(\rho \pi_\alpha) + \partial_n(\rho \pi_\alpha u + a^2 \frac{\rho_\alpha}{\rho} u) \end{array} \right. = 0,$$

- Projection step:

$$\left\{ \begin{array}{l} \frac{d\rho}{dt} = \frac{du}{dt} = \frac{dv}{dt} = 0, \\ \pi_\alpha = p_\alpha, \quad \alpha = e, i, \\ C_{v,e} \frac{dT_e}{dt} = \nu_{ei}(T_i - T_e), \\ C_{v,i} \frac{dT_i}{dt} = -\nu_{ei}(T_i - T_e), \end{array} \right.$$

Implosion, Initial data

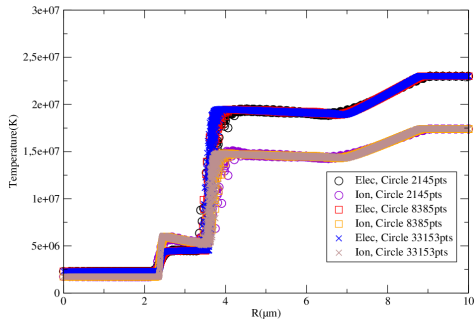
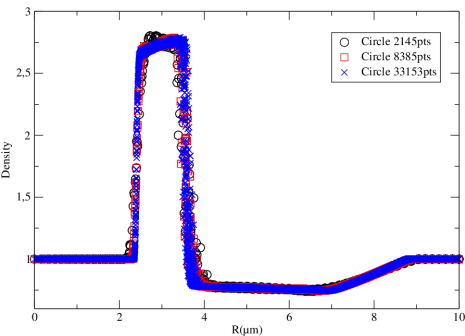


$$\begin{aligned} \rho_L &= \rho_R &= 1, \\ \mathbf{V}_L &= \mathbf{V}_R &= \mathbf{0}, \end{aligned}$$

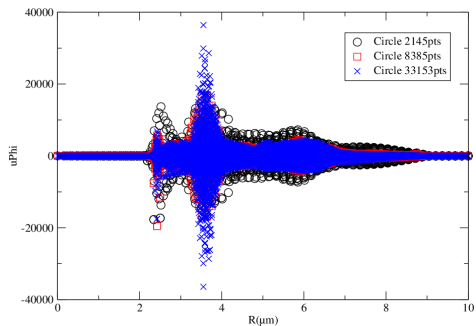
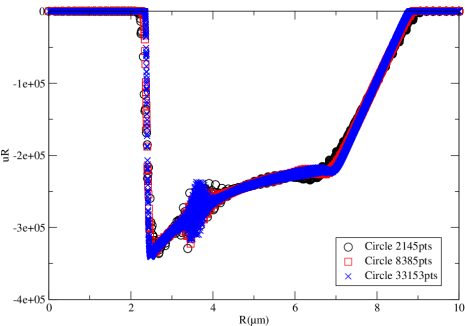
$$\begin{aligned} T_i^L &= 150\text{eV} &= 1.7406 \cdot 10^6 \text{K}, \\ T_i^R &= 1500\text{eV} &= 1.7406 \cdot 10^7 \text{K}, \end{aligned}$$

$$\begin{aligned} T_e^L &= 198,2\text{eV} &= 2,3 \cdot 10^6 \text{K}, \\ T_e^R &= 1982\text{eV} &= 2,3 \cdot 10^7 \text{K}. \end{aligned}$$

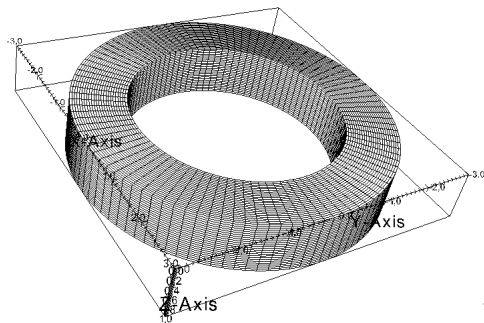
Density and Temperatures at $t = 5\mu s$



Velocity at $t = 5ps$



Torus model



Coordinate system: cylindrical.
Input: 2D mesh.
Simulation: 2D axisymmetric and 3D.
2D mesh: plan (R, Z) *i.e.* poloidal plan.
3rd direction: ϕ , toroidal direction.

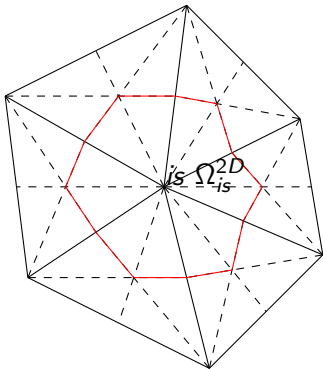
Numerical tools

- Time step:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|\Omega_i|} \sum_{j \in \mathcal{V}(i)} (F(U_{ij}^*, \mathbf{n}_{ij}) \mathbf{n}_{ij} l_{ij}).$$

- Numerical approach: Vertex Centered.
- Numerical tools:
 - 1 $|\Omega_i|$: volume of control cell.
 - 2 \mathbf{n}_{ij} : unit normal. 2 cases:
 - Normal between 2 points of the same poloidal plan.
 - Normal between 2 points which are not inside the poloidal plan
 - 3 l_{ij} : length of segments.

Computing of $|\Omega_{is}|$

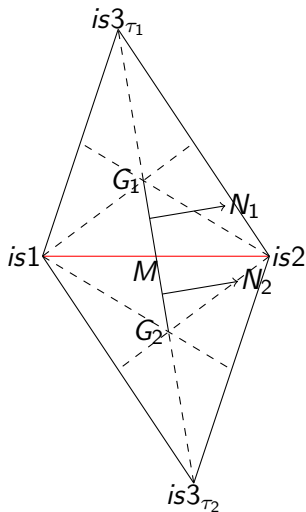


$$\Delta\phi_{iplan} = \phi_{iplan+1/2} - \phi_{iplan-1/2}.$$

$$\begin{aligned} |\Omega_{is}| &= \int_{\Omega_{is}^{2D}} RdRdZd\phi, \\ &= \int_{\phi_{iplan-1/2}}^{\phi_{iplan+1/2}} d\phi \int_{\Omega_{is}^{2D}} RdRdZ, \\ &= \Delta\phi_{iplan} |\Omega_{is}^{2D}|, \\ &= \Delta\phi_{iplan} \sum_{\tau \in \Omega_{is}^{2D}} R^\tau \frac{|\tau|}{6}. \end{aligned}$$

A. Bonnement, T. Fajraoui, H. Guillard, M. Martin, A. Mouton, B. Nkonga, and A. Sangam, *Finite Volume Method in Curvilinear Coordinates*, ESAIM Proc., Vol. 32, 163-176, 2011.

$l_{ij} \mathbf{n}_{ij}$ in poloidal plane



$$\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 = \begin{bmatrix} n_R \\ n_Z \end{bmatrix}_{(R,Z)},$$

$$= \begin{bmatrix} n_R \cos(\phi) \\ n_R \sin(\phi) \\ n_Z \end{bmatrix}_{(x,y,z)},$$

$$l_{ij} \mathbf{n}_{ij} = \int_{\partial \Omega_{is}} \mathbf{N} R dR dZ,$$

$$= \|n\| \begin{bmatrix} 2n_R \sin\left(\frac{\Delta\phi_{iplan}}{2}\right) \\ n_Z \Delta\phi_{iplan} \end{bmatrix}_{(R,Z)}.$$

l_{ij} and \mathbf{n}_{ij} in toroidal direction

- \mathbf{n}_{ij} : easy to compute, the points are not in the same poloidal plan,

$$\mathbf{n}_{ij} = \mathbf{e}_{\phi_{i\text{plan}+1/2}}.$$

- l_{ij} computing:

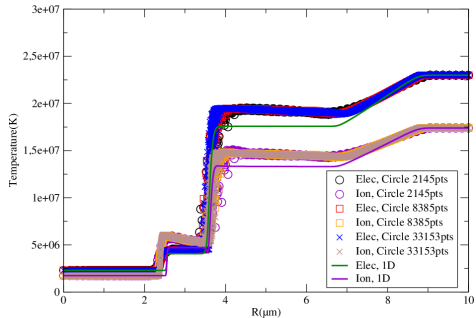
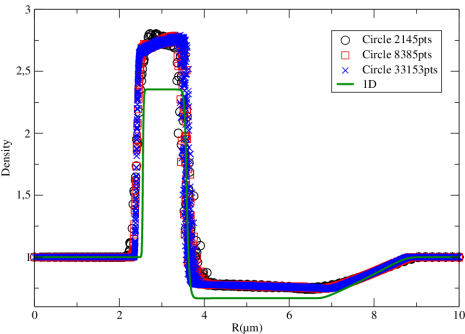
$$l_{ij} = \int_{\partial\Omega_{is}^{\pm}} R dR dZ = \sum_{\tau \in \Omega_{is}^{2D}} R^{\tau} \frac{|\tau|}{6}.$$

Summary and perspectives

- Summary:
 - Bi-temperature model tested in 2D geometry.
 - Spatial discretization of a torus.
- Perspective:
 - Need a numeric test with the toroidal geometry.
 - Extension to a bi-temperature MHD model.

Merci de votre attention!

Density and Temperatures at $t = 5\text{ps}$



Velocity at $t = 5ps$

