Focused and Synthetic Nested Sequents (Extended Technical Report)

Kaustuv Chaudhuri, Sonia Marin, Lutz Strassburger

Focused and Synthetic Nested Sequents

Kaustuv Chaudhuri, Sonia Marin, and Lutz Straßburger

Inria & LIX/École polytechnique, France
{kaustuv.chaudhuri,sonia.marin,lutz.strassburger}@inria.fr

January 6, 2016

Abstract. Focusing is a general technique for transforming a sequent proof system into one with a syntactic separation of non-deterministic choices without sacrificing completeness. This not only improves proof search, but also has the representational benefit of distilling sequent proofs into synthetic normal forms. We show how to apply the focusing technique to nested sequent calculi, a generalization of ordinary sequent calculi to tree-like instead of list-like structures. We thus improve the reach of focusing to the most commonly studied modal logics, the logics of the modal S5 cube. Among our key contributions is a focused cut-elimination theorem for focused nested sequents.

This is an extended version of a paper with the same title and authors that will appear in the Proceedings of the 19th International Conference on Foundations of Software Science and Computation Structures (FoSSaCS), Eindhoven, the Netherlands, April 2016.

1 Introduction

The *focusing* technique has its origin in the foundations of logic programming [22, 1] and is now increasingly relevant in *structural proof theory* because it improves proof search procedures [10, 21] and because focused proofs have clearly identifiable and semantically meaningful *synthetic* normal forms [31, 6, 9, 8]. The essential idea of focusing is to identify and coalesce the non-deterministic choices in a proof, so that a proof can be seen as an alternation of *synchronous* and *asynchronous* phases, in turn letting us abstract from the usual unary and binary logical connectives to n-ary *synthetic connectives*. The full theory of focusing was initially developed for the sequent calculus for linear logic [1], but it has since been extended to a wide variety of logics [10, 19, 27] and proof systems [7, 4]. This generality suggests that the ability to transform a proof system into a focused form is a good indication of its syntactic quality, in a manner similar to how admissibility of cut shows that a proof system is syntactically consistent.

It is natural to ask whether the focusing technique works as well for modal logics. Traditionally, modal logics are specified in terms of Hilbert-style axiomatic systems, but such systems are not particularly suitable since axioms reveal none of the structure of logical reasoning. It is well known that certain modal logics, S5 in particular, are not representable in a variant of Gentzen's sequent calculus without sacrificing analyticity. There are two principal ways to overcome this problem. The first is based on *labeled proof systems* that reify the Kripke semantics—the frame conditions—directly as formulas in the sequents [24, 29]. These "semantic formulas" are not subformulas of the end-sequent and they cause the interpretation of sequents to fall outside the class of propositional modal formulas, and for this reason, such calculi are also called *external*.

The second way is to use so-called *internal* calculi, that enrich the sequent structure such that analyticity is preserved and such that every sequent has an interpretation that stays inside the language of the modal formulas. Well-known examples are *hypersequents* [2] and *display calculi* [3]. A more recent development are *nested sequents* [15, 5, 26, 12], which generalize the notion of context from a list-like structure (familiar from Gentzen's sequent calculus) to a tree-like structure. Like ordinary sequents, nested sequents have a straightforward interpretation in the language of the formulas of the logic, and enjoy cut admissibility (with a cut-elimination proof that stays wholly internal to the system) and hence the usual subformula property. Moreover, nested proof systems can be built *modularly* for every modal logic in the S5 cube, in both classical and intuitionistic variants [5, 20].

In this paper, we build a focused variant, with its concomitant benefits, for all modal logics of the classical S5 cube. For hygienic reasons we choose to use a polarized syntax [16] consisting of two classes of positive and negative formulas and a pair of *shift* connectives to move back and forth between the classes. Crucially, we interpret $\Diamond$ as positive and $\Box$ as negative, which differs from the polarity that would be assigned to these connectives if they were interpreted in terms of ? and !, respectively, from linear logic [25]. Our key technical contributions are: (1) a purely *internal* proof of cut-elimination for the focused nested calculus, given in terms of a traditional rewriting procedure to eliminate cuts, and (2) a proof of completeness of the focused system with respect to the non-focused system (and hence to the Kripke semantics) by showing that the focused system admits the rules of the non-focused system. It generalizes similar proofs of cut-elimination and focusing completeness for (non-nested) sequent calculi [10, 19].

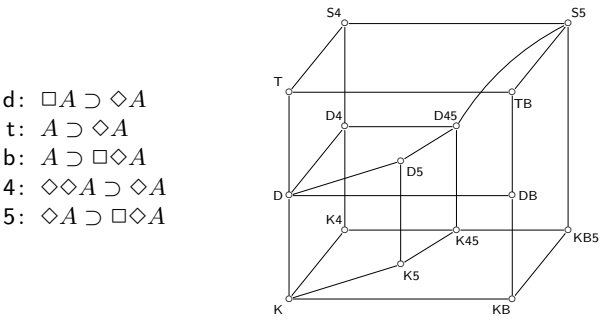

Fig. 1. Left: Some standard modal axioms **Right:** Modal S5 cube

To our knowledge there exist only two other approaches to applying the focusing technique to modal logics. In [23] the authors use a labeled system, using the work in [24, 11] on geometric axioms to obtain systems that extend the basic modal logic K. The cut-elimination and completeness results in [23] are a reduction to LKF, a focused system for first-order predicate logic [19]. The second approach [18] also uses nested sequents, but in a restricted form, in which the tree-structure is reduced to a single branch.

In our approach we use the full power of nested sequents. Our intention is to present the development of the focused system together with its internal cut-elimination proof as a prototype for how similar focused systems may be built for other modal logic formalisms. After giving some preliminaries on modal logics (in Section 2) and nested sequents (in Section 3), we start with a *weakly focused* proof system (in Section 4), where asynchronous rules may be applied everywhere, including in the middle of focused phases. From this system, we extract a *strongly focused* system (also in Section 4) and a *synthetic* system (in Section 5) where the logical content of the phases of focusing are abstracted from the level of formulas to the level of nested sequents. We also sketch the cut-elimination theorem for this synthetic variant. The synthetic design generalizes similar designs for the sequent calculus [6, 31, 14].

2 Preliminaries: Modal Logics and the S5 Cube

Classical modal logic is obtained from classical propositional logic by adding the modal connectives $\Box$ and $\Diamond$. Starting with a countable set of atoms $(a, b, \dots)$, the formulas $(A, B, \dots)$ of modal logic are given by the following grammar:

$$A, B, \ldots ::= a \mid \bar{a} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A \tag{1}$$

To avoid excessive syntax, formulas are kept in negation-normal form, so the only formally negated formulas are the atoms. The negation $\bar{A}$ of an arbitrary formula A is given by the De Morgan laws: $\bar{\bar{A}} = A$, $\overline{A \wedge B} = \bar{A} \vee \bar{B}$ and $\overline{\Box A} = \Diamond \bar{A}$. We also define $A \supset B$ as $\bar{A} \vee B$, $A \equiv B$ as $(A \supset B) \wedge (B \supset A)$, $\top$ as $a \vee \bar{a}$, and $\bot$ as $a \wedge \bar{a}$ (for some atom a).

Modal logics are traditionally specified using Hilbert-style axiom schemata. The basic modal logic K, for instance, is obtained by adding the following k axiom to the ordinary Hilbert axioms for propositional logic.

$$\mathsf{k}\colon \Box(A \supset B) \supset (\Box A \supset \Box B) \tag{2}$$

To obtain the theorems of K, we then also add two inference rules of *modus ponens* and *necessitation*.

$$\mathsf{mp}\ \frac{A \quad A \supset B}{B} \qquad\qquad \mathsf{nec}\ \frac{A}{\Box A} \tag{3}$$

Stronger modal logics can be obtained by adding to K other axioms mentioning the modal connectives. In this paper, we consider the most common five axioms d, t, b, 4 and 5, which are shown on the left in Figure 1. Picking subsets of these axioms lets us define thirty-two modal logics, but only fifteen of them are non-redundant. For example, the sets $\{\mathsf{b}, \mathsf{4}\}$ and $\{\mathsf{t}, \mathsf{5}\}$ both yield the modal logic S5. The fifteen distinct modal logics follow chains of extension from K to S5 and can be arranged as a pair of nested cubes depicted on the right in Figure 1; this is sometimes called the S5 *cube* [13].

3 Preliminaries: The Nested Sequent Calculus KN

First, let us recall the notion of *nested sequents*, first defined by Kashima [15] and then independently rediscovered by Poggiolesi [26] (who called them *tree-hypersequents*) and Brünnler [5]. In Gentzen's one-sided sequent calculus, a sequent is just a multiset of formulas; nested sequents generalize this notion to a multiset of formulas and boxed sequents, resulting in a tree structure.

$$\mathsf{id}\ \frac{}{\Gamma\{a,\bar{a}\}} \quad \vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \quad \wedge\ \frac{\Gamma\{A\}\quad\Gamma\{B\}}{\Gamma\{A\wedge B\}} \quad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \mathsf{k}^{\Diamond}\ \frac{\Gamma\{\Diamond A,[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

$$\mathsf{d}^{\Diamond}\ \frac{\Gamma\{\Diamond A,[A]\}}{\Gamma\{\Diamond A\}} \quad \mathsf{t}^{\Diamond}\ \frac{\Gamma\{\Diamond A,A\}}{\Gamma\{\Diamond A\}} \quad \mathsf{b}^{\Diamond}\ \frac{\Gamma\{[\Delta,\Diamond A],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

$$\mathsf{4}^{\Diamond}\ \frac{\Gamma\{\Diamond A,[\Diamond A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}} \quad \mathsf{5}^{\Diamond}\ \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathrm{dp}(\Gamma\{\ \}\{\emptyset\})\geq 1$$

Fig. 2. Rules for $\mathsf{KN}+\mathsf{X}^{\Diamond}$. The first row constitutes KN.

Definition 3.1 A *nested sequent* $(\Gamma,\Delta,\dots)$ is a finite multiset of formulas and *boxed sequents* of the form $[\Delta]$, where Δ is itself a nested sequent. In other words, nested sequents have the following grammar:

$$\Gamma,\Delta,\dots ::= A_1,\dots,A_m,[\Gamma_1],\dots,[\Gamma_n] \tag{4}$$

Both m and n may be 0, in which case the sequent is *empty*; when we need to be explicit, we will use the notation $\emptyset$ to stand for an empty sequent. As is usual in sequent calculi, we consider the comma to be associative and commutative.

Definition 3.2 (Corresponding Formulas) For any nested sequent Γ, a *corresponding formula*, written $\mathrm{fm}(\Gamma)$, gives an interpretation of Γ as a modal logic formula. Corresponding formulas obey the following equivalences: $\mathrm{fm}(\emptyset)\equiv\bot$, $\mathrm{fm}(A)\equiv A$, $\mathrm{fm}([\Gamma])\equiv\Box\mathrm{fm}(\Gamma)$, and $\mathrm{fm}(\Gamma_1,\Gamma_2)\equiv\mathrm{fm}(\Gamma_1)\vee\mathrm{fm}(\Gamma_2)$.

Definition 3.3 (Context) An *n-holed context* is like a nested sequent but contains n pairwise distinct numbered *holes* of the form $\{\ \}_i$ (for $1\leq i\leq n$) in place of formulas. (No hole can occur inside a formula.) We depict such a context as $\Gamma\{\ \}_1\cdots\{\ \}_n$. Given such a context and n nested sequents $\Delta_1,\dots,\Delta_n$, we write $\Gamma\{\Delta_1\}_1\cdots\{\Delta_n\}_n$ to stand for the nested sequent where the hole $\{\ \}_i$ (for $1\leq i\leq n$) in the context has been replaced by Δ_i, with the understanding that if Δ_i is empty then the hole is simply removed. Unless there is any ambiguity, we will omit the hole index subscripts in this paper to keep the notation light.

Definition 3.4 The *depth* of $\Gamma\{\ \}$, written $\mathrm{dp}(\Gamma\{\ \})$, is given inductively by: $\mathrm{dp}(\{\ \})=0$, $\mathrm{dp}(\Delta,\Gamma\{\ \})=\mathrm{dp}(\Gamma\{\ \})$, and $\mathrm{dp}([\Gamma\{\ \}])=\mathrm{dp}(\Gamma\{\ \})+1$.

Example 3.5 Let $\Gamma\{\ \}\{\ \}=A,[B,\{\ \},[\{\ \}],C]$. For the sequents $\Delta_1=D$ and $\Delta_2=A,[C]$, we get: $\Gamma\{\Delta_1\}\{\Delta_2\}=A,[B,D,[A,[C]],C]$ and $\Gamma\{\emptyset\}\{\Delta_2\}=A,[B,[A,[C]],C]$. We also have that $\mathrm{dp}(\Gamma\{\ \}\{\Delta_1\})=1$ and $\mathrm{dp}(\Gamma\{\Delta_1\}\{\ \})=2$.

The basic modal logic K (as presented in [5]) is captured using nested sequents as the cut-free proof system KN shown in the first row in Figure 2. The deductive system corresponding to each normal extension $\mathsf{K}+\mathsf{X}$, where $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ is a set of modal axioms (Figure 1), can be obtained by adding the corresponding *diamond rules* $\mathsf{X}^{\Diamond}\subseteq\{\mathsf{d}^{\Diamond},\mathsf{t}^{\Diamond},\mathsf{b}^{\Diamond},\mathsf{4}^{\Diamond},\mathsf{5}^{\Diamond}\}$ (final two rows of Figure 2) to KN. The $\mathsf{5}^{\Diamond}$ rule has a side condition that the context in which the principal $\Diamond$-formula occurs has non-zero *depth*, i.e., that it does not occur at the root of the $[\ ]$-tree.

To make the correspondence between extensions of K and the proof systems precise, we need the following additional notion:

Definition 3.6 (45-Closure) We say that $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ is *45-closed*, if:

- whenever $\mathsf{4}$ is derivable in $\mathsf{K}+\mathsf{X}$, $\mathsf{4}\in\mathsf{X}$
- whenever $\mathsf{5}$ is derivable in $\mathsf{K}+\mathsf{X}$, $\mathsf{5}\in\mathsf{X}$.

In this paper, we will always work with 45-closed axiom sets. The reason is that, for example, the axiom $\mathsf{4}$ is not provable (without cut) in $\mathsf{KN}+\{\mathsf{t}^{\Diamond},\mathsf{5}^{\Diamond}\}$ even though $\mathsf{4}$ is a theorem of the logic $\mathsf{K}+\{\mathsf{t},\mathsf{5}\}$ (which is $\mathsf{S5}$). Note that this is not a real restriction, since for every logic in the modal cube (Figure 1) there is a 45-closed set of axioms defining it (see [5] for details). We can now state the soundness, completeness, and cut-admissibility for KN and its extensions.

Theorem 3.7 *Let* cut *be the following rule:*

$$\mathsf{cut}\ \frac{\Gamma\{A\}\quad\Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

Let $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *be 45-closed. For any formula A, the following are equivalent.*

$$\mathsf{sto}\,\frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}} \quad \mathsf{rel}\,\frac{\Gamma\{N\}}{\Gamma\{\langle\downarrow N\rangle\}} \quad \mathsf{dec}\,\frac{\Gamma\{P, \langle P\rangle\}}{\Gamma\{P\}}$$

$$\bar{\wedge}\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \mathbin{\bar{\wedge}} B\}} \quad \bar{\vee}\,\frac{\Gamma\{A, B\}}{\Gamma\{A \mathbin{\bar{\vee}} B\}} \quad \Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}$$

$$\mathsf{id}\,\frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \quad \dot{\wedge}\,\frac{\Gamma\{\langle A\rangle\} \quad \Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \mathbin{\dot{\wedge}} B\rangle\}} \quad \dot{\vee}_1\,\frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle A \mathbin{\dot{\vee}} B\rangle\}} \quad \dot{\vee}_2\,\frac{\Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \mathbin{\dot{\vee}} B\rangle\}} \quad \mathsf{k}_\mathsf{f}^\Diamond\,\frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}}$$

$$\mathsf{d}_\mathsf{f}^\Diamond\,\frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\Diamond A\rangle\}} \quad \mathsf{t}_\mathsf{f}^\Diamond\,\frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}} \quad \mathsf{b}_\mathsf{f}^\Diamond\,\frac{\Gamma\{[\Delta, \langle A\rangle]\}}{\Gamma\{[\Delta, \langle\Diamond A\rangle]\}}$$

$$\mathsf{4}_\mathsf{f}^\Diamond\,\frac{\Gamma\{[\langle\Diamond A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \quad \mathsf{5}_\mathsf{f}^\Diamond\,\frac{\Gamma\{\emptyset\}\{\langle\Diamond A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}\{\emptyset\}}\ \mathrm{dp}(\Gamma\{\ \}\{\emptyset\}) \geq 1$$

Fig. 3. Focused Rules for $\mathsf{KNF} + \mathsf{X}_\mathsf{f}^\Diamond$ and $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond$. The first three rows constitute KNwF and KNF.

1. *A is a theorem of* $\mathsf{K} + \mathsf{X}$.
2. *A is provable in* $\mathsf{KN} + \mathsf{X}^\Diamond + \mathsf{cut}$.
3. *A is provable in* $\mathsf{KN} + \mathsf{X}^\Diamond$. □

The proof that $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 1$ can be found in [5].

4 The Focused Systems KNwF and KNF

The essence of the focusing technique [1] is to classify formulas into *positive* formulas, whose rules are not invertible, and *negative* whose rules are invertible. (As usual, we consider a rule to be invertible if whenever the conclusion of the rule is derivable then so are each of its premises.) Due to invertibility, when searching for a proof it is always safe to apply—reading from conclusion to premises—a rule for a negative formula, so these may be applied at any time. On the other hand, rules for positive formulas may require rules on other formulas to be applied first. For example, the KN sequent $\Box\bar{a}, \Diamond a$ can only be proved by first applying the $\Box$ rule, showing that $\Diamond$-formulas are positive. A focused proof is one where the *decision* to apply a rule to a positive formula has to be explicitly taken, which then commits the proof to continue applying rules to this *focused* positive formula and its immediate positive descendants (and no other formula in the sequent), which drastically reduces the search space. The main theorem of focusing is that this strategy is complete, i.e., every theorem has a focused proof.

We will now build such a focused version of KN. However, to simplify the meta-theorems about this system, we will adopt a *polarized syntax* where the positive and negative formulas are grouped together in different syntactic categories and explicitly mediated by *shift* connectives ($\uparrow$ and $\downarrow$). As already mentioned, $\Diamond$ is in the positive class, and its dual $\Box$ is in the negative class. The rest of the formulas have ambiguous polarities; we arbitrarily assign all atoms to be positive (and their negations to be negative), and split the conjunctions and disjunctions into positive and negative versions. Thus, polarized formulas have this grammar:

$$\begin{array}{ll} \text{positive:} & P, Q, \ldots ::= a \mid P \mathbin{\dot{\wedge}} Q \mid P \mathbin{\dot{\vee}} Q \mid \Diamond P \mid \downarrow N \\ \text{negative:} & N, M, \ldots ::= \bar{a} \mid N \mathbin{\bar{\vee}} M \mid N \mathbin{\bar{\wedge}} M \mid \Box N \mid \uparrow P \end{array} \tag{5}$$

Each column in the grammar above defines a De Morgan dual pair; note that the negation of a positive formula is a negative formula, and vice versa. When the polarity of a formula is not important, we write it as $A, B, \ldots$. A polarized nested sequent is the same as in the non-focused setting, with the difference that all formulas are polarized. Likewise, a polarized context is a polarized nested sequent where some formulas have been replaced by holes. In the rest of this paper, we will drop the adjective "polarized" and treat all constructs implicitly as polarized, unless otherwise indicated.

Definition 4.1 (Neutral) A formula is said to be *neutral* if it is a positive formula or a negated atom. A nested sequent is *neutral* if it is built from multisets of neutral formulas and boxed neutral sequents. A context $\Gamma\{\ \}\cdots\{\ \}$ is *neutral* if $\Gamma\{\emptyset\}\cdots\{\emptyset\}$ is neutral.

Definition 4.2 A *focused sequent* is of the form $\Gamma\{\langle P\rangle\}$ where $\Gamma\{\ \}$ is a context and P is a positive formula. The formula P is called its *focus*. The notion of corresponding formula (Def. 3.2) is extended with $\mathrm{fm}(\langle P\rangle) \equiv \mathrm{fm}(P)$.

The inference rules of the focused system KNwF (w for "weak") are shown in the first three rows in Figure 3 (the basic system for K), while the last two rows contain the focused variants of the diamond rules of Figure 2. Observe that the rules for negative formulas are exactly the same as in KN, while the rules for positive formulas

$$\mathsf{weak}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{weak_f}\ \frac{\Gamma\{\emptyset\}\{\langle P\rangle\}}{\Gamma\{\Delta\}\{\langle P\rangle\}} \qquad \mathsf{cont}\ \frac{\Gamma\{\bar{a},\bar{a}\}}{\Gamma\{\bar{a}\}} \qquad \mathsf{m}^{[]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[\Delta,\Sigma]\}}$$

Fig. 4. Structural rules.

$$\mathsf{d}^{[]}\ \frac{\Gamma\{[\,]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{[\Sigma],\Delta\}} \qquad \mathsf{4}^{[]}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \qquad \mathsf{5}^{[]}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}$$

Fig. 5. Structural modal rules for axioms $\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}$ (where $\mathrm{dp}(\Gamma\{\,\}\{[\Delta]\}) \geq 1$ in $\mathsf{5}^{[]}$)

can only be applied if the principal formula is in focus. Mediating between ordinary and focused sequents are the rules dec ("decide"), that chooses a positive formula in the conclusion and focuses on a copy of it in the premise, and rel ("release") that drops the focus on a shifted formula. Since dec keeps the original positive formula around, there is no need to incorporate contraction in every positive rule, like in KN. The sto ("store") rule removes a shift in front of a positive formula and is used to produce neutral premises from non-neutral conclusions.

We define the system KNF to be a restriction of KNwF where the conclusion of the dec rule is required to be neutral, as are the contexts surrounding the focus in all rules involving focused sequents. Thus, in KNF, the dec rule is only applicable when no other rule is applicable, and hence we sometimes call it *strongly focused*. Then, for a set $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$, we write $\mathsf{X}_\mathsf{f}^\diamond \subseteq \{\mathsf{d}_\mathsf{f}^\diamond,\mathsf{t}_\mathsf{f}^\diamond,\mathsf{b}_\mathsf{f}^\diamond,\mathsf{4}_\mathsf{f}^\diamond,\mathsf{5}_\mathsf{f}^\diamond\}$ for the corresponding subset of the focused diamond rules in the last two rows of Figure 3. We immediately have the following proposition:

Proposition 4.3 *Let $X \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. A formula A is provable in $\mathsf{KNF}+\mathsf{X}_\mathsf{f}^\diamond$ if and only if it is provable in $\mathsf{KNwF}+\mathsf{X}_\mathsf{f}^\diamond$.*

Proof A derivation in $\mathsf{KNF}+\mathsf{X}_\mathsf{f}^\diamond$ is by definition also a derivation in $\mathsf{KNwF}+\mathsf{X}_\mathsf{f}^\diamond$. Conversely, to convert a derivation in $\mathsf{KNwF}+\mathsf{X}_\mathsf{f}^\diamond$ into one in $\mathsf{KNF}+\mathsf{X}_\mathsf{f}^\diamond$, we first have to replace all instances of id with a sequence of applications of $\{\Box,\bar{\wedge},\dot{\wedge},\mathsf{sto}\}$ followed (reading from conclusion upwards) by id, to ensure that the conclusion of the id rule is neutral. Then, the negative rules $\{\bar{\wedge},\bar{\vee},\Box\}$ can be permuted down by straightforward rule permutations to ensure that dec only applies to neutral sequents. □

In order to establish the soundness and completeness of KNwF, we use the obvious forgetful injection of the polarized syntax into the unpolarized syntax.

Definition 4.4 (Depolarization) If A is a polarized formula, then we write $\lfloor A \rfloor$ for the unpolarized formula obtained from A by erasing the shifts $\uparrow$ and $\downarrow$, collapsing $\dot{\wedge}$ and $\bar{\wedge}$ into $\wedge$, and collapsing $\dot{\vee}$ and $\bar{\vee}$ into $\vee$.

Theorem 4.5 (Soundness) *Let $X \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. If a formula A is provable in $\mathsf{KNF}+\mathsf{X}_\mathsf{f}^\diamond$, then $\lfloor A \rfloor$ is provable in $\mathsf{KN}+\mathsf{X}^\diamond$.*

Proof By forgetting the polarity information, every $\mathsf{KNF}+\mathsf{X}_\mathsf{f}^\diamond$ proof of A is transformed into a $\mathsf{KN}+\mathsf{X}^\diamond$ proof of $\lfloor A \rfloor$. □

Completeness is considerably trickier. We use a technique pioneered by Laurent for linear logic [16] and proceed via cut-elimination in KNwF.

4.1 Cut Elimination

In this section we will show that a collection of cuts is admissible for $\mathsf{KNwF}+\mathsf{X}_\mathsf{f}^\diamond$. As usual, a rule is said to be admissible if it is the case that whenever any instance of all its premises are derivable, so is the corresponding instance of the conclusion. In order to show the admissibility of the cut rules, it will be very useful to appeal to a collection of other admissible and invertible rules.

Lemma 4.6 *Let $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. The rules weak, $\mathsf{weak_f}$, cont, and $\mathsf{m}^{[]}$ (shown in Figure 4) are admissible for $\mathsf{KNwF}+\mathsf{X}_\mathsf{f}^\diamond$. Moreover, the rules sto, $\Box$, $\bar{\wedge}$, and $\bar{\vee}$ are invertible for $\mathsf{KNwF}+\mathsf{X}_\mathsf{f}^\diamond$.*

Proof By straightforward induction on the height of the derivation. □

Note that here we use contraction only on negated atoms because that is all that is needed in the cut-elimination proof below. One can indeed show that the general contraction rule on arbitrary sequents (and not just formulas) is admissible, but this requires a complicated argument for focused sequents. The corresponding result for KN is shown in [5].

Lemma 4.7 *Let $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$. If X is 45-closed, then any rule $\mathsf{x}^{[]}$ in $\mathsf{X}^{[]}$ (shown in Figure 5) is admissible for $\mathsf{KNwF}+\mathsf{X}_\mathsf{f}^\diamond$.*

Proof We reason by induction on the height of the derivation above the rule $\mathsf{x}^{[]}$. If $\mathsf{x}^{[]}$ is applied at height 1, we are in the following case:

$$\mathsf{x}^{[]}\ \dfrac{\mathsf{id}\ \dfrac{}{\Gamma'\{\langle a\rangle, \bar{a}\}}}{\Gamma\{\langle a\rangle, \bar{a}\}}$$

which is simple to solve since $\Gamma\{\langle a\rangle, \bar{a}\}$ is an axiom as well.

If $\mathsf{x}^{[]}$ is applied at height ≥ 1 then we carry out a case analysis on the last rule r which appears before $\mathsf{x}^{[]}$ in the derivation and we use the induction hypothesis to conclude. If $\mathsf{r} \in \{\mathsf{cut}, \bar{\wedge}, \bar{\wedge}, \bar{\vee}, \bar{\vee}, \Box, \mathsf{d}^{\diamond}, \mathsf{d}^{[]}, \mathsf{t}^{\diamond}\}$, since only the context is affected by the rule, $\mathsf{x}^{[]}$ easily permutes with the given rule. Otherwise, there are specific cases depending on each rule $\mathsf{x}^{[]}$.

– For $\mathsf{x}^{[]} = \mathsf{t}^{[]}$
  - $\mathsf{r} = \mathsf{k}^{\diamond}_{\mathsf{f}}$:

$$\begin{array}{l}\mathsf{k}^{\diamond}_{\mathsf{f}}\\ \mathsf{t}^{[]}\end{array}\ \dfrac{\dfrac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, [\Delta]\}}}{\Gamma\{\langle\diamond A\rangle, \Delta\}} \quad\rightsquigarrow\quad \begin{array}{l}\mathsf{t}^{[]}\\ \mathsf{t}^{\diamond}_{\mathsf{f}}\end{array}\ \dfrac{\dfrac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle A\rangle, \Delta\}}}{\Gamma\{\langle\diamond A\rangle, \Delta\}}$$

  - $\mathsf{r} = \mathsf{b}^{\diamond}_{\mathsf{f}}$: similar to the case for $\mathsf{k}^{\diamond}_{\mathsf{f}}$
  - $\mathsf{r} = 4^{\diamond}_{\mathsf{f}}$:

$$\begin{array}{l}4^{\diamond}_{\mathsf{f}}\\ \mathsf{t}^{[]}\end{array}\ \dfrac{\dfrac{\Gamma\{[\langle\diamond A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, [\Delta]\}}}{\Gamma\{\langle\diamond A\rangle, \Delta\}} \quad\rightsquigarrow\quad \mathsf{t}^{[]}\ \dfrac{\Gamma\{[\langle\diamond A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, \Delta\}}$$

  - $\mathsf{r} = 5^{\diamond}_{\mathsf{f}}$: the only nontrivial cases appear also when $\mathrm{dp}(\Gamma\{\ \}\{\emptyset\}) = 1$ in the conclusion, i.e. either $\Gamma\{\diamond A\}\{\emptyset\} = [\diamond A, \Delta], \Sigma\{\emptyset\}$ or $\Gamma\{\diamond A\}\{\emptyset\} = [\diamond A, \Delta\{\emptyset\}], \Sigma$. We show the first case, using the fact that $4^{\diamond}$ is in $\mathsf{X}^{\diamond}$ when $\{\mathsf{t}, 5\} \subseteq \mathsf{X}$ by 45-closure. The second case is similar.

$$\begin{array}{l}5^{\diamond}_{\mathsf{f}}\\ \mathsf{t}^{[]}\end{array}\ \dfrac{[\Delta], \Sigma\{\langle\diamond A\rangle\}}{[\Delta, \langle\diamond A\rangle], \Sigma\{\emptyset\}}}{\Delta, \langle\diamond A\rangle, \Sigma\{\emptyset\}} \quad\rightsquigarrow\quad \begin{array}{c}\mathsf{t}^{[]}\ \dfrac{[\Delta], \Sigma\{\langle\diamond A\rangle\}}{\Delta, \Sigma\{\langle\diamond A\rangle\}}\\ 4^{\diamond}_{\mathsf{f}}\ \Big\|\\ \Delta, \langle\diamond A\rangle, \Sigma\{\emptyset\}\end{array}$$

– For $\mathsf{x}^{[]} = \mathsf{b}^{[]}$:
  - $\mathsf{r} = \mathsf{k}^{\diamond}_{\mathsf{f}}$:

$$\begin{array}{l}\mathsf{k}^{\diamond}_{\mathsf{f}}\\ \mathsf{b}^{[]}\end{array}\ \dfrac{\dfrac{\Gamma\{[\Sigma, [\langle A\rangle, \Delta]]\}}{\Gamma\{[\Sigma, \langle\diamond A\rangle, [\Delta]]\}}}{\Gamma\{[\Sigma, \langle\diamond A\rangle], \Delta\}} \quad\rightsquigarrow\quad \begin{array}{l}\mathsf{b}^{[]}\\ \mathsf{b}^{\diamond}_{\mathsf{f}}\end{array}\ \dfrac{\dfrac{\Gamma\{[\Sigma, [\langle A\rangle, \Delta]]\}}{\Gamma\{[\Sigma], \langle A\rangle, \Delta\}}}{\Gamma\{[\Sigma, \langle\diamond A\rangle], \Delta\}}$$

  - $\mathsf{r} = \mathsf{b}^{\diamond}_{\mathsf{f}}$:

$$\begin{array}{l}\mathsf{b}^{\diamond}_{\mathsf{f}}\\ \mathsf{b}^{[]}\end{array}\ \dfrac{\dfrac{\Gamma\{[\Sigma, \langle A\rangle, [\Delta]]\}}{\Gamma\{[\Sigma, [\langle\diamond A\rangle, \Delta]]\}}}{\Gamma\{[\Sigma], \langle\diamond A\rangle, \Delta\}} \quad\rightsquigarrow\quad \begin{array}{l}\mathsf{b}^{[]}\\ \mathsf{k}^{\diamond}_{\mathsf{f}}\end{array}\ \dfrac{\dfrac{\Gamma\{[\Sigma, \langle A\rangle, [\Delta]]\}}{\Gamma\{\Delta, [\Sigma, \langle A\rangle]\}}}{\Gamma\{[\Sigma], \langle\diamond A\rangle, \Delta\}}$$

  - $\mathsf{r} = 4^{\diamond}_{\mathsf{f}}$: we use the fact that $5^{\diamond}_{\mathsf{f}}$ is in $\mathsf{X}^{\diamond}_{\mathsf{f}}$ when $\{\mathsf{b}, 4\} \subseteq \mathsf{X}$, by 45-closure of X.

$$\begin{array}{l}4^{\diamond}_{\mathsf{f}}\\ \mathsf{b}^{[]}\end{array}\ \dfrac{\dfrac{\Gamma\{[\Sigma, [\langle\diamond A\rangle, \Delta]]\}}{\Gamma\{[\Sigma, \langle\diamond A\rangle, [\Delta]]\}}}{\Gamma\{[\Sigma, \langle\diamond A\rangle], \Delta\}} \quad\rightsquigarrow\quad \begin{array}{l}\mathsf{b}^{[]}\\ 5^{\diamond}_{\mathsf{f}}\end{array}\ \dfrac{\dfrac{\Gamma\{[\Sigma, [\diamond A, \Delta]]\}}{\Gamma\{[\Sigma], \langle\diamond A\rangle, \Delta\}}}{\Gamma\{[\Sigma, \langle\diamond A\rangle], \Delta\}}$$

  - $\mathsf{r} = 5^{\diamond}_{\mathsf{f}}$: similarly, the nontrivial case appears when $\mathrm{dp}(\Gamma\{\ \}\{\emptyset\}) = 2$ in the conclusion and we use the fact that $4^{\diamond}_{\mathsf{f}}$ is in $\mathsf{X}^{\diamond}_{\mathsf{f}}$ when $\{\mathsf{b}, 5\} \subseteq \mathsf{X}$.

$$\begin{array}{l}5^{\diamond}_{\mathsf{f}}\\ \mathsf{b}^{[]}\end{array}\ \dfrac{\dfrac{[\Sigma, [\Delta]], \Gamma'\{\langle\diamond A\rangle\}}{[\Sigma, [\langle\diamond A\rangle, \Delta]], \Gamma'\{\emptyset\}}}{\langle\diamond A\rangle, \Delta, [\Sigma], \Gamma'\{\emptyset\}} \quad\rightsquigarrow\quad \begin{array}{c}\mathsf{b}^{[]}\ \dfrac{[\Sigma, [\Delta]], \Gamma'\{\langle\diamond A\rangle\}}{[\Sigma], \Delta, \Gamma'\{\diamond A\}}\\ \Big\|\ 4^{\diamond}_{\mathsf{f}}\\ [\Sigma], \Delta, \langle\diamond A\rangle, \Gamma'\{\emptyset\}\end{array}$$

$$\mathsf{cut}_1\ \dfrac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}} \qquad \mathsf{cut}_2\ \dfrac{\Gamma\{\langle P\rangle\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}} \qquad \mathsf{cut}_3\ \dfrac{\Gamma\{\langle Q\rangle\}\{P\} \quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}$$

Fig. 6. The various cuts in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond$

– For $\mathsf{x}^{[]} = \mathsf{4}^{[]}$:
  - $\mathsf{r} = \mathsf{k}_\mathsf{f}^\Diamond$:

$$\begin{array}{l}\mathsf{k}_\mathsf{f}^\Diamond\\ \mathsf{4}^{[]}\end{array}\dfrac{\dfrac{\Gamma\{[\langle A\rangle, \Delta], [\Sigma]\}}{\Gamma\{\langle \Diamond A\rangle, [\Delta], [\Sigma]\}}}{\Gamma\{\langle \Diamond A\rangle, [[\Delta], \Sigma]\}} \quad \rightsquigarrow \quad \begin{array}{l}\mathsf{4}^{[]}\\ \mathsf{k}_\mathsf{f}^\Diamond\\ \mathsf{4}_\mathsf{f}^\Diamond\end{array}\dfrac{\dfrac{\Gamma\{[\langle A\rangle, \Delta], [\Sigma]\}}{\Gamma\{[[\langle A\rangle, \Delta], \Sigma]\}}}{\Gamma\{[\langle \Diamond A\rangle, [\Delta], \Sigma]\}}}{\Gamma\{\langle \Diamond A\rangle, [[\Delta], \Sigma]\}}$$

  - $\mathsf{r} = \mathsf{b}_\mathsf{f}^\Diamond$: we need to use the fact that $\mathsf{5}_\mathsf{f}^\Diamond$ is in $\mathsf{X}_\mathsf{f}^\Diamond$ when $\{\mathsf{b}, \mathsf{4}\} \subseteq \mathsf{X}$ by 45-closure.

$$\begin{array}{l}\mathsf{b}_\mathsf{f}^\Diamond\\ \mathsf{4}^{[]}\end{array}\dfrac{\dfrac{\Gamma\{\langle A\rangle, [\Delta], [\Sigma]\}}{\Gamma\{[\Delta, \langle \Diamond A\rangle], [\Sigma]\}}}{\Gamma\{[[\Delta, \langle \Diamond A\rangle], \Sigma]\}} \quad \rightsquigarrow \quad \begin{array}{l}\mathsf{4}^{[]}\\ \mathsf{b}_\mathsf{f}^\Diamond\\ \mathsf{5}_\mathsf{f}^\Diamond\end{array}\dfrac{\dfrac{\Gamma\{\langle A\rangle, [\Delta], [\Sigma]\}}{\Gamma\{\langle A\rangle, [[\Delta], \Sigma]\}}}{\Gamma\{[\langle \Diamond A\rangle, [\Delta], \Sigma]\}}}{\Gamma\{[[[\Delta, \langle \Diamond A\rangle]], \Sigma]\}}$$

  - $\mathsf{r} = \mathsf{4}_\mathsf{f}^\Diamond$: similar to the case for the $\mathsf{k}_\mathsf{f}^\Diamond$ rule.
  - $\mathsf{r} = \mathsf{5}_\mathsf{f}^\Diamond$: trivial. Only the context can be affected by the $\mathsf{5}_\mathsf{f}^\Diamond$ rule.

– For $\mathsf{x}^{[]} = \mathsf{5}^{[]}$:
  - $\mathsf{r} = \mathsf{k}_\mathsf{f}^\Diamond$:

$$\begin{array}{l}\mathsf{k}_\mathsf{f}^\Diamond\\ \mathsf{5}^{[]}\end{array}\dfrac{\dfrac{\Gamma\{[\langle A\rangle, \Delta]\}\{\emptyset\}}{\Gamma\{\langle \Diamond A\rangle, [\Delta]\}\{\emptyset\}}}{\Gamma\{\langle \Diamond A\rangle\}\{[\Delta]\}} \quad \rightsquigarrow \quad \begin{array}{l}\mathsf{5}^{[]}\\ \mathsf{k}_\mathsf{f}^\Diamond\\ \mathsf{5}_\mathsf{f}^\Diamond\end{array}\dfrac{\dfrac{\Gamma\{[\langle A\rangle, \Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\langle A\rangle, \Delta]\}}}{\Gamma\{\emptyset\}\{\langle \Diamond A\rangle, [\Delta]\}}}{\Gamma\{\langle \Diamond A\rangle\}\{[\Delta]\}}$$

  - $\mathsf{r} = \mathsf{b}_\mathsf{f}^\Diamond$: we write $\Gamma\{\langle A\rangle, [\Delta]\}\{\emptyset\} = \Gamma'\{[\Sigma, \langle A\rangle, [\Delta]]\}\{\emptyset\}$.

$$\begin{array}{l}\mathsf{b}_\mathsf{f}^\Diamond\\ \mathsf{5}^{[]}\end{array}\dfrac{\dfrac{\Gamma'\{[\Sigma, \langle A\rangle, [\Delta]]\}\{\emptyset\}}{\Gamma'\{[\Sigma, [\langle \Diamond A\rangle, \Delta]]\}\{\emptyset\}}}{\Gamma'\{[\Sigma]\}\{[\langle \Diamond A\rangle, \Delta]\}} \quad \rightsquigarrow \quad \begin{array}{l}\mathsf{5}^{[]}\\ \mathsf{k}_\mathsf{f}^\Diamond\\ \mathsf{5}_\mathsf{f}^\Diamond\end{array}\dfrac{\dfrac{\Gamma'\{[\Sigma, \langle A\rangle, [\Delta]]\}\{\emptyset\}}{\Gamma'\{[\Sigma], \langle A\rangle\}\{[\Delta]\}}}{\Gamma'\{[\Sigma, \langle \Diamond A\rangle]\}\{[\Delta]\}}}{\Gamma'\{[\Sigma]\}\{[\langle \Diamond A\rangle, \Delta]\}}$$

  - $\mathsf{r} = \mathsf{4}_\mathsf{f}^\Diamond$:

$$\begin{array}{l}\mathsf{4}_\mathsf{f}^\Diamond\\ \mathsf{5}^{[]}\end{array}\dfrac{\dfrac{\Gamma\{[\langle \Diamond A\rangle, \Delta]\}\{\emptyset\}}{\Gamma\{\langle \Diamond A\rangle, [\Delta]\}\{\emptyset\}}}{\Gamma\{\langle \Diamond A\rangle\}\{[\Delta]\}} \quad \rightsquigarrow \quad \begin{array}{l}\mathsf{5}^{[]}\\ \mathsf{5}_\mathsf{f}^\Diamond\end{array}\dfrac{\dfrac{\Gamma\{[\langle \Diamond A\rangle, \Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\langle \Diamond A\rangle, \Delta]\}}}{\Gamma\{\langle \Diamond A\rangle\}[\Delta]}$$

  - $\mathsf{r} = \mathsf{5}_\mathsf{f}^\Diamond$: trivial, only the context can be affected by the $\mathsf{5}_\mathsf{f}^\Diamond$ rule. □

We are now ready to prove the admissibility of cuts. Specifically, we show all the cuts in Figure 6 are simultaneously admissible. The cut_1 rule is our *standard cut* between ordinary nested sequents, while cut_2 defines a *principal cut* between a focus and its dual as cut formulas. Finally, cut_3 is a *commutative cut* for situations where the positive cut formula is not principal. Note that this collection of cuts is just sufficiently large to make the standard cuts admissible. It is easy to imagine many other cut-like rules, but it is not necessary—and may not even be possible—to admit them.

Definition 4.8 The *height* of a formula A, written $\mathrm{ht}(A)$, is computed inductively as follows: $\mathrm{ht}(a) = \mathrm{ht}(\bar{a}) = 1$, $\mathrm{ht}(A \star B) = \max(\mathrm{ht}(A), \mathrm{ht}(B)) + 1$ where $\star \in \{\dagger, \bar{\wedge}, \ddagger, \bar{\vee}\}$, and $\mathrm{ht}(\bigcirc A) = \mathrm{ht}(A) + 1$ where $\bigcirc \in \{\Diamond, \Box, \uparrow, \downarrow\}$. The *rank* of an instance of one of the cut rules is the height of its cut formula (the P in Figure 6).

Lemma 4.9 (Cut Reduction) *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ be 45-closed. For every derivation*

$$\mathsf{cut}_i\ \dfrac{\overset{\mathcal{D}_1}{\Gamma_1} \qquad \overset{\mathcal{D}_2}{\Gamma_2}}{\Gamma_0} \tag{6}$$

in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond + \{\mathsf{cut}_1, \mathsf{cut}_2, \mathsf{cut}_3\}$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are cut-free, there is a cut-free derivation of Γ_0 in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond$.

Proof Let $\mathcal{D}_1$ always stand for the derivation with the positive cut formula. We proceed by lexicographic induction: the induction hypothesis may be applied whenever (1) the rank of the cut decreases, or (2) the rank stays the same and a cut_1 is replaced by a cut_2, or (3) the rank stays the same and the height of $\mathcal{D}_1$ decreases. The height of $\mathcal{D}_2$ does not matter for the induction. The proof is then given in terms of a terminating rewrite sequence, written with $\rightsquigarrow$, that reduces the topmost instances of cut_i.

– We start with the cases of cut_1. There are only three possibilities.
  • The first case is the ordinary commutative case, where we apply a negative rule in the context.

$$\mathsf{cut}_1\dfrac{\mathsf{r}\dfrac{\mathcal{D}_1'::\Gamma'\{P\}}{\Gamma\{P\}}\quad\mathcal{D}_2::\Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{r}\dfrac{\mathsf{cut}_1\dfrac{\mathcal{D}_1'::\Gamma'\{P\}\quad\mathsf{r}^{-1}\dfrac{\mathcal{D}_2::\Gamma\{\bar{P}\}}{\Gamma'\{\bar{P}\}}}{\Gamma'\{\emptyset\}}}{\Gamma\{\emptyset\}}$$

  Here r is invertible (Lemma 4.6) and we can appeal to the inductive hypothesis because the height is reduced. If r is a binary rule the situation is similar:

$$\mathsf{cut}_1\dfrac{\bar{\wedge}\dfrac{\mathcal{D}_1'::\Gamma\{N\}\{P\}\quad\mathcal{D}_1''::\Gamma\{M\}\{P\}}{\Gamma\{N\,\bar{\wedge}\,M\}\{P\}}\quad\mathcal{D}_2::\Gamma\{N\,\bar{\wedge}\,M\}\{\bar{P}\}}{\Gamma\{N\,\bar{\wedge}\,M\}\{\emptyset\}}$$

$$\rightsquigarrow\quad\bar{\wedge}\dfrac{\mathsf{cut}_1\dfrac{\mathcal{D}_1'::\Gamma\{N\}\{P\}\quad\bar{\wedge}^{-1}\dfrac{\mathcal{D}_2::\Gamma\{N\,\bar{\wedge}\,M\}\{\bar{P}\}}{\Gamma\{N\}\{\bar{P}\}}}{\Gamma\{N\}\{\emptyset\}}\quad\mathsf{cut}_1\dfrac{\mathcal{D}_1'::\Gamma\{M\}\{P\}\quad\bar{\wedge}^{-1}\dfrac{\mathcal{D}_2::\Gamma\{N\,\bar{\wedge}\,M\}\{\bar{P}\}}{\Gamma\{M\}\{\bar{P}\}}}{\Gamma\{M\}\{\emptyset\}}}{\Gamma\{N\,\bar{\wedge}\,M\}\{\emptyset\}}$$

  • The second case is similar. We decide on a positive formula in the context.

$$\mathsf{cut}_1\dfrac{\mathsf{dec}\dfrac{\mathcal{D}_1'::\Gamma\{Q,\langle Q\rangle\}\{P\}}{\Gamma\{Q\}\{P\}}\quad\mathcal{D}_2::\Gamma\{Q\}\{\bar{P}\}}{\Gamma\{Q\}\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{dec}\dfrac{\mathsf{cut}_3\dfrac{\mathcal{D}_1'::\Gamma\{Q,\langle Q\rangle\}\{P\}\quad\mathcal{D}_2::\Gamma\{Q\}\{\bar{P}\}}{\Gamma\{Q,\langle Q\rangle\}\{\emptyset\}}}{\Gamma\{Q\}\{\emptyset\}}$$

  We can appeal to the inductive hypothesis because the height is reduced.
  • The last case occurs when we decide on the cut formula.

$$\mathsf{cut}_1\dfrac{\mathsf{dec}\dfrac{\mathcal{D}_1'::\Gamma\{P,\langle P\rangle\}}{\Gamma\{P\}}\quad\mathcal{D}_2::\Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{cut}_2\dfrac{\mathsf{cut}_3\dfrac{\mathcal{D}_1'::\Gamma\{P,\langle P\rangle\}\quad\mathcal{D}_2::\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}}\quad\mathcal{D}_2::\Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

  We first appeal to the inductive hypothesis for the upper cut, which is possible because the height is reduced. Then we appeal to the inductive hypothesis to the lower cut, which is allowed because a cut_2-reduction can justify a cut_1-reduction.

– In the case of cut_3 there are two principal possibilities to consider: First, the commutative cases which are similar to the ones for cut_1, and second, the following three cases, where the last rule in $\mathcal{D}_1$ operates on the formula with the focus and can simply be permuted under the cut:
  • If $\mathsf{r} = \mathsf{id}$:

$$\mathsf{cut}_3\dfrac{\mathsf{id}\dfrac{}{\Gamma\{\langle a\rangle\}\{P\}}\quad\mathcal{D}_2::\Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle a\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{id}\dfrac{}{\Gamma\{\langle a\rangle\}\{\emptyset\}}$$

  since $\bar{a}$ has to be in the context.

- If the rule keeps the focus we proceed as follows:

$$\mathsf{cut}_3\,\dfrac{\mathsf{r}\,\dfrac{\mathcal{D}_1'\quad \Gamma'\{\langle Q'\rangle\}\{P\}}{\Gamma\{\langle Q\rangle\}\{P\}}\qquad \mathcal{D}_2\quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad \mathsf{r}\,\dfrac{\mathsf{cut}_3\,\dfrac{\mathcal{D}_1'\quad \Gamma'\{\langle Q'\rangle\}\{P\}\qquad \mathcal{D}_2\quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma'\{\langle Q'\rangle\}\{\emptyset\}}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}$$

  and similarly if r is a binary rule. We can appeal to the inductive hypothesis because the height is reduced.
- If the focus is released, we proceed similarly and use Lemma 4.6:

$$\mathsf{cut}_3\,\dfrac{\mathsf{rel}\,\dfrac{\mathcal{D}_1'\quad \Gamma\{N\}\{P\}}{\Gamma\{\langle\downarrow N\rangle\}\{P\}}\qquad \mathcal{D}_2\quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle\downarrow N\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad \mathsf{cut}_1\,\dfrac{\mathcal{D}_1'\quad \Gamma\{N\}\{P\}\qquad \mathsf{weak}\,\dfrac{\mathcal{D}_2\quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{N\}\{\bar{P}\}}}{\mathsf{rel}\,\dfrac{\Gamma\{N\}\{\emptyset\}}{\Gamma\{\langle\downarrow N\rangle\}\{\emptyset\}}}$$

– Finally, we consider the cases for cut_2, which correspond to the key cases (because the bottom-most rule in $\mathcal{D}_1$ works on the cut formula which has the focus):
- In the axiom case we use admissibility of contraction (Lemma 4.6):

$$\mathsf{cut}_2\,\dfrac{\mathsf{id}\,\dfrac{}{\Gamma\{\bar{a},\langle a\rangle\}}\qquad \mathcal{D}_2\quad \Gamma\{\bar{a},\bar{a}\}}{\Gamma\{\bar{a}\}}\quad\rightsquigarrow\quad \mathsf{cont}\,\dfrac{\mathcal{D}_2\quad \Gamma\{\bar{a},\bar{a}\}}{\Gamma\{\bar{a}\}}$$

- If we release the focus we use invertibility of sto (Lemma 4.6):

$$\mathsf{cut}_2\,\dfrac{\mathsf{rel}\,\dfrac{\mathcal{D}_1'\quad \Gamma\{N\}}{\Gamma\{\langle\downarrow N\rangle\}}\qquad \mathcal{D}_2\quad \Gamma\{\uparrow\bar{N}\}}{\Gamma\{\emptyset\}}\quad\rightsquigarrow\quad \mathsf{cut}_1\,\dfrac{\mathsf{sto}^{-1}\,\dfrac{\mathcal{D}_2\quad \Gamma\{\uparrow\bar{N}\}}{\Gamma\{\bar{N}\}}\qquad \mathcal{D}_1'\quad \Gamma\{N\}}{\Gamma\{\emptyset\}}$$

  We can appeal to the inductive hypothesis because the cut rank is reduced.
- The cases of the binary connectives are standard:

$$\mathsf{cut}_2\,\dfrac{\barwedge\,\dfrac{\mathcal{D}_1'\quad \Gamma\{\langle P\rangle\}\qquad \mathcal{D}_1''\quad \Gamma\{\langle Q\rangle\}}{\Gamma\{\langle P\barwedge Q\rangle\}}\qquad \mathcal{D}_2\quad \Gamma\{\bar{P}\barvee\bar{Q}\}}{\Gamma\{\emptyset\}}\quad\rightsquigarrow\quad \mathsf{cut}_2\,\dfrac{\mathcal{D}_1'\quad \Gamma\{\langle P\rangle\}\qquad \mathsf{weak}_{\mathsf{f}}\,\dfrac{\mathcal{D}_1''\quad \Gamma\{\langle Q\rangle\}}{\Gamma\{\bar{P},\langle Q\rangle\}}\qquad \barvee^{-1}\,\dfrac{\mathcal{D}_2\quad \Gamma\{\bar{P}\barvee\bar{Q}\}}{\Gamma\{\bar{P},\bar{Q}\}}}{\Gamma\{\bar{P}\}}}{\Gamma\{\emptyset\}}$$

  and

$$\mathsf{cut}_2\,\dfrac{\overset{+}{\vee}_1\,\dfrac{\mathcal{D}_1'\quad \Gamma\{\langle P\rangle\}}{\Gamma\{\langle P\overset{+}{\vee}Q\rangle\}}\qquad \mathcal{D}_2\quad \Gamma\{\bar{P}\bar{\wedge}\bar{Q}\}}{\Gamma\{\emptyset\}}\quad\rightsquigarrow\quad \mathsf{cut}_2\,\dfrac{\mathcal{D}_1'\quad \Gamma\{\langle P\rangle\}\qquad \bar{\wedge}^{-1}\,\dfrac{\mathcal{D}_2\quad \Gamma\{\bar{P}\bar{\wedge}\bar{Q}\}}{\Gamma\{\bar{P}\}}}{\Gamma\{\emptyset\}}$$

  and similarly for $\overset{+}{\vee}_2$. We can appeal to the inductive hypothesis because the cut rank is reduced.
- If the cut formula is a $\Diamond$-formula, there are six cases because we have six rules that work on the $\Diamond$:

$$\mathsf{cut}_2\,\dfrac{\mathsf{k}^{\Diamond}\,\dfrac{\mathcal{D}_1'\quad \Gamma\{[\langle P\rangle,\Delta]\}}{\Gamma\{\langle\Diamond P\rangle,[\Delta]\}}\qquad \mathcal{D}_2\quad \Gamma\{\Box\bar{P},[\Delta]\}}{\Gamma\{[\Delta]\}}\quad\rightsquigarrow\quad \mathsf{cut}_2\,\dfrac{\mathcal{D}_1'\quad \Gamma\{[\langle P\rangle,\Delta]\}\qquad \begin{array}{c}\mathcal{D}_2\\ \Box^{-1}\,\Gamma\{\Box\bar{P},[\Delta]\}\\ \mathsf{m}^{[]}\,\Gamma\{[\bar{P}],[\Delta]\}\end{array}}{\Gamma\{[\bar{P},\Delta]\}}}{\Gamma\{[\Delta]\}}$$

$$\mathsf{d}^{\Diamond}\dfrac{\mathcal{D}_1'::\dfrac{\Gamma\{[\langle P\rangle]\}}{\Gamma\{\langle\Diamond P\rangle\}}\quad \mathcal{D}_2::\Gamma\{\Box\bar{P}\}}{\mathsf{cut}_2\dfrac{}{\Gamma\{\emptyset\}}} \quad\rightsquigarrow\quad \mathsf{cut}_2\dfrac{\mathcal{D}_1'::\Gamma\{[\langle P\rangle]\}\qquad \Box^{-1}\dfrac{\mathcal{D}_2::\Gamma\{\Box\bar{P}\}}{\Gamma\{[\bar{P}]\}}}{\mathsf{d}^{[]}\dfrac{\Gamma\{[\,]\}}{\Gamma\{\emptyset\}}}$$

$$\mathsf{t}^{\Diamond}\dfrac{\mathcal{D}_1'::\dfrac{\Gamma\{\langle P\rangle\}}{\Gamma\{\langle\Diamond P\rangle\}}\quad \mathcal{D}_2::\Gamma\{\Box\bar{P}\}}{\mathsf{cut}_2\dfrac{}{\Gamma\{\emptyset\}}} \quad\rightsquigarrow\quad \mathsf{cut}_2\dfrac{\mathcal{D}_1'::\Gamma\{\langle P\rangle\}\qquad \Box^{-1}\dfrac{\mathcal{D}_2::\Gamma\{\Box\bar{P}\}}{\mathsf{t}^{[]}\dfrac{\Gamma\{[\bar{P}]\}}{\Gamma\{\bar{P}\}}}}{\Gamma\{\emptyset\}}$$

$$\mathsf{b}^{\Diamond}\dfrac{\mathcal{D}_1'::\dfrac{\Gamma\{\langle P\rangle,[\Delta]\}}{\Gamma\{[\langle\Diamond P\rangle,\Delta]\}}\quad \mathcal{D}_2::\Gamma\{[\Box\bar{P},\Delta]\}}{\mathsf{cut}_2\dfrac{}{\Gamma\{[\Delta]\}}} \quad\rightsquigarrow\quad \mathsf{cut}_2\dfrac{\mathcal{D}_1'::\Gamma\{\langle P\rangle,[\Delta]\}\qquad \Box^{-1}\dfrac{\mathcal{D}_2::\Gamma\{[\Box\bar{P},\Delta]\}}{\mathsf{b}^{[]}\dfrac{\Gamma\{[[\bar{P}],\Delta]\}}{\Gamma\{\bar{P},[\Delta]\}}}}{\Gamma\{[\Delta]\}}$$

$$\mathsf{4}^{\Diamond}\dfrac{\mathcal{D}_1'::\dfrac{\Gamma\{[\langle\Diamond P\rangle,\Delta]\}}{\Gamma\{\langle\Diamond P\rangle,[\Delta]\}}\quad \mathcal{D}_2::\Gamma\{\Box\bar{P},[\Delta]\}}{\mathsf{cut}_2\dfrac{}{\Gamma\{[\Delta]\}}} \quad\rightsquigarrow\quad \mathsf{cut}_2\dfrac{\mathcal{D}_1'::\Gamma\{[\langle\Diamond P\rangle,\Delta]\}\qquad \Box^{-1}\dfrac{\mathcal{D}_2::\Gamma\{\Box\bar{P},[\Delta]\}}{\mathsf{4}^{[]}\dfrac{\Gamma\{[\bar{P}],[\Delta]\}}{\Box\dfrac{\Gamma\{[[\bar{P}],\Delta]\}}{\Gamma\{[\Box\bar{P},\Delta]\}}}}}{\Gamma\{[\Delta]\}}$$

$$\mathsf{5}^{\Diamond}\dfrac{\mathcal{D}_1'::\dfrac{\Gamma\{\emptyset\}\{\langle\Diamond P\rangle\}}{\Gamma\{\langle\Diamond P\rangle\}\{\emptyset\}}\quad \mathcal{D}_2::\Gamma\{\Box\bar{P}\}\{\emptyset\}}{\mathsf{cut}_2\dfrac{}{\Gamma\{\emptyset\}\{\emptyset\}}} \quad\rightsquigarrow\quad \mathsf{cut}_2\dfrac{\mathcal{D}_1'::\Gamma\{\emptyset\}\{\langle\Diamond P\rangle\}\qquad \Box^{-1}\dfrac{\mathcal{D}_2::\Gamma\{\Box\bar{P}\}\{\emptyset\}}{\mathsf{5}^{[]}\dfrac{\Gamma\{[\bar{P}]\}\{\emptyset\}}{\Box\dfrac{\Gamma\{\emptyset\}\{[\bar{P}]\}}{\Gamma\{\emptyset\}\{\Box\bar{P}\}}}}}{\Gamma\{\emptyset\}\{\emptyset\}}$$

In all six cases we use the admissibility of the structural rules (Lemma 4.7), which is where we need the 45-closure, and the invertibility of the $\Box$ rule (Lemma 4.6). In the first four cases we can apply the inductive hypothesis because the cut rank is reduced. In the last two cases we can proceed by appealing to the inductive hypothesis because of a smaller height. $\Box$

Theorem 4.10 *Let $\mathsf{X}\subseteq\{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ be 45-closed. If a sequent Γ is provable in $\mathsf{KNF}+\mathsf{X}_{\mathsf{f}}^{\Diamond}+\{\mathsf{cut}_1,\mathsf{cut}_2,\mathsf{cut}_3\}$, then it is also provable in $\mathsf{KNF}+\mathsf{X}_{\mathsf{f}}^{\Diamond}$.*

Proof Apply Lemma 4.9 to all cut instances in the derivation, starting with a topmost one. This gives us a cut-free proof in $\mathsf{KNwF}+\mathsf{X}_{\mathsf{f}}^{\Diamond}$, from which we get a cut-free proof in $\mathsf{KNF}+\mathsf{X}_{\mathsf{f}}^{\Diamond}$ using Proposition 4.3. $\Box$

4.2 Completeness

We can now use Theorem 4.10 to show completeness of the focused systems $\mathsf{KNF}+\mathsf{X}_{\mathsf{f}}^{\Diamond}$ (and hence $\mathsf{KNwF}+\mathsf{X}_{\mathsf{f}}^{\Diamond}$) with respect to $\mathsf{KN}+\mathsf{X}^{\Diamond}$. As an intermediate step, we consider a variant of KN that can deal with polarized formulas. Let KN' denote the system that is obtained from KN by adding the rules

$$\mathsf{rel}\,\frac{\Gamma\{N\}}{\Gamma\{{\downarrow}N\}}\qquad\qquad \mathsf{sto}\,\frac{\Gamma\{P\}}{\Gamma\{{\uparrow}P\}}$$

and by duplicating the rules for $\wedge$ and $\vee$ such that there is a variant for each of $\mathbin{\dot\wedge}$ and $\mathbin{\bar\wedge}$, and $\mathbin{\dot\vee}$ and $\mathbin{\bar\vee}$, respectively. We immediately have the following lemma:

Lemma 4.11 *A formula A is provable in $\mathsf{KN}' + \mathsf{X}^\Diamond$ if and only if $\lfloor A \rfloor$ is provable in $\mathsf{KN} + \mathsf{X}^\Diamond$.* □

We are now going to simulate $\mathsf{KN}' + \mathsf{X}^\Diamond$ in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond$. For this, we need another property of KNwF:

Lemma 4.12 (Identity Reduction) *The following rule is derivable in* KNF*:*

$$\mathsf{gid}\ \frac{}{\Gamma\{\langle P\rangle, \bar{P}\}}$$

Proof We proceed by induction on the height of the formula P. When P is not an atom, we can derive $\Gamma\{\langle P\rangle, \bar{P}\}$ using the induction hypothesis applied to the subformulas of P.

- $P = a$: $\Gamma\{\langle a\rangle, \bar{a}\}$ is an axiom.
- $P = P_1 \mathbin{\dot\wedge} P_2$:

$$\begin{array}{c} \bar\vee\ \dfrac{\Gamma\{\langle P_1\rangle, \bar{P}_1, \bar{P}_2\}}{\Gamma\{\langle P_1\rangle, \bar{P}_1 \mathbin{\bar\vee} \bar{P}_2\}} \qquad \bar\vee\ \dfrac{\Gamma\{\langle P_2\rangle, \bar{P}_1, \bar{P}_2\}}{\Gamma\{\langle P_2\rangle, \bar{P}_1 \mathbin{\bar\vee} \bar{P}_2\}} \\ \dot\wedge\ \dfrac{}{\Gamma\{\langle P_1 \mathbin{\dot\wedge} P_2\rangle, \bar{P}_1 \mathbin{\bar\vee} \bar{P}_2\}} \end{array}$$

- $P = P_1 \mathbin{\dot\vee} P_2$:

$$\begin{array}{c} \dot\vee\ \dfrac{\Gamma\{\langle P_1\rangle, \bar{P}_1\}}{\Gamma\{\langle P_1 \mathbin{\dot\vee} P_2\rangle, \bar{P}_1\}} \qquad \dot\vee\ \dfrac{\Gamma\{\langle P_2\rangle, \bar{P}_2\}}{\Gamma\{\langle P_1 \mathbin{\dot\vee} P_2\rangle, \bar{P}_2\}} \\ \bar\wedge\ \dfrac{}{\Gamma\{\langle P_1 \mathbin{\dot\vee} P_2\rangle, \bar{P}_1 \mathbin{\bar\wedge} \bar{P}_2\}} \end{array}$$

- $P = \Diamond Q$:

$$\begin{array}{c} \mathsf{k}^\Diamond\ \dfrac{\Gamma\{[\langle Q\rangle, \bar{Q}]\}}{\Gamma\{\langle \Diamond Q\rangle, [\bar{Q}]\}} \\ \Box\ \dfrac{}{\Gamma\{\langle \Diamond Q\rangle, \Box\bar{Q}\}} \end{array}$$

- $P = \downarrow N$:

$$\begin{array}{c} \mathsf{dec}\ \dfrac{\Gamma\{N, \bar{N}, \langle \bar{N}\rangle\}}{\Gamma\{N, \bar{N}\}} \\ \mathsf{sto}\ \dfrac{}{\Gamma\{N, \uparrow\bar{N}\}} \\ \mathsf{rel}\ \dfrac{}{\Gamma\{\langle\downarrow N\rangle, \uparrow\bar{N}\}} \end{array}$$

□

Lemma 4.13 (Simulation) *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, and let A be a formula that is provable in $\mathsf{KN}' + \mathsf{X}^\Diamond$. If $\mathsf{d} \notin \mathsf{X}$, then A is provable in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond + \mathsf{cut}_1$, otherwise A is provable in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond + \mathsf{cut}_1 + \mathsf{d}^{[]}$.*

Proof We show that each rule in $\mathsf{KN}' + \mathsf{X}^\Diamond \setminus \{\mathsf{d}^\Diamond\}$ is derivable in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond + \mathsf{cut}_1$, and that $\mathsf{d}^\Diamond$ is derivable in $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond + \mathsf{cut}_1 + \mathsf{d}^{[]}$. Then the lemma follows by induction on the height of the proof in $\mathsf{KN}' + \mathsf{X}^\Diamond$. For the rules $\mathsf{sto}, \Box, \bar\wedge, \bar\vee$ this is trivial.

- $\mathsf{r} = \mathsf{id}$: $\mathsf{dec}\ \dfrac{\mathsf{id}\ \dfrac{}{\Gamma\{p, \langle p\rangle, \bar{p}\}}}{\Gamma\{p, \bar{p}\}}$
- $\mathsf{r} = \mathsf{rel}$:

$$\mathsf{cut}_1\ \frac{\mathsf{weak}\ \dfrac{\Gamma\{N\}}{\Gamma\{\downarrow N, N\}} \qquad \mathsf{dec}\ \dfrac{\mathsf{rel}\ \dfrac{\mathsf{dec}\ \dfrac{\Gamma\{\downarrow N, N, \langle \bar{N}\rangle\}}{\Gamma\{\downarrow N, N, \bar{N}\}}}{\Gamma\{\downarrow N, \langle\downarrow N\rangle, \bar{N}\}}}{\Gamma\{\downarrow N, \bar{N}\}}}{\Gamma\{\downarrow N\}}$$

- $\mathsf{r} = \vee$ when applied to an occurrence of $\dot\vee$:

$$\mathsf{cut}_1\ \frac{\mathsf{weak}\ \dfrac{\Gamma\{P, Q\}}{\Gamma\{P \mathbin{\dot\vee} Q, P, Q\}} \qquad \dot\vee\ \dfrac{\mathsf{dec}\ \dfrac{\Gamma\{P \mathbin{\dot\vee} Q, \langle Q\rangle, P, \bar{Q}\}}{\Gamma\{P \mathbin{\dot\vee} Q, \langle P \mathbin{\dot\vee} Q\rangle, P, \bar{Q}\}}}{\Gamma\{P \mathbin{\dot\vee} Q, P, \bar{Q}\}} \qquad \dot\vee\ \dfrac{\mathsf{dec}\ \dfrac{\Gamma\{P \mathbin{\dot\vee} Q, \langle P\rangle, \bar{P}\}}{\Gamma\{P \mathbin{\dot\vee} Q, \langle P \mathbin{\dot\vee} Q\rangle, \bar{P}\}}}{\Gamma\{P \mathbin{\dot\vee} Q, \bar{P}\}}}{\Gamma\{P \mathbin{\dot\vee} Q\}}$$

– $\mathsf{r} = \wedge$ when applied to an occurrence of $\dagger\wedge$:

$$\dfrac{\dfrac{\dfrac{\Gamma\{P\}}{\Gamma\{P, P \dagger\wedge Q\}}\,\mathsf{weak} \qquad \dfrac{\dfrac{\Gamma\{Q\}}{\Gamma\{\bar{P}, P \dagger\wedge Q, Q\}}\,\mathsf{weak} \qquad \dfrac{\dfrac{\Gamma\{\bar{A}, \bar{B}, A \dagger\wedge B, \langle A\rangle\} \quad \Gamma\{\bar{A}, \bar{B}, A \dagger\wedge B, \langle B\rangle\}}{\Gamma\{\bar{P}, \bar{Q}, P \dagger\wedge Q, \langle P \dagger\wedge Q\rangle\}}\,\dagger\wedge}{\Gamma\{\bar{P}, P \dagger\wedge Q, \bar{Q}\}}\,\mathsf{dec}}{\Gamma\{\bar{P}, P \dagger\wedge Q\}}\,\mathsf{cut}}{\Gamma\{P \dagger\wedge Q\}}\,\mathsf{cut}$$

– $\mathsf{r} = \Diamond$:

$$\dfrac{\Gamma\{\Diamond A, [A, \Delta]\} \qquad \dfrac{\dfrac{\Gamma\{\Diamond A, [\langle A\rangle, \bar{A}, \Delta]\}}{\Gamma\{\Diamond A, \langle \Diamond A\rangle, [\bar{A}, \Delta]\}}\,\mathsf{k}^{\Diamond}_{\mathsf{f}}}{\Gamma\{\Diamond A, [\bar{A}, \Delta]\}}\,\mathsf{dec}}{\Gamma\{\Diamond A, [\Delta]\}}\,\mathsf{cut}_1$$

– $\mathsf{r} = \mathsf{d}^{\Diamond}$:

$$\dfrac{\dfrac{\Gamma\{\Diamond A, [A]\} \qquad \dfrac{\dfrac{\Gamma\{\Diamond A, [\langle A\rangle, \bar{A}]\}}{\Gamma\{\Diamond A, \langle \Diamond A\rangle, [\bar{A}]\}}\,\mathsf{k}^{\Diamond}_{\mathsf{f}}}{\Gamma\{\Diamond A, [\bar{A}]\}}\,\mathsf{dec}}{\Gamma\{\Diamond A, [\,]\}}\,\mathsf{d}^{[\,]}}{\Gamma\{\Diamond A\}}\,\mathsf{cut}_1$$

– $\mathsf{r} = \mathsf{t}^{\Diamond}$:

$$\dfrac{\Gamma\{\Diamond A, A\} \qquad \dfrac{\dfrac{\Gamma\{\Diamond A, \langle A\rangle, \bar{A}\}}{\Gamma\{\Diamond A, \langle \Diamond A\rangle, \bar{A}\}}\,\mathsf{t}^{\Diamond}_{\mathsf{f}}}{\Gamma\{\Diamond A, \bar{A}\}}\,\mathsf{dec}}{\Gamma\{\Diamond A\}}\,\mathsf{cut}_1$$

– $\mathsf{r} = \mathsf{b}^{\Diamond}$:

$$\dfrac{\Gamma\{[\Delta, \Diamond A], A\} \qquad \dfrac{\dfrac{\Gamma\{[\Delta, \Diamond A], \langle A\rangle, \bar{A}\}}{\Gamma\{[\Delta, \Diamond A, \langle \Diamond A\rangle, \bar{A}]\}}\,\mathsf{b}^{\Diamond}_{\mathsf{f}}}{\Gamma\{[\Delta, \Diamond A], \bar{A}\}}\,\mathsf{dec}}{\Gamma\{[\Delta, \Diamond A]\}}\,\mathsf{cut}_1$$

– $\mathsf{r} = 4^{\Diamond}$:

$$\dfrac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\} \qquad \dfrac{\dfrac{\Gamma\{\Diamond A, [\langle \Diamond A\rangle, \Box\bar{A}, \Delta]\}}{\Gamma\{\Diamond A, \langle \Diamond A\rangle, [\Box\bar{A}, \Delta]\}}\,4^{\Diamond}_{\mathsf{f}}}{\Gamma\{\Diamond A, [\Box\bar{A}, \Delta]\}}\,\mathsf{dec}}{\Gamma\{\Diamond A, [\Delta]\}}\,\mathsf{cut}_1$$

– $\mathsf{r} = 5^{\Diamond}$:

$$\dfrac{\Gamma\{\Diamond A\}\{\Diamond A\} \qquad \dfrac{\dfrac{\Gamma\{\Diamond A\}\{\langle \Diamond A\rangle, \Box\bar{A}\}}{\Gamma\{\Diamond A, \langle \Diamond A\rangle\}\{\Box\bar{A}\}}\,5^{\Diamond}_{\mathsf{f}}}{\Gamma\{\Diamond A\}\{\Box\bar{A}\}}\,\mathsf{dec}}{\Gamma\{\Diamond A\}\{\emptyset\}}\,\mathsf{cut}_1$$

□

Theorem 4.14 (Completeness) *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ *be 45-closed. For any* A*, if* $\lfloor A \rfloor$ *is provable in* $\mathsf{KN} + \mathsf{X}^{\Diamond}$*, then* A *is provable in* $\mathsf{KNF} + \mathsf{X}^{\Diamond}_{\mathsf{f}}$*.*

$$\preccurlyeq\bar{\vee}\ \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \mathbin{\bar{\vee}} N} \qquad \preccurlyeq\bar{\wedge}_1\ \frac{\Gamma \preccurlyeq M}{\Gamma \preccurlyeq M \mathbin{\bar{\wedge}} N} \qquad \preccurlyeq\bar{\wedge}_2\ \frac{\Gamma \preccurlyeq N}{\Gamma \preccurlyeq M \mathbin{\bar{\wedge}} N} \qquad \preccurlyeq\Box\ \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N}$$

$$\preccurlyeq\uparrow\ \frac{}{P \preccurlyeq \uparrow P} \qquad \preccurlyeq\mathsf{id}\ \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Fig. 7. Synthetic substructure matching

Proof Suppose that we have a proof of $\lfloor A \rfloor$ in $\mathsf{KN} + \mathsf{X}^\diamond$. By Lemma 4.11, there is a proof in $\mathsf{KN}' + \mathsf{X}^\diamond$. Then, by Lemma 4.13, there is also a proof in $\mathsf{KNwF} + \mathsf{X}^\diamond_\mathsf{f} + \mathsf{cut}_\mathsf{1}$ (or in $\mathsf{KNwF} + \mathsf{X}^\diamond_\mathsf{f} + \mathsf{cut}_\mathsf{1} + \mathsf{d}^{[]}$ if $\mathsf{d} \in \mathsf{X}$). Using Theorem 4.10 and Lemma 4.7, we get a proof in $\mathsf{KNwF} + \mathsf{X}^\diamond_\mathsf{f}$. Finally, by Proposition 4.3 we get a proof in $\mathsf{KNF} + \mathsf{X}^\diamond_\mathsf{f}$. □

5 The Synthetic System

As already mentioned, the strongly focused system KNF is given as a restriction of KNwF where the dec rule is restricted to neutral contexts. However, the cut-elimination and admissibility theorems (4.10 and 4.14) were proved in the KNwF system and made essential use of the admissibility of weakening by arbitrary formulas, including negative formulas. This freedom simplifies the proofs of the meta-theorems, and leaves them at least a recognizable variant of similar proofs in the non-focused system KN. Of course, thanks to Proposition 4.3, we also have a cut-elimination proof for $\mathsf{KNF} + \mathsf{X}^\diamond_\mathsf{f}$, but this is not entirely satisfactory: it is not an internal proof, i.e., a sequence of cut reductions for $\mathsf{KNF} + \mathsf{X}^\diamond_\mathsf{f} + \{\mathsf{cut}_1, \mathsf{cut}_2, \mathsf{cut}_3\}$ that stays inside the system.

One possible response to this issue might be to try to redo the cut-elimination using just $\mathsf{KNF} + \mathsf{X}^\diamond_\mathsf{f}$, but this quickly gets rather complicated because we no longer have access to the weakening rules (Figure 4) in the case where the weakened structure contains negative formulas. Indeed, published proofs of similar attempts for the sequent calculus usually solve this problem by adding additional cut rules, which greatly complicates the cut-elimination argument [10, 19, 28].

To avoid this complexity, it is better to consider the focused proof system in a *synthetic* form where the logical inference rules for the various connectives are composed as much as possible, so that the proof system itself contains exactly two logical rules: one for a positive and one for a negative *synthetic connective* [31, 6]. This synthetic view moreover improves the concept of focusing itself by showing that a focused proof consists of: (1) a selection of a certain *substructure* (the generalization of *subformula*) of the principal formula, and (2) the *contextualization* of that substructure. For positive principal formulas, this contextualization is in the form of a *check* in the surrounding context for other structures, such as dual atoms or nested sequents. For negative principal formulas, on the other hand, contextualization amounts to *asserting* the presence of additional structure in the surrounding context. The duality between positive and negative synthetic rules then amounts to a meta-quantification over these substructures: the positive rule quantifies existentially over substructures, while the negative rule quantifies universally. This design will become clear in the explicit formulation of the synthetic version of KNF—which we call KNS—in the rest of this section.

5.1 Synthetic Substructures

For any negative formula, there is a collection of corresponding nested sequents that represents one of the possible branches taken in a sequence of negative rules applied to the formula. This correspondence is formally given below.

Definition 5.1 (Matching) The nested sequence Γ *matches* the negative formula N, written $\Gamma \preccurlyeq N$, if it is derivable from the rules in Figure 7.

For the system to follow, we will use two additional sequent-like structures that are not themselves neutral sequents.

Definition 5.2 (Extended Sequents)

- An *inversion sequent* is a structure of the form $\Gamma\{N\}$ where $\Gamma\{\ \}$ is a neutral sequent context.
- A *focused sequent* is a structure of the form $\Gamma\{\langle\Delta\rangle\}$ where $\Gamma\{\ \}$ is a neutral sequent context and Δ is a neutral sequent.

Note that in $\Gamma\{N\}$, there is exactly one occurrence of N as a top-level formula anywhere; likewise, in $\Gamma\{\langle\Delta\rangle\}$, there is a single occurrence of the sub-structure $\langle\Delta\rangle$. Hence, these extended sequent forms uniquely determine their decomposition into context (the $\Gamma\{\ \}$) and the extended entity (the N or the $\langle\Delta\rangle$).

$$\mathsf{pos}^{\langle\rangle}\ \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad \mathsf{neg}^{\langle\rangle}\ \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{split}^{\langle\rangle}\ \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathsf{id}^{\langle\rangle}\ \frac{}{\Gamma\{\bar{a}, \langle\bar{a}\rangle\}} \qquad \mathsf{rel}^{\langle\rangle}\ \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}} \qquad \mathsf{k}^{\langle\rangle}\ \frac{\Gamma\{[\Omega, \langle\Delta\rangle]\}}{\Gamma\{[\Omega], \langle[\Delta]\rangle\}}$$

$$\mathsf{d}^{\langle\rangle}\ \frac{\Gamma\{[\langle\Delta\rangle]\}}{\Gamma\{\langle[\Delta]\rangle\}} \qquad \mathsf{t}^{\langle\rangle}\ \frac{\Gamma\{\langle\Delta\rangle\}}{\Gamma\{\langle[\Delta]\rangle\}} \qquad \mathsf{b}^{\langle\rangle}\ \frac{\Gamma\{[\Omega, \langle\Delta\rangle]\}}{\Gamma\{[\Omega, \langle[\Delta]\rangle]\}}$$

$$\mathsf{4}^{\langle\rangle}\ \frac{\Gamma\{[\Omega, \langle[\Delta]\rangle]\}}{\Gamma\{[\Omega], \langle[\Delta]\rangle\}} \qquad \mathsf{5}^{\langle\rangle}\ \frac{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}}\ \mathrm{dp}(\Gamma\{\emptyset\}\{\ \}) \geq 1$$

Fig. 8. Synthetic rules for $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$. The first two rows constitute KNS.

$$\mathsf{cut}_1^{\langle\rangle}\ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}} \qquad \mathsf{cut}_2^{\langle\rangle}\ \frac{\Gamma\{\langle\Delta\rangle\} \quad \Gamma\{\Delta\}}{\Gamma\{\emptyset\}} \qquad \mathsf{cut}_3^{\langle\rangle}\ \frac{\Gamma\{\langle\Delta\rangle\}\{P\} \quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle\Delta\rangle\}\{\emptyset\}}$$

Fig. 9. Synthetic variants of the cut rule (cf. Figure 6)

Definition 5.3 (Corresponding Formula) A *corresponding formula* of a neutral or extended sequent Γ, denoted as $\mathrm{f\bar{m}}(\Gamma)$, is a negative formula satisfying:

$$\mathrm{f\bar{m}}(\Gamma, \Delta) \equiv \mathrm{f\bar{m}}(\Gamma) \mathbin{\bar{\vee}} \mathrm{f\bar{m}}(\Delta) \qquad \mathrm{f\bar{m}}(P) \equiv \uparrow P \qquad \mathrm{f\bar{m}}(N) \equiv N$$

$$\mathrm{f\bar{m}}([\Gamma]) \equiv \Box \mathrm{f\bar{m}}(\Gamma) \qquad \mathrm{f\bar{m}}(\langle\Delta\rangle) \equiv \uparrow \left(\overline{\mathrm{f\bar{m}}(\Delta)}\right)$$

where $N \equiv M$ stands for $(\uparrow\bar{N} \mathbin{\bar{\vee}} M) \mathbin{\bar{\wedge}} (\uparrow\bar{M} \mathbin{\bar{\vee}} N)$.

The system KNS consists of the rules in the first two lines of Fig. 8. For a set $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, we write $\mathsf{X}^{\langle\rangle} \subseteq \{\mathsf{d}^{\langle\rangle}, \mathsf{t}^{\langle\rangle}, \mathsf{b}^{\langle\rangle}, \mathsf{4}^{\langle\rangle}, \mathsf{5}^{\langle\rangle}\}$ for the corresponding structural rules in Figure 8, and $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$ for the corresponding system. The $\mathsf{neg}^{\langle\rangle}$ rule of KNS requires a bit of explanation. The rule actually consists of one premise for each way in which to prove $\Delta \preccurlyeq N$. Thus, for example, if N is $\bar{a} \mathbin{\bar{\wedge}} \Box(\bar{b} \mathbin{\bar{\vee}} \uparrow P)$, then we know that $\bar{a} \preccurlyeq N$ and $[\bar{b}, P] \preccurlyeq N$, so the rule instance in this case is:

$$\mathsf{neg}^{\langle\rangle}\ \frac{\Gamma\{\bar{a}\} \quad \Gamma\{[\bar{b}, P]\}}{\Gamma\{\bar{a} \mathbin{\bar{\wedge}} \Box(\bar{b} \mathbin{\bar{\vee}} \uparrow P)\}}$$

It is instructive to compare $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$ with $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond$ (and hence also $\mathsf{KNF} + \mathsf{X}_\mathsf{f}^\Diamond$). In the latter system, the focus $\langle\Diamond P\rangle$ is used to drive the modal rules $\{\mathsf{k}_\mathsf{f}^\Diamond, \mathsf{d}_\mathsf{f}^\Diamond, \mathsf{t}_\mathsf{f}^\Diamond, \mathsf{b}_\mathsf{f}^\Diamond, \mathsf{4}_\mathsf{f}^\Diamond, \mathsf{5}_\mathsf{f}^\Diamond\}$. Such modal rules can be applied only finitely many times before $\langle\Diamond P\rangle$ needs to be reduced to $\langle P\rangle$, which is necessary to finish the proof since foci can never be weakened. Thus, the analysis of P is forced to be interleaved with the modal rules for $\Diamond P$. In $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$, in contrast, the $\mathsf{pos}^{\langle\rangle}$ rule itself performs the analysis of P up front to produce a synthetic substructure; the modal rules $\{\mathsf{k}^{\langle\rangle}, \mathsf{d}^{\langle\rangle}, \mathsf{t}^{\langle\rangle}, \mathsf{b}^{\langle\rangle}, \mathsf{4}^{\langle\rangle}, \mathsf{5}^{\langle\rangle}\}$ then work entirely at the level of focused substructures. Thus, the modal rules of $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$ are properly seen as *structural* rules rather than logical rules.

5.2 Cut Elimination

Cut elimination for $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$ is achieved in a similar way to that for $\mathsf{KNwF} + \mathsf{X}_\mathsf{f}^\Diamond$.

Lemma 5.4 (Admissible Rules) *The rules* weak, weak_f, cont, $\mathsf{m}^{[]}$ *(shown in Figure 4), restricted to neutral and extended sequents (as appropriate) are admissible in* $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$. *Moreover, if* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *is 45-closed, then any rule* $\mathsf{x}^{[]} \in \mathsf{X}^{[]}$ *(see Figure 5) is admissible in* $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$.

Proof By induction on the height of the derivation, analogous to the proofs of Lemmas 4.6 and 4.7. □

Note that we do not allow, e.g., weakening $\Gamma\{\langle\Delta\rangle\}\{\emptyset\}$ to $\Gamma\{\langle\Delta\rangle\}\{N\}$; the latter is, in fact, not even a well-formed KNS focused sequent. Like with KNwF, we have three cut rules for KNS, which are shown in Figure 9. We can now give the synthetic variant of the cut-elimination theorem.

Theorem 5.5 *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *be 45-closed. Each rule of* $\{\mathsf{cut}_1^{\langle\rangle}, \mathsf{cut}_2^{\langle\rangle}, \mathsf{cut}_3^{\langle\rangle}\}$ *is admissible in* $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$.

Proof (Sketch) The idea of the proof is similar to that of Theorem 4.10, using a reduction similar to that of Lemma 4.9. However, in the synthetic case there is a complication because the inverse of the negative rule $\mathsf{neg}^{\langle\rangle}$ is not admissible in $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$. The cut reduction argument therefore has to work at the level of synthetic derivations. We write the proof using a rewrite $\rightsquigarrow$ and the same measure as in the proof of Lemma 4.9; as a visual aid, we use the following color code to justify the decreasing measure:

- a cut of a strictly lower rank;
- a $\mathsf{cut}_2^{\langle\rangle}$ justifying a $\mathsf{cut}_1^{\langle\rangle}$ of the same rank; and
- a cut of a strictly lower height but of the same rank

The following are all the possible instances of $\{\mathsf{cut}_1^{\langle\rangle}, \mathsf{cut}_2^{\langle\rangle}, \mathsf{cut}_3^{\langle\rangle}\}$ in $\mathsf{KNS} + \mathsf{X}^{\langle\rangle}$.

- Principal cuts (all cases of $\mathsf{cut}_2^{\langle\rangle}$)
  - $\mathsf{split}^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\mathsf{cut}_2^{\langle\rangle}\dfrac{\mathsf{split}^{\langle\rangle}\dfrac{\mathcal{D}_1\ \Gamma\{\langle\Delta_1\rangle\} \quad \mathcal{D}_2\ \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1,\langle\Delta_2\rangle\rangle\}} \quad \mathcal{D}_3\ \Gamma\{\Delta_1,\Delta_2\}}{\Gamma\{\emptyset\}} \rightsquigarrow \mathsf{cut}_2^{\langle\rangle}\dfrac{\mathcal{D}_1\ \Gamma\{\langle\Delta_1\rangle\} \quad \mathsf{weak}\dfrac{\mathcal{D}_2\ \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\Delta_1,\langle\Delta_2\rangle\}} \quad \mathcal{D}_3\ \Gamma\{\Delta_1,\Delta_2\}}{\Gamma\{\Delta_1\}}}{\Gamma\{\emptyset\}}$$

  - $\mathsf{id}^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\mathsf{cut}_2^{\langle\rangle}\dfrac{\mathsf{id}^{\langle\rangle}\dfrac{}{\Gamma\{\bar{a},\langle\bar{a}\rangle\}} \quad \mathcal{D}\ \Gamma\{\bar{a},\bar{a}\}}{\Gamma\{\bar{a}\}} \rightsquigarrow \mathsf{cont}\dfrac{\mathcal{D}\ \Gamma\{\bar{a},\bar{a}\}}{\Gamma\{\bar{a}\}}$$

  - $\mathsf{rel}^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\mathsf{cut}_2^{\langle\rangle}\dfrac{\mathsf{rel}^{\langle\rangle}\dfrac{\mathcal{D}_1\ \Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}} \quad \mathcal{D}_2\ \Gamma\{P\}}{\Gamma\{\emptyset\}} \rightsquigarrow \mathsf{cut}_1^{\langle\rangle}\dfrac{\mathcal{D}_2\ \Gamma\{P\} \quad \mathcal{D}_1\ \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

- The "decision cut", which is a cut between a matching pair of $\mathsf{pos}^{\langle\rangle}$ and $\mathsf{neg}^{\langle\rangle}$

$$\mathsf{cut}_1^{\langle\rangle}\dfrac{\mathsf{pos}^{\langle\rangle}\dfrac{\Delta \preccurlyeq \bar{P} \quad \mathcal{D}\ \Gamma\{P,\langle\Delta\rangle\}}{\Gamma\{P\}} \quad \mathsf{neg}^{\langle\rangle}\dfrac{\left\{\mathcal{D}_\Delta\ \Gamma\{\Delta\}\right\}_{\Delta\preccurlyeq\bar{P}}}{\Gamma\{\bar{P}\}}}{\Gamma\{\emptyset\}} \rightsquigarrow \mathsf{cut}_2^{\langle\rangle}\dfrac{\mathsf{cut}_3^{\langle\rangle}\dfrac{\mathcal{D}\ \Gamma\{P,\langle\Delta\rangle\} \quad \mathsf{neg}^{\langle\rangle}\dfrac{\left\{\mathcal{D}_\Delta\ \Gamma\{\Delta\}\right\}_{\Delta\preccurlyeq\bar{P}}}{\Gamma\{\bar{P}\}}}{\Gamma\{\langle\Delta\rangle\}} \quad \mathcal{D}_\Delta\ \Gamma\{\Delta\}}{\Gamma\{\emptyset\}}$$

- Modal instances of $\mathsf{cut}_2^{\langle\rangle}$
  - $\mathsf{k}^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\mathsf{cut}_2^{\langle\rangle}\dfrac{\mathsf{k}^{\langle\rangle}\dfrac{\mathcal{D}_1\ \Gamma\{[\Omega,\langle\Delta\rangle]\}}{\Gamma\{[\Omega],\langle[\Delta]\rangle\}} \quad \mathcal{D}_2\ \Gamma\{[\Omega],[\Delta]\}}{\Gamma\{[\Omega]\}} \rightsquigarrow \mathsf{cut}_2^{\langle\rangle}\dfrac{\mathcal{D}_1\ \Gamma\{[\Omega,\langle\Delta\rangle]\} \quad \mathsf{k}^{[]}\dfrac{\mathcal{D}_2\ \Gamma\{[\Omega],[\Delta]\}}{\Gamma\{[\Omega,\Delta]\}}}{\Gamma\{[\Omega]\}}$$

  - $\mathsf{d}^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\mathsf{cut}_2^{\langle\rangle}\dfrac{\mathsf{d}^{\langle\rangle}\dfrac{\mathcal{D}_1\ \Gamma\{[\langle\Delta\rangle]\}}{\Gamma\{\langle[\Delta]\rangle\}} \quad \mathcal{D}_2\ \Gamma\{[\Delta]\}}{\Gamma\{\emptyset\}} \rightsquigarrow \mathsf{cut}_2^{\langle\rangle}\dfrac{\mathcal{D}_1\ \Gamma\{[\langle\Delta\rangle]\} \quad \mathcal{D}_2\ \Gamma\{[\Delta]\}}{\mathsf{d}^{[]}\dfrac{\Gamma\{[\,]\}}{\Gamma\{\emptyset\}}}$$

- $\mathsf{t}^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle\}}\,\mathsf{t}^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Delta]\}}\,\mathsf{cut}_2^{\langle\rangle}}{\Gamma\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{cut}_2^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle\}}\quad\dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{[\Delta]\}}\,\mathsf{t}^{[]}}{\Gamma\{\Delta\}}}{\Gamma\{\emptyset\}}$$

- $\mathsf{b}^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle,[\Omega]\}}\,\mathsf{b}^{\langle\rangle}}{\Gamma\{[\langle[\Delta]\rangle,\Omega]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[[\Delta],\Omega]\}}\,\mathsf{cut}_2^{\langle\rangle}}{\Gamma\{[\Omega]\}}\quad\rightsquigarrow\quad\mathsf{cut}_2^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle,[\Omega]\}}\quad\dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{[[\Delta],\Omega]\}}\,\mathsf{b}^{[]}}{\Gamma\{\Delta,[\Omega]\}}}{\Gamma\{[\Omega]\}}$$

- $4^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle[\Delta]\rangle,\Omega]\}}\,4^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Delta],[\Omega]\}}\,\mathsf{cut}_2^{\langle\rangle}}{\Gamma\{[\Omega]\}}\quad\rightsquigarrow\quad\mathsf{cut}_2^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle[\Delta]\rangle,\Omega]\}}\quad\dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{[\Delta],[\Omega]\}}\,4^{[]}}{\Gamma\{[[\Delta],\Omega]\}}}{\Gamma\{[\Omega]\}}$$

- $5^{\langle\rangle}/\mathsf{cut}_2^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}}\,5^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Delta]\}\{\emptyset\}}\,\mathsf{cut}_2^{\langle\rangle}}{\Gamma\{\emptyset\}\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{cut}_2^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}}\quad\dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{[\Delta]\}\{\emptyset\}}\,5^{[]}}{\Gamma\{\emptyset\}\{[\Delta]\}}}{\Gamma\{\emptyset\}\{\emptyset\}}$$

– Ordinary (non-modal) commutative cases

- $\mathsf{pos}^{\langle\rangle}/\mathsf{cut}_1^{\langle\rangle}$ where the cut formula is a side formula in $\mathsf{pos}^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Delta\preccurlyeq\bar{Q}\quad\Gamma\{Q,\langle\Delta\rangle\}\{P\}}\,\mathsf{pos}^{\langle\rangle}}{\Gamma\{Q\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{Q\}\{\bar{P}\}}\,\mathsf{cut}_1^{\langle\rangle}}{\Gamma\{Q\}\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{pos}^{\langle\rangle}\dfrac{\Delta\preccurlyeq\bar{Q}\quad\mathsf{cut}_3^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{Q,\langle\Delta\rangle\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{Q\}\{\bar{P}\}}}{\Gamma\{Q,\langle\Delta\rangle\}\{\emptyset\}}}{\Gamma\{Q\}\{\emptyset\}}$$

- $\mathsf{split}^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{\langle\Delta_2\rangle\}\{P\}}\quad\dfrac{\mathcal{D}_3}{\Gamma\{\emptyset\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle\Delta_1,\Delta_2\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{cut}_3^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta_1\rangle\}\{P\}}\quad\dfrac{\mathcal{D}_3}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\mathsf{split}^{\langle\rangle}\dfrac{\Gamma\{\langle\Delta_1\rangle\}\{\emptyset\}}{\Gamma\{\langle\Delta_1,\Delta_2\rangle\}\{\emptyset\}}}\quad\mathsf{cut}_3^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{\langle\Delta_2\rangle\}\{P\}}\quad\dfrac{\mathcal{D}_3}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\Gamma\{\langle\Delta_2\rangle\}\{\emptyset\}}$$

- $\mathsf{id}^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\mathsf{id}^{\langle\rangle}\,\dfrac{}{\Gamma\{\bar{a},\langle\bar{a}\rangle\}\{P\}}\quad\dfrac{\mathcal{D}}{\Gamma\{\bar{a}\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\bar{a},\langle\bar{a}\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{id}^{\langle\rangle}\dfrac{}{\Gamma\{\bar{a},\langle\bar{a}\rangle\}\{\emptyset\}}$$

- $\mathsf{rel}^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\left\{\dfrac{\mathcal{D}_\Omega}{\Gamma\{\Omega\}\{P\}}\right\}_{\Omega\preccurlyeq\bar{Q}}\,\mathsf{neg}^{\langle\rangle}}{\Gamma\{\bar{Q}\}\{P\}}\,\mathsf{rel}^{\langle\rangle}}{\Gamma\{\langle Q\rangle\}\{P\}}\quad\dfrac{\mathcal{D}}{\Gamma\{\emptyset\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad\mathsf{neg}^{\langle\rangle}\dfrac{\left\{\mathsf{cut}_1^{\langle\rangle}\dfrac{\dfrac{\mathcal{D}_\Omega}{\Gamma\{\Omega\}\{P\}}\quad\mathsf{weak}\dfrac{\dfrac{\mathcal{D}}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\Gamma\{\Omega\}\{\bar{P}\}}}{\Gamma\{\Omega\}\{\emptyset\}}\right\}_{\Omega\in\bar{Q}}}{\mathsf{rel}^{\langle\rangle}\dfrac{\Gamma\{\bar{Q}\}\{\emptyset\}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}}$$

– Modal commutative cases

- $\mathsf{k}^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle\Delta\rangle,\Omega]\}\{P\}}\,\mathsf{k}^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega]\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega]\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega]\}\{\emptyset\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle\Delta\rangle,\Omega]\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega]\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{[\langle\Delta\rangle,\Omega]\}\{\emptyset\}}{\Gamma\{\langle[\Delta]\rangle,[\Omega]\}\{\emptyset\}}\,\mathsf{k}^{\langle\rangle}}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle\Delta\rangle,\Omega\{P\}]\}}\,\mathsf{k}^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega\{P\}]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega\{\bar{P}\}]\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega\{\emptyset\}]\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle\Delta\rangle,\Omega\{P\}]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega\{\bar{P}\}]\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{[\langle\Delta\rangle,\Omega\{\emptyset\}]\}}{\Gamma\{\langle[\Delta]\rangle,[\Omega\{\emptyset\}]\}}\,\mathsf{k}^{\langle\rangle}}$$

- $\mathsf{d}^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle\Delta\rangle]\}\{P\}}\,\mathsf{d}^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle\Delta\rangle]\}\{P\}}\quad\mathsf{weak}\,\dfrac{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}}{\Gamma\{[\emptyset]\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{[\langle\Delta\rangle]\}\{\emptyset\}}{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}\,\mathsf{d}^{\langle\rangle}}$$

- $\mathsf{t}^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle\}\{P\}}\,\mathsf{t}^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{\langle\Delta\rangle\}\{\emptyset\}}{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}\,\mathsf{t}^{\langle\rangle}}$$

- $\mathsf{b}^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle,[\Omega]\}\{P\}}\,\mathsf{b}^{\langle\rangle}}{\Gamma\{[\langle[\Delta]\rangle,\Omega]\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega]\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{[\Omega,\langle[\Delta]\rangle]\}\{\emptyset\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle,[\Omega]\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega]\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{\langle\Delta\rangle,[\Omega]\}\{\emptyset\}}{\Gamma\{[\Omega,\langle[\Delta]\rangle]\}\{\emptyset\}}\,\mathsf{b}^{\langle\rangle}}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle,[\Omega\{P\}]\}}\,\mathsf{b}^{\langle\rangle}}{\Gamma\{[\langle[\Delta]\rangle,\Omega\{P\}]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega\{\bar{P}\}]\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{[\langle[\Delta]\rangle,\Omega\{\emptyset\}]\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\langle\Delta\rangle,[\Omega\{P\}]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega\{\bar{P}\}]\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{\langle\Delta\rangle,[\Omega\{\emptyset\}]\}}{\Gamma\{[\langle[\Delta]\rangle,\Omega\{\emptyset\}]\}}\,\mathsf{b}^{\langle\rangle}}$$

- $4^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle[\Delta]\rangle,\Omega]\}\{P\}}\,4^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega]\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega]\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega]\}\{\emptyset\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle[\Delta]\rangle,\Omega]\}\{P\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega]\}\{\bar{P}\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{[\langle[\Delta]\rangle,\Omega]\}\{\emptyset\}}{\Gamma\{\langle[\Delta]\rangle,[\Omega]\}\{\emptyset\}}\,4^{\langle\rangle}}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle[\Delta]\rangle,\Omega\{P\}]\}}\,4^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega\{P\}]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega\{\bar{P}\}]\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle,[\Omega\{\emptyset\}]\}}\quad\rightsquigarrow\quad\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{[\langle[\Delta]\rangle,\Omega\{P\}]\}}\quad\dfrac{\mathcal{D}_2}{\Gamma\{[\Omega\{\bar{P}\}]\}}\,\mathsf{cut}_3^{\langle\rangle}}{\dfrac{\Gamma\{[\langle[\Delta]\rangle,\Omega\{\emptyset\}]\}}{\Gamma\{\langle[\Delta]\rangle,[\Omega\{\emptyset\}]\}}\,4^{\langle\rangle}}$$

- $5^{\langle\rangle}/\mathsf{cut}_3^{\langle\rangle}$

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}\{P\}}\,5^{\langle\rangle} \quad \dfrac{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\emptyset\}\{\bar{P}\}}}{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}\{\emptyset\}}\,\mathsf{cut}_3^{\langle\rangle} \quad\leadsto\quad \dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}\{P\}} \quad \dfrac{\mathcal{D}_2}{\Gamma\{\emptyset\}\{\emptyset\}\{\bar{P}\}}}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}\{\emptyset\}}\,\mathsf{cut}_3^{\langle\rangle}}{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}\{\emptyset\}}\,5^{\langle\rangle}$$

□

It is worth remarking that this cut-elimination proof did not have to mention any logical connectives, and was instead able to factorize all logical reasoning in terms of the matching. This means that the matching judgement can be modified at will without affecting the nature of the cut argument, as long as it leaves the structure of nested sequents untouched. This makes our result modular in yet another way, in addition to the modularity obtained by means of the structural rules for foci. Indeed, we can obtain a similarly synthetic version of identity reduction (Lemma 4.12).

Lemma 5.6 (Identity Reduction) *The following rule is derivable in* KNS*.*

$$\mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle\Delta\rangle,\Delta\}}$$

Proof By induction on the structure of the focus. We present it in the form of an expansionary rewrite, $\leadsto$.

$$\mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle\Delta_1,\Delta_2\rangle,\Delta_1,\Delta_2\}} \quad\leadsto\quad \mathsf{split}^{\langle\rangle}\,\dfrac{\mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle\Delta_1\rangle,\Delta_1,\Delta_2\}} \quad \mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle\Delta_2\rangle,\Delta_1,\Delta_2\}}}{\Gamma\{\langle\Delta_1,\Delta_2\rangle,\Delta_1,\Delta_2\}}$$

$$\mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle P\rangle,P\}} \quad\leadsto\quad \mathsf{neg}^{\langle\rangle}\,\dfrac{\left\{\mathsf{pos}^{\langle\rangle}\,\dfrac{\Delta \preccurlyeq \bar{P} \quad \mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\Delta,P,\langle\Delta\rangle\}}}{\Gamma\{\Delta,P\}}\right\}_{\Delta\preccurlyeq\bar{P}}}{\mathsf{rel}^{\langle\rangle}\,\dfrac{\Gamma\{\bar{P},P\}}{\Gamma\{\langle P\rangle,P\}}}$$

$$\mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle\bar{a}\rangle,\bar{a}\}} \quad\leadsto\quad \mathsf{id}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle\bar{a}\rangle,\bar{a}\}}$$

$$\mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{\langle[\Delta]\rangle,[\Delta]\}} \quad\leadsto\quad \mathsf{k}^{\langle\rangle}\,\dfrac{\mathsf{sid}^{\langle\rangle}\,\dfrac{}{\Gamma\{[\langle\Delta\rangle,\Delta]\}}}{\Gamma\{\langle[\Delta]\rangle,[\Delta]\}}$$

□

Showing $\mathsf{KNS}+\mathsf{X}^{\langle\rangle}$ sound and complete with respect to $\mathsf{KN}+\mathsf{X}^{\diamond}$ is a fairly simple matter. Soundness follows directly from replacing the KNS sequent $\Gamma\{\langle\Delta\rangle\}$ with the KNF sequent $\Gamma\{\langle\overline{\mathrm{fm}}(\Delta)\rangle\}$ and then interpreting the $\mathsf{KNS}+\mathsf{X}^{\langle\rangle}$ proof in $\mathsf{KNF}+\mathsf{X}^{\diamond}_{\mathrm{f}}$, using matching derivation (Figure 7) to determine how to choose between the two $\dot{\vee}$ rules. It is a fairly mundane proof and the details are omitted here. For completeness, we can follow the strategy of Lemma 4.13 nearly unchanged. However, like with the cut-elimination proof, we have to avoid appeals to weakening with negative formulas by using the synthetic form of $\mathsf{neg}^{\langle\rangle}$.

6 Perspectives

We have presented strongly focused and synthetic systems for all modal logics in the classical S5-cube. We used the formalism of nested sequents as carrier, but we are confident that something similar can be achieved for hypersequents, for example based on the work of Lellmann [17]. We chose nested sequents over hypersequents for two reasons. First, the formula interpretation of a nested sequent is the same for all logics in the S5-cube, which simplifies the presentation of the meta-theory. Second, due to the close correspondence between nested sequents and prefixed tableaux [12] we can from our work directly extract *focused tableau systems* for modal logics. Furthermore, even though we spoke in this paper only about classical modal logic, we are confident that the same results can also be obtained for the intuitionistic variant of the modal S5-cube [29], if we start from the non-focused systems in [30].

One extension that would be worth considering would be relaxing the restriction that there can be at most one focus in a KNF or KNS proof. Allowing multiple foci would take us from ordinary focusing to multi-focusing, which is well known to reveal more parallelism in sequent proofs [9]. It has been shown that a certain well chosen multi-focusing system can yield syntactically canonical representatives of equivalence classes of sequent proofs for classical predicate logic [8]. Extending this approach to the modal case seems promising.

References

1. J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
2. A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *Logic: from foundations to applications: European logic colloquium*, pages 1–32. Clarendon Press, 1996.
3. N. D. Belnap, Jr. Display logic. *Journal of Philosophical Logic*, 11:375–417, 1982.
4. T. Brock-Nannestad and C. Schürmann. Focused natural deduction. In *LPAR 17*, volume 6397 of *Lecture Notes in Computer Science*, pages 157–171, Yogyakarta, Indonesia, 2010. Springer.
5. K. Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009.
6. K. Chaudhuri. Focusing strategies in the sequent calculus of synthetic connectives. In *LPAR: International Conference on Logic, Programming, Artificial Intelligence and Reasoning*, volume 5330 of *Lecture Notes in Computer Science*, pages 467–481. Springer, Nov. 2008.
7. K. Chaudhuri, N. Guenot, and L. Straßburger. The Focused Calculus of Structures. In *Computer Science Logic: 20th Annual Conference of the EACSL*, Leibniz International Proceedings in Informatics (LIPIcs), pages 159–173. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Sept. 2011.
8. K. Chaudhuri, S. Hetzl, and D. Miller. A multi-focused proof system isomorphic to expansion proofs. *Journal of Logic and Computation*, June 2014.
9. K. Chaudhuri, D. Miller, and A. Saurin. Canonical sequent proofs via multi-focusing. In *Fifth International Conference on Theoretical Computer Science*, volume 273 of *IFIP*, pages 383–396. Springer, Sept. 2008.
10. K. Chaudhuri, F. Pfenning, and G. Price. A logical characterization of forward and backward chaining in the inverse method. *J. of Automated Reasoning*, 40(2-3):133–177, Mar. 2008.
11. R. Dyckhoff and S. Negri. Proof analysis in intermediate logics. *Arch. Math. Log.*, 51(1-2):71–92, 2012.
12. M. Fitting. Prefixed tableaus and nested sequents. *Annals of Pure and Applied Logic*, 163:291–313, 2012.
13. J. Garson. Modal logic. In *The Stanford Encyclopedia of Philosophy*. Stanford University, 2008.
14. S. Graham-Lengrand. *Polarities & Focussing: a journey from Realisability to Automated Reasoning*. Habilitation thesis, Université Paris-Sud, 2014.
15. R. Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1):119–136, 1994.
16. O. Laurent. *Étude de la Polarisation en Logique*. PhD thesis, Univiversit Aix-Marseille II, 2002.
17. B. Lellmann. Axioms vs hypersequent rules with context restrictions: Theory and applications. In *International Joint Conference on Automated Reasoning IJCAR*, volume 8562 of *Lecture Notes in Computer Science*, pages 307–321. Springer, 2014.
18. B. Lellmann and E. Pimentel. Proof search in nested sequent calculi. In *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, Nov. 2015.
19. C. Liang and D. Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009.
20. S. Marin and L. Straßburger. Label-free modular systems for classical and intuitionistic modal logics. In *Advances in Modal Logic (AIML-10)*, 2014.
21. S. McLaughlin and F. Pfenning. Imogen: Focusing the polarized focused inverse method for intuitionistic propositional logic. In *15th International Conference on Logic, Programming, Artificial Intelligence and Reasoning (LPAR)*, volume 5330 of *Lecture Notes in Computer Science*, pages 174–181, Nov. 2008.
22. D. Miller, G. Nadathur, F. Pfenning, and A. Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51:125–157, 1991.
23. D. Miller and M. Volpe. Focused labeled proof systems for modal logic. In *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, 2015.
24. S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5-6):507–544, 2005.
25. F. Pfenning and R. Davies. A judgmental reconstruction of modal logic. *Mathematical Structures in Computer Science*, 11:511–540, 2001. Notes to an invited talk at the Workshop on Intuitionistic Modal Logics and Applications (IMLA'99).
26. F. Poggiolesi. The method of tree-hypersequents for modal propositional logic. In *Towards Mathematical Philosophy*, volume 28 of *Trends in Logic*, pages 31–51. Springer, 2009.
27. J. Reed and F. Pfenning. Focus-preserving embeddings of substructural logics in intuitionistic logic. Draft manuscript, Jan. 2010.
28. R. J. Simmons. Structural focalization. *ACM Trans. Comput. Log.*, 15(3):21:1–21:33, 2014.
29. A. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, University of Edinburgh, 1994.
30. L. Straßburger. Cut elimination in nested sequents for intuitionistic modal logics. In *Foundations of Software Science and Computation Structures, 16th International Conference (FOSSACS)*, volume 7794 of *Lecture Notes in Computer Science*, pages 209–224. Springer, 2013.
31. N. Zeilberger. Focusing and higher-order abstract syntax. In *Proceedings of the 35th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2008, San Francisco, California, USA, January 7-12, 2008*, pages 359–369. ACM, 2008.