Hilbert and Thompson geometries isometric to infinite-dimensional Banach spaces

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CORMAC WALSH

Abstract. We study the horofunction boundaries of Hilbert and Thompson geometries, and of Banach spaces, in arbitrary dimension. By comparing the boundaries of these spaces, we show that the only Hilbert and Thompson geometries that are isometric to Banach spaces are the ones defined on the cone of positive continuous functions on a compact space.

1. Introduction

It was observed by Nussbaum [13, pages 22–23] and de la Harpe [7] that the Hilbert geometry on a finite-dimensional simplex is isometric to a normed space. Later, Foertsch and Karlsson [8] proved the converse, that is, if a Hilbert geometry on a finite-dimensional convex domain is isometric to a normed space, then the domain is a simplex.

In this paper, we extend this result to infinite dimension.

The natural setting is that of order units spaces. An order unit space is a triple $(X, \overline{C}, u)$ consisting of a vector space X, an Archimedean convex cone $\overline{C}$ in X, and an order unit u. Let C be the interior of $\overline{C}$ with respect to the topology on X coming from the order unit norm. Define, for each x and y in C,

$$M(x,y) := \inf\{\lambda > 0 \mid x \le \lambda y\}.$$

Since every element of C is an order unit, this quantity is finite. Hilbert's projective metric is defined to be

$$d_H(x,y) := \log M(x,y)M(y,x), \qquad \text{for each } x, y \in C.$$

It satisfies $d_H(\lambda x, \nu y) = d_H(x,y)$, for all $x, y \in C$ and $\lambda, \nu > 0$, and is a metric on the projective space $P(C)$ of the cone.

In infinite dimension the role of the simplex will be played by the cone $C^+(K)$ of positive continuous functions on a compact topological space K. This cone lives in the linear space $C(K)$ of continuous functions on K, and is the interior of $\overline{C^+(K)}$ the cone of non-negative continuous functions on K. The triple $(C(K), \overline{C^+(K)}, u)$ forms an order unit space, where u is the function on K that is identically 1.

It is not hard to show that the Hilbert metric on $C^+(K)$ is the following:

$$d_H(x,y) = \log \sup_{k,k' \in K} \frac{x(k)}{y(k)} \frac{y(k')}{x(k')}, \qquad \text{for } x, y \in C^+(K),$$

The map $\log\colon C^+(K) \to C(K)$ that takes the logarithm coordinate-wise is an isometry when $C(K)$ is equipped with the semi-norm

$$||z||_H := \sup_{k \in K} z_k - \inf_{k \in K} z_k.$$

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Denote by $\equiv$ the equivalence relation on $C(K)$ where two functions are equivalent if they differ by a constant, that it, $x \equiv y$ if $x = y + c$ for some constant c. The seminorm $||x||_H$ is a norm on the quotient space $C(K)/\equiv$. This space is a Banach space, and we denote it by $H(K)$. The coordinate-wise logarithm map induces an isometry from the projective space $P(C(K))$ to $H(K)$.

We show that every Hilbert geometry isometric to a Banach space arises in this way. When we talk about the Hilbert geometry on a cone C, we are assuming that C is the interior of the cone of an order unit space.

Theorem 12.6. *If a Hilbert geometry on cone C is isometric to a Banach space, then C is linearly isomorphic to $C^+(K)$ for some compact Hausdorff space K.*

We also prove a similar result for another metric related to the Hilbert metric, the Thompson metric. This is defined to be

$$d_T(x,y) := \log \max\big(M(x,y), \log M(x,y)\big), \qquad \text{for each } x, y \in C.$$

Theorem 11.3. *If a Thompson geometry on a cone C is isometric to a Banach space, then C is linearly isomorphic to $C^+(K)$, for some compact Hausdorff space K.*

The main technique we use in both cases is to compare the horofunction boundary of the Banach space with that of the Hilbert or Thompson geometry. The horofunction boundary was first introduced by Gromov [10]. Since it is defined purely in terms of the metric structure, it is useful for studying isometries of metric spaces.

In finite dimension, the horofunction boundary of normed spaces was studied in [16] and that of Hilbert geometries was studied in [17]. The results for the Hilbert geometry were used to study the isometry group of this geometry in the polyhedral case [12] and more generally [15]. See [20] for a survey of these results.

Usually, when one develops the theory of the horofunction boundary, one makes the assumption that the space is proper, that is, that closed balls are compact. For normed spaces and Hilbert geometries, this is equivalent to the dimension being finite. To deal with infinite dimensional spaces, we are forced to extend the framework. For example, we must use nets rather than sequences. In section 2, we reprove some basic results concerning the horofunction boundary in this setting. We study the boundary of normed spaces in section 3. In this and later sections, We make extensive use of the theory of affine functions on a compact set, including some Choquet Theory. To demonstrate the usefulness of the horofunction boundary, we give a short proof of the Masur–Ulam theorem in section 4. As an example, we determine explicitly the Busemann points in the boundary of the important Banach space $(C(K), ||\cdot||_\infty)$ in section 5. Important to our method will be to consider the Hilbert and Thompson metrics as symmetrisations of a non-symmetric metric, the *Funk metric*. In sections 6, 7, and 8, we study the boundaries of, respectively, the reverse of the Funk metric, the Funk itself, and Hilbert metric. Again, we take a closer look in section 9 at an example, here the cone $C^+(K)$. We study the boundary of the Thompson geometry in section 10, which allows us to prove Theorem 11.3 in section 11. We prove Theorem 12.6 in section 12.

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2. Preliminaries

2.1. **Hilbert's metric.** Let $\overline{C}$ be cone in a real vector space X. In other words, $\overline{C}$ is closed under addition and multiplication by non-negative real numbers, and $\overline{C} \cap -\overline{C} = 0$.

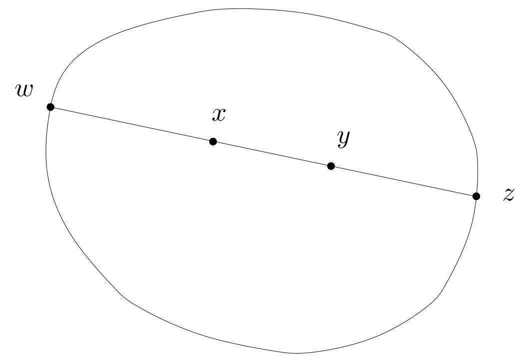

FIGURE 1. Hilbert's definition of a distance.

The cone $\overline{C}$ induces a partial ordering $\leq$ on X by $x \leq y$ if $y - x \in \overline{C}$. We say that $\overline{C}$ is *Archimedean* if, whenever $x \in X$ and $y \in \overline{C}$ satisfy $nx \leq y$ for all $n \in \mathbb{N}$, we have $x \leq 0$. An order unit is an element $u \in \overline{C}$ such that for each $x \in X$ there is some $\lambda > 0$ such that $x \leq \lambda u$. An order unit space $(X, \overline{C}, u)$ is a vector space X equipped with a cone $\overline{C}$ containing an order unit u.

We define the *order unit norm* on X:

$$||x||_u := \inf\{\lambda > 0 \mid -\lambda u \leq x \leq \lambda u\}.$$

We use on X the topology induced by $|| \cdot ||_u$. It is known that, under this topology, $\overline{C}$ is closed [4, Theorem 2.55] and has non-empty interior. Indeed, the interior of $\overline{C}$ is precisely the set of its order units; see [11].

On C, we define Hilbert's projective metric as in the introduction.

Hilbert originally defined his metric on bounded open convex sets. Suppose D is such a set, and that we are given two distinct points x and y in D. Define w and z to be the points in the boundary ∂D of D such that w, x, y, and z are collinear and arranged in this order along the line in which they lie. Hilbert defined the distance between x and y to be the logarithm of the cross ratio of these four points:

$$d_H(x, y) := \log \frac{|zx|\,|wy|}{|zy|\,|wx|}.$$

In the case of an infinite-dimensional cone, one may recover Hilbert's original definition if the cone has a strictly positive state, that is, if there exists a continuous linear functional ψ that is positive everywhere on C. In this situation, Hilbert's definition applied to the cross section $\{x \in C \mid \psi(x) = 1\}$ agrees with the definition in the introduction. It was shown in [11] that Hilbert's definition makes sense if and only if the convex domain is affinely isomorphic to a cross section of the cone of an order unit space.

Not every order unit space has a strictly positive state however. For example, take $X := \mathbb{R}^D$, the space of real-valued bounded functions on some uncountable set D. The subset of these functions that are non-negative is an Archimedean cone, and the function that is identically 1 is an order unit. On spaces such as these, the metric d_H is well-defined even though Hilbert's original construction is not.

2.2. **The Funk and reverse-Funk metrics.** Essential to our method will be to consider the Hilbert and Thompson metrics as symmetrisations of the Funk metric, which is defined as follows:

$$d_F(x,y) := \log M_C(x,y), \qquad \text{for all } x \in X \text{ and } y \in C.$$

This metric first appeared in [9]. We call its reverse $d_R(x,y) := d_F(y,x)$ the *reverse-Funk metric*.

Like Hilbert's metric, the Funk metric was first defined on bounded open convex sets. On a cross section D of a cone C, one can show that

$$d_F(x,y) = \log \frac{|xz|}{|yz|} \qquad \text{and} \qquad d_R(x,y) = \log \frac{|wy|}{|wx|},$$

for all $x, y \in D$. Here w and z are the points of the boundary ∂D shown in Figure 1.

On D, the Funk metric is a *quasi-metric*, in other words, it satisfies the usual metric space axioms except that of symmetry. On C, it satisfies the triangle inequality but is not non-negative. It has the following homogeneity property:

$$d_F(\alpha x, \beta y) = d_F(x,y) + \log \alpha - \log \beta, \qquad \text{for all } x, y \in C \text{ and } \alpha, \beta > 0.$$

Observe that both the Hilbert and Thompson metrics are symmetrisations of the Funk metric: for all $x, y \in C$,

$$\begin{aligned} d_H(x,y) &= d_F(x,y) + d_R(x,y) \qquad \text{and} \\ d_T(x,y) &= \max\big(d_F(x,y), d_R(x,y)\big). \end{aligned}$$

2.3. **The horofunction boundary.** Let (X,d) be a metric space. Associate to each point $z \in X$ the function $\psi_z \colon X \to \mathbb{R}$,

$$\psi_z(x) := d(x,z) - d(b,z),$$

where $b \in X$ is some base-point. It can be shown that the map $\psi \colon X \to C(X)$, $z \mapsto \psi_z$ is injective and continuous. Here, $C(X)$ is the space of continuous real-valued functions on X, with the topology of uniform convergence on compact sets. We identify X with its image under ψ.

Let cl denote the topological closure operator. Since elements of $\operatorname{cl}\psi(X)$ are equi-Lipschitzian, uniform convergence on compact sets is equivalent to pointwise convergence, by the Ascoli–Arzelà theorem. Also, from the same theorem, the set $\operatorname{cl}\psi(X)$ is compact. We call it the *horofunction compactification*. We define the *horofunction boundary* of (X,d) to be

$$X(\infty) := \big(\operatorname{cl}\psi(X)\big)\backslash\psi(X),$$

The elements of this set are the *horofunctions* of (X,d).

Although this definition appears to depend on the choice of base-point, one may verify that horofunction boundaries coming from different base-points are homeomorphic, and that corresponding horofunctions differ only by an additive constant.

In the finite-dimensional setting one commonly considers geodesics parameterised by $\mathbb{Z}$ or $\mathbb{R}$. In infinite dimension, however, one must use nets. Recall that a directed set is a nonempty pre-ordered set such that every pair of elements has an upper bound in the set, and that a net in a topological space is a function from a directed set to the space.

Definition 2.1. A net of real-valued functions f_α is *almost non-decreasing* if, for any $\varepsilon > 0$, there exists A such that $f_\alpha \le f_{\alpha'} + \varepsilon$, for all α and α' greater than A, with $\alpha < \alpha'$.

An almost non-increasing net is defined similarly.

Definition 2.2. A net of real-valued functions f_α is *almost non-increasing* if, for any $\varepsilon > 0$, there exists A such that $f_\alpha \geq f_{\alpha'} - \varepsilon$, for all α and α' greater than A, with $\alpha < \alpha'$.

Observe that if f_α is an almost non-increasing net of functions, and m_α is a net (on the same directed set) of real numbers converging to zero, then $f_\alpha + m_\alpha$ is also almost non-increasing.

Definition 2.3. A net in a metric space is *almost geodesic* if, for all $\varepsilon > 0$,

$$d(b, z_{\alpha'}) \geq d(b, z_\alpha) + d(z_\alpha, z_{\alpha'}) - \varepsilon,$$

for α and α' large enough, with $\alpha < \alpha'$.

This definition is similar to Rieffel's [14], except that here we use nets rather than sequences and the almost geodesics are unparameterised.

Proposition 2.4. *Let z_α be a net in a metric space. Then, z_α is an almost geodesic if and only if $\varphi_{z_\alpha} := d(\cdot, z_\alpha) - d(b, z_\alpha)$ is an almost non-increasing net.*

Proof. Let $\varepsilon > 0$ be given. Assume z_α is almost geodesic. So, for α and α' large enough, with $\alpha < \alpha'$,

$$d(b, z_{\alpha'}) \geq d(b, z_\alpha) + d(z_\alpha, z_{\alpha'}) - \varepsilon.$$

Let x be a point in the metric space. Combining the above inequality with the triangle inequality concerning the points x, z_α, and $z_{\alpha'}$, we get

$$d(x, z_\alpha) - d(b, z_\alpha) \geq d(x, z_{\alpha'}) - d(b, z_{\alpha'}) - \varepsilon. \tag{1}$$

We conclude that the net φ_{z_α} is almost non-increasing.

Now assume that φ_{z_α} is almost non-increasing, in other words that (1) holds when α and α' are large enough, with $\alpha < \alpha'$, for all points x. Taking x equal to z_α, we get that z_α is an almost geodesic. □

Proposition 2.5. *Let (X, d) be a metric space. Let z_α and z'_α be an almost geodesics in X converging to the same Busemann point. Then, there exists almost-geodesics x_β, x'_β, and y_β such that x_β is a subnet of both z_α and y_β, and x'_β is a subnet of both z'_α and y_β.*

Proof. Denote by ξ the common limit of z_α and z'_α, and by $\mathcal{D}$ and $\mathcal{D}'$ the directed sets of z_α and z'_α, respectively, which we may consider to be disjoint. Let $\mathcal{N}$ be the set of neighbourhoods of ξ in the horofunction compactification of X.

To be able to handle the two nets simultaneously, we write $\bar{z}_\alpha := z_\alpha$ when $\alpha \in \mathcal{D}$, and $\bar{z}_\alpha := z'_\alpha$ when $\alpha \in \mathcal{D}'$.

Define $\mathcal{Y}$ be the set of elements (W, α, ε) of $\mathcal{N} \times (\mathcal{D} \cup \mathcal{D}') \times (0, \infty)$ such that $\bar{z}_\alpha \in W$ and

$$0 \leq d(b, \bar{z}_\alpha) + \xi(\bar{z}_\alpha) < \varepsilon. \tag{2}$$

We may write $\mathcal{Y} := \mathcal{E} \cup \mathcal{E}'$, where $\mathcal{E}$ is the set of elements (W, α, ε) of $\mathcal{Y}$ where $\alpha \in \mathcal{D}$ and $\mathcal{E}'$ is the set where $\alpha \in \mathcal{D}'$.

Define on this set an order by $(W, \alpha, \varepsilon) \leq (W', \alpha', \varepsilon')$ if either the two elements are identical, or if $W \supset W'$, $\varepsilon \geq \varepsilon'$, and

$$|d(\bar{z}_\alpha, y) - d(b, y) - \xi(\bar{z}_\alpha)| < \varepsilon, \qquad \text{for all } y \in W'. \tag{3}$$

We further require that $\alpha \leq \alpha'$ if α and α' either both belong to $\mathcal{E}$ or both belong to $\mathcal{E}'$. It is not hard to verify that this order is reflexive and transitive.

To show that every pair of elements (W,α,ε) and $(W',\alpha',\varepsilon')$ has an upper bound, take $\varepsilon'' := \min(\varepsilon,\varepsilon')$, and choose W'' to be a neighbourhood of ξ smaller than both W and W', small enough that (3) holds for both $\bar{z}_\alpha$ and $\bar{z}_{\alpha'}$, for all $y \in W'''$. Finally, α''' must be chosen large enough in $\mathcal{D} \cup \mathcal{D}'$ that $\bar{z}_{\alpha'''} \in W'''$ and (2) holds.

Define the net $y_\beta := \bar{z}_\alpha$, for all $\beta := (W,\alpha,\varepsilon)$ in $\mathcal{Y}$.

Suppose some $\lambda > 0$ is given. If $\beta := (W,\alpha,\varepsilon) \in \mathcal{Y}$ is large enough that $\varepsilon < \lambda/2$, and $\beta' := (W',\alpha',\varepsilon') \in \mathcal{Y}$ is such that $\beta \le \beta'$, then combining 2 and (3) we get $d(b,y_\beta) + d(y_\beta,y_{\beta'}) - d(b,y_{\beta'}) < \lambda$. This proves that the net y_β is an almost geodesic.

We define the nets

$$\begin{aligned} x_\beta &:= \bar{z}_\alpha, \qquad \text{for } \beta := (W,\alpha,\varepsilon) \text{ in } \mathcal{E}, \qquad \text{and} \\ x'_\beta &:= \bar{z}_\alpha, \qquad \text{for } \beta := (W,\alpha,\varepsilon) \text{ in } \mathcal{E}'. \end{aligned}$$

These two nets are clearly subnets of y_β since both $\mathcal{E}$ and $\mathcal{E}'$ are cofinal subsets of $\mathcal{Y}$. As subnets of an almost geodesic, they are themselves almost geodesics.

For $\beta := (W,\alpha,\varepsilon)$ in $\mathcal{E}$, we may write $x_\beta = z_{a(\beta)}$, where $a(\beta) := \alpha$. □

Proposition 2.6. *Let z_α be an almost geodesic in a complete metric space (X,d) with basepoint b. If $d(b,z_\alpha)$ is bounded, then z_α converges to a point in X.*

Proof. For any $\varepsilon > 0$, we have, for α and α' large enough with $\alpha < \alpha'$,

$$d(b,z_\alpha) \le d(b,z_\alpha) + d(z_\alpha,z_{\alpha'}) \le d(b,z_{\alpha'}) + \varepsilon.$$

Hence, $d(b,z_\alpha)$ is an almost non-decreasing net of real numbers. By assumption, it is bounded above. We deduce that it converges to some real number, as α tends to infinity. But again, for any $\varepsilon > 0$, we have, for α and α' large enough with $\alpha < \alpha'$, $d(z_\alpha,z_{\alpha'}) \le d(b,z_{\alpha'}) - d(b,z_\alpha) + \varepsilon$. So, we see that z_α is a Cauchy net, and hence converges since X is assumed complete. □

2.4. **The detour cost.** Let (X,d) be a metric space with base-point b. One defines the *detour cost* for any two horofunctions ξ and η in $X(\infty)$ to be

$$H(\xi,\eta) := \sup_{W \ni \xi} \inf_{x \in W \cap X} \big(d(b,x) + \eta(x)\big),$$

where the supremum is taken over all neighbourhoods W of ξ in $X \cup X(\infty)$. This concept first appeared in [1] in a slightly different setting. More detail about it can be found in [19].

Lemma 2.7. *Let ξ and η be horofunctions of a metric space (X,d). Then, there exists a net converging to ξ such that*

$$H(\xi,\eta) = \lim_\alpha \big(d(b,z_\alpha) + \eta(z_\alpha)\big).$$

Proof. To ease notation, write $f(x) := d(b,x) + \eta(x)$, for all $x \in X$.

Let $\mathcal{N}$ be the set of neighbourhoods of ξ in $X \cup X(\infty)$. Define an order on the set

$$\mathcal{D} := \{(W,x) \in \mathcal{N} \times X \mid x \in W \cap X\}$$

by $(W_1,x_1) \le (W_2,x_2)$ if $W_1 \supset W_2$. This order makes $\mathcal{D}$ into a directed set.

For each $\alpha := (W,x) \in \mathcal{D}$, let $z_\alpha := x$. Clearly, the net z_α converges to ξ.

Let E be an open neighbourhood of $H(\xi,\eta)$ in $[0,\infty]$, and let $(W',x') \in \mathcal{D}$. Take $W \in \mathcal{N}$ such that $W \subset W'$ and

$$\inf_{y \in W \cap X} f(y) \in E.$$

We can then take $x \in W \cap X$ such that $f(x) \in E$. So, $\alpha := (W, x)$ satisfies $\alpha \geq (W', x')$ and $f(z_\alpha) \in E$. This shows that $H(\xi, \eta)$ is a cluster point of the net $f(z_\alpha)$.

Therefore, there is some subnet w_β of z_α such that $f(w_\beta)$ converges to $H(\xi, \eta)$. □

The following was proved in [18, Lemma 3.3] in a slightly different setting. There is a proof in [19] that works with very little modification in the present setting with nets.

Lemma 2.8. *Let z_α be an almost-geodesic net converging to a Busemann point ξ, and let $y \in X$. Then,*

$$\lim_\alpha \big(d(y, z_\alpha) + \xi(z_\alpha)\big) = \xi(y).$$

Moreover, for any horofunction η,

$$H(\xi, \eta) = \lim_\alpha \big(d(b, z_\alpha) + \eta(z_\alpha)\big).$$

Theorem 2.9. *A horofunction ξ is a Busemann point if and only if $H(\xi, \xi) = 0$.*

Proof. If ξ is a Busemann point, then it follows from Lemma 2.8 that $H(\xi, \xi) = 0$.

Now assume that ξ is a horofunction satisfying $H(\xi, \xi) = 0$. By Lemma 2.7, there is a net $z_\alpha : \mathcal{D} \to X$ in the metric space (X, d) converging to ξ such that $d(b, z_\alpha) + \xi(z_\alpha)$ converges to zero.

Define the set

$$\mathcal{D}' := \Big\{(\alpha, \beta, \varepsilon) \in \mathcal{D} \times \mathcal{D} \times \mathbb{R} \mid \varepsilon > 0,\ \alpha \leq \beta,\ |d(z_\alpha, z_\gamma) - d(b, z_\gamma) - \xi(z_\alpha)| < \varepsilon \text{ for all } \gamma \geq \beta\Big\}$$

Observe that, for any $\alpha \in \mathcal{D}$ and $\varepsilon > 0$, there exists $\beta \in \mathcal{D}$ such that $(\alpha, \beta, \varepsilon) \in \mathcal{D}'$, because $d(\cdot, z_\gamma) - d(b, z_\gamma)$ converges to $\xi(\cdot)$ pointwise.

Define on $\mathcal{D}'$ the order relation $\leq$, where $(\alpha, \beta, \varepsilon) \leq (\alpha', \beta', \varepsilon')$ if either $(\alpha, \beta, \varepsilon)$ and $(\alpha', \beta', \varepsilon')$ are identical, or if $\beta \leq \alpha'$ and $\varepsilon \geq \varepsilon'$. This relation is easily seen to be reflexive and transitive. Also, it is not hard to show that $\mathcal{D}'$ is directed by $\leq$.

The map $h\colon (\alpha, \beta, \varepsilon) \mapsto \alpha$ is monotone, and its image is cofinal, that is, for any $\alpha \in \mathcal{D}$, there exists $(\alpha', \beta', \varepsilon') \in \mathcal{D}'$ such that $h(\alpha', \beta', \varepsilon') \geq \alpha$.

So, the net y_κ defined by $y_\kappa := z_{h(\kappa)}$, for all $\kappa \in \mathcal{D}'$, is a subnet of z_α. In particular, it converges to ξ. Moreover, $d(b, y_\kappa) + \xi(y_\kappa)$ converges to zero.

Let $\kappa := (\alpha, \beta, \varepsilon)$ and $\kappa' := (\alpha', \beta', \varepsilon')$ be elements of $\mathcal{D}'$, satisfying $\kappa \leq \kappa'$. So, $\alpha' \geq \beta$, which implies that $|d(z_\alpha, z_{\alpha'}) - d(b, z_{\alpha'}) - \xi(z_\alpha)| < \varepsilon$.

Hence, for κ and κ' large enough, with $\kappa < \kappa'$,

$$\begin{aligned} d(y_\kappa, y_{\kappa'}) - d(b, y_{\kappa'}) &< \xi(y_\kappa) + \varepsilon \\ &< -d(b, y_\kappa) + 2\varepsilon, \end{aligned}$$

which proves that y_κ is an almost-geodesic. □

The detour cost satisfies the triangle inequality and is non-negative. By symmetrising the detour cost, we obtain a metric on the set of Busemann points:

$$\delta(\xi, \eta) := H(\xi, \eta) + H(\eta, \xi), \qquad \text{for all Busemann points } \xi \text{ and } \eta.$$

We call δ the *detour metric*. It is possibly infinite valued, so it actually an *extended metric*. One may partition the set of Busemann points into disjoint subsets in such a way that $\delta(\xi, \eta)$ is finite if and only if ξ and η lie in the same subset. We call these subsets the *parts* of the horofunction boundary.

The following expression for the detour cost will prove useful. See [15, Prop. 4.5].

Proposition 2.10. *Let ξ be a Busemann point, and η a horofunction of a metric space (X,d). Then,*

$$H(\xi,\eta) = \sup_{x\in X}\big(\eta(x)-\xi(x)\big) = \inf\big\{\lambda\in\mathbb{R} \mid \eta(\cdot)\le\xi(\cdot)+\lambda\big\}.$$

3. The Busemann points of a normed space

In this section, we determine the Busemann points of an arbitrary normed space.

Let K be a convex subset of a locally-convex topological vector space E. A function f defined on K is said to be affine if

$$f\big((1-\lambda)x+\lambda y\big) = (1-\lambda)f(x)+\lambda f(y),$$

for all $x, y \in K$ and $\lambda\in[0,1]$. We denote by $A(K,E)$ the set of affine functions on K that are the restrictions of continuous affine functions on the whole of E. The following is [2, Cor. I.1.4]

Lemma 3.1. *If K is a compact convex subset of E and $a\colon K\to(-\infty,\infty]$ is a lower semicontinuous affine function, then there is a non-decreasing net in $A(K,E)$ converging pointwise to a.*

The following lemma extends Dini's theorem.

Lemma 3.2. *Let g_α be an almost non-increasing net of functions on a Hausdorff space D. Then, g_α converges pointwise to a function g. If the g_α are upper-semicontinuous, then so is the limit. If furthermore D is compact, then $\sup g_\alpha$ converges to $\sup g$.*

Proof. Let g_α be an almost non-increasing net of functions on D, and choose $x\in D$. It is clear that for each $\varepsilon>0$, we have $\liminf_\alpha g_\alpha(x)\ge\limsup_\alpha g_\alpha(x)-\varepsilon$, from which it follows that $g_\alpha(x)$ converges.

Denote by g the pointwise limit of g_α. Let x_β be a net in D converging to a point $x\in D$, and let $\varepsilon>0$. Assume that each g_α is upper semicontinuous, and so $g_\alpha(x)\ge\limsup_\beta g_\alpha(x_\beta)$. That g_α is almost non-increasing implies that $g\le g_\alpha+\varepsilon$ for α large enough. So, choose α large enough that this holds and $g_\alpha(x)\le g(x)+\varepsilon$. Putting all this together, we get $g(x)\ge\limsup_\beta g(x_\beta)-2\varepsilon$, and we conclude that g is upper semicontinuous.

Now assume that D is compact. So, for each α, since g_α is upper semicontinuous, it attains its supremum at some point x_α, and furthermore the net x_α has a cluster point x in D. By passing to a subnet if necessary, we may assume that $g_\alpha(x_\alpha)$ converges to a limit l, and that x_α converges to x.

Let $\varepsilon>0$. For α large enough, $g_{\alpha'}(x_{\alpha'})\le g_\alpha(x_{\alpha'})+\varepsilon$ for all $\alpha'\ge\alpha$. Taking the limit supremum in α', using the upper semicontinuity of g_α, and then taking the limit in α we get that $l\le g(x)+\varepsilon\le\sup g+\varepsilon$, and hence that $l\le\sup g$, since ε was chosen arbitrarily. The opposite inequality comes from the fact that, in general, the limit of a supremum is greater than or equal to the supremum of the limit. □

Let $(X,||\cdot||)$ be a normed space. We denote by B the unit ball of X, and by B° the dual ball. The topological dual space of X is denoted by X^*, and we take the weak* topology on this space. The dual ball is compact in the weak* topology, by the Banach–Alaoglu theorem.

Recall that the Legendre–Fenchel transform of a function f on X is defined to be

$$f^*(y) := \sup_{x\in X}\big(\langle y,x\rangle - f(x)\big),\qquad \text{for all } y\in X^*.$$

Since it is a supremum of weak* continuous affine functions, f^* is weak* lower semi-continuous and convex. One may also define the transform of a function g on X^* as follows:

$$g^*(x) := \sup_{y \in X^*} \big(\langle y, x\rangle - g(x)\big), \qquad \text{for all } x \in X.$$

These maps are inverses of one another in the following sense. A function taking values in $(-\infty, \infty]$ is said to be *proper* if it not identically ∞. Recall that a lower semicontinuous convex function is automatically weakly lower semicontinuous. Denoting by $\Gamma(X)$ the proper lower semicontinuous convex functions on X, and by $\Gamma^*(X^*)$ the proper weak* lower semicontinuous convex functions on X^*, we have

$$f^{**} = f \quad \text{for } f \in \Gamma(X) \qquad \text{and} \qquad g^{**} = g \quad \text{for } g \in \Gamma^*(X^*).$$

We use the notation $f|_G$ to denote the restriction of a function f to a set G. Also, we denote by $X_B(\infty)$ the set of Busemann points of a metric space X.

Theorem 3.3. *Let $(X, ||\cdot||)$ be a normed space. A function on X is in $X_B(\infty) \cup X$ if and only if it is the Legendre–Fenchel transform of a function that is affine on the dual ball, infinite outside the dual ball, and weak* lower semi-continuous, and has infimum 0.*

Proof. The Legendre–Fenchel transform of any function is automatically weak* lower semi-continuous. Every Busemann point is 1-Lipschitz, and so its transform takes the value ∞ outside the dual ball. Since each Busemann point takes the value 0 at the origin, the transform has infimum 0. That the transform of a Busemann point must be affine on the dual ball was proved in [16, Lemma 3.1]; the theorem is stated there for finite dimensional spaces, but the proof works in infinite dimension as well.

Now let f be a real-valued function on X such that its transform f^* has the properties stated. By Lemma 3.1, there exists a non-decreasing net g_α of elements of $A(B^\circ, X^*)$ that converges pointwise to f^*. For each α, we may write $g_\alpha = \langle \cdot, z_\alpha\rangle|_{B^\circ} + c_\alpha$, where $z_\alpha \in X$ and $c_\alpha \in \mathbb{R}$.

Let $m_\alpha := \inf g_\alpha$, for each α. So, m_α is a non-decreasing net of real numbers, and by Lemma 3.2 it converges to $\inf f^* = 0$. It is not too hard to calculate that, for each α, the transform of $\varphi_{z_\alpha}(\cdot) := ||z_\alpha - \cdot|| - ||z_\alpha||$ is $\varphi^*_{z_\alpha} = g_\alpha - m_\alpha$; see [16].

The Legendre–Fenchel transform is order-reversing, and so the net $(g_\alpha)^*$ is non-increasing. So, by the observation after the Definition 2.2, $(g_\alpha)^* + m_\alpha$ is almost non-increasing. But

$$(g_\alpha)^* + m_\alpha = (g_\alpha - m_\alpha)^* = \varphi_{z_\alpha}, \qquad \text{for all } \alpha.$$

Therefore, by Proposition 2.4, z_α is an almost geodesic in $(X, ||\cdot||)$.

Let x be a point in X. We have

$$\varphi_{z_\alpha}(x) = m_\alpha + \sup_{y \in B^\circ} \big(\langle y, x\rangle - g_\alpha(y)\big), \qquad \text{for all } \alpha.$$

Since g_α is non-decreasing, the net of functions $\langle \cdot, x\rangle - g_\alpha(\cdot)$ is non-increasing. So, by Lemma 3.2, its supremum over the dual ball B° converges to the supremum of the pointwise limit $\langle \cdot, x\rangle - f^*(\cdot)$. We deduce that φ_{z_α} converges pointwise to f. We have thus proved that f is either a Busemann point or a point in the horofunction compactification corresponding to an element of X. □

We now determine the detour metric on the boundary of a normed space.

Theorem 3.4. *Let ξ_1 and ξ_2 be Busemann points of a normed space, having Legendre–Fenchel transforms g_1 and g_2, respectively. Then, the distance between them in the detour metric is*

$$\delta(\xi_1, \xi_2) = \sup_{y \in B^\circ} \big(g_1(y) - g_2(y)\big) + \sup_{y \in B^\circ} \big(g_2(y) - g_1(y)\big).$$

Proof. By the properties of the Legendre–Fenchel transform, we have, for any $\lambda \in \mathbb{R}$, that $\xi_2 \leq \xi_1 + \lambda$ if and only if $g_2 \geq g_1 - \lambda$. So, applying Proposition 2.10, we get

$$H(\xi_1, \xi_2) = \inf \big\{\lambda \in \mathbb{R} \mid g_2 \geq g_1 - \lambda\big\} = \sup_{y \in B^\circ} \big(g_1(y) - g_2(y)\big).$$

The result is now obtained upon symmetrising. □

Corollary 3.5. *Two Busemann points of a normed space are in the same part if and only if their respective Legendre–Fenchel transforms g_1 and g_2 satisfy*

$$g_1 - c \leq g_2 \leq g_1 + c, \qquad \text{for some } c \in \mathbb{R}. \tag{4}$$

Corollary 3.6. *A function ξ is a singleton Busemann point of a normed space if and only if it is an extreme point of the dual ball.*

Proof. If ξ_1 is an extreme point of the dual ball, then its transform g_1 takes the value zero at ξ_1 and infinity everywhere else. Let ξ_2 be another Busemann point in the same part, which implies that its transform g_2 satisfies (4). So, g_2 is finite at ξ_1 and infinite everywhere else, and since, by Theorem 3.3, it has infimum zero, we get that $g_2 = g_1$. Hence ξ_2 is identical to ξ_1.

Now let ξ_1 be a Busemann point that is not an extreme point of the dual ball, and let g_1 be its transform. Since g_1 is affine on the dual ball, the set on which it is finite is an extreme set of this ball, and therefore must contain at least two points, for otherwise ξ_1 would be an extreme point. Choose an element x of the normed space such that $\langle \cdot, x\rangle$ separates these two points, that is, does not take the same value at the two points. The function

$$g_2 := g_1 + \langle \cdot, x\rangle - \inf_{B^\circ}(g_1 + \langle \cdot, x\rangle)$$

satisfies all the conditions of Theorem 3.3, and so its transform ξ_2 is a Busemann point. Moreover, g_1 and g_2 satisfy (4), which implies that ξ_2 is in the same part as ξ_1. But, by construction, g_2 differs from g_1, and so ξ_2 differs from ξ_1. □

4. The Masur–Ulam Theorem

The techniques developed so far allow us to write a short proof of the Masur–Ulam theorem.

Recall that, according to Corollary 3.6, the singleton Busemann points of a normed space are exactly the extreme points of the dual ball. Recall also that any surjective isometry between metric spaces can be extended to a homeomorphism between their horofunction boundaries, which maps singletons to singletons.

Theorem 4.1 (Masur–Ulam)**.** *Let $\varphi\colon X \to Y$ be a surjective isometry between two normed spaces. Then, φ is affine.*

Proof. It will suffice to assume that φ maps the origin of X to the origin of Y, and show that it is linear.

Let f be an extreme point of the dual ball of Y. So, f is a singleton of the horofunction boundary of Y. Therefore, $f \circ \varphi$ is a singleton of the horofunction boundary of X, and hence an extreme point of the dual ball of X, and hence linear. So,

$$\begin{aligned} f\big(\varphi(\alpha x + \beta y)\big) &= \alpha f(\varphi(x)) + \beta f(\varphi(y)) \\ &= f(\alpha\varphi(x) + \beta\varphi(y)), \qquad \text{for all } \alpha, \beta \in \mathbb{R} \text{ and } x, y \in X. \end{aligned}$$

Since this is true for every extreme point f of the dual ball of Y, we have $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$. □

5. The horofunction boundary of $(C(K), ||\cdot||_\infty)$

In this section we look in more detail at the space $C(K)$ with the supremum norm, where K is an arbitrary compact Hausdorff space. Here we can describe explicitly the Busemann points.

Theorem 5.1. *The Busemann points of $(C(K), ||\cdot||_\infty)$ are the functions of the following form*

$$\Phi(g) := \sup_{x\in K} \big(-u(x) - g(x)\big) \vee \sup_{x\in K} \big(-v(x) + g(x)\big), \qquad \text{for all } g \in C(K), \tag{5}$$

where u and v are two lower-semicontinuous functions from $K \to [0,\infty]$, such that $\inf u \wedge \inf v = 0$, and such that $u(x) \vee v(x) = \infty$ for all $x \in K$.

The proof will use the characterisation in the previous section of the Legendre–Fenchel transforms of the Busemann points of a normed space. Recall that these were shown to be the functions that are affine on the dual ball, infinite outside the dual ball, weak* lower semi-continuous, and have infimum 0. We will identify all such functions on the dual space of $C(K)$.

Recall that the dual space of $C(K)$ is $ca_r(K)$, the set of regular signed Borel measures on K of bounded variation. Any element μ of $ca_r(K)$ can be written $\mu = \mu^+ - \mu^-$, where μ^- and μ^+ are non-negative measures. This is called the Jordan decomposition. The dual norm is the *total variation norm*, which satisfies $||\mu|| = ||\mu^-|| + ||\mu^+||$.

Proposition 5.2. *Consider a function from $\Theta\colon ca_r(K) \to \mathbb{R}_+$ that is not the restriction to the dual ball of a continuous affine function. Then, Θ is affine on the dual ball, infinite outside the dual ball, weak* lower semi-continuous, and has infimum 0 if and only if it can be written*

$$\Theta(\mu) = \Xi(\mu) := \begin{cases} +\infty, & ||\mu|| \neq 1; \\ \displaystyle\int u \, d\mu^- + \int v \, d\mu^+, & ||\mu|| = 1, \end{cases} \tag{6}$$

where u and v are as in the statement of Theorem 5.1.

The proof if this proposition will require several lemmas.

Lemma 5.3. *The function Ξ in (6) is lower semicontinuous.*

Proof. Let μ_α be a net in $ca_r(K)$ converging in the weak* topology to $\mu \in ca_r(K)$. We must show that $\liminf_\alpha \Xi(\mu_\alpha) \geq \Xi(\mu)$. By taking a subnet if necessary, we may suppose that $\Xi(\mu_\alpha)$ converges to a limit, which we assume to be finite. This implies that $||\mu_\alpha|| = 1$, eventually. Since the dual unit ball is compact, we may, by taking a further subnet if necessary, assume that μ_α^+ and μ_α^- converge, respectively, to non-negative measures ν and ν'. These measures satisfy $\mu = \mu^+ - \mu^- = \nu - \nu'$, and, so, from the minimality property of the Jordan decomposition, $\nu \geq \mu^+$ and $\nu' \geq \mu^-$.

Since u and v are lower-semicontinuous, we get from the Portmanteau theorem that

$$\begin{aligned}\liminf_{\alpha} \Xi(\mu_\alpha) &\geq \liminf_{\alpha} \Big(\int u\, d\mu_\alpha^-\Big) + \liminf_{\alpha} \Big(\int v\, d\mu_\alpha^+\Big) \\ &\geq \int u\, d\nu' + \int v\, d\nu. \qquad (7)\end{aligned}$$

Consider the case where $\bar{\mu} := \nu - \mu^+ = \nu' - \mu^-$ is non-zero. Since $u + v$ is identically infinity, either $\int u\, d\bar{\mu}$ or $\int v\, d\bar{\mu}$ must equal infinity. This implies that the right-hand-side of (7) is equal to infinity.

On the other hand, if $\nu = \mu^+$ and $\nu' = \mu^-$, then

$$||\mu|| = ||\mu^+|| + ||\mu^-|| = ||\nu|| + ||\nu'|| = \lim_{\alpha} ||\mu_\alpha|| = 1.$$

So, in this case, the right-hand-side of (7) is equal to $\Xi(\mu)$. □

Lemma 5.4. *Let μ_1 and μ_2 be in the unit ball of $ca_r(K)$. Let $\mu := (1-\lambda)\mu_1 + \lambda\mu_2$, for some $\lambda \in (0,1)$. If $||\mu|| = 1$, then $||\mu_1|| = ||\mu_2|| = 1$, and $\mu^+ = (1-\lambda)\mu_1^+ + \lambda\mu_2^+$ and $\mu^- = (1-\lambda)\mu_1^- + \lambda\mu_2^-$.*

Proof. Observe that the functions $\mu \mapsto ||\mu^+||$ and $\mu \mapsto ||\mu^-||$ are both convex, and hence

$$||\mu^+|| \leq (1-\lambda)||\mu_1^+|| + \lambda||\mu_2^+|| \qquad \text{and} \qquad (8)$$

$$||\mu^-|| \leq (1-\lambda)||\mu_1^-|| + \lambda||\mu_2^-||. \qquad (9)$$

Moreover, the sum of these two functions is $\mu \mapsto ||\mu||$. Using that $||\mu_1|| \leq 1$ and $||\mu_2|| \leq 1$, and $||\mu|| = 1$, we deduce that inequalities (8) and (9) are actually equalities.

Since $\mu \mapsto \mu^+$ is also convex, we have $\mu^+ \leq (1-\lambda)\mu_1^+ + \lambda\mu_2^+$. Combining this with the equalities just established, we get that $\mu^+ = (1-\lambda)\mu_1^+ + \lambda\mu_2^+$, since the norm is additive on non-negative measures. The equation involving μ^- is proved similarly. □

Lemma 5.5. *The function Ξ is affine on the unit ball of $ca_r(K)$.*

Proof. Let μ, μ_1, and μ_2 be in the unit ball of $ca_r(K)$, such that $\mu = (1-\lambda)\mu_1 + \lambda\mu_2$, for some $\lambda \in (0,1)$. We wish to show that $\Xi(\mu) = (1-\lambda)\Xi(\mu_1) + \lambda\Xi(\mu_2)$.

Consider the case when $||\mu|| = 1$. By Lemma 5.4, $||\mu_1|| = ||\mu_2|| = 1$, and $\mu^+ = (1-\lambda)\mu_1^+ + \lambda\mu_2^+$ and $\mu^- = (1-\lambda)\mu_1^- + \lambda\mu_2^-$. We deduce that the second case in the definition of Ξ is the relevant one, for each of μ, μ_1, and μ_2, and furthermore that the affine relation holds.

Now consider the case where $||\mu|| < 1$. So, $\Xi(\mu) = \infty$. To prove the affine relation, we must show that either $\Xi(\mu_1)$ or $\Xi(\mu_2)$ is infinite.

Assume for the sake of contradiction that both $\Xi(\mu_1)$ and $\Xi(\mu_2)$ are finite. Denote by U and V the subsets of K where, respectively, u and v are finite. So, U and V are disjoint. From the definition of Ξ, we see that $||\mu_1|| = ||\mu_2|| = 1$, that μ_1^+ and μ_2^+ are concentrated on V, and that μ_1^- and μ_2^- are concentrated on U. It follows that $||\mu|| = (1-\lambda)(||\mu_1^+|| + ||\mu_1^-||) + \lambda(||\mu_2^+|| + ||\mu_2^-||) = 1$, which contradicts our assumption. □

The following result is due to Choquet; see [2, Theorem I.2.6].

Theorem 5.6. *If f is a real-valued affine function of the first Baire class on a compact convex set K in a locally convex Hausdorff space, then f is bounded and $f(x) = \int f\, d\mu$, where μ is any probability measure on K and x is the barycenter of μ.*

We will need a version of Lebesgue's monotone convergence theorem for *nets* of functions. The following was proved in [5, Proposition 2.13].

Lemma 5.7. *Let X be a locally compact and σ-compact Hausdorff space, and let λ be a positive Borel measure that is complete and regular and satisfies $\lambda(K) < \infty$ for all compact sets $K \subset X$.*

Let I be a directed set, and let $f_i\colon X \to [0,\infty]$, $i \in I$, be a family of lower semicontinuous functions that is monotone non-decreasing. Set $f(x) := \sup_{i\in I} f_i(x)$ for all $x \in X$. Then,

$$\int_X f \, d\lambda = \sup_{i\in I} \int_X f_i \, d\lambda.$$

Lemma 5.8. *If a function $\Xi\colon ca_r(K) \to \mathbb{R}_+$ is affine on the dual ball, infinite outside the dual ball, weak* lower semi-continuous, and has infimum 0, then it either can be written in the form (6) or is the restriction to the dual ball of a continuous affine function on the dual space.*

Proof. Denote by δ_x the measure consisting of an atom of mass one at a point x. Define the functions

$$v\colon K \to [0,\infty], \qquad v(x) := \Xi(\delta_x), \qquad \text{and}$$
$$u\colon K \to [0,\infty], \qquad u(x) := \Xi(-\delta_x).$$

These two functions are non-negative because $\inf \Xi = 0$. Moreover, the dual ball is weak* compact, and so, as a lower semicontinuous affine function, Ξ attains its infimum over it at an extreme point. Recall that, in the present case, the extreme points are exactly the positive and negative Dirac masses; denote the set of these by $\partial_e := \partial_e^+ \cup \partial_e^-$, where $\partial_e^+ := \{\delta_x \mid x \in K\}$ and $\partial_e^- := \{-\delta_x \mid x \in K\}$. Thus, $\inf u \wedge \inf v = 0$.

Also observe that u and v are lower semicontinuous.

Consider the case where $u(x)$ and $v(x)$ are both finite for some $x \in K$. This implies that $\Xi(0)$ is finite since Ξ is affine. It follows from this that Ξ is finite on the whole of the dual ball. Using the fact that the dual ball is balanced, that is, closed under multiplication by scalars of absolute value less than or equal to 1, we can reflect about the origin to get that Ξ is upper semicontinuous. So, Ξ is continuous on the dual ball. It is hence the restriction of a continuous affine function on the whole dual space; see [2, Cor. I.1.9].

So, from now on, assume that $u(x) \vee v(x) = \infty$, for all $x \in K$

Since Ξ is affine on the dual ball and infinite outside it, the set where Ξ is finite is an extreme set of the dual ball. Note that, given any distinct points μ_1 and μ_2 in the dual ball such that $||\mu_2|| < 1$, there is a line segment in the dual ball having μ_1 as a endpoint and μ_2 as a point in the relative interior. It follows that if Ξ is finite at some point μ_2 with $||\mu_2|| < 1$, then Ξ is finite everywhere in the dual ball. But this contradicts what we have just assumed, and we conclude that $\Xi(\mu)$ takes the value $+\infty$ if $||\mu|| < 1$.

Let μ be in the dual ball such that $||\mu|| = 1$. By Choquet theory, there is a probability measure $\overline{\mu}$ on the dual ball that has barycenter μ and is pseudo-concentrated on the extreme points of the dual ball. In the present case, since the set of extreme points is closed, and hence measurable, $\overline{\mu}$ is concentrated on the extreme points.

In fact, we have the following description of $\overline{\mu}$: for any measurable subset

$$U = \{\delta_x \mid x \in U^+ \subset K\} \cup \{-\delta_x \mid x \in U^- \subset K\}$$

of ∂_e, we have $\overline{\mu}[U] := \mu^+[U^+] + \mu^-[U^-]$.

By Lemma 3.1, there is a non-decreasing net g_α of continuous affine functions on the dual space converging pointwise to Ξ on the dual ball. Applying Theorem 5.6 and

Lemma 5.7, we get

$$\begin{aligned}\Xi(\mu) &= \lim_\alpha g_\alpha(\mu) \\ &= \lim_\alpha \int g_\alpha \, d\overline{\mu} \\ &= \int \Xi \, d\overline{\mu} \\ &= \int_K \Xi(\delta_x)\, d\mu^+ + \int_K \Xi(-\delta_x)\, d\mu^- \\ &= \int_K v(x)\, d\mu^+ + \int_K u(x)\, d\mu^- .\end{aligned}$$

Finally, it's not hard to show that $\Xi(\mu)$ takes the value $+\infty$ if $||\mu|| < 1$. □

Proof of Proposition 5.2. Any function Ξ of the given form is clearly infinite outside the dual ball and has infimum zero. The rest was proved in Lemmas 5.3, 5.5, and 5.8. □

Lemma 5.9. *The function Ξ is the Legendre–Fenchel transform of the function Φ in Theorem 5.1.*

Proof. Fix $g \in C(K)$, and let $\Psi \colon ca_r(K) \to \mathbb{R}_+$ be defined by $\Psi(\mu) := \langle \mu, g\rangle - \Xi(\mu)$. By Lemmas 5.3 and 5.5, Ψ is upper-semicontinuous and affine on the unit ball of $ca_r(K)$. Outside the unit ball, Ψ takes the value $-\infty$. So, the supremum of Ψ is attained at an extreme point of the unit ball. The set of these extreme points is $\{\delta_x \mid x \in K\} \cup \{-\delta_x \mid x \in K\}$. For all $x \in K$, we have that $\Psi(\delta_x) = g(x) - v(x)$ and $\Psi(-\delta_x) = -g(x) - u(x)$. It follows that the Legendre-Fenchel transform of Ξ is the function Φ given in (5).

Since Ξ is a lower-semicontinuous proper convex function, it is equal to the transform of its transform. □

We can now prove Theorem 5.1.

Proof of Theorem 5.1. We combine Theorem 3.3, Proposition 5.2, and Lemma 5.9. □

6. The horofunction boundary of the reverse-Funk geometry

Although they are not strictly speaking metric spaces, the reverse-Funk and Funk geometries retain enough of the properties of metric spaces for the definition of the horofunction boundary, and of Busemann points, to make sense. In this and the following section, we study the boundary of these two geometries.

Recall that the indicator function I_E of a set E is defined to take the value 1 on E and the value 0 everywhere else.

Lemma 6.1. *Let D be a compact convex subset of a locally-convex Hausdorff space E, and let f_1 and f_2 be upper-semicontinuous real-valued non-negative affine functions on D. If $\sup_D f_1/g \le \sup_D f_2/g$ for each continuous real-valued affine function g on E that is positive on D, then $f_1 \le f_2$ on D.*

Proof. Let y be an extreme point of D. The function $1/\mathrm{I}_{\{y\}}$, which takes the value 1 at y and the value ∞ everywhere else, is a weak*-lower-semicontinuous affine function on D. Therefore, there exists a non-decreasing net g_α of continuous real-valued affine functions that are positive on D such that g_α converges pointwise on D to $1/\mathrm{I}_{\{y\}}$. So, both f_1/g_α and f_2/g_α are non-increasing nets of real-valued upper-semicontinuous functions on D

converging pointwise, respectively, to $f_1 \, \mathrm{I}_{\{y\}}$ and $f_2 \, \mathrm{I}_{\{y\}}$. By Lemma 3.2, $\sup_D f_1/g_\alpha$ and $\sup_D f_2/g_\alpha$ converge respectively to $f_1(y)$ and $f_2(y)$. We conclude that $f_1(y) \leq f_2(y)$.

This is true for any extreme point of D, and the conclusion follows upon applying Choquet theory. □

The next lemma is similar to the previous one.

Lemma 6.2. *Let D be a compact convex subset of a locally-convex Hausdorff space E, and let f_1 and f_2 be lower-semicontinuous positive affine functions on D. If $\sup_D g/f_1 \leq \sup_D g/f_2$ for each continuous real-valued positive affine function g on D, then $f_1 \geq f_2$ on D.*

Proof. For each extreme point y of D, the indicator function $\mathrm{I}_{\{y\}}$ is a weak*-upper-semicontinuous affine function on D. Therefore, there exists a non-increasing net g_α of continuous real-valued positive affine functions on D that converges pointwise to $\mathrm{I}_{\{y\}}$. So, both g_α/f_1 and g_α/f_2 are non-increasing nets of functions converging pointwise, respectively, to $\mathrm{I}_{\{y\}}/f_1$ and $\mathrm{I}_{\{y\}}/f_2$. By Lemma 3.2, $\sup_D g_\alpha/f_1$ and $\sup_D g_\alpha/f_2$ converge respectively to $1/f_1(y)$ and $1/f_2(y)$. We conclude that $f_1(y) \geq f_2(y)$.

This is true for any extreme point of D, and the conclusion follows upon applying Choquet theory. □

Theorem 6.3. *The Busemann points of the reverse-Funk geometry on C are the functions of the following form:*

$$\xi(x) := \log \sup_{y \in C^* \setminus \{0\}} \frac{g(y)}{\langle y, x \rangle}, \qquad \text{for all } x \in C, \tag{10}$$

where g is a weak-upper-semicontinuous non-negative linear functional on C^*, normalised so that* $\sup_{y \in C^* \setminus \{0\}} g(y)/\langle y, b \rangle = 1$.

Proof. We consider the cross-section $D := \{y \in C^* \mid \langle y, b \rangle = 1\}$, which is compact. Each linear functional on C^* corresponds to an affine function on D.

Let ξ be of the above form. Take a net g_α of elements of C that, when viewed as a net of continuous affine functions on D, is non-increasing and converges pointwise to g. Fix $x \in C$. So, the function $y \mapsto g_\alpha(y)/\langle y, x \rangle$ defined on D is non-increasing and converges pointwise to $g(y)/\langle y, x \rangle$. Therefore, by Lemma 3.2, the net

$$d_R(x, g_\alpha) := \log \sup_{y \in D} \frac{g_\alpha(y)}{\langle y, x \rangle}$$

converges to $\xi(x)$. In particular, $d_R(b, g_\alpha)$ converges to zero. It follows that g_α converges to ξ in the reverse-Funk horofunction boundary. Moreover, the monotonicity of the convergence implies that $d_R(\cdot, g_\alpha) - d_R(b, g_\alpha)$ is an almost non-increasing net of functions; see the observation after Definition 2.2. So, by Proposition 2.4, g_α is an almost-geodesic and ξ is a Busemann point.

Now let g_α be an almost-geodesic net in C converging to a Busemann point ξ. So, $d_R(\cdot, g_\alpha) - d_R(b, g_\alpha)$ is an almost non-increasing net of functions converging to ξ. By scaling g_α if necessary, we may assume that $d_R(b, g_\alpha) = 0$, for all α.

So, for any $\varepsilon > 0$, there exists an index A such that

$$\sup_{y \in D} \frac{g_{\alpha'}(y)}{\langle y, x \rangle} \leq e^\varepsilon \sup_{y \in D} \frac{g_\alpha(y)}{\langle y, x \rangle}, \qquad \text{for all } x \in C,$$

whenever α and α' satisfy $A \leq \alpha \leq \alpha'$. But this implies by Lemma 6.1 that $g_{\alpha'} \leq e^\varepsilon g_\alpha$ on D, whenever $A \leq \alpha \leq \alpha'$. We conclude that $\log g_\alpha|_D$ is an almost non-increasing net.

Applying Lemma 3.2 and exponentiating, we get that g_α converges pointwise on D to an upper semicontinuous function g, which is necessarily affine and non-negative. We extend g to a linear functional on C^* using homogeneity.

By applying Lemma 3.2 to the function $\log g_\alpha(\cdot) - \log\langle\cdot, x\rangle$ on D, we get that ξ, which is the pointwise limit of $d_R(\cdot, g_\alpha)$, has the form given in the statement of the theorem.

The normalisation can be verified by evaluating at b. □

Theorem 6.4. *Let ξ_1 and ξ_2 be Busemann points of the reverse-Funk geometry, corresponding via (10) to linear functionals g_1 and g_2, respectively, with the properties specified in Lemma 6.3. Then, the distance between them in the detour metric is*

$$\delta(\xi_1, \xi_2) = \log \sup_{y\in C^*} \frac{g_1(y)}{g_2(y)} + \log \sup_{y\in C^*} \frac{g_2(y)}{g_1(y)}.$$

(The supremum is always taken only over those points where the ratio is well-defined).

Proof. Let $D := \{y \in C^* \mid \langle y, b\rangle = 1\}$, which is a compact set. Restricted to D, each linear functional on C^* is affine. We have $\xi_1(x) = \log\sup_D g_1/\langle\cdot, x\rangle$ for all $x \in C$, and a similar formula holds for ξ_2.

For any $\lambda \in \mathbb{R}$, we have that $\xi_2 \le \xi_1 + \lambda$ if and only if

$$\sup_{y\in D} \frac{g_2(y)}{\langle y, x\rangle} \le \sup_{y\in D} \frac{g_1(y)}{\langle y, x\rangle} e^\lambda, \qquad \text{for all } x \in C.$$

By Lemma 6.1, this is equivalent to $g_2 \le g_1 \exp(\lambda)$.

It follows using Proposition 2.10 that

$$H(\xi_1, \xi_2) = \inf\big\{\lambda \in \mathbb{R} \mid \xi_2(\cdot) \le \xi_1(\cdot) + \lambda\big\} = \log\sup_{y\in D} \frac{g_2(y)}{g_1(y)}.$$

The result is now obtained upon symmetrising. □

Corollary 6.5. *The two reverse-Funk Busemann points ξ_1 and ξ_2 are in the same part if and only if $g_2/\lambda \le g_1 \le \lambda g_2$, for some $\lambda > 0$.*

Questions 6.6. *Is it possible for reverse-Funk geometries to have non-Busemann horofunctions? This is not the case in finite dimension* [17]*, and we will see in section 9 that it is not the case either for the positive cone $C^+[0, 1]$.*

The linear functional g in the statement of the theorem is always bounded on D and therefore continuous in the norm topology of the dual. Do singleton Busemann points correspond exactly the extreme rays of the bidual cone?

We have some partial results concerning the singletons Busemann points of this geometry.

Proposition 6.7. *Let ξ be a Busemann point of the reverse-Funk geometry, and let g be as in (10). If g is in an extremal ray of the bidual cone C^{**}, then ξ is a singleton Busemann point.*

Proof. Let ξ_1 and ξ_2 be Busemann points in the same part. By Theorem 6.3, we may write both of these points in the form (10), with g_1 and g_2, respectively, substituted in for g. Both g_1 and g_2 are bounded on the cross-section D of the dual cone, and therefore continuous in the norm topology of the dual. Thus, they are both elements of the bidual.

By Corollary 6.5, there exists $\lambda > 0$ such that $g_2/\lambda \le g_1 \le \lambda g_2$. Define $f = g_1 + g_2/\lambda$ and $h = g_1 - g_2/\lambda$. Both f and h are linear functionals on the dual cone that are continuous in the norm topology of the dual space. Moreover, they are non-negative. We conclude that f and h are in the bidual cone. But we have $g_1 = f/2 + h/2$, which shows that g_1 is not in an extremal ray of the bidual cone. □

Let U be the cone of non-negative finite weak*-upper-semicontinuous linear functionals on the dual cone C^*.

Proposition 6.8. *Let ξ be a Busemann point of the reverse-Funk geometry, and let g be as in (10). If ξ is a singleton, then g is in an extremal ray of the cone U.*

Proof. Let $g = g_1 + g_2$, with g_1 and g_2 in U, and write $g' := g_1 + 2g_2$. By normalising g', the conditions of Theorem 6.3 are met, so we obtain a Busemann point ξ'. Moreover, $g'/2 \le g \le g'$, and so according to Theorem 6.4, ξ and ξ' lie in the same part of the boundary. Therefore, $\xi = \xi'$, which implies that g' is a multiple of g, which further implies that g_1 is a multiple of g_2. □

7. The horofunction boundary of the Funk geometry

The proof of the following theorem parallels that of the corresponding result for the reverse-Funk geometry.

Theorem 7.1. *The Busemann points of the Funk geometry on C are the functions of the following form:*

$$\xi(x) := \log \sup_{y \in C^* \backslash \{0\}} \frac{\langle y, x \rangle}{f(y)}, \qquad \text{for all } x \in C. \tag{11}$$

where f is a weak-lower-semicontinuous non-negative linear function on the dual cone C^*, normalised so that $\sup_{y \in C^* \backslash \{0\}} \langle y, b\rangle / f(y) = 1$.*

Proof. We consider the cross-section $D := \{y \in C^* \mid \langle y, b \rangle = 1\}$, which is compact. Each linear functional on C^* corresponds to an affine function on D.

Let ξ be of the above form. Take a net f_α of elements of C that, when viewed as a net of continuous affine functions on D, is non-decreasing and converges pointwise to f. Fix $x \in C$. So, the function $y \mapsto \langle y, x \rangle / f_\alpha(y)$ is non-increasing and converges pointwise to $\langle y, x \rangle / f(y)$. Therefore, by Lemma 3.2, the net

$$d_F(x, f_\alpha) := \log \sup_{y \in D} \frac{\langle y, x \rangle}{f_\alpha(y)}$$

converges to $\xi(x)$. In particular, $d_F(b, f_\alpha)$ converges to zero. It follows that f_α converges to ξ in the Funk horofunction boundary. Moreover, the monotonicity of the convergence implies that $d_F(\cdot, f_\alpha) - d_F(b, f_\alpha)$ is an almost non-increasing net of functions; see the observation after Definition 2.2). So, by Proposition 2.4, f_α is an almost-geodesic and ξ is a Busemann point.

Now let f_α be an almost-geodesic net in C converging to a Busemann point ξ. So, $d_F(\cdot, f_\alpha) - d_F(b, f_\alpha)$ is an almost non-increasing net of functions converging to ξ. By scaling f_α if necessary, we may assume that $d_F(b, f_\alpha) = 0$, for all α.

So, for any $\varepsilon > 0$, there exists an index A such that

$$\sup_{y \in D} \frac{\langle y, x \rangle}{f_{\alpha'}(y)} \le e^\varepsilon \sup_{y \in D} \frac{\langle y, x \rangle}{f_\alpha(y)}, \qquad \text{for all } x \in C,$$

whenever α and α' satisfy $A \le \alpha \le \alpha'$. But this implies by Lemma 6.2 that $e^\varepsilon f_{\alpha'} \ge f_\alpha$ on D, whenever $A \le \alpha \le \alpha'$. We conclude that $\log f_\alpha|_D$ is an almost non-decreasing net.

Applying Lemma 3.2 and exponentiating, we get that f_α converges pointwise on D to a lower-semicontinuous function f, which is necessarily affine and non-negative. We extend f to a linear functional on C^* using homogeneity.

By applying Lemma 3.2 to the function $\log\langle\cdot, x\rangle - \log f_\alpha(\cdot)$ on D, we get that ξ, which is the pointwise limit of $d_F(\cdot, f_\alpha)$, has the form given in the statement of the theorem.

The normalisation can be verified by evaluating at b. □

The proofs of the following theorem and corollary are similar to those of the equivalent results in the case of the reverse-Funk geometry.

Theorem 7.2. *Let ξ_1 and ξ_2 be Busemann points of the reverse-Funk geometry, corresponding via (11) to linear functionals f_1 and f_2, respectively, with the properties specified in Lemma 7.1. Then, the distance between them in the detour metric is*

$$\delta(\xi_1, \xi_2) = \log \sup_{y \in C^*} \frac{f_1(y)}{f_2(y)} + \log \sup_{y \in C^*} \frac{f_2(y)}{f_1(y)}.$$

(The supremum is always taken only over those points where the ratio is well-defined).

Corollary 7.3. *The two Funk Busemann points ξ_1 and ξ_2 are in the same part if and only if $f_2/\lambda \le f_1 \le \lambda f_2$, for some $\lambda > 0$.*

Unlike in the case of the reverse-Funk geometry, we can determine explicitly the singleton Busemann points of the Funk geometry. Recall that we have defined the cross section $D := \{y \in C^* \mid \langle y, b\rangle = 1\}$.

Corollary 7.4. *A function ξ is a singleton Busemann point of the Funk geometry if and only if it is an extreme point of D.*

Proof. The proof is the same as that of Corollary 3.6, when one considers the cross-section D instead of the dual ball. □

8. The horofunction boundary of the Hilbert geometry

In this section, we relate the boundary of the Hilbert geometry to those of the reverse-Funk and Funk geometries.

We denote by $P(C)$ the projective space of the cone C, and by $[h]$ the projective class of an element h of C. Recall that we may regard the elements of C as positive continuous linear functionals of C^*.

Proposition 8.1. *Let z_α be a net in C. Then, z_α is an almost-geodesic in the Hilbert geometry if and only if it is an almost-geodesic in both the Funk and reverse-Funk geometries.*

Proof. For α and α' satisfying $\alpha < \alpha'$, define

$$\begin{aligned} R(\alpha, \alpha') &:= d_R(b, z_\alpha) + d_R(z_\alpha, z_{\alpha'}) - d_R(b, z_{\alpha'}), \\ F(\alpha, \alpha') &:= d_F(b, z_\alpha) + d_F(z_\alpha, z_{\alpha'}) - d_F(b, z_{\alpha'}), \qquad \text{and} \\ H(\alpha, \alpha') &:= d_H(b, z_\alpha) + d_H(z_\alpha, z_{\alpha'}) - d_H(b, z_{\alpha'}). \end{aligned}$$

Clearly $H = R + F$. Also, by the triangle inequality, R, F, and H are all non-negative.

For any α and α' with $\alpha < \alpha'$, and any $\varepsilon > 0$, we have that $H(\alpha, \alpha') < \varepsilon$ implies $R(\alpha, \alpha') < \varepsilon$ and $F(\alpha, \alpha') < \varepsilon$. Conversely, we have that $R(\alpha, \alpha') < \varepsilon/2$ and $F(\alpha, \alpha') < \varepsilon/2$ implies $H(\alpha, \alpha') < \varepsilon$. The conclusion follows easily. □

We define the following compatibility relation between reverse-Funk and Funk horofunctions. We write $\xi_R \sim \xi_F$, when there exists a net in C that is an almost-geodesic in both the reverse-Funk and the Funk geometry, and converges to ξ_R in the former and to ξ_F in the latter.

Theorem 8.2. *The Busemann points of the Hilbert geometry are the functions of the form $\xi_H := \xi_R + \xi_F$, where ξ_R and ξ_F are, respectively, reverse-Funk and Funk Busemann points, satisfying $\xi_R \sim \xi_F$. Each Hilbert Busemann point may we written in a unique way in this form.*

Proof. Let ξ_R and ξ_F be as in the statement. So, there exists a net z_α in $P(C)$ that is an almost-geodesic in both reverse-Funk geometry and in the Funk geometry, and converges to ξ_R in the former and to ξ_F in the latter. Applying Proposition 8.1, we get that z_α is an almost-geodesic in the Hilbert geometry, and it must necessarily converge to $\xi_R + \xi_F$.

To prove the converse, let ξ_H be a Busemann point of the Hilbert geometry, and let z_α be an almost-geodesic net converging to it. By Proposition 8.1, z_α is an almost-geodesic in both the Funk and reverse-Funk geometries, and so converges to a Busemann point ξ_F in the former and to a Busemann point ξ_R in the latter. So, $\xi_R \sim \xi_F$, and we also have that $\xi_H = \xi_R + \xi_F$.

To prove the last part, suppose that $\xi_H = \xi_R + \xi_F = \xi'_R + \xi'_F$, where ξ_R and ξ'_R are reverse-Funk Busemann points and ξ_F and ξ'_F are Funk Busemann points, with $\xi_R \sim \xi_F$ and $\xi'_R \sim \xi'_F$. So, there exist nets z_α and z'_α in $P(C)$ that are almost geodesic in both the reverse-Funk and Funk geometries, and converge in the former to ξ_R and ξ'_R, respectively, and in the latter to ξ_F and ξ'_F, respectively. By Proposition 8.1, both z_α and z'_α are almost geodesic in the Hilbert geometry, in which they converge to ξ_H.

By Proposition 2.5, we can find Hilbert almost-geodesics x_β, x'_β, and y_β such that x_β is a subnet of both z_α and y_β, and x'_β is a subnet of both z'_α and y_β. Each of these three nets is an almost-geodesic in the Funk geometry, by Proposition 8.1, and so must have a limit in this geometry. Recalling that a subnet of a convergent net has the same limit, it is not too hard to see that the three nets above and the nets z_α and z'_α must all have the same limit in the Funk geometry. The same reasoning also holds for the reverse-Funk geometry. It follows that $\xi_R = \xi'_R$ and $\xi_F = \xi'_F$. □

Our next goal is to make explicit the meaning of the relation $\sim$.

Suppose we are given two non-negative linear functionals f and g on C^* with the following properties. We assume that g is upper semicontinuous and has supremum 1 on the cross-section D, whereas f is lower semicontinuous and has infimum 1 on the same set. We assume further that g takes the value zero everywhere that f is finite.

Denote by $\mathcal{C}$ the set

$$\mathcal{C} := \big\{(h, h') \in C \times C \mid [h] = [h'] \text{ and } g < h \le h' < f\big\}.$$

We define a relation $\preceq$ on $\mathcal{C}$ in the following way: we say that $(h_1, h'_1) \preceq (h_2, h'_2)$ if $h_2 \le h_1$ and $h'_1 \le h'_2$. It is clear that this ordering is reflexive, transitive, and antisymmetric.

Lemma 8.3. *We have $f = \sup\{h' \mid (h, h') \in \mathcal{C}\}$ and $g = \inf\{h \mid (h, h') \in \mathcal{C}\}$.*

Proof. Denote by D the cross-section of the dual cone C^*, and by E the affine hull of D.

Define the epigraph of f and the (truncated) hypograph of g:

$$\operatorname{epi} f := \big\{(x, \lambda) \in D \times \mathbb{R} \mid f(x) \le \lambda\big\} \qquad \text{and}$$
$$\operatorname{hyp} g := \big\{(x, \lambda) \in D \times \mathbb{R} \mid g(x) \ge \lambda \ge 0\big\}.$$

Both of these sets are closed and convex, and $\operatorname{hyp} g$ is compact.

Let $x \in D$, and let $\lambda < f(x)$. Let K be the convex hull of the union of $\{(x, \lambda)\}$ and $\operatorname{hyp} g$. It is not hard to check that K is compact and disjoint from $\operatorname{epi} f$. Therefore, by the Hahn–Banach separation theorem, there is a closed hyperplane H in $E \times \mathbb{R}$ that strongly separates $\operatorname{epi} f$ and K. Note that the strong separation implies that H can not be of the form $H' \times \mathbb{R}$, where H' is a hyperplane of E. It follows that H is the graph of a continuous affine function $h\colon E \to \mathbb{R}$, satisfying $g < h < f$ and $h(x) > \lambda$.

We can extend h to the whole of the dual space in a unique way by requiring homogeneity. Since h is strictly positive on D, this gives us an element of C, which we denote again by h. So $(h, h) \in \mathcal{C}$. We have established that $f(x) \geq \sup\{h'(x) \mid (h, h') \in \mathcal{C}\}$ when $x \in D$. The same is true for any $x \in C^*$, by homogeneity. The opposite inequality is trivial.

The second part is similar, but we must be careful because the epigraph of f is not necessarily compact. So, this time, we choose λ arbitrarily so that $\lambda > g(x)$, and separate $\operatorname{hyp} g$ from the convex hull of the following three sets,

$$\operatorname{epi} f \cap \{(y, \beta) \in D \times \mathbb{R} \mid 0 \leq \beta \leq 2\}, \quad \{(y, \beta) \in D \times \mathbb{R} \mid \beta = 2\}, \quad \text{and} \quad \{(x, \lambda)\}.$$

All three of these sets are compact, and, since none of them intersect $\operatorname{hyp} g$, neither does the convex hull of their union.

In the same manner as before, we obtain an element h of C satisfying $g < h < \min(2, f)$ on D, and $h(x) < \lambda$, and the rest of the proof is the same. □

Lemma 8.4. *Let f be a lower-semicontinuous linear functional on C^*, and let $\{h_i\}_i$ be a finite collection of upper-semicontinuous linear functionals on C^* satisfying $h_i < f$ on the cross-section D, for each i. Then there exists a $h' \in C$ such that $\max_i h_i < h' < f$ on D.*

Proof. The proof is similar to that of the previous lemma. We chose $I \in (-\infty, \inf f)$, and separate $\operatorname{epi} f$ from the convex hull of the union of the compact sets

$$\operatorname{hyp} g_i \cap \{(y, \beta) \in D \times \mathbb{R} \mid I \leq \beta\}, \quad \text{for all } i, \quad \text{and } \{(y, \beta) \in D \times \mathbb{R} \mid \beta = I\}.$$

We obtain $h \in C$ satisfying $I < h < f$, and $h(y) > g_i(y)$ for all i and $y \in D$ such that $g_i(y) \geq I$. The conclusion follows. □

Lemma 8.5. *The set $\mathcal{C}$ is a directed set under the ordering $\preceq$.*

Proof. Let (h_1, h'_1) and (h_2, h'_2) be in $\mathcal{C}$.

By Lemma 8.4, there is a continuous real-valued linear functional h' satisfying

$$\max(h'_1, h'_2) < h' < f.$$

Restrict attention to the compact cross-section D of C^*, on which the linear functionals are affine functions. Since g is upper-semicontinuous, and h_1 and h_2 are continuous, the function $\min(h_1, h_2) - g$ attains its minimum over D. This minimum is positive. Choose an $\varepsilon \in (0, 1)$ strictly smaller than this minimum. Let $\lambda \in (0, 1)$ be such that $0 < \lambda h' < \varepsilon$. So, $k := g + \lambda h'$ is a non-negative upper-semicontinuous linear functional. We have that

$$\max(g, \lambda h'_1, \lambda h'_2) < k < \min(h_1, h_2).$$

Also, since g takes the value zero everywhere that f is finite, we have $k < \lambda f$.

We deduce using Lemma 8.4 that there exists a real-valued continuous linear functional l satisfying $k < l < \min(h_1, h_2, \lambda f)$. Hence $g < l$. Moreover,

$$\max(h_1', h_2') < \frac{l}{\lambda} < f.$$

We have thus proved that $(l, l/\lambda)$ is in $\mathcal{C}$, and that

$$(h_1, h_1') \preceq \Big(l, \frac{l}{\lambda}\Big) \qquad \text{and} \qquad (h_2, h_2') \preceq \Big(l, \frac{l}{\lambda}\Big). \qquad \square$$

Proposition 8.6. *Let C be a cone giving rise to a complete Hilbert geometry. Let ξ_R and ξ_F be, respectively, a reverse-Funk Busemann point (of the form (10) and a Funk Busemann point (of the form (11). Then, $\xi_R \sim \xi_F$ if and only if, for each $y \in C^*\backslash\{0\}$, either $g(y) = 0$ or $f(y) = \infty$.*

Proof. Assume that g and f satisfy the stated condition. The set $\mathcal{C}$ with the ordering $\preceq$ defined using g and f is a directed set, by Lemma 8.5. Consider the net z_α defined on the directed set $\mathcal{C}$ by $z_\alpha := \alpha$, for all $\alpha \in \mathcal{C}$. Write $(g_\alpha, f_\alpha) := z_\alpha$, for each $\alpha \in \mathcal{C}$. Observe that g_α is non-increasing, and f_α is non-decreasing.

Combining Lemmas 3.2 and 8.3 applied to a cross section D of C^*, we get that the net f_α converges to f.

Fix $x \in C$. So, the net of functions $y \mapsto \langle y, x\rangle / f_\alpha(y)$ is non-increasing and converges pointwise to $\langle y, x\rangle / f(y)$. Therefore, by Lemma 3.2, the net

$$d_F(x, f_\alpha) = \log \sup_{y \in D} \frac{\langle y, x\rangle}{f_\alpha(y)}$$

converges to $\xi_F(x)$. In particular, $d_F(b, f_\alpha)$ converges to zero. It follows that f_α converges to ξ_F in the compactification of the Funk geometry. Moreover, the monotonicity of the convergence implies that $d_F(\cdot, f_\alpha) - d_F(b, f_\alpha)$ is an almost non-increasing net of functions (see the observation after Definition 2.2). So, by Proposition 2.4, f_α is an almost-geodesic.

Recall that convergence in the Funk geometry is a property of the projective class of the points rather that the points themselves. So, $[f_\alpha]$ converges in the Funk geometry to the Funk Busemann point ξ_F.

The same method works to show that $[g_\alpha]$ converges in the reverse-Funk geometry to the reverse-Funk Busemann point ξ_R. Recall, moreover, that $[f_\alpha] = [g_\alpha]$, for all α.

To prove the converse, assume that ξ_R and ξ_F satisfy $\xi_R \sim \xi_F$. So, there is a net z_α in C that is an almost-geodesic in both the Funk and reverse-Funk geometries, and converges to ξ_F in the former and to ξ_R in the latter.

By using reasoning similar to that in second part of the proof of Theorem 6.3, we get that $z_\alpha / \exp(d_R(b, z_\alpha))$ converges pointwise to g on C^*. Similarly, $z_\alpha \exp(d_F(b, z_\alpha))$ converges pointwise to f.

It follows that $d_H(b, z_\alpha) := d_R(b, z_\alpha) + d_F(b, z_\alpha)$ converges to $\log(f(y)/g(y))$, for all $y \in C^*$. But this net grows without bound according to Proposition 2.6, and so the latter function is identically infinity. We have shown that, at each point of $C^*\backslash\{0\}$, either g is zero, or f is infinite. $\square$

Theorem 8.7. *Let $\xi_H = \xi_R + \xi_F$ and $\xi_H' := \xi_R' + \xi_F'$ be Busemann points of a Hilbert geometry, each written as the sum of a reverse-Funk Busemann point and a Funk Busemann point that are compatible with one another. Then, the distance between them in the detour metric is*

$$\delta_H(\xi_H, \xi_H') = \delta_R(\xi_R, \xi_R') + \delta_H(\xi_R, \xi_R'),$$

where δ_R and δ_F denote, respectively, the detour metrics in the reverse-Funk and Funk geometries.

Proof. Let z_α be a net in $P(C)$ that is an almost-geodesic in the Hilbert geometry and converges in this geometry to ξ_H. By Proposition 8.1, z_α is also an almost-geodesic in both the reverse-Funk and Funk geometries. Moreover, it converges in the former geometry to ξ_R and in the latter to ξ_F. By Lemma 2.8, we have

$$H_R(\xi_R, \xi'_R) = \lim_\alpha \big(d_R(b, z_\alpha) + \xi'_R(z_\alpha)\big) \qquad \text{and}$$
$$H_F(\xi_F, \xi'_F) = \lim_\alpha \big(d_F(b, z_\alpha) + \xi'_F(z_\alpha)\big),$$

where H_R and H_F denote, respectively, the detour cost in the reverse-Funk and Funk geometry. Adding, and using Lemma 2.8 again, we get that $H_R(\xi_R, \xi'_R) + H_F(\xi_F, \xi'_F) = H_H(\xi_H, \xi'_H)$, where H_H is the Hilbert detour cost. The result follows upon symmetrising. □

Proposition 8.8. *Assume the Hilbert geometry is complete. Let ξ_R and ξ'_R be reverse-Funk Busemann points in the same part, and let ξ_F and ξ'_F be Funk Busemann points in the same part. If $\xi_R \sim \xi_F$, then $\xi'_R \sim \xi'_F$.*

Proof. This follows from combining Proposition 8.6 with Theorems 6.4 and 7.2. □

Corollary 8.9. *A Busemann point $\xi_H = \xi_R + \xi_F$ of a complete Hilbert geometry, with $\xi_R \sim \xi_F$, is a singleton if and only if ξ_R and ξ_F are singleton Busemann points of, respectively, the reverse-Funk and Funk geometries.*

Proof. Assume that ξ_R is not a singleton, that is, there exists another reverse-Funk Busemann point ξ'_R in the same part as it. By, Proposition 8.8, $\xi'_R \sim \xi_F$, and so, by Theorem 8.2, $\xi'_R + \xi_F$ is a Busemann point of the Hilbert geometry. Hence, ξ_H is not a singleton. One may also prove in the same way that if ξ_F is not a singleton, then neither is ξ_H.

Assume now that there exists a Busemann point $\xi'_H = \xi'_R + \xi'_F$, of the Hilbert geometry, with $\xi'_R \sim \xi'_F$, that is distinct from ξ_H but in the same part as it. So, either ξ_R and ξ'_R are distinct, or ξ_F and ξ'_F are. By Theorem 8.7, ξ'_R is in the same part as ξ_R, and ξ'_F is in the same part as ξ_F. This shows that either ξ_R or ξ_F is not a singleton. □

9. The Hilbert geometry on the cone $C^+(K)$

In this section, we study the positive cone $C^+(K)$, that is, the cone of positive continuous functions on a compact Hausdorff space K. We take the basepoint b to be the function that is identically equal to 1. The dual cone of $C^+(K)$ is the cone $ca_r^+(K)$ of regular Borel measures on K. The cross section of the dual cone consisting of the probability measures on K is a Bauer simplex. The extreme points of this cross section are the Dirac masses. Every semicontinuous function on K can be extended to an semicontinuous linear functional on the dual cone:

$$\bar{g}(\mu) := \int_K g \, d\mu, \qquad \text{for } \mu \in ca_r^+(K).$$

In the opposite direction, any semicontinuous linear functional on the dual cone gives rise to a semicontinuous function of K by being evaluated at the Dirac masses: $g(x) = \bar{g}(\delta_x)$, for $x \in K$.

9.1. **The boundary of the reverse-Funk geometry on $C^+(K)$.** The reverse-Funk metric on $C^+(K)$ is given by

$$d_R(f,g) = \log \sup_{x\in K} \frac{g(x)}{f(x)}, \qquad \text{for all } g,\ f \text{ in } C^+(K).$$

Recall that the hypograph of a function $f\colon X \to [-\infty,\infty]$ is the set $\operatorname{hyp} f := \{(x,\alpha) \in X\times\mathbb{R} \mid \alpha \le f(x)\}$. A net of functions is said to converge in the hypograph topology if their hypographs converge in the Kuratowski–Painlevé topology.

Proposition 9.1. *The horofunctions of the reverse-Funk geometry on the positive cone $C^+(K)$ are the functions of the form*

$$\xi_R(f) := \log \sup_{x\in K} \frac{g(x)}{f(x)}, \qquad \text{for all } f\in C^+(K), \tag{12}$$

where $g\colon K\to[0,1]$ is an upper-semicontinuous function with supremum 1. All these horofunctions are Busemann points.

Proof. Let ξ_R be of the above form, extend g as above to get $\bar g$, a non-negative upper-semicontinuous linear functional on the dual cone. Let $f\in C^+(K)$, and denote by D the set of elements μ of the dual cone such that $\int_K f\,d\mu = 1$. So D is a cross-section of the dual cone. Observe that $\bar g$ attains its supremum over D at an extreme point of D, and that the set of these extreme points is $\{\delta_x/f(x) \mid x\in K\}$, where δ_x denotes the unit atomic mass at a point x. So,

$$\xi_R(f) = \log\sup_{\mu\in D} \bar g(\mu) = \log \sup_{\mu\in C^*\backslash\{0\}} \frac{\bar g(\mu)}{\int_K f\,d\mu}.$$

Therefore, by Theorem 6.3, the function ξ_R is a Busemann point of the reverse-Funk geometry of $C^+(K)$.

Now let g_α be a net in $C^+(K)$ converging to a horofunction. By scaling if necessary, we may assume that the supremum of each g_α is 1. Consider the hypographs $\operatorname{hyp}(g_\alpha)$ of these functions. This is a net of closed subsets of $K\times\mathbb{R}$. By the theorem of Mrowka (see [6, Theorem 5.2.11, page 149]), this net has a subnet that converges in the Kuratowski–Painlevé topology. Therefore, g_α has a subnet that converges in the hypograph topology to a proper upper-semicontinuous function g. We reuse the notation g_α to denote this subnet.

Since g_α takes values in $[0,1]$, so also does g. From [6, Theorem 5.3.6, page 160], we get that $\sup g_\alpha$ converges to $\sup g$. Thus, $\sup g = 1$.

Fix $f\in C^+(K)$. We have that g_α/f converges in the hypograph topology to the proper upper semicontinuous function g/f. Applying [6, Theorem 5.3.6] again, we get that

$$\lim_\alpha \sup_{x\in K} \frac{g_\alpha(x)}{f(x)} = \sup_{x\in K}\frac{g(x)}{f(x)}.$$

Since f was chosen arbitrarily, we see that g_α converges in the horofunction boundary to a point of the required form. ∎

9.2. **The boundary of the Funk geometry on $C^+(K)$.**

Proposition 9.2. *The horofunctions of the Funk geometry on the positive cone $C^+(K)$ are the functions of the form*

$$\xi_F(f) := \log\sup_{x\in K}\frac{f(x)}{g(x)}, \qquad \text{for all } f\in C^+(K), \tag{13}$$

where $g\colon K \to [1,\infty]$ *is a lower-semicontinuous function with infimum* 1. *All these horofunctions are Busemann points.*

Proof. Use Proposition 9.1 and the fact that the pointwise reciprocal map is an isometry taking the reverse-Funk metric to the Funk metric. □

9.3. **The boundary of the Hilbert geometry on $C^+(K)$.**

Proposition 9.3. *The Busemann points of the Hilbert geometry on the positive cone $C^+(K)$ are the functions of the form*

$$\xi_H(h) := \log \sup_{x\in K} \frac{g(x)}{h(x)} + \log \sup_{x\in K} \frac{h(x)}{f(x)}, \qquad \textit{for all } h \in C^+(K),$$

where $g\colon K \to [0,1]$ *is an upper-semicontinuous function with supremum* 1, *and* $f\colon K \to [1,\infty]$ *is a lower-semicontinuous function with infimum* 1, *and, for each* $x \in K$, *either* $g(x) = 0$ *or* $f(x) = \infty$.

Proof. By Theorem 8.2, the Busemann points of the Hilbert geometry are exactly the functions of the form $\xi_R + \xi_F$, with $\xi_R \sim \xi_F$, where ξ_R and ξ_F are Busemann points of, respectively, the reverse-Funk and Funk geometries. The Busemann points of these geometries were described in Propositions 9.1 and 9.2. Let g and f be as in those propositions. Proposition 8.6 states that $\xi_R \sim \xi_F$ if and only if $\bar{g}$ and $\bar{f}$ are not both positive and finite at any point of $ca_r^+(K)\backslash\{0\}$. It is not to hard to show that this condition is equivalent to g and f not being both positive and finite at any point of K. □

We have seen that all reverse-Funk horofunctions and all Funk horofunctions on the cone $C^+(K)$ are Busemann points. However, it is not necessarily true that all Hilbert horofunctions on this cone are Busemann. Indeed, consider the case where $K := [0,1]$ and take for example $g := I_{[0,1/2)}/2 + I_{[1/2,1]}$ and $f := I_{[0,1/2]} + 2I_{(1/2,1]}$. By the propositions above, ξ_R is a Busemann point of the reverse-Funk geometry and ξ_F is a Busemann point of the Funk geometry, where ξ_R and ξ_F are defined as in (12) and (13), respectively. Observe that if h_n is a non-increasing sequence of continuous functions on $C^+(K)$ that converges pointwise to g, then h_n is an almost-geodesic and converges to g in the reverse-Funk geometry; see Figure 2. Moreover, it converges to ξ_F in the Funk geometry, although it is not an almost geodesic in this geometry. This shows that $\xi_R + \xi_F$ is a horofunction of the Hilbert geometry. However, $\xi_R + \xi_F$ is not a Busemann of this geometry, according to Proposition 9.3, since f and g do not satisfy the compatibility condition.

Thus, the situation differs from the finite-dimensional case. There, the reverse-Funk horofunctions are all automatically Busemann, and every Hilbert horofunction is Busemann if and only if every Funk horofunctions is.

10. The horofunction boundary of the Thompson geometry

Here we study the boundary of the Thompson geometry.

Recall that reverse-Funk Busemann points are of the form (10 and Funk Busemann points are of the form (11.

It was shown in Proposition 8.6, that if ξ_R and ξ_F are Busemann points of their respective geometries, then $\xi_R \sim \xi_F$ if and only if, for each $y \in C^*\backslash\{0\}$, either $g(y) = 0$ or $f(y) = \infty$. Here, g and f are the functionals appearing in (10) and (11).

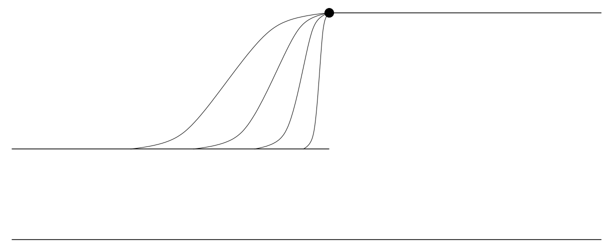

FIGURE 2. A sequence converging to a horofunction

For each $x \in D$, define the following functions on C:

$$r_x(\cdot) := \log \frac{M_C(x,\cdot)}{M_C(x,b)} \qquad \text{and} \quad f_x(\cdot) := \log \frac{M_C(\cdot,x)}{M_C(b,x)}.$$

Let $\vee$ and $\wedge$ denote, respectively, maximum and minimum. We use the convention that addition and subtraction take precedence over these operators. We write $x^+ := x \vee 0$ and $x^- := x \wedge 0$. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Given two real-valued functions f_1 and f_2, and $c \in \overline{\mathbb{R}}$, define

$$[f_1, f_2, c] := f_1 + c^- \vee f_2 - c^+.$$

Observe that if $c = \infty$, then $[f_1, f_2, c] = f_1$, whereas if $c = -\infty$, then $[f_1, f_2, c] = f_2$.

Let B_R and B_F be the set of Busemann points of the reverse-Funk and Funk geometries, respectively.

Proposition 10.1. *Let C be a cone giving rise to a complete Thompson geometry. The set of Busemann points of this geometry is*

$$\{r_z \mid z \in C\} \cup \{f_z \mid z \in C\} \cup B_R \cup B_F \cup \{[r, f, c] \mid r \in B_R, f \in B_F, r \sim f, c \in \mathbb{R}\}.$$

Proof. Let ξ_T be a Busemann point of the Thompson geometry, and let z_α be an almost-geodesic net in C converging to it.

By taking subnets if necessary, we may assume that z_α converges in both the Funk and reverse-Funk horofunction compactifications, to the limits ξ_R and ξ_F, respectively, and furthermore that $d_R(b, z_\alpha) - d_F(b, z_\alpha)$ converges to a limit c in $\overline{\mathbb{R}}$. So, as α tends to infinity,

$$\begin{aligned} &d_T(b, z_\alpha) - d_R(b, z_\alpha) \to -c^-, \quad \text{and} \\ &d_T(b, z_\alpha) - d_F(b, z_\alpha) \to c^+. \end{aligned}$$

Therefore,

$$\begin{aligned} d_T(y, z_\alpha) - d_T(b, z_\alpha) &= \Big(d_R(y, z_\alpha) \vee d_F(y, z_\alpha)\Big) - d_T(b, z_\alpha) \\ &= \Big(d_R(y, z_\alpha) - d_R(b, z_\alpha) + c^-\Big) \vee \Big(d_F(y, z_\alpha) - d_F(b, z_\alpha) - c^+\Big) \\ &\to [\xi_R, \xi_F, c](y). \end{aligned}$$

Consider the case when $c < \infty$. Let $\lambda > 0$ be such that $c < 2 \log \lambda$. For α large enough,

$$d_R(\lambda b, z_\alpha) - d_F(\lambda b, z_\alpha) = d_R(b, z_\alpha) - d_F(b, z_\alpha) - 2 \log \lambda < 0,$$

and hence $d_T(\lambda b, z_\alpha) = d_F(\lambda b, z_\alpha)$. Note also that $d_T(z_\alpha, z_{\alpha'}) \geq d_F(z_\alpha, z_{\alpha'})$, for all α and α'.

Recall that an almost-geodesic remains an almost-geodesic when the basepoint is changed. It will be convenient to consider almost-geodesics with respect to the basepoint λb.

Let $\varepsilon > 0$ be given. Since z_α is an almost geodesic in the Thompson geometry, we have, for α and α' large enough, with $\alpha < \alpha'$,

$$d_T(\lambda b, z_{\alpha'}) \geq d_T(\lambda b, z_\alpha) + d_T(z_\alpha, z_{\alpha'}) - \varepsilon,$$

and hence

$$d_F(\lambda b, z_{\alpha'}) \geq d_F(\lambda b, z_\alpha) + d_F(z_\alpha, z_{\alpha'}) - \varepsilon.$$

We deduce that z_α is an almost-geodesic in the Funk geometry, and so ξ_F is either of the form f_z, with $z \in C$, or a Funk Busemann point.

Similarly, when $c > -\infty$, z_α is an almost-geodesic in the reverse-Funk geometry and ξ_R is either of the form r_z, with $z \in C$, or a reverse-Funk Busemann point.

So, if $c = \infty$, then ξ_T is in $\{r_z \mid z \in C\} \cup B_R$. On the other hand, if $c = -\infty$, then ξ_T is in $\{f_z \mid z \in C\} \cup B_F$.

There remains the case where c is finite. Since ξ_T was assumed to be in the horofunction boundary, we have, by Proposition 2.6, that $d_T(b, z_\alpha)$ converges to infinity, and so both $d_R(b, z_\alpha)$ and $d_F(b, z_\alpha)$ do too, since their difference remains bounded. Therefore both ξ_R and ξ_F are Busemann points. We have shown that $\xi_T = [\xi_R, \xi_F, c]$, with $\xi_R \in B_R$ and $\xi_F \in B_F$ such that $\xi_R \sim \xi_F$. □

We extended the definition of H by setting $H(\xi + u, \eta + v) := H(\xi, \eta) + v - u$ for all Busemann points ξ and η, and $u, v \in [-\infty, 0]$. Here, we use the convention that $-\infty$ is absorbing for addition. The following proposition was proved in [15] in the finite-dimensional case, but the proof carries over to infinite dimension.

Proposition 10.2. *The distance in the detour metric between two Busemann points ξ_T and ξ'_T in a complete Thompson geometry is $\delta(\xi_T, \xi'_T) = d_H(x, x')$ if $\xi_T = r_x$ and $\xi'_T = r'_x$, with $x, x' \in D$. The same formula holds when $\xi_T = f_x$ and $\xi_T = f_{x'}$, with $x, x' \in D$. If $\xi_T = [\xi_R, \xi_F, c]$ and $\xi'_T = [\xi'_R, \xi'_F, c']$, with $\xi_R, \xi'_R \in B_R$, $\xi_F, \xi'_F \in B_F$, $\xi_R \sim \xi_F$, $\xi'_R \sim \xi'_F$, $c, c' \in \mathbb{R}$, then*

$$\delta(\xi_T, \xi'_T) = \max\Big(H(\bar{\xi}_R, \bar{\xi}'_R), H(\bar{\xi}_F, \bar{\xi}'_F)\Big) + \max\Big(H(\bar{\xi}'_R, \bar{\xi}_R), H(\bar{\xi}'_F, \bar{\xi}_F)\Big),$$

where

$$\bar{\xi}_R := \xi_R + c^-, \qquad \bar{\xi}_F := \xi_F - c^+,$$
$$\bar{\xi}'_R := \xi'_R + c'^-, \qquad \bar{\xi}'_F := \xi'_F - c'^+.$$

In all other cases, $\delta(\xi_T, \xi'_T) = \infty$.

Corollary 10.3. *The set of singletons of a complete Thompson geometry is exactly the union of the Funk singletons and the reverse-Funk singletons.*

11. Thompson geometries isometric to Banach spaces

In this section we determine which Thompson geometries are isometric to Banach spaces.

We start with a technical lemma.

Lemma 11.1. *For all $\alpha, \beta \in \mathbb{R}$, the sequence $p_n := n^{-1} \log(\exp(n\alpha) + \exp(n\beta))$ is non-increasing.*

Proof. Fix $n \in \mathbb{N}$. Observe that, for any $r \in \{0, \dots, n\}$,

$$e^{(\alpha-\beta)(n-r)} + e^{(\beta-\alpha)r} \geq 1,$$

since one or other of the terms is greater than or equal to 1. Equivalently,

$$e^{\alpha n} + e^{\beta n} \geq e^{\alpha r} e^{\beta(n-r)}.$$

By considering binomial coefficients and using the previous inequality, we get that

$$(e^{\alpha n} + e^{\beta n})(e^{\alpha n} + e^{\beta n})^n \geq \big(e^{\alpha(n+1)} + e^{\beta(n+1)}\big)^n$$

Taking logarithms and rearranging, we get $p_n \geq p_{n+1}$. □

Recall that a linear subspace of a Riesz space (vector lattice) E is a *Riesz subspace* if it is closed under the lattice operations on E. We will need the lattice version of the Stone–Weierstrass theorem. This theorem states [3] that if K is a compact space, then any Reisz subspace of $C(K)$ that separates the points of K and contains the constant function 1 is uniformly dense in $C(K)$.

The setting for the next lemma is an order unit space $(V, \overline{C}, b)$.

Lemma 11.2. *Let K be the pointwise closure of a set of Funk singletons of the cone, and let Φ be a bijection from a linear space X to C, such that the pullback $f \circ \Phi$ of each element f of K is linear. Then, the map $\varphi \colon X \to C(K)$, $x \mapsto \varphi_x$, where*

$$\varphi_x(f) := f(\Phi(x)), \qquad \text{for all } f \in K,$$

is linear and its image is dense in $C(K)$.

Proof. Observe that, as a closed subset of a compact set, K is compact.

We have, for all $\alpha, \beta \in \mathbb{R}$, and $x, y \in X$, and $f \in K$,

$$\varphi_{\alpha x+\beta y}(f) = f \circ \Phi(\alpha x + \beta y) = \alpha f \circ \Phi(x) + \beta f \circ \Phi(y) = \alpha\varphi_x(f) + \beta\varphi_y(f).$$

Therefore, φ is linear. Furthermore, the image of φ is a linear subspace of $C(K)$.

Define the following operation on X:

$$x \oplus_n y := \frac{1}{n}\Phi^{-1}(\Phi(nx) + \Phi(ny)), \qquad \text{for } x, y \in X.$$

Fix x and y in X. Let $f \in K$, and write $g := f \circ \Phi$. So, g is a linear functional on X. Recall that f is the logarithm of a linear functional l on the cone, that is, $f = \log \circ\, l$ for some linear functional l on C. Note that $l = \exp \circ f$. We have

$$\begin{aligned} g(x \oplus_n y) &= \frac{1}{n} g\big(\Phi^{-1}(\Phi(nx) + \Phi(ny))\big) \\ &= \frac{1}{n} f(\Phi(nx) + \Phi(ny)) \\ &= \frac{1}{n}\log(l \circ \Phi(nx) + l \circ \Phi(ny)) \\ &= \frac{1}{n}\log\big(\exp\circ f \circ \Phi(nx) + \exp\circ f \circ \Phi(ny)\big) \\ &= \frac{1}{n}\log\big(\exp(ng(x)) + \exp(ng(y))\big). \end{aligned}$$

Using Lemma 11.1, we get that the sequence $g(x \oplus_n y) = \varphi_{x\oplus_n y}(f)$ converges monotonically to its limit, $\max\{g(x), g(y)\}$, as n tends to infinity. Since this is true for every $f \in K$, we have that $\varphi_{x\oplus_n y}$ converges monotonically and pointwise to $\varphi_x \vee \varphi_y$, and hence converges to this limit uniformly, by Dini's theorem. Therefore, $\varphi_x \vee \varphi_y$ is in $\operatorname{cl}\operatorname{Im}\varphi$, the closure of the image of φ. It follows that $\operatorname{cl}\operatorname{Im}\varphi$ is a Reisz subspace of $C(K)$.

Let f_1 and f_2 be distinct elements of K. So, there is some $y \in C$ such that $f_1(y) \neq f_2(y)$. Setting $x := \Phi^{-1}(y)$, we get $\varphi_x(f_1) \neq \varphi_x(f_2)$. This shows that $\varphi(X)$ separates the points of K.

Recall that we have chosen the basepoint b of the cone so that $b = \Phi(0)$. Write $x := \Phi^{-1}(eb)$, where e is Euler's number. Let $f \in K$. Since f is the logarithm of a linear functional, $f(\alpha z) = \log \alpha + f(z)$ for all points z in the cone, and all $\alpha > 0$. So, since $f(b) = 0$, we get $f(eb) = 1$. Hence, $\varphi_x(f) = f(eb) = 1$. We have shown that the constant function 1 is in the image of φ.

Applying the Stone–Weierstrass theorem to $\operatorname{cl} \operatorname{Im} \varphi$, we get that $\operatorname{cl} \operatorname{Im} \varphi$ is dense in $C(K)$, and so $\operatorname{Im} \varphi$ is dense in $C(K)$. □

Theorem 11.3. *If a Thompson geometry is isometric to a Banach space, then the cone is linearly isomorphic to $C^+(K)$, for some compact Hausdorff space K.*

Proof. Assume that $\Phi\colon X \to C$ is an isometry from a Banach space $(X, ||\cdot||)$ to a Thompson geometry on a cone C. We choose the basepoint b of the cone so that $b = \Phi(0)$.

Let K be the pointwise closure of the set of Funk singletons of C. Each element f of K can be written $f = \log y|_C$, where y is in the weak* closure of the set of extreme points of the cross-section $\{y \in C^* \mid \langle y, b\rangle = 1\}$ of the dual cone C^*. Since it is a closed subset of a compact set, K is compact.

By Corollary 10.3, the pull back $f \circ \Phi$ of each Funk singleton f is a singleton Busemann point of the Banach space X, and is therefore linear, by Corollary 3.6. It follows that $f \circ \Phi$ is linear for all $f \in K$.

Let φ be defined as in Lemma 11.2. By Lemma 11.2, the image of the Banach space X under φ is a uniformly dense subspace of $C(K)$.

Let F denote the set of Funk singletons. Recall that the following formula for the Funk metric holds for an arbitrary cone (see [15, Proposition 4.4]):

$$d_F(w, z) = \sup_{f \in F} \big(f(w) - f(z)\big), \qquad \text{for } w, z \in C.$$

So, the Thompson metric is given by

$$d_T(w, z) = d_F(w, z) \vee d_F(z, w) = \sup_{f \in F} \big|f(w) - f(z)\big|, \qquad \text{for } w, z \in C.$$

The same formula holds with F replaced by K since the former is dense in the latter. We conclude that, for all $x, y \in X$,

$$||y - x|| = d_T(\Phi(x), \Phi(y)) = \sup_{f \in K} |f(\Phi(x)) - f(\Phi(y))| = \sup_{f \in K} |\varphi_x(f) - \varphi_y(f)|.$$

Therefore φ is an isometry from $(X, ||\cdot||)$ to $C(K)$ with the supremum norm.

But we have assumed that X is complete, and so its image under φ is complete. We conclude that this image is the whole of $C(K)$.

We have shown that φ is an isometric linear-isomorphism from X to $C(K)$.

Define the map $\Theta := \exp \circ \varphi \circ \Phi^{-1}$ from C to $C_+(K)$. For each $p \in C$, we have

$$(\Theta p)(f) = \exp \circ \varphi_{\Phi^{-1} p}(f) = e^{f(p)}, \quad \text{for all } f \in K.$$

Since f is the logarithm of a linear functional on C, it follows that Θ is linear. So, Θ is a linear isomorphism between C and $C^+(K)$. □

12. Hilbert geometries isometric to Banach spaces

Here we prove that the only Hilbert geometries isometric to Banach spaces are the ones on the cones $C^+(K)$, for some compact space K.

We first require some lemmas concerning singleton Busemann points in cones geometries and in Banach spaces.

Lemma 12.1. *Let z_α be an almost-geodesic net in the Funk geometry on a cone C, converging to a Funk Busemann point ξ_F and normalised so that $d_F(b, z_\alpha) = 0$ for all α. If ξ_R is a reverse-Funk Busemann point such that $\xi_R(z_\alpha)$ converges to $-\infty$, then $\xi_R \sim \xi_F$.*

Proof. Write ξ_R and ξ_F in the form (10) and (11), respectively, for some functionals g and f.

Choose $\varepsilon > 0$. Since z_α is a Funk almost-geodesic, normalised in the way we have assumed, we have, for α large enough, that $d_F(x, z_\alpha) > \xi_F(x) - \varepsilon$, for all $x \in C$. This is equivalent to

$$\sup_{y\in C^*\backslash\{0\}} \frac{\langle y, x\rangle}{\langle y, z_\alpha\rangle} > e^{-\varepsilon} \sup_{y\in C^*\backslash\{0\}} \frac{\langle y, x\rangle}{f(y)}, \qquad \text{for all } x \in C.$$

Applying Lemma 6.2, we get that $\langle\cdot, z_\alpha\rangle < \exp(\varepsilon) f$, for α large enough.

Since $\xi_R(z_\alpha)$ converges to $-\infty$, we have that $\sup_{y\in C^*\backslash\{0\}} g(y)/\langle y, z_\alpha\rangle$ converges to zero, which implies that $g(y)/\langle y, z_\alpha\rangle$ converges to zero for all $y \in C^*\backslash\{0\}$. We deduce that, if $g(y)$ is non-zero for some $y \in C^*\backslash\{0\}$, then $\langle y, z_\alpha\rangle$ converges to infinity, and so $f(y)$ is infinite. The conclusion follows on applying Proposition 8.6. □

Lemma 12.2. *Let s be a singleton of a Banach space, and let $\varphi\colon x \mapsto x + v$ be the translation by some vector v. Then, φ leaves s invariant.*

Proof. By Corollary 3.6, s is linear, and the conclusion follows easily. □

Lemma 12.3. *Let s and s' be singletons of a Banach space X. Then, there exists an almost-geodesic net z_α converging to s such that $s'(z_\alpha)$ converges to either ∞ or $-\infty$.*

Proof. Let z_α be an almost geodesic converging to s, and denote by $\mathcal{D}$ be the directed set on which the net z_α is based. By taking a subnet if necessary, we may assume that $s'(z_\alpha)$ converges to a limit in $[-\infty, \infty]$. If this limit is infinite, then the conclusion of the lemma holds, so assume the contrary.

Take a point v in the Banach space such that $s'(v) < 0$. Observe that for each $n \in \mathbb{N}$, the net $z_\alpha + nv$ is an almost geodesic, and by Lemma 12.2 it converges to s.

We denote by $d(\cdot, \cdot)$ the metric coming from the norm $||\cdot||$.

Denote by $\mathcal{N}$ the set of neighbourhoods of s in the horofunction compactification of X. Let $\mathcal{D}'$ be the set of elements $(W, x, n, \varepsilon, A)$ of $\mathcal{N} \times X \times \mathbb{N} \times (0, \infty) \times \mathcal{D}$ such that

(i) $x = z_\alpha + nv$ for some $\alpha \geq A$;
(ii) $x \in W$;
(iii) $0 \leq d(b, x) + s(x) < \varepsilon$.

Note that, given any $W \in \mathcal{N}$, $n \in \mathbb{N}$, $\varepsilon \in (0, \infty)$, and $A \in \mathcal{D}$, we can find an $x \in X$ satisfying (i), (ii), and (iii).

Define on $\mathcal{D}'$ an order by $(W, x, n, \varepsilon, A) \leq (W', x', n', \varepsilon', A')$ if the two elements are the same, or if $W \supset W'$, $n \leq n'$, $\varepsilon \geq \varepsilon'$, $A \leq A'$, and

$$|d(x, y) - d(b, y) - s(x)| < \varepsilon, \qquad \text{for all } y \in W'. \tag{14}$$

It is not hard to verify that this order makes $\mathcal{D}'$ into a directed set.

Define the net $y_\beta := x$, for $\beta := (W, x, n, \varepsilon, A)$. It is clear that y_β converges to s.

Suppose some $\lambda > 0$ is given. If $\beta := (W, x, n, \varepsilon, A)$ is large enough that $\varepsilon < \lambda/2$, and $\beta' := (W', x', n', \varepsilon', A')$ is such that $\beta \leq \beta'$, then combining (iii) and (14) we get

$$d(b, y_\beta) + d(y_\beta, y_{\beta'}) - d(b, y_{\beta'}) < \lambda.$$

This proves that the net y_β is an almost geodesic.

Since s' is linear, $s'(y_\beta) = s'(z_\alpha) + ns'(v)$, for all β, where α is as in (i). Both α and n can be made as large as we wish by taking β large enough, and so $s'(y_\beta)$ converges to $-\infty$. □

Observe that if r and f are reverse-Funk and Funk horofunctions, respectively, then $r + f$ is constant on each projective class of the cone, so we may consider it to be defined on $P(C)$.

Lemma 12.4. *Let $\Phi\colon X \to P(C)$ be an isometry from a Banach space to the Hilbert geometry on a cone. Let s be a singleton of the Banach space. Write $s = (r + f) \circ \Phi$ and $-s = (r' + f') \circ \Phi$, where r and r' are singletons of the reverse-Funk geometry, and f and f' are singletons of the Funk geometry. Then, $r = -f'$ and $r' = -f$.*

Proof. Let p and q be points in the projective space $P(C)$ of the cone.

Let z_β be an almost geodesic in the Banach space converging to s. Taking the reflection $z'_\beta := 2\Phi^{-1}(p) - z_\beta$ in the point $\Phi^{-1}(p)$, we get an almost geodesic converging to $-s$.

Let $y_\beta := \Phi(z_\beta)$, for all β. This is an almost geodesic in the Hilbert geometry, and so it is also an almost geodesic in both the reverse-Funk and Funk geometries. Hence, it converges to a Busemann point in both of these geometries. Moreover, the sum of the two Busemann points is equal to the Hilbert geometry Busemann point $r + f$. Since, by Theorem 8.2, a Hilbert geometry Busemann point can be written in a unique way as the sum of a reverse-Funk Busemann point and a Funk Busemann point, we see that the limit of y_β in the reverse-Funk geometry is r and that the limit of y_β in the Funk geometry is f.

Likewise, let $y'_\beta := \Phi(z'_\beta)$, for all β. Again, this is an almost geodesic in the Hilbert geometry, converging this time to r' in the reverse-Funk geometry and to f' in the Funk geometry.

For all β, since $\Phi^{-1}(p)$ is the midpoint of z'_β and z_β, we have

$$d_H(y'_\beta, p) + d_H(p, y_\beta) = d_H(y'_\beta, y_\beta).$$

This implies that

$$d_F(y'_\beta, p) + d_F(p, y_\beta) = d_F(y'_\beta, y_\beta), \qquad \text{for all } \beta. \tag{15}$$

Recall that

$$f(q) - f(p) = \lim_\beta \big(d_F(q, y_\beta) - d_F(p, y_\beta)\big) \qquad \text{and}$$

$$r'(q) - r'(p) = \lim_\beta \big(d_F(y'_\beta, q) - d_F(y'_\beta, p)\big).$$

Combining these two equations with (15), and using the Funk metric triangle inequality applied to points y'_β, p, and y_β, we get $f(q) - f(p) + r'(q) - r'(p) \geq 0$.

But this holds for arbitrary p and q in $P(C)$, and so $r' + f$ must be constant on $P(C)$. Since this function takes the value zero at b, we see that it is zero everywhere.

The proof that $r + f' = 0$ is similar. □

Given a cone C, we use F to denote the set of singletons f of the Funk geometry on C such that there exists a reverse-Funk singleton r satisfying $r \sim f$.

Lemma 12.5. *Let C be a cone whose Hilbert geometry is isometric to a Banach space. Let $\Phi' \colon X' \to C$ be a bijection from a linear space to C, such that the pullback $h \circ \Phi'$ of every Hilbert geometry singleton h is a linear functional on X'. If there exists $f_0 \in F$ whose pullback is linear, then the pullback of every element of F is linear.*

Proof. Since f_0 is in F, there is some reverse-Funk singleton r_0 satisfying $r_0 \sim f_0$, and so $h_0 := r_0 + f_0$ is a Hilbert singleton, by Corollary 8.9. By assumption, $s := h_0 \circ \Phi'$ is linear. We have also assumed that $f_0 \circ \Phi'$ is linear, so we deduce that $r_0 \circ \Phi'$ is linear.

Let $f \in F$. So, there is some singleton r of the reverse-Funk geometry such that $h := r + f$ is a singleton of the Hilbert geometry. From the isometry between the Banach space and the Hilbert geometry, we get that there is another singleton h' of the Hilbert geometry satisfying $h' = -h$. We may write $h' = r' + f'$, where r' and f' are singletons of the reverse-Funk and Funk geometries, respectively.

Using Lemma 12.3 and the isometry between the Banach space and the Hilbert geometry, we get that there exists an almost geodesic net $[y_\alpha]$ in $P(C)$ converging in the Hilbert geometry to h_0 such that either $h(y_\alpha)$ or $h'(y_\alpha)$ converges to $-\infty$.

Consider the former case. For all α, take a representative y_α of the projective class $[y_\alpha]$ so that $d_F(b, y_\alpha) = 0$. This implies, for each α, that $\langle \cdot, b \rangle \leq \langle \cdot, y_\alpha \rangle$ on C, from which it follows that $f(y_\alpha) \geq 0$. Since we are supposing that $h(y_\alpha) = r(y_\alpha) + f(y_\alpha)$ converges to $-\infty$, we must have that $r(y_\alpha)$ converges to $-\infty$. From Proposition 8.1, we get that y_α is an almost geodesic in the Funk geometry. In this geometry, it converges to f_0. Applying Lemma 12.1, we get that $r \sim f_0$. This means that $r + f_0$ is a singleton of the Hilbert geometry on C, and so, by assumption, its pullback $(r + f_0) \circ \Phi'$ is linear. We deduce that $r \circ \Phi'$ is linear, and hence that $f \circ \Phi' = (h - r) \circ \Phi'$ is linear.

Now consider the second case, that is, where $h'(y_\alpha) = r'(y_\alpha) + f'(y_\alpha)$ converges to $-\infty$. Using reasoning similar to that in the previous paragraph, we get that $r' \circ \Phi'$ is linear. But, according to Lemma 12.4, $r' = -f$, and so $f \circ \Phi'$ is linear. $\square$

Let K be a compact space. Define the following seminorm on $C(K)$.

$$||x||_H := \sup_{f \in K} x(f) - \inf_{f \in K} x(f).$$

Denote by $\equiv$ the equivalence relation on $C(K)$ where two functions are equivalent if they differ by a constant, that it, $x \equiv y$ if $x = y + c$ for some constant c. The seminorm $||x||_H$ is a norm on the quotient $C(K)/\equiv$. This space is a Banach space, and we denote it by $H(K)$.

Theorem 12.6. *If a Hilbert geometry on cone C is isometric to a Banach space, then C is linearly isomorphic to $C^+(K)$ for some compact Hausdorff space K.*

Proof. Let $\Phi \colon X \to P(C)$ be an isometry from a Banach space $(X, || \cdot ||)$ to the Hilbert geometry on a cone C.

Each singleton of the Hilbert geometry may be written $r + f$, where r and f are singletons of the reverse-Funk and Funk geometries, respectively. Let F denote the set of Funk singletons that appear in this way, and denote by K the pointwise closure of this set of functions.

Recall that each element f of K can be written $f = \log y|_C$, where y is in the weak* closure of the set of extreme points of the cross-section $\{y \in C^* \mid \langle y, b \rangle = 1\}$ of the dual cone C^*. Since K is a closed subset of a compact set, it is compact.

Consider the linear space $X' := X \times \mathbb{R}$, and define a map $\Phi' \colon X' \to C$ in the following way. Fix a choice of a particular $f_0 \in F$. For each $x \in X$, the projective class $\Phi(x)$ is a ray in C, and along this ray the function f_0 is monotonically increasing, taking values from $-\infty$ to ∞. So, for each $x \in X$ and $\alpha \in \mathbb{R}$, we may define $\Phi'(x, \alpha)$ to be the representative of $\Phi(x)$ where f_0 takes the value α.

Observe that, since by definition

$$(f_0 \circ \Phi')(x, \alpha) = \alpha, \qquad \text{for all } x \in X \text{ and } \alpha \in \mathbb{R}, \tag{16}$$

we have that $f_0 \circ \Phi'$ is linear.

Consider the pull back $h \circ \Phi'$ of a Hilbert singleton h. This is a function on X' that is independent of the second coordinate. Moreover, the dependence on the first coordinate is linear since each singleton of the Banach space X is linear. So, $h \circ \Phi'$ is linear on X'.

Applying Lemma 12.5, we get that $f \circ \Phi'$ is linear for all $f \in F$, and it follows that the same is true for all $f \in K$.

Define the map $\varphi \colon X' \to C(K)$, $x \mapsto \varphi_x$ as in Lemma 11.2. That is,

$$\varphi_x(f) := f(\Phi'(x)), \qquad \text{for all } f \in K.$$

According to Lemma 11.2, φ is linear and its image is a uniformly dense subspace of $C(K)$.

We make the following claim, the proof of which we postpone.

Claim 1. *The image of X' under the map φ is a complete subset of $(C(K), ||\cdot||_\infty)$.*

Using this claim, we conclude that the image of φ is the whole of $C(K)$.

Extend the norm on X to a seminorm on X' by ignoring the second coordinate. We again use $||\cdot||$ to denote this seminorm. We make a second claim.

Claim 2. *The map φ is an isometry between $(X', ||\cdot||)$ and $(C(K), ||\cdot||_H)$.*

It follows from this, upon quotienting on each side by the respective subspace where the seminorm is zero, that X is isometric to $H(K)$.

Define the map $\Theta := \exp \circ \varphi \circ \Phi'^{-1}$ from C to $C^+(K)$. For each $p \in C$, we have

$$(\Theta p)(f) = \exp \circ \varphi_{\Phi'^{-1}p}(f) = e^{f(p)}, \quad \text{for all } f \in K.$$

Since f is the logarithm of a linear functional on C, it follows that Θ is linear. So, Θ is a linear isomorphism between C and $C^+(K)$. $\square$

Proof of Claim 2. Let S be the set of singletons of the Banach space $(X, ||\cdot||)$. Extend each $s \in S$ to X' by $s(x, \alpha) := s(x)$, for all $x \in X$ and $\alpha \in \mathbb{R}$.

We have that $||p|| = \sup_{s \in S} s(p)$, for all p in X. We have seen that a function s is in S if and only if it is of the form $s = f \circ \Phi' - f' \circ \Phi'$, with f and f' distinct elements of F. Note that if f are f' were identical, then $f \circ \Phi' - f' \circ \Phi'$ would be zero. So, for any $p \in X'$,

$$\begin{aligned} ||p|| &= \sup_{f, f' \in F} \big(f \circ \Phi'(p) - f' \circ \Phi'(p) \big) \\ &= \sup_{f \in F} \varphi_p(f) - \inf_{f' \in F} \varphi_p(f') \\ &= ||\varphi_p||_H, \end{aligned}$$

since F is dense in K. $\square$

Proof of Claim 1. Recall that we extend the norm on X to a seminorm $||\cdot||$ on X' by ignoring the second coordinate. Consider another norm on X' defined by

$$||p||' := ||x|| + |\alpha|, \qquad \text{for all } p := (x, \alpha) \text{ in } X'.$$

This is the ℓ_1-product of two complete norms, and so makes X' into a Banach space.

From Claim 2 and (16), we see that φ induces an isometry between the norm $||\cdot||'$ on X' and the norm $||\cdot||''$ on $C(K)$ defined by

$$||g||'' := ||g||_H + |g(f_0)|, \qquad \text{for all } g \text{ in } C(K).$$

But it is not hard to show that $||\cdot||_\infty \leq ||\cdot||'' \leq 3||\cdot||_\infty$, and so $||\cdot||''$ and $||\cdot||_\infty$ are equivalent norms on $C(K)$. It follows that the image of φ is complete in $C(K)$. □

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INRIA & CMAP, École Polytechnique,, 91128 Palaiseau, France
E-mail address: `cormac.walsh@inria.fr`