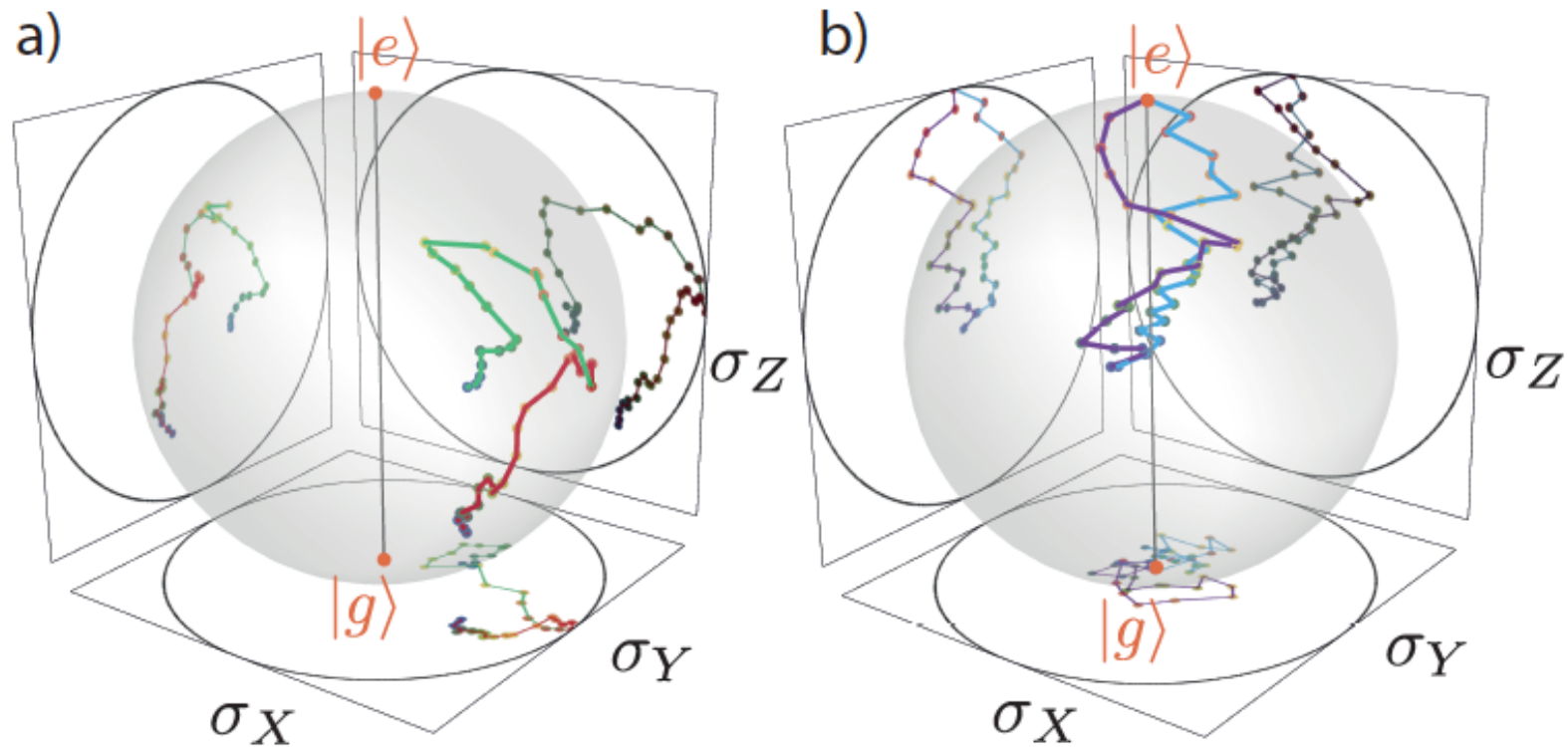


Exact solution for the evolution of a qubit under continuous measurement

Alain Sarlette, INRIA Paris
Pierre Rouchon, Mines ParisTech

joint work with B.Huard, F.Mallet, Ph.Campagne (LPA, ENS Paris)

Exact solution for



Outline

Open quantum systems

Explicit solution of Fokker-Planck PDE for $\eta = 1$

Explicit solution of Fokker-Planck PDE : general case

Link with control theory

Open quantum systems

$|\psi\rangle$ “wave function”



$\rho = |\psi\rangle\langle\psi|$: $\rho \in \mathbb{C}^{n \times n}$, $\rho = \rho^*$ positive definite

$\text{trace}(\rho) = 1$ “density operator”

$\text{rank}(\rho) = 1$

Open quantum systems

$|\psi\rangle$ “wave function”



$\rho = |\psi\rangle\langle\psi|$: $\rho \in \mathbb{C}^{n \times n}$, $\rho = \rho^*$ positive definite

$\text{trace}(\rho) = 1$ “density operator”

~~$\text{rank}(\rho) = 1$~~

Represents probability distribution over possible $|\psi\rangle$ [not unique]

Origin: - Coupling to other quantum systems [entanglement]

- Measurement with fidelity $\eta \in [0,1)$

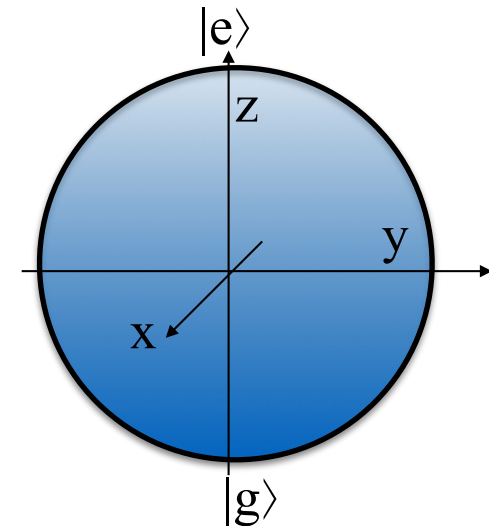
Open quantum systems

Qubit example: $|\psi\rangle \in \mathbb{C}^2 = \text{span}(|g\rangle, |e\rangle)$



$$\rho \in \mathbb{C}^{2 \times 2}$$

$$\rho = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + z \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}{2}$$



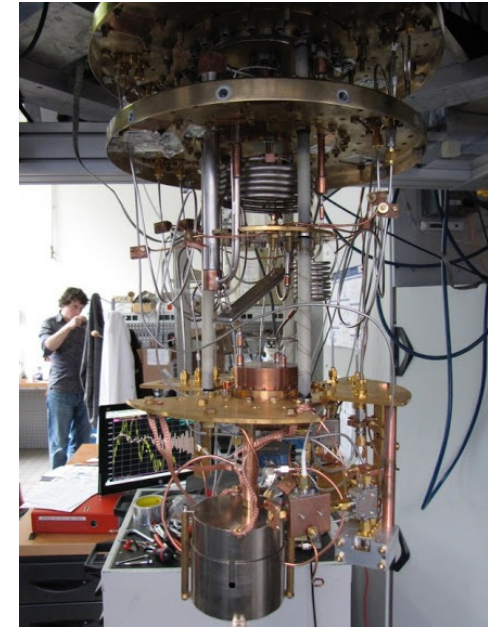
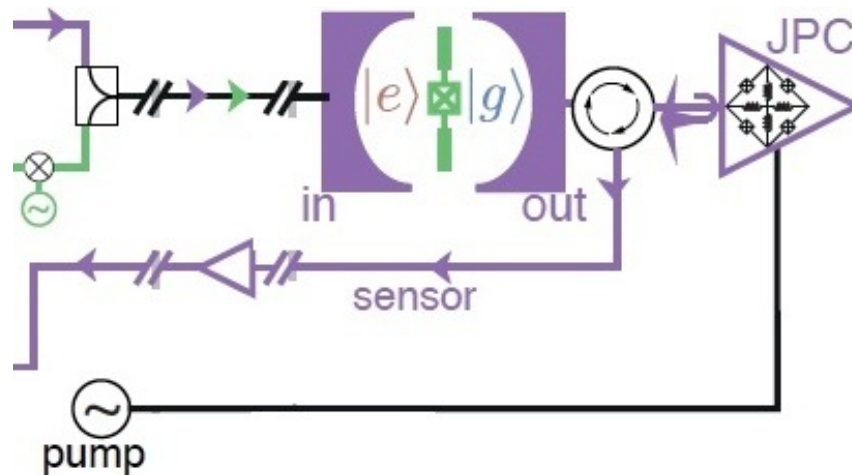
“Bloch sphere”

$x^2 + y^2 + z^2 = 1$: $\rho = |\psi\rangle\langle\psi|$, ideal system

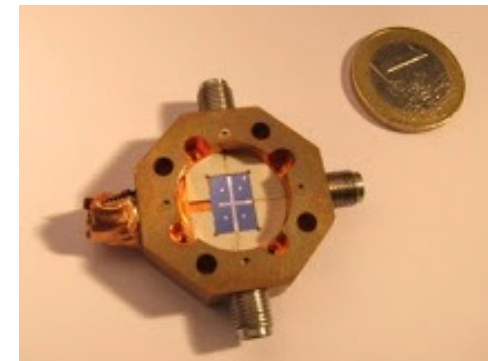
$x^2 + y^2 + z^2 < 1$: $\text{rank}(\rho) > 1$, open system

We address an actual experiment

Cavity/Circuit quantum electrodynamics (CQED)
B.Huard et al, LPA, Ecole Normale Supérieure, Paris



Heterodyne Measurement:
two quadratures of the outgoing EM field



Evolution of open quantum system [Ito]

Stochastic master equation [under standard Markovian assumption]

$$d\rho_t = \left(-\frac{i}{\hbar} [\mathbf{H}, \rho_t] + \sum_{\nu} \mathbf{L}_{\nu} \rho_t \mathbf{L}_{\nu}^{\dagger} - \frac{1}{2} (\mathbf{L}_{\nu}^{\dagger} \mathbf{L}_{\nu} \rho_t + \rho_t \mathbf{L}_{\nu}^{\dagger} \mathbf{L}_{\nu}) \right) dt$$
$$+ \sum_{\nu} \sqrt{\eta_{\nu}} \left(\mathbf{L}_{\nu} \rho_t + \rho_t \mathbf{L}_{\nu}^{\dagger} - \text{Tr} \left((\mathbf{L}_{\nu} + \mathbf{L}_{\nu}^{\dagger}) \rho_t \right) \rho_t \right) dW_{\nu,t},$$
$$dy_{\nu,t} = dW_{\nu,t} + \sqrt{\eta_{\nu}} \text{Tr} \left((\mathbf{L}_{\nu} + \mathbf{L}_{\nu}^{\dagger}) \rho_t \right) dt$$

i.i.d. $N(0,1)$

[Filter describing the estimated state of the quantum system.

No way to get “real ρ ” by measuring something]

A note about numerical simulation

[P.Rouchon & J.Ralf]

Up to order dt this is Ito equivalent to the discrete time system:

$$\rho_{t+dt} = \frac{\mathbf{M}_{dy_t} \rho_t \mathbf{M}_{dy_t}^\dagger + \sum_{\nu} (1 - \eta_{\nu}) \mathbf{L}_{\nu} \rho_t \mathbf{L}_{\nu}^\dagger dt}{\text{Tr} \left(\mathbf{M}_{dy_t} \rho_t \mathbf{M}_{dy_t}^\dagger + \sum_{\nu} (1 - \eta_{\nu}) \mathbf{L}_{\nu} \rho_t \mathbf{L}_{\nu}^\dagger dt \right)}$$

with

$$\mathbf{M}_{dy_t} = \mathbf{I} + \left(-\frac{i}{\hbar} \mathbf{H} - \frac{1}{2} \mathbf{L}_{\nu}^\dagger \mathbf{L}_{\nu} \right) dt + \sum_{\nu} \sqrt{\eta_{\nu}} dy_{\nu,t} \mathbf{L}_{\nu}$$

$$dy_t = s_t \sqrt{dt}$$

$$\mathbb{P}(s_t \in [s, s+ds] | \rho_t) = \frac{\text{Tr} \left(\mathbf{M}_{s\sqrt{dt}} \rho_t \mathbf{M}_{s\sqrt{dt}}^\dagger + \sum_{\nu} (1 - \eta_{\nu}) \mathbf{L}_{\nu} \rho_t \mathbf{L}_{\nu}^\dagger dt \right)}{\sqrt{2\pi}} e^{-\frac{|s|^2}{2}} ds$$

Goal: characterize stochastic trajectories

Case A:

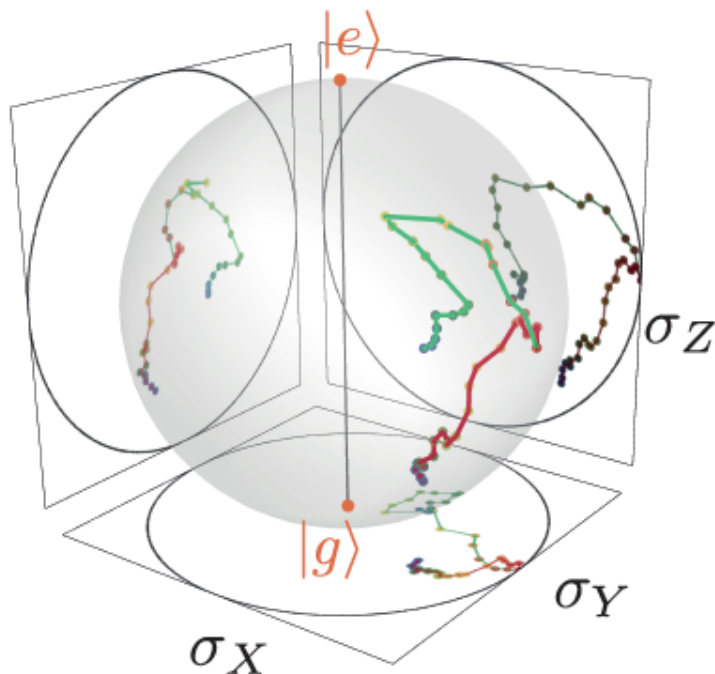
$$L_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$$

field takes energy from qubit

Case B:

$$L_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

qubit modifies field refraction index



B.Huard et al, LPA,
Ecole Normale Supérieure,
Paris

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Link with control theory

Stochastic evolution equations

On Bloch sphere, cylindrical coordinates

Case A:

$$dr = \frac{\gamma}{2} \left(\frac{\eta(1+z)^2}{2r} - r \right) dt + \sqrt{\frac{\eta\gamma}{2}} (1+z-r^2) dW_r$$
$$dz = -\gamma(1+z) dt - \sqrt{\frac{\eta\gamma}{2}} r(1+z) dW_r$$
$$d\theta = \sqrt{\frac{\eta\gamma}{2}} \frac{1+z}{r} dW_\theta$$

Case B:

$$dr = -\gamma \left(1 - \frac{\eta}{2} \right) r dt - \sqrt{\eta\gamma} r z dW_r$$
$$dz = \sqrt{\eta\gamma} (1-z^2) dW_r$$
$$d\theta = \sqrt{\eta\gamma} dW_\theta$$

θ coordinate undergoes independent random walk — consider solved

The case $\eta = 1$ (perfect experiment)

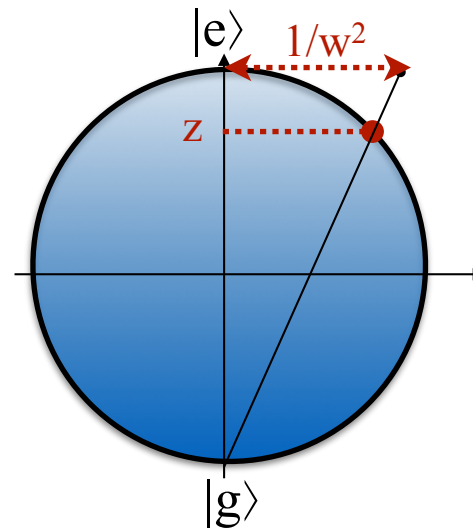
We know physically that the system stays on Bloch sphere $r^2 + z^2 = 1$

[confirmed by Ito calculus: $d(r^2 + z^2) = 0$ if $(r^2 + z^2) = \eta = 1$]

→ **Fokker-Planck PDE in z** from (case A)

$$dz = -\gamma(1+z)dt - \sqrt{\gamma/2}(1+z)\sqrt{1-z^2}dW_r$$

→ variable change $w = \frac{1-z}{1+z}$



The case $\eta = 1$ (perfect experiment)

→ Fokker-Planck PDE in w

$$\frac{\partial P_w}{\partial \tau} = \frac{\partial^2}{\partial w^2}(wP_w) - \frac{\partial}{\partial w} \left(\left((w+1) + \frac{2w}{w+1} \right) P_w \right)$$

→ variable change $F(w, \tau) = P_w(w, \tau) / (w+1)$

$$\frac{\partial F}{\partial \tau} = w \frac{\partial^2 F}{\partial w^2} + (1-w) \frac{\partial F}{\partial w} - 2F$$

→ Ansatz $F(w, \tau) = \int_s q_s F_s(w) \exp(-s\tau)$ must satisfy

$$w \frac{\partial^2 F_s}{\partial w^2} + (b-w) \frac{\partial F_s}{\partial w} - a F_s = 0$$

Kummer type

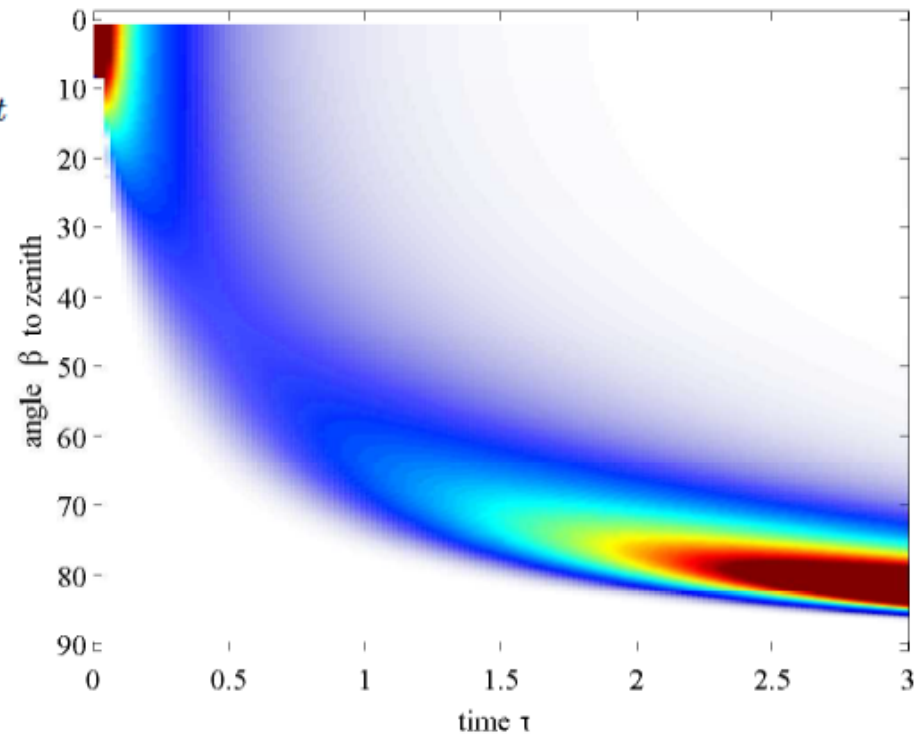
with $b=1, a=2-s$

Explicit solution via Kummer ODE

Restriction $a \in \{-1, -2, -3, \dots\} \rightarrow$ Laguerre polynomials $L_n(w)$

e.g. state x in $|e\rangle$: $\sum_{n=0}^{+\infty} d^{n+1} L_n(x) = \frac{d}{1-d} e^{\frac{-d}{1-d}x}$ and $d \xrightarrow{\lim} 1$

$$P_w(w, t) = \frac{1}{N} (w + 1) \sum_{s=2}^{+\infty} d^{s-1} L_{(s-2)}(w) e^{-st}$$



But in reality $\eta = 0.28$

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Case A:

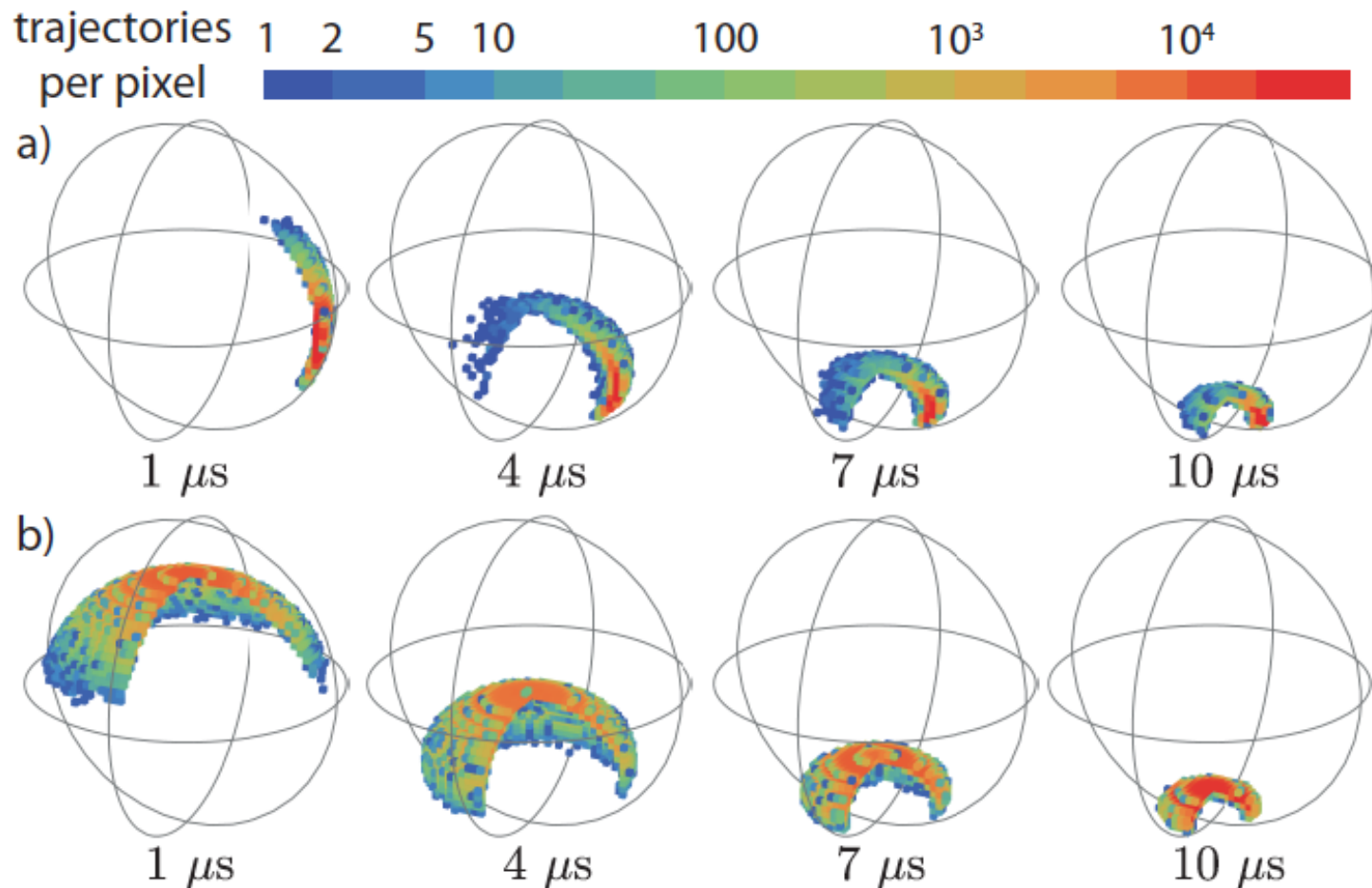
$$dr = \frac{\gamma}{2} \left(\frac{\eta(1+z)^2}{2r} - r \right) dt + \sqrt{\frac{\eta\gamma}{2}} (1+z-r^2) dW_r$$
$$dz = -\gamma(1+z) dt - \sqrt{\frac{\eta\gamma}{2}} r(1+z) dW_r$$
$$d\theta = \sqrt{\frac{\eta\gamma}{2}} \frac{1+z}{r} dW_\theta$$

Case B:

$$dr = -\gamma \left(1 - \frac{\eta}{2} \right) r dt - \sqrt{\eta\gamma} r z dW_r$$
$$dz = \sqrt{\eta\gamma} (1-z^2) dW_r$$
$$d\theta = \sqrt{\eta\gamma} dW_\theta$$

θ coordinate undergoes independent random walk — consider solved

Let's have a look at the (filtered) data

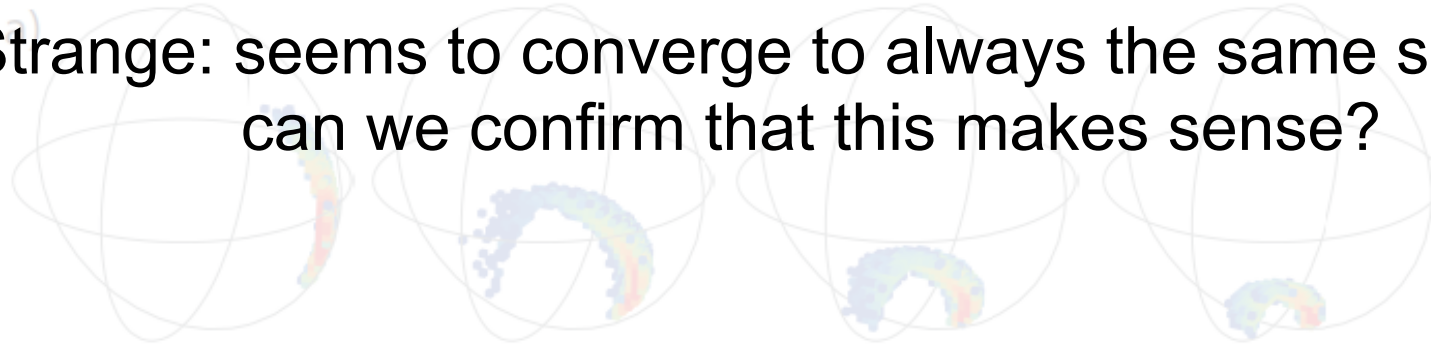


[Case A: B.Huard et al, LPA, Ecole Normale Supérieure, Paris]

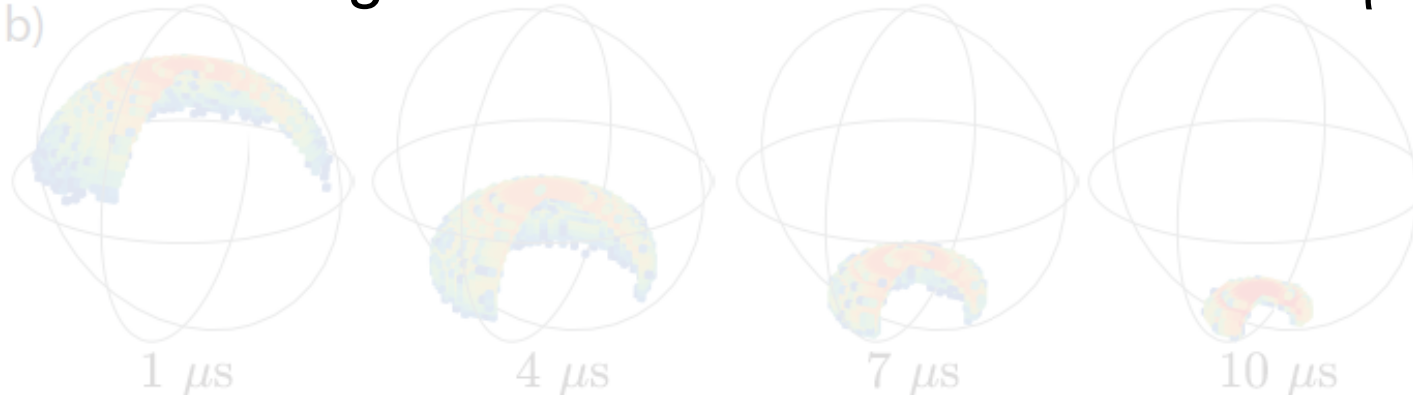
Let's have a look at the (filtered) data



Strange: seems to converge to always the same surface
can we confirm that this makes sense?



Are we seeing a “hole” in the middle or not for $\eta=0.28$?



A deterministic evolution of the supporting surface for all η and t

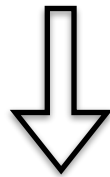
(Case A) consider the surfaces with
for $c \in [1/2, +\infty]$

$$\frac{r^2}{2} + c(1+z)^2 - (1+z) = 0$$

A deterministic evolution of the supporting surface for all η and t

(Case A) consider the surfaces with
for $c \in [1/2, +\infty]$

$$\frac{r^2}{2} + c(1+z)^2 - (1+z) = 0$$



$$dc = (c - \frac{\eta}{2}) \gamma dt \quad \text{i.e.}$$

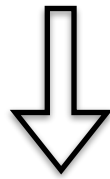
deterministically

$$c(t) = (c(0) - \frac{\eta}{2}) e^{\gamma t} + \frac{\eta}{2}.$$

A deterministic evolution of the supporting surface for all η and t

(Case A) consider the surfaces with
for $c \in [1/2, +\infty]$

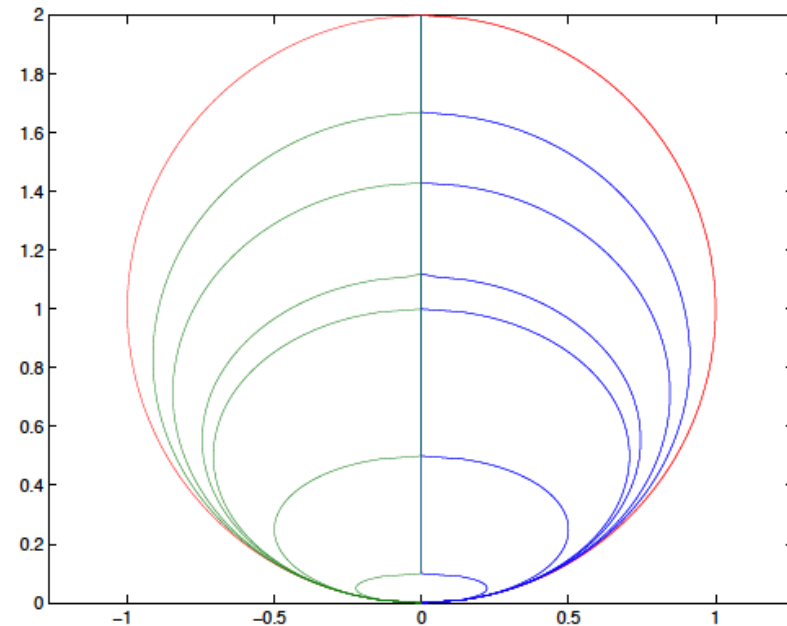
$$\frac{r^2}{2} + c(1+z)^2 - (1+z) = 0$$



$$dc = \left(c - \frac{\eta}{2}\right) \gamma dt \quad \text{i.e.}$$

deterministically

$$c(t) = \left(c(0) - \frac{\eta}{2}\right) e^{\gamma t} + \frac{\eta}{2}.$$



[helping to guess this: we know that if $\rho(0) = |g\rangle\langle g|$ then $\rho(t) = |g\rangle\langle g|$]

On the ellipsoidal cap: Fokker-Planck can be reduced to 1D Kummer-type

Change of variables becomes

$$\phi = \frac{2}{\eta} \left(\frac{1}{z+1} - c \right) = \frac{r^2}{\eta(1+z)^2} \quad \rightarrow \quad \boxed{\frac{\partial F_\phi}{\partial \tau} = \phi \frac{\partial^2 F_\phi}{\partial \phi^2} + (1-\phi) \frac{\partial F_\phi}{\partial \phi} - 2F_\phi.}$$

i.e. same probabilistic evolution as before,
but on deterministically shrinking ellipsoid

Conclusion: We get a fully explicit expression of $\mathbf{P}(r,z,t)$ for all $\eta < 1$

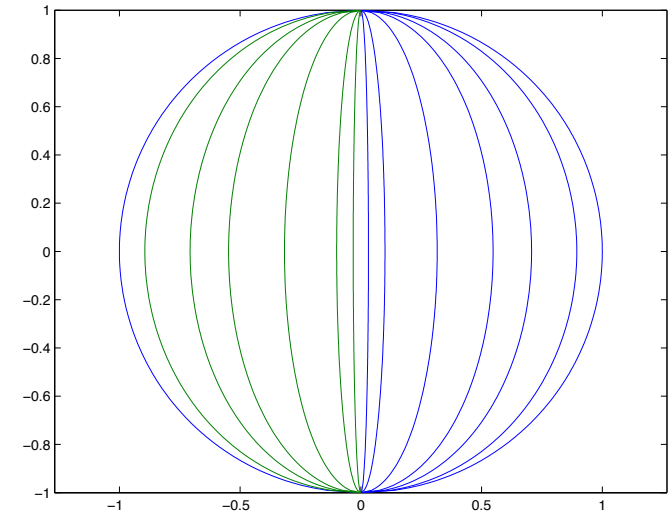
Similar treatment applies to refraction index measurements

Surfaces: $r^2 + c(1 - z^2) = 0$ for $c \in [0,1]$

$$\rightarrow c(t) = c(0) e^{-2(1-\eta)\gamma t}$$

→ variable change $w = a \sinh\left(\frac{z}{1-z^2}\right)$
yields the PDE

$$\frac{\partial P_w}{\partial \tau} = \frac{\eta}{2} \frac{\partial^2 P_w}{\partial w^2} - \eta \frac{\partial}{\partial w} (\tanh(w) P_w)$$



→ Ansatz with basis solutions $P_s(w) = \cosh(w) e^{\pm \sqrt{1-2s} w}$



Strange: seems to converge to always the same surface
 can we confirm that this makes sense?

YES

Are we seeing a “hole” in the middle or not for $\eta=0.28$?

YES, exactly the same as for $\eta=1$

How general is this deterministic surface evolution?

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Noise is like an uncontrolled control input

$$dx = f(x) dt + \sum g_i(x) dW_i, \quad x \in \mathbf{R}^n$$

features an embedded manifold with deterministic evolution



The noises dW_i cannot steer $x(0)$ towards an open set of $x(T)$
 \equiv control through $\{dW_i\}$ does not satisfy “strong accessibility”

[1970s]: strong accessibility **iff** $\text{Lie}([f, \text{Lie}(\{g_i\})])$ has dimension n

Strong accessibility: the quantum case

$$d\rho_t = \left(-\frac{i}{\hbar}[\mathbf{H}, \rho_t] + \sum_{\nu} \mathbf{L}_{\nu} \rho_t \mathbf{L}_{\nu}^{\dagger} - \frac{1}{2}(\mathbf{L}_{\nu}^{\dagger} \mathbf{L}_{\nu} \rho_t + \rho_t \mathbf{L}_{\nu}^{\dagger} \mathbf{L}_{\nu}) \right) dt \\ + \sum_{\nu} \sqrt{\eta_{\nu}} \left(\mathbf{L}_{\nu} \rho_t + \rho_t \mathbf{L}_{\nu}^{\dagger} - \text{Tr} \left((\mathbf{L}_{\nu} + \mathbf{L}_{\nu}^{\dagger}) \rho_t \right) \rho_t \right) dW_{\nu,t},$$

f(x) linear, g(x) quadratic in state

f(x) quadratic, g(x) linear in physical parameters \mathbf{L}_{ν}

Examples: measured qubit

- **Case A:**

$$L_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$$

- **Case B:**

$$L_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

both satisfy sufficient condition for rank-deficiency

$$L_i = a L_j \quad \text{and} \quad [L_i, L_i^*] L_i = b L_i \quad a, b \in \mathbf{R}$$

- **Case C:** $L_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $L_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$, $L_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ full rank even if $\eta_3=0$
except at few places that we directly leave

- For $\eta=1$ we expect **always** rank-deficiency on $x^2 + y^2 + z^2 = 1$

- Simple characterization of general case: ongoing work

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