



# Proper orientation of cacti

Julio Araujo, Frédéric Havet , Claudia Linhares Sales , Ana Silva

**RESEARCH  
REPORT**

**N° 8833**

December 2015

Project-Teams Coati





## Proper orientation of cacti

Julio Araujo<sup>\*</sup>, Frédéric Havet<sup>† ‡</sup>, Claudia Linhares Sales<sup>§ ¶</sup>,  
Ana Silva<sup>\* ||</sup>

Project-Teams Coati

Research Report n° 8833 — December 2015 — 17 pages

**Abstract:** An orientation of a graph is *proper* if two adjacent vertices have different indegrees. We prove that every cactus admits a proper orientation with maximum indegree at most 7. We also prove that the bound 7 is tight by showing a cactus having no proper orientation with maximum indegree less than 7. We also prove that any planar claw-free graph has a proper orientation with maximum indegree at most 6 and that this bound can also be attained.

**Key-words:** proper orientation, graph coloring, cactus graph, claw-free graph

---

This work was partially supported by the FUNCAP/CNRS project GAIATO INC-0083-00047.01.00/13 and by the CNPq-Brazil Universal project 459466/2014-3.

\* ParGO, Departamento de Matemática, Universidade Federal do Ceará, Fortaleza, Brazil

† Projet COATI, I3S (CNRS, UNS) and INRIA, Sophia-Antipolis, France

‡ Partially supported by ANR under contract STINT ANR-13-BS02-0007.

§ ParGO, Departamento de Computação, Universidade Federal do Ceará, Fortaleza, Brazil

¶ Partially supported by CAPES 99999.000458/2015-05 and CNPq 307252/2013-2.

|| Partially supported by CNPq PDE 232891/2014-1.

**RESEARCH CENTRE  
SOPHIA ANTIPOLIS – MÉDITERRANÉE**

2004 route des Lucioles - BP 93  
06902 Sophia Antipolis Cedex

## Orientation propre des cactus

**Résumé :** Une orientation d'un graphe est *propre* si deux sommets adjacents ont des degrés entrants différents. Nous montrons que tous les cactus admettent une orientation propre de degré entrant maximum au plus 7. Nous prouvons également que cette borne est serrée.

**Mots-clés :** orientation propre, coloration de graphes, graphe cactus, graphe sans griffe

## 1 Introduction

For basic notions and notations on Graph Theory and Computational Complexity, the reader is referred to [3, 5]. All graphs in this work are considered to be simple.

An *orientation*  $D$  of a graph  $G = (V, E)$  is a digraph obtained from  $G$  by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each  $v \in V(G)$ , the *indegree* of  $v$  in  $D$ , denoted by  $d_D^-(v)$ , is the number of arcs with root  $v$  in  $D$ . We use the notation  $d^-(v)$  when the orientation  $D$  is clear from the context. An orientation  $D$  of  $G$  is *proper* if  $d^-(u) \neq d^-(v)$ , for all  $uv \in E(G)$ . An orientation with maximum indegree at most  $k$  is called a *k-orientation*. The *proper-orientation number* of a graph  $G$ , denoted by  $\overrightarrow{\chi}(G)$ , is the minimum integer  $k$  such that  $G$  admits a proper  $k$ -orientation.

This graph parameter was introduced by Ahadi and Dehghan [1]. They observed that this parameter is well-defined for any graph  $G$  since one can always obtain a proper  $\Delta(G)$ -orientation. Note that every proper orientation of a graph  $G$  induces a proper vertex coloring of  $G$ . Hence, we have the following sequence of inequalities:

$$\omega(G) - 1 \leq \chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G).$$

These inequalities are best possible since, for a complete graph  $K_n$ :

$$\omega(K_n) - 1 = \chi(K_n) - 1 = \overrightarrow{\chi}(K_n) = \Delta(K_n) = n - 1.$$

In [1], the authors characterize the proper-orientation number of regular bipartite graphs, study other particular subclasses of regular graphs and prove the NP-hardness of the problem even when restricted to planar graphs.

Recently, it has been shown that the problem remains NP-hard for subclasses of planar graphs that are also bipartite and of bounded degree [2]. In the same paper, it is proved that the proper-orientation number of a tree is at most 4.

**Theorem 1** ([2]). *Every tree has proper-orientation number at most 4.*

A natural question is to ask how this theorem can be generalized.

**Problem 2.** *Which graph classes containing the trees have bounded proper-orientation number?*

In [2], several generalizations are suggested: on the one hand, the authors ask whether the proper-orientation number of planar graphs is bounded; on the other hand, they asked whether the proper-orientation number can be bounded by a function of the treewidth. We pose a similar, but simpler, question.

**Problem 3.** *Is there a constant  $c$  such that  $\overrightarrow{\chi}(G) \leq c$ , for every outerplanar graph  $G$ ?*

Already this question seems highly non-trivial. One of the reasons is that, contrary to many other parameters like the chromatic number, the proper-orientation number is not monotonic. Recall that a graph parameter  $\gamma$  is *monotonic* if  $\gamma(H) \leq \gamma(G)$  for every (induced) subgraph  $H$  of  $G$ . For example, the tree  $T^*$ , depicted in Figure 1, satisfies  $\overrightarrow{\chi}(T^*) = 2$ , while  $\overrightarrow{\chi}(T^* \setminus \{x\}) = 3$  as  $T^* \setminus \{x\}$  is exactly the tree  $T_3$  mentioned in [2]. Its non-monotonicity makes it difficult to handle the proper orientation number.

In this paper, we consider a standard graph class containing the trees, namely the cacti. A graph  $G$  is a *cactus* if every 2-connected component of  $G$  is either an edge or a cycle. Clearly, every cactus is an outerplanar graph. We prove that the proper orientation of such graphs is bounded by 7.

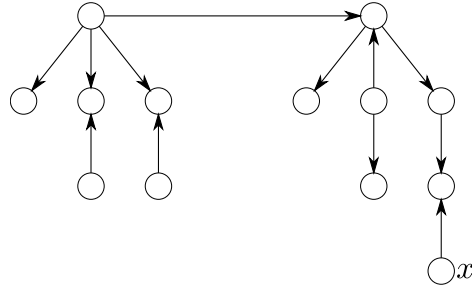


Figure 1: Tree  $T^*$  and a proper 2-orientation of it.

**Theorem 4.** *If  $G$  is a cactus, then  $\vec{\chi}(G) \leq 7$ .*

Furthermore, we show in Proposition 17 that this upper bound 7 is attained.

We conclude this section by introducing some definitions and previous results that we need in different sections of this work.

Let  $S \subseteq V(G)$  be a subset of vertices of  $G$  and  $F \subseteq E(G)$  be a subset of its edges. We denote by  $G[S]$  the subgraph of  $G$  induced by  $S$ , by  $G \setminus F$  the graph obtained from  $G$  by removing the edges in  $F$  from its edge set  $E(G)$ , and by  $G - S$  the graph  $G[V(G) \setminus S]$ .

For any two adjacent vertices  $u$  and  $v$  of  $G$ , the edge  $(u, v)$  is denoted by  $uv$ . Given an orientation  $D$  of  $G$ , we denote the orientation of  $uv$  towards  $v$  by  $(u, v)$ .

Let  $T$  be a tree. A *leaf* of  $T$  is a vertex with degree 1. A *twig* of  $T$  is a vertex which is not a leaf and whose neighbors are all leaves except possibly one. A *bough* of  $T$  is a vertex which is neither a leaf nor a twig and whose neighbors are all leaves or twigs except possibly one. A *branch* of  $T$  is a vertex which is neither a leaf nor a twig nor a bough and whose neighbors are all leaves or twigs or boughs except possibly one.

The definitions above are the same as the ones used in [2] and we borrow from them. Let  $G$  be a graph. The *block tree* associated to  $G$  is the tree  $T(G)$  with vertex set the set of blocks of  $G$  such that two vertices are adjacent in  $T(G)$  if and only if the blocks intersect. A block of order  $i$  is said to be an  *$i$ -block*. A *leaf block* (resp. *twig block*, *bough block*, *branch block*) is a block which is a leaf (resp. twig, bough, branch) in  $T(G)$ . By the definitions in the previous paragraph, observe that if  $B$  is of one of these types of blocks, then  $B$  may have a neighbor in  $T(G)$  that is an exception in its neighborhood. If such a neighbor  $B'$  exists and  $u \in B$  separates  $B$  from  $B'$ , then we call  $u$  the *root* of  $B$ . Otherwise, we pick any vertex of  $B$  to be the root of  $B$ . If  $B$  is a twig block with root  $r$ , then the *twig subgraph of  $G$  with root  $r$*  is the union of  $B$  and all leaf blocks with root in  $V(B) \setminus \{r\}$ . If  $B$  is a bough block with root  $r$ , then the *bough subgraph of  $G$  with root  $r$*  is the union of  $B$  and all twig subgraphs with root in  $V(B) \setminus \{r\}$ . Observe that twig and bough subgraphs are connected.

Let  $B$  be a block in  $G$ . For any vertex  $v \in B$  we denote by  $G_B \langle v \rangle$  the connected component of  $G \setminus E(B)$  containing  $v$ . If the block  $B$  is clear from the context, we often drop the subscript  $B$ .

## 2 Proper 7-orientation of cacti

In this section, we prove Theorem 4 by considering a minimum counter-example. Such a counter-example is a cactus  $G$  that admits no proper 7-orientation, and such that every cactus  $H$  with

fewer vertices than  $G$  has a proper 7-orientation. Observe that such a counter-example  $G$  is clearly a connected graph.

The idea of the proof is to analyse the structure of the leaf, twig and bough subgraphs of  $G$  and observe that there is always one such subgraph in  $G$  with root  $r$  such that any proper 7-orientation of  $G(r)$  (which exists by the minimality of  $G$ ) can be extended in a proper 7-orientation of  $G$ , which is a contradiction.

If  $B$  is a block of  $G$  with vertex set  $\{v_1, \dots, v_p\}$  appearing in this order on the cycle (or edge), then we write  $B$  as  $\langle v_1, \dots, v_p \rangle$ .

**Lemma 5.** *Let  $P = (v_1, \dots, v_n)$  be a path on  $n$  vertices,  $n \neq 2$ . Then, there exists a proper 2-orientation of  $P$  such that  $v_1$  and  $v_n$  have indegree 0.*

*Proof.* If  $n$  is odd, it suffices to orient the arcs of  $P$  from vertices with odd indices towards vertices with even indices. This yields an alternating indegree sequence of 0's and 2's that starts and ends with 0. If  $n$  is even, orient  $(v_1, \dots, v_{n-1})$  as above and  $v_{n-1}v_n$  towards  $v_{n-1}$  in order to obtain the desired orientation.  $\square$

Now we show that, in  $G$ , every vertex of small degree has a neighbor of higher degree.

**Proposition 6.** *Let  $u$  be a vertex of  $G$ . If  $d(u) \leq 7$ , then there exists  $v \in N(u)$  such that  $d(v) > d(u)$ .*

*Proof.* Suppose for a contradiction that  $d(u) \leq 7$  and all vertices in  $N(u)$  have degree at most  $d(u)$ . Let  $D$  be a proper 7-orientation of  $G - u$ . For each  $v \in N_G(u)$ , since  $d_{G-u}(v) = d_G(v) - 1 \leq d_G(u) - 1$ , we know that  $d_D^-(v) < d_G(u)$ . Therefore, because  $d_G(u) \leq 7$ , one can extend  $D$  by orienting every edge incident to  $u$  in  $G$  towards  $u$  to obtain a proper 7-orientation of  $G$ , a contradiction.  $\square$

**Proposition 7.** *Every leaf block of  $G$  is either a 2-block or a 3-block.*

*Proof.* Observe that, for any leaf block with at least four vertices, there must be at least one vertex of degree 2 whose neighbors also have degree 2, contradicting Proposition 6.  $\square$

Proposition 7 implies that a leaf block is either a 1-path (i.e. a path of length 1) or a triangle (i.e. a cycle of length 3). In Figure 2, we present every possible proper orientation of a leaf block.

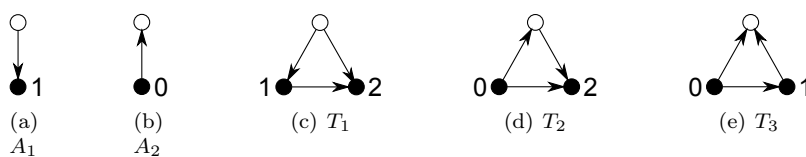


Figure 2: Leaf blocks and their possible proper orientations.

**Proposition 8.** *Every vertex of  $G$  is contained in at most one leaf 2-block.*

*Proof.* By contradiction, suppose that it is not the case and let  $\langle u, v \rangle, \langle u, w \rangle$  be two leaf 2-blocks containing  $u$ . Let  $D$  be a proper 7-orientation of  $G - w$ . If  $d_D^-(u) \neq 1$ , orienting  $uw$  towards  $w$  extends  $D$  into a proper 7-orientation of  $G$ , a contradiction. Hence  $d_D^-(u) = 1$ . Since  $D$  is proper, the edge  $uw \in E(G)$  must be the only one oriented towards  $u$  in  $D$ . Therefore all neighbors of  $u$  distinct from  $v$  and  $w$  have indegree greater than 1 in  $D$ . Reverting the orientation of  $uw$  in  $D$  and orienting  $uw$  towards  $w$ , we obtain a 7-orientation of  $G$ , which is proper because the indegree of  $u$  is 0, hence different from the indegree of all of its neighbors. This is a contradiction.  $\square$

**Proposition 9.** *Every twig block is a 2-block or a 3-block.*

*Proof.* Let  $B$  be a twig block of order  $q$  at least 4, say  $B = \langle u_1, \dots, u_q \rangle$  with  $u_1$  the root of  $B$ .

**Claim 9.1.**  $d(u_i) \neq 3$ , for every  $i \in \{2, \dots, q\}$ .

*Subproof.* By contradiction, suppose that there exists a vertex  $u_i \in \{u_2, \dots, u_q\}$  of degree 3 in  $G$ . Note that  $u_i$  is contained in the block  $B$  and in a leaf 2-block, say  $\langle u_i, v \rangle$ .

First suppose that  $i \notin \{2, q\}$  and let  $G' = G - \{u_i, v\}$ . By the minimality of  $G$ , there exists a proper 7-orientation  $D$  of  $G'$ . If  $\{d_D^-(u_{i-1}), d_D^-(u_{i+1})\} \neq \{2, 3\}$ , then one could extend  $D$  to a proper 7-orientation of  $G$  by orienting  $u_i u_{i-1}$  and  $u_i u_{i+1}$  towards  $u_i$  and choosing the orientation of  $u_i v$  according to the indegrees of  $u_{i-1}$  and  $u_{i+1}$  in  $D$ , a contradiction. Hence, without loss of generality, consider that  $d_D^-(u_{i-1}) = 2$  and  $d_D^-(u_{i+1}) = 3$ .

Let us extend  $D$  by orienting all the arcs incident to  $u_i$  away from this vertex. The resulting orientation  $D'$  is not yet proper but we shall prove how to change it into a proper 7-orientation of  $G$ . Problems could only appear in edges incident to  $u_{i-1}$  or  $u_{i+1}$  which had indegree 3 and 4 respectively in  $D$ . Observe that these two vertices have degree more than 2 and thus belong to some other blocks which must be leaf blocks since  $u_1$  is the root of  $B$ . One can reorient the leaf blocks containing  $u_{i+1}$  using the orientations of Figure 2 so that the indegree of  $u_{i+1}$  becomes 3 again. Similarly, if  $d(u_{i-1}) = 4$ , one can reorient the leaf blocks containing  $u_{i-1}$  so that the indegree of  $u_{i-1}$  is in  $\{3, 4\} \setminus \{d_D^-(u_{i-2})\}$ , and if  $d(u_{i-1}) = 3$  (that is  $u_{i-1}$  is in a unique leaf 2-block), one can reorient the leaf block containing  $u_{i-1}$  so that the indegree of  $u_{i-1}$  becomes 2 again. The resulting orientation is then a proper 7-orientation of  $G$ , a contradiction.

Suppose now that  $i \in \{2, q\}$ . Without loss of generality, we may assume that  $i = 2$ . Let  $G'$  be the connected component of  $G - u_3$  containing  $u_2$ . Let  $D'$  be a proper 7-orientation of  $G'$ . Clearly,  $d_{D'}^-(u_2) \leq 2$ . By the previous paragraphs and because  $q \geq 4$ , we know that  $d_G(u_3) \neq 3$ . If  $d_G(u_3) > 3$ , we can obtain a proper orientation of  $G$  by orienting edges  $u_2 u_3$  and  $u_3 u_4$  towards  $u_3$  and orienting the leaf blocks containing  $u_3$  in such a way that  $d^-(u_3) \in \{3, 4\} \setminus d_D^-(u_4)$ ; this is a contradiction. Consequently,  $d(u_3) = 2$ , and we can suppose that  $2 \in \{d_D^-(u_2), d_D^-(u_4)\}$ , as otherwise we get a contradiction by adding  $(u_2, u_3)$  and  $(u_4, u_3)$ .

First, suppose that  $d_D^-(u_2) \neq 2$ , in which case one can verify that we can suppose that  $d_D^-(u_2) = 0$ . If  $d(u_4) > 3$ , we reorient the leaf blocks and  $u_3 u_4$  so that  $u_4$  has indegree in  $\{3, 4\} \setminus \{d_D^-(u_5)\}$ , then we let  $u_3$  have indegree 1 or 2, depending on the orientation of  $u_3 u_4$ . This gives us a contradiction, and, because  $d_D^-(u_4) = 2$ , we get that  $d(u_4) = 3$  and, by the previous paragraphs, that  $q = 4$ . Let  $v' \in N(u_4) \setminus B$ . We get a contradiction by reversing  $(v', u_4)$  and adding  $(u_2, u_3)$  and  $(u_3, u_4)$ .

Finally, suppose that  $d_D^-(u_2) = 2$ . Then we can also suppose that  $d_D^-(u_4) = 1$  as otherwise we can reverse  $(v, u_2)$ , orient  $u_2 u_3$  towards  $u_2$ , and orient  $u_3 u_4$  towards  $u_3$  to obtain a proper 7-orientation of  $G$ . By similar arguments, if  $d(u_4) > 3$ , then we can change its color to some  $c \in \{3, 4\} \setminus \{d_D^-(u_5)\}$ ; hence  $d(u_4) \in \{2, 3\}$  and we analyse the cases below:

- $d(u_4) = 3$ : let  $v' \in N(u_4) \setminus B$ . Because  $d^-(u_4) = 1$ , we know that  $(v', u_4), (u_4, u_1) \in D$ . Reverse  $(v, u_2)$  and  $(v', u_4)$ , and add  $(u_3, u_2)$  and  $(u_4, u_3)$  to obtain a contradiction;
- $d(u_4) = 2$ : if  $q = 4$ , because  $d_D^-(u_2) = 2$  we know that  $d_D^-(u_1) \neq 2$ . Reverse  $(v, u_2)$  and add  $(u_3, u_2)$  and  $(u_3, u_4)$  to obtain a contradiction. Otherwise, by similar arguments we can suppose that  $d_D^-(u_5) = 2$ . Suppose that  $d(u_5) > 3$  and reorient the leaf blocks containing  $u_5$  and  $u_4 u_5$  so that  $u_5$  has indegree in  $\{3, 4\} \setminus \{d_D^-(u_6)\}$ . After this,  $u_4$  has indegree either 0 or 1, in which case we reverse  $(v, u_2)$ , add  $(u_3, u_2)$  and either  $(u_4, u_3)$  or  $(u_3, u_4)$ , depending on  $u_4$ . Finally, we can suppose that  $d(u_5) = 3$  and  $q = 5$ . Let  $v' \in N(u_5) \setminus B$ . Reverse  $(v, u_2)$  and  $(v', u_5)$ , and add  $(u_3, u_2)$ ,  $(u_4, u_5)$  and  $(u_3, u_4)$  to get a contradiction.



◇

Now we return to the proof of the lemma. By the minimality of  $G$ , there is a proper 7-orientation  $D$  of  $G\langle u_1 \rangle$ .

We shall extend  $D$  into a proper 7-orientation of  $G$ , which gives us the desired contradiction. We first add  $(u_1, u_2), (u_1, u_q)$ . We then distinguish some cases according to  $d_D^-(u_1)$ .

Assume first  $d_D^-(u_1) \notin \{2, 4\}$ . Add  $(u_3, u_2), (u_{q-1}, u_q)$  and orient the path  $(u_3, \dots, u_{q-1})$  according to Lemma 5. So far the vertices  $u_2, \dots, u_q$  have indegree 0, 1, or 2 in  $B$ . For each  $i \in \{2, \dots, q\}$ , if  $u_i$  is contained in some leaf block, then by Claim 9.1  $d(u_i) \geq 4$ . Thus, by Proposition 8,  $u_i$  is in at least one leaf 3-block. If  $u_i$  has indegree 0 in  $B$ , then we orient all the leaf blocks containing  $u_i$  with  $A_1$  or  $T_1$ , so that  $u_i$  still has indegree 0. If  $u_i$  has indegree 1 (resp. 2) in  $B$ , we orient one leaf 3-block according to  $T_3$  and all other blocks according to  $A_1$  and  $T_1$ , so that its indegree is 3 (resp. 4). It is now a simple matter to check that the obtained orientation is a proper 7-orientation of  $G$ .

Assume now  $d_D^-(u_1) \in \{2, 4\}$ . If  $q = 4$ , add  $(u_2, u_3)$  and  $(u_4, u_3)$ , and one can verify that we can get a contradiction again by orienting the leaf blocks containing vertices in  $B$  in the same way as above. So, suppose that  $q \geq 6$ . Add  $(u_2, u_3), (u_4, u_3), (u_q, u_{q-1})$ , and  $(u_{q-2}, u_{q-1})$ . Furthermore, if  $q = 7$  then add  $(u_4, u_5)$ , and if  $q > 7$  apply Lemma 5 to orient the path  $(u_4, \dots, u_{q-2})$ . We then orient the leaf blocks containing vertices in  $B$  in the same way as above to get a contradiction.

Therefore, we can consider  $q = 5$ . Add the arcs  $(u_1, u_2), (u_1, u_5), (u_3, u_4)$ , and  $(u_5, u_4)$  to  $D$ .

If  $d(u_2) > 2$ , then  $u_2$  is in a leaf 3-block. Add  $(u_3, u_2)$ , and orient one leaf 3-block containing  $u_2$  with  $T_2$  and the other leaf blocks with  $A_1$  or  $T_1$  so that  $u_2$  has indegree 3. For  $j \in \{3, 4, 5\}$ , if  $u_j$  is contained in some leaf block, orient its leaf blocks so that the indegree of  $u_j$  increases by 2 (using one  $T_3$  and possibly some  $A_1$  and  $T_1$ ). It is simple matter to check that it gives a proper 7-orientation of  $G$ . By symmetry, we get the result if  $d(u_5) = 2$ .

Finally, consider  $d(u_2) = d(u_5) = 2$ , and since  $B$  is not a leaf block, we can suppose, without loss of generality, that  $d(u_3) > 2$ . In this case, add  $(u_2, u_3), (u_3, u_4)$  and  $(u_5, u_4)$ , orient the leaf block(s) containing  $u_3$  so that its indegree is 3 and, if  $d(u_4) > 2$ , orient the leaf block(s) containing  $u_4$  so that its indegree is 4.

□

**Proposition 10.** *Let  $B$  be a twig block with root  $u_1$ .*

- (a) *If  $B = \langle u_1, u_2 \rangle$ , then either  $d(u_2) = 2$  or  $u_2$  belongs exactly to  $B$  and to a leaf 3-block.*
- (b) *If  $B = \langle u_1, u_2, u_3 \rangle$ , then, for each  $j \in \{2, 3\}$ ,  $u_j$  belongs exactly to  $B$  and either a leaf 2-block or a leaf 3-block.*

*Proof.* (a) Assume that  $d(u_2) > 2$ . Let  $D$  be a proper 7-orientation of  $G\langle u_1 \rangle$ . We can suppose that  $d(u_2) = 3$ , as otherwise we extend  $D$  to a proper 7-orientation of  $G$  by orienting  $u_1 u_2$  towards  $u_2$  and orienting the leaf blocks with root  $u_2$  in such a way that its indegree belongs to  $\{3, 4\} \setminus \{d_D^-(u_1)\}$ . Consequently, by Proposition 7 and Proposition 8, we obtain that  $u_2$  is contained exactly in  $B$  and in a leaf 3-block.

(b) Suppose first that one vertex of  $\{u_2, u_3\}$ , say  $u_3$ , is in no leaf block.

Suppose  $d(u_2) > 3$  and let  $D$  be a proper 7-orientation of  $G\langle u_1 \rangle$ . One can orient the edges  $u_1 u_2$  and  $u_1 u_3$  from  $u_1$  to its neighbors and then orient the leaf block(s) containing  $u_2$  and the

edge  $u_2u_3$  in such a way that the indegree of the pair  $(u_2, u_3)$  is  $(3, 2)$ , in case  $d_D^-(u_1) \notin \{2, 3\}$ , or  $(4, 1)$ , otherwise. This results in a proper 7-orientation of  $G$ , a contradiction.

If  $d(u_2) = 3$ , then let  $D$  be a proper 7-orientation of  $G - v$ , where  $v$  is the neighbor of  $u_1$  not in  $B$ . Since  $u_2$  and  $u_3$  are symmetric in  $G - v$ , we can suppose that  $d_D^-(u_2) \neq 1$ , in which case we can extend  $D$  into a proper 7-orientation of  $G$  by orienting  $u_2v$  towards  $v$ . This is a contradiction.

Suppose now that  $d(u_2) > 2$ , and  $d(u_3) > 2$ . If  $d(u_2) \geq 5$ , let  $G'$  be the component of  $G - u_2$  containing  $u_1$ . Let  $D$  be a proper 7-orientation of  $G'$ . One could then extend  $D$  to a proper 7-orientation of  $G$  by orienting the edges  $u_1u_2$  and  $u_2u_3$  towards  $u_2$  and orienting the leaf blocks containing  $u_2$  in such a way that its indegree belongs to  $\{3, 4, 5\} \setminus \{d_D^-(u_1), d_D^-(u_3)\}$ . By symmetry, we get a contradiction in the same way if  $d(u_3) \geq 5$ . Therefore  $d(u_2) \leq 4$  and  $d(u_3) \leq 4$ . Then, the proposition follows by Proposition 7 and by Proposition 8.  $\square$

The *2-path*, the *kite*, the *bull*, the *elk*, and the *moose* are the rooted graphs depicted in Figure 3 where the root is the white vertex.

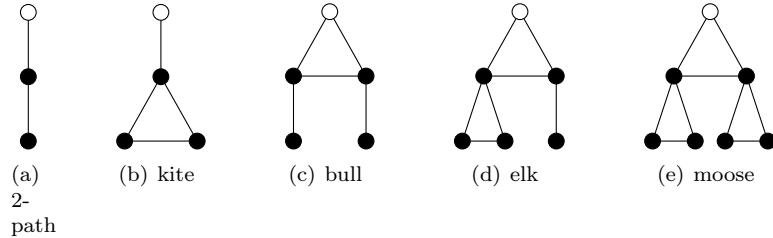


Figure 3: The five possible twig subgraphs.

Propositions 8, 9, and 10 imply directly the following.

**Corollary 11.** *Every twig subgraph in  $G$  is either a 2-path, or a kite, or a bull, or an elk, or a moose.*

In the following we will very often use this corollary without referring explicitly to it.

All the possible (partial) proper orientations of the twig subgraphs are depicted in Figures 4 to 8. In these figures, the notation  $i - j$  means that the corresponding vertex can have any indegree in this range, depending on the orientation given to the non-oriented edges.

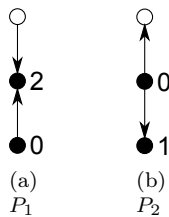


Figure 4: Proper orientations of the 2-path.

**Proposition 12.** *Let  $B$  be a bough block with root  $u$ . Every vertex  $v$  in  $V(B) \setminus \{u\}$  with degree at least 3 is the root of a twig subgraph or a leaf block that is neither a kite nor a moose.*

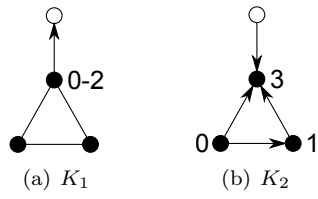


Figure 5: Proper orientations of the kite.

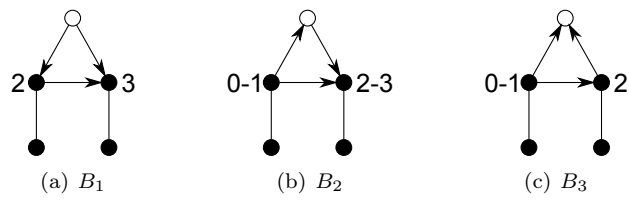


Figure 6: Proper orientations of the bull.

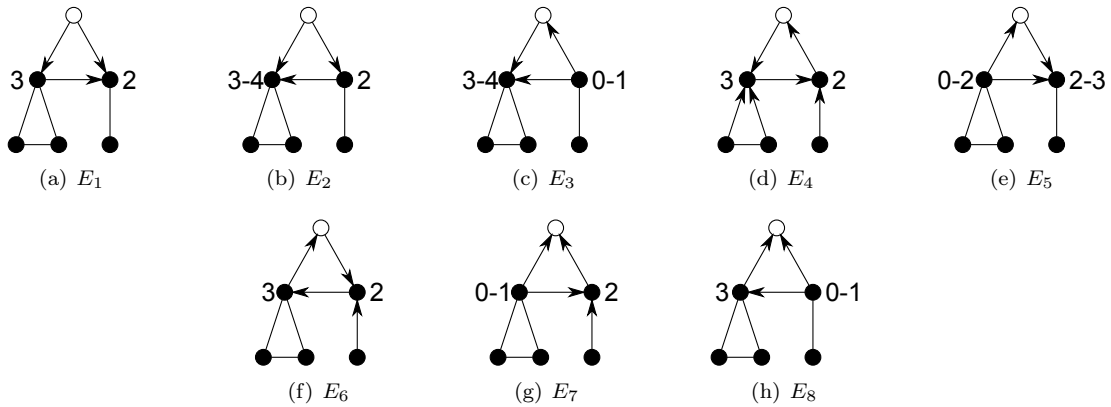


Figure 7: Proper orientations of the elk.

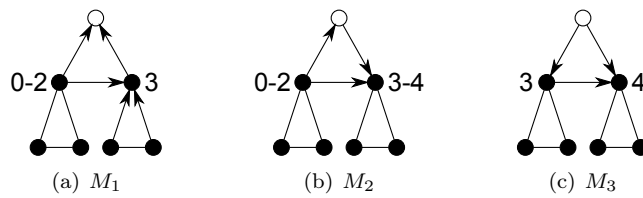


Figure 8: Proper orientations of the moose.

*Proof.* Let  $v$  be a vertex in  $V(B) \setminus \{u\}$  with degree at least 3. It must be the root of at least one twig subgraph or leaf block. Suppose the contrary that  $v$  is only root of kites and moose. Let  $D$  be a proper 7-orientation of  $G \setminus v$ . Observe that  $d_D^-(v) \leq 2$ . Thus, one could extend  $D$  to  $G$  by orienting the kites and moose rooted at  $v$  according to  $K_2$  or  $M_3$ .  $\square$

**Proposition 13.** *Let  $B = \langle u_1, \dots, u_q \rangle$  be a bough block with root  $u_1$ . For all  $i \in \{2, \dots, q\}$ ,  $d(u_i) \leq 4$ .*

*Proof.* Let  $i \in \{2, \dots, q\}$ . Let  $G'$  be the connected component containing  $u_1$  in  $G - u_i$ . By the minimality of  $G$ ,  $G'$  admits a proper 7-orientation  $D$ . Set  $F = \{d_D^-(u_{i-1}), d_D^-(u_{i+1})\}$ . Add the arcs  $(u_{i-1}, u_i)$  and  $(u_{i+1}, u_i)$ .

If  $d(u_i) \geq 7$ , then one can properly orient the twig subgraphs and leaf blocks with root  $u_i$  in such a way that  $u_i$  has indegree in  $\{5, 6, 7\} \setminus F$ . Observe that all other vertices of those graphs have indegree at most 4, so we obtain a proper 7-orientation of  $G$ , a contradiction.

If  $d(u_i) = 6$ , by Proposition 12, it is contained in at most one moose. Therefore, one can orient the twig and leaf subgraphs containing  $u_i$  so that  $u_i$  has indegree in  $\{4, 5, 6\} \setminus F$ , taking care to use  $M_1$  for the possible moose. Observe that every other possible twig can avoid a 4 from appearing in  $N(u_i)$ . Hence, we have a proper 7-orientation of  $G$ , a contradiction.

Thus, we can suppose that  $d(u_i) \leq 5$ , for all  $i \in \{2, \dots, q\}$ .

Assume now for a contradiction that there is some  $i \in \{2, \dots, q\}$  such that  $d(u_i) = 5$ .

If  $q = 2$ , then  $|F| = 1$  and one can extend  $D$  to a proper 7-orientation of  $G$  by orienting the twig and leaf blocks containing  $u_i$  so that the indegree of  $u_i$  belongs to  $\{4, 5\} \setminus F$ . This is a contradiction so  $q \geq 3$ .

Observe that if  $\{4, 5\} \neq F$ , then one can extend  $D$  to  $G$  by orienting the twig and leaf blocks containing  $u_i$  in such a way that its indegree belong to  $\{4, 5\} \setminus F$ . Consequently, we can assume that  $F = \{4, 5\}$ . But  $d(u_j) \geq d_D^-(u_j) + 1$ . So one vertex in  $\{u_{i-1}, u_{i+1}\}$  is  $u_1$ . Free to relabel the vertices in the other sense around  $B$ , we may assume that  $i = 2$ . Hence  $d_D^-(u_1) = 5$  and  $d_D^-(u_3) = 4$ . So  $d(u_3) = 5$ . Applying the same reasoning to  $u_3$ , we obtain that  $q = 3$ .

**Claim 13.1.** *There is a proper 7-orientation  $D'$  of  $G'$  such that  $d_{D'}^-(u_3) \in \{2, 3\}$ .*

*Subproof.* The idea is to start from  $D$  and to reorient the edges of the leaf blocks and twig subgraphs with root  $u_3$ . Observe that in  $D$  all the edges incident to  $u_3$  are directed towards  $u_3$ . In particular  $(u_1, u_3)$  is an arc of  $D$ .

By Propositions 7, 9 and 10,  $u_3$  is the root of:

1. two subgraphs,  $H_1$  and  $H_2$ , with  $H_1$  being a triangle, a bull, an elk or a moose, and  $H_2$  being a 1-path, a 2-path, or a kite; or
2. three subgraphs,  $H_1$ ,  $H_2$  and  $H_3$ , each of them being a 1-path, a 2-path, or a kite.

If Case 1 occurs, then we are in one of the following subcases.

- 1.1.  $H_1$  is a moose. Orient it using  $M_3$  and  $H_2$  using  $A_2$ ,  $P_2$  or  $K_1$  (with the in degree of its neighbor 0). This yields the desired proper orientation  $D'$  with  $d_{D'}^-(u_3) = 2$ .
- 1.2.  $H_1$  is an elk or a bull. If  $H_2$  is a 1-path or 2-path, then orient  $H_1$  with  $E_7$  or  $B_3$  and  $H_2$  with  $A_1$  or  $P_1$  to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 3$ . If not, then  $H_2$  is a kite. Orient  $H_1$  with  $E_3$  or  $B_2$  (with the neighbor of  $u_3$  having in degree different from 2) and  $H_2$  with  $K_2$  to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 2$ .
- 1.3  $H_1$  is a triangle. Orient  $H_1$  with  $T_2$  and  $H_2$  with  $A_2$ ,  $P_2$  or  $K_1$  to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 3$ .

If Case 2 occurs, without loss of generality, we are in one of the following subcases.

- 2.1  $H_1$  and  $H_2$  are kites. Orient  $H_1$  and  $H_2$  using  $K_2$  and  $H_3$  using  $A_2$ ,  $P_2$  or  $K_1$ , to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 2$ .
- 2.2  $H_1$  is a kite or a 1-path or a 2-path, and  $H_2$  and  $H_3$  are 1-path or a 2-path. Orient  $H_1$  using  $K_1$  or  $A_2$  or  $P_2$ ,  $H_2$  using  $A_2$  or  $P_2$ , and  $H_3$  using  $A_1$  or  $P_1$ , to obtain the desired orientation  $D'$  with  $d_{D'}^-(u_3) = 3$ .

◇

Now apply the above reasoning with the orientation  $D'$  given by Claim 13.1: we have  $F \neq \{4, 5\}$  because  $d_{D'}^-(u_3) \in \{2, 3\}$ . Therefore, we obtain a proper 7-orientation of  $G$ , a contradiction. □

Proposition 6 imply the following.

**Proposition 14.** *Let  $u$  be a vertex in  $G$ .*

- (a) *if  $u$  is the root of a kite or a bull, then  $d(u) \geq 4$ ;*
- (b) *if  $u$  is the root of an elk or a moose, then  $d(u) \geq 5$ .*

**Proposition 15.** *Every bough block is a 3-block.*

*Proof.* Let  $B = \langle u_1, \dots, u_q \rangle$  be a block with root  $u_1$ .

Assume first that  $q = 2$ . Let  $D$  be a proper 7-orientation of  $G \langle u_1 \rangle$ . By Proposition 13, we know that  $d(u_2) \leq 4$ . If  $d(u_2) = 4$ , we can orient the remaining edges in such a way that  $u_2$  has indegree in  $\{3, 4\} \setminus \{d_D^-(u_1)\}$  taking care that all kites are oriented using  $K_1$ . This is possible because  $u_2$  is the root of at most two kites thanks to Proposition 12. This yields a proper 7-orientation of  $G$ , a contradiction.

Henceforth, since  $B$  is a bough block,  $u_2$  is the root of a twig subgraph  $H_1$ . In particular,  $d(u_2) = 3$ , and by Proposition 14,  $H_1$  is a 2-path, say  $(u_2, x, x')$ . Vertex  $u_2$  must also be the root of another subgraph  $H_2$  that is either a 2-path  $(u_2, y, y')$  or a 1-path  $(u, y)$ . Add the arc  $(u_1, u_2)$ . If  $d_D^-(u_1) \neq 3$ , one can orient  $H_1$  and  $H_2$  using  $P_2$  and  $A_2$  so that  $u_2$  get indegree 3. This yields a proper 7-orientation of  $G$ , a contradiction. Assume  $d_D^-(u_1) = 3$ . If  $H_2$  is a 2-path, then orient  $H_1$  and  $H_2$  using  $P_1$  so that  $u_2$  get indegree 1. If  $H_2$  is a 1-path, then orient  $H_1$  using  $P_2$  and  $H_2$  using  $A_1$  so that  $u_2$  get indegree 2. In both cases, it results in a proper 7-orientation of  $G$ , a contradiction.

Now, suppose that  $q \geq 4$ . Note that Propositions 6 and 13 imply  $d(u_3) \leq 3$ , and that Proposition 14 implies that  $u_3$  is not root of a kite. So, either  $d(u_3) = 2$  or  $u_3$  is the root of a 1-path or a 2-path.

Suppose first that  $d(u_2) = 4$ . Let  $D$  be a proper orientation of  $G \langle u_2 \rangle - u_2$ . Because  $d(u_3) \leq 3$ , we get that  $d_D^-(u_3) \leq 2$ . Add the arcs  $(u_1, u_2)$  and  $(u_3, u_2)$ . By Proposition 12,  $u_2$  is neither the root of a moose nor of two kites. Therefore, one can orient the twig subgraphs and leaf blocks with root  $u_2$  so that its indegree belongs to  $\{3, 4\} \setminus \{d_D^-(u_1)\}$ . This results in a proper 7-orientation of  $G$ , a contradiction.

Similarly, we get a contradiction if  $d(u_{q-1}) = 4$ , so we can assume that:  $(\star)$   $d(u_i) \leq 3$ , for all  $i \in \{2, \dots, q\}$ .

Now by Proposition 14-(a), if  $d(u_i) = 3$  for some  $i \in \{2, \dots, q\}$ , it is the root of a 1-path or a 2-path. Consequently, by  $(\star)$ , for all  $i \in \{3, \dots, q-1\}$ , it has no neighbor of degree more than 3.

Thus, by Proposition 6, we get  $d(u_i) = 2$ , for every  $i \in \{3, \dots, q-1\}$ , and  $q \leq 5$ , for otherwise  $u_4$  has degree 2 and no neighbor of degree more than 2.

Since  $B$  is a bough block and not a twig block, one of its vertices distinct from the root  $u_1$  must be the root of a twig subgraph. Necessarily, it must be  $u_2$  or  $u_q$  as all other vertices have degree 2. By symmetry, we may assume that it is  $u_2$ . Furthermore, since  $d(u_2) = 3$ , by Proposition 14-(a),  $u_2$  is necessarily the root of a 2-path, say  $(u_2, x, x')$ .

Let  $D$  be a proper 7-orientation of  $G \setminus \langle u_1 \rangle$ . Orient the edges  $u_1 u_2$ ,  $u_1 u_q$  and  $u_2 u_3$  towards  $u_2$ ,  $u_q$  and  $u_3$ , respectively. We now describe how to extend this orientation in a proper 7-orientation of  $G$ , yielding the contradiction. We distinguish two cases depending on whether  $q = 4$  or  $q = 5$ .

- $q = 4$ . Assume first  $d(u_4) = 2$ . If  $d_D^-(u_1) \neq 2$ , add  $(u_3, u_4)$ ,  $(x, u_2)$  and  $(x, x')$ ; otherwise, add their reverses. So suppose that  $d(u_4) = 3$ . Then  $u_4$  is the root of either a 1-path  $(u_4, y)$  or a 2-path  $(u_4, y, y')$  by Proposition 14. If  $d_D^-(u_1) \neq 3$ , then  $D$  can be extended to  $G$  by reversing  $u_2 u_3$  and orienting the remaining edges so that the indegrees of  $u_2$  and  $u_4$  will be 3. If  $d_D^-(u_1) = 3$ . Add  $(u_4, y)$ . If  $u_4$  is the root of a 1-path, add  $(u_3, u_4)$ ,  $(x, u_2)$  and  $(x, x')$ . Otherwise,  $u_4$  is the root of a 2-path : add  $(u_4, u_3)$ ,  $(u_2, x)$ ,  $(x', x)$ , and  $(y', y)$ .
- $q = 5$ . By Proposition 6, we have  $d(u_5) = 3$ . So  $u_5$  is the root of either a 1-path  $(u_5, y)$  or a 2-path  $(u_5, y, y')$  by Proposition 14. If  $d_D^-(u_1) \neq 3$ , reverse  $u_2 u_3$  and orient properly the remaining edges in a way that the indegrees of  $u_2$  and  $u_5$  is 3. If  $d_D^-(u_1) = 3$ , first add  $(u_2, x)$ ,  $(x', x)$  and  $(u_4, u_3)$  to  $D$ . If  $u_5$  is the root of a 1-path, then add  $(u_5, y)$  and  $(u_4, u_5)$ ; otherwise,  $u_5$  is the root of a 2-path : add  $(y, u_5)$ ,  $(u_5, u_4)$  and  $(y, y')$ .

□

A *reindeer* is the graph depicted in Figure 9, where the root is the white vertex. It also depicts all possible orientations of the reindeer.

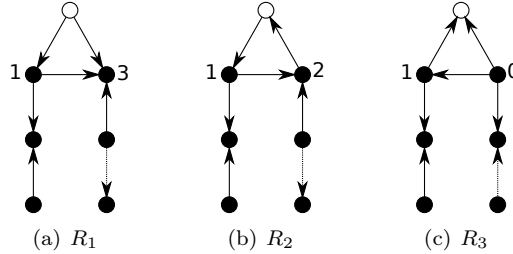


Figure 9: The reindeer and its possible orientations. The dashed edge may or may not exist.

**Proposition 16.** *Every bough subgraph is a reindeer.*

*Proof.* Let  $H$  be a bough subgraph rooted at  $u_1$ . It contains a bough block  $B$ . By Proposition 15,  $B$  is a 3-block, say  $B = \langle u_1, u_2, u_3 \rangle$ . By Proposition 13,  $d(u_2) \leq 4$  and  $d(u_3) \leq 4$ .

Let  $G'$  be the connected component of  $G - u_2$  containing  $u_1$ . Let  $D$  be a proper 7-orientation of  $G'$ .

Assume  $d(u_2) = 4$ . By Proposition 14,  $u$  is the root of no moose nor elk, and by Proposition 12, it is the root of at most one kite. If  $\{d_D^-(u_1), d_D^-(u_3)\} \neq \{3, 4\}$ , then adding  $(u_1, u_2)$  and  $(u_3, u_2)$  and using appropriate orientations of the twig subgraphs and leaf blocks with root  $u_2$ , one can get an orientation of  $D$  such that  $d^-(u_2) \in \{3, 4\} \setminus \{d_D^-(u_1), d_D^-(u_3)\}$ . This is a proper 7-orientation of  $D$ , a contradiction. Consequently,  $\{d_D^-(u_1), d_D^-(u_3)\} = \{3, 4\}$ , and so

$d_D^-(u_3) = d_G(u_3) - 1 = 3$ . Let  $x$  be a neighbor of  $u_3$  not in  $B$  and let  $H$  be the twig subgraph or leaf block with root  $u_3$  containing  $x$ . By Proposition 12, one can choose  $x$  so that  $H$  is not in a kite. Add  $(u_2, u_3)$  and use  $A_1, T_2, P_1$ , or  $B_2$  to reverse  $(x, u_3)$ . If  $u_2$  is not the root of two 2-paths, we can orient the twig subgraphs and leaf blocks with root  $u_2$  so that its indegree becomes 2 by using orientations  $A, T_2, P_2, K$  or  $B_2$ . If  $u_2$  is the root of two 2-paths, we can orient these 2-paths using  $P_2$  so that  $u_2$  gets indegree 1. In both cases, we obtain a proper 7-orientation of  $D$ , a contradiction.

Similarly, we get a contradiction if  $d(u_3) = 4$ . Therefore  $d(u_2) \leq 3$  and  $d(u_3) \leq 3$ . Since  $B$  is a bough block,  $u_2$  or  $u_3$  must be the root of a twig subgraph. Without loss of generality, we may assume that  $u_2$  is. By Proposition 14,  $u_2$  must be the root of a 2-path, say  $(u_2, x, x')$ .

Assume  $d(u_3) = 2$ . If  $d^-(u_1) \notin \{1, 2\}$ , add  $(u_2, x), (u_2, u_3), (x', x)$ , and if  $(u_3, u_1) \in D$ , reverse it and add  $(u_2, u_1)$ ; otherwise, add  $(u_1, u_2)$ . And if  $d^-(u_1) \in \{1, 2\}$ , add  $(u_1, u_2), (u_3, u_2), (x, u_2)$  and  $(x, x')$ . In both cases, it results in a proper 7-orientation of  $D$ , a contradiction.

Hence  $d(u_3) = 3$ , which by Proposition 12 implies that  $u_3$  is the root of either a 2-path or a 1-path. Therefore  $H$  is a reindeer.  $\square$

We can finally prove the main result of this paper.

*Proof of Theorem 4.* If  $G$  has no branch blocks, then there exists a vertex  $u$  such that  $G$  is the union of bough subgraphs, twig subgraphs and leaf blocks with root  $u$ . In this case, one may obtain a proper 4-orientation of  $G$  by orienting all bough subgraphs, twig subgraphs and leaf blocks so that the indegree of  $u$  is 0.

Thus,  $G$  contains a branch block  $B$ . It must contain a vertex  $u$  which is the root of a bough subgraph  $R$ . By Proposition 16,  $R$  is a reindeer, and by Proposition 6, we have  $d(u) \geq 4$ . Denote by  $Q$  the subgraph rooted at  $u$  containing exactly all the bough, twig and leaf blocks rooted at  $u$ .

Let  $H$  be the component of  $G - u$  that contains  $B - u$ ; then  $u$  has at most 2 neighbors in  $H$ . By minimality of  $G$ ,  $H$  has a proper 7-orientation  $D$ . Let  $F$  be the set of indegrees of neighbors of  $u$  in  $H$ . Orient the edges of  $H$  incident to  $u$  towards  $u$ .

If  $d(u) \geq 7$ , we can orient  $G \setminus u$  in such a way that  $u$  has indegree in  $\{5, 6, 7\} \setminus F$  and no vertex in  $Q$  has indegree more than 4. This gives a proper 7-orientation of  $G$ , a contradiction.

Assume  $d(u) = 6$ . Let  $\alpha$  be an integer in  $\{4, 5, 6\} \setminus F$ . We can orient  $Q$  in such a way that  $u$  has indegree  $\alpha$  and no vertex of  $Q - u$  has indegree  $\alpha$ . This is possible because no vertex has indegree 5 in the orientations depicted in Figures 2, 4–8 and 9 and  $u$  is in at most two moose, so if  $\alpha = 4$ , we can orient the moose first using  $M_1$  or  $M_2$ . This gives a proper 7-orientation of  $G$ , a contradiction.

Assume  $d(u) = 4$ . If  $u$  has two neighbors in  $H$ , then  $Q = R$ . Let  $\alpha$  be an integer in  $\{2, 3, 4\} \setminus F$ . If  $\alpha = 2$ , then orient  $R$  with  $R_1$ ; if  $\alpha = 3$ , then orient  $R$  with  $R_2$ ; if  $\alpha = 4$ , then orient  $R$  with  $R_3$ . In each case, this yields a proper 7-orientation of  $G$ , a contradiction. If  $u$  has a unique neighbor in  $H$ , then  $Q$  is the union of  $R$  and either a 1-path, or a 2-path, or a kite. Orient that subgraph using  $A_2, P_2$  or  $K_1$ . Now, since  $|F| = 1$ , we can orient  $R$  using  $R_2$  or  $R_3$  so that the indegree of  $u$  in  $\{3, 4\} \setminus F$ . This yields a proper 7-orientation of  $G$ , a contradiction.

Finally assume  $d(u) = 5$ . If  $F \neq \{4, 5\}$ , we can orient the edges of  $Q$  so that the indegree of  $u$  is some  $\alpha \in \{4, 5\} \setminus F$ , and no vertex of  $Q - u$  has indegree  $\alpha$ . If  $\alpha = 4$ , this is possible because  $u$  is in at most one moose, and we can start orienting the moose with  $M_2$ . This yields a proper 7-orientation of  $G$ , a contradiction. If  $F = \{4, 5\}$ , then  $Q$  is the union of  $R$  and either a 1-path or a 2-path or a kite. In the first two cases, orient the 1-path or 2-path by using  $A_1$  or  $P_1$ , and  $R$  with  $R_2$ , so that vertex  $u$  has indegree 3. In the latter case, orient the kite with  $K_2$  and  $R$  with  $R_1$ , so that vertex  $u$  has indegree 2. In both cases, we obtain a proper 7-orientation of  $G$ , a contradiction.  $\square$

### 3 A tight example

In this section, we prove that the bound of Theorem 4 cannot be improved.

**Proposition 17.** *There exists a cactus  $G$  such that  $\vec{\chi}(G) = 7$ .*

*Proof.*

**Claim 17.1.** *Let  $G$  be a graph.*

(i) *If  $v$  is the root of three triangles, then, any proper orientation  $D$  of  $G$  satisfies  $d_D^-(v) \notin \{1, 2\}$ .*

(ii) *If  $v$  is the root of five moose, then, any proper orientation  $D$  of  $G$  satisfies  $d_D^-(v) \notin \{3, 4\}$ .*

*Subproof.* (i) By contradiction, suppose that there is a proper orientation  $D$  of  $G$  such that  $d_D^-(v) \in \{1, 2\}$ . Since  $v$  is the root of three triangles, one of these must be oriented as in  $T_1$ . Consequently,  $D$  is not proper.

(ii) By contradiction, suppose that there is a proper orientation  $D$  of  $G$  such that  $d_D^-(v) \in \{3, 4\}$ . Since  $v$  is the root of five moose, one of these must be oriented as in  $M_3$ . Consequently,  $D$  is not proper.  $\diamond$

Intuitively,  $v$  being the root of three triangles forbids the indegree of  $v$  to be 1 and 2 in any proper orientation of the input graph, and  $v$  being the root of five moose forbids the indegree of  $v$  to be 3 or 4.

Let  $H$  be the graph which is the union of sixteen complete graphs of order 3 :

- $K$  with vertex set  $\{v_1, v_2, v_3\}$ ;
- $K_i^j$  with vertex set  $(v_i, y_i^j, z_i^j)$  for  $i \in \{1, 2, 3\}$  and  $j \in \{1, \dots, 5\}$ .

Let  $G$  be the graph obtained from  $H$  by adding three triangles and five moose with root  $v$  at every vertex  $v \in V(H)$ .

Let us prove that  $\vec{\chi}(G) \geq 7$ . Suppose for a contradiction, that  $G$  has a proper 6-orientation  $D$ . By Claim 17.1, all vertices of  $H$  have indegree in  $\{0, 5, 6\}$ . Since  $K$  is complete and  $D$  is proper, a vertex of  $K$ , say  $v_1$ , has indegree 5. For the same reason each complete graph in  $K \cup \{K_1^j \mid 1 \leq j \leq 5\}$  has a vertex of indegree 0 which must dominate  $v_1$ . Hence  $v_1$  has indegree 6, a contradiction.  $\square$

## 4 Further Research

### 4.1 Proper-orientation number of outerplanar graphs

We believe that Problem 3 must be answered in the affirmative: outerplanar graphs have proper-orientation number bounded by a constant  $c$ . If such a  $c$  exists, then  $c \geq 7$ , since cacti (and in particular, the one described in Section 3) are outerplanar. A first step would be to established the result for 2-connected outerplanar graphs. We actually believe that in this case this constant should be smaller than 7 and that it should not be much greater than 3. One can easily attain 3 as a lower bound using the following lemma.

**Lemma 18 [2].** *Let  $k$  be a positive integer, and let  $G$  be a graph containing a clique  $K$  of size  $k + 1$ . In any proper  $k$ -orientation of  $G$ , all edges between  $V(K)$  and  $V(G) \setminus V(K)$  are oriented from  $V(K)$  to  $V(G) \setminus V(K)$ .*



**Proposition 19.** *There exists a 2-connected outerplanar graph  $G$  such that  $\vec{\chi}(G) = 3$ .*

*Proof.* Let  $G$  be the graph on six vertices defined by  $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  and  $E(G) = \{v_1v_2, v_2v_3, v_1v_3, v_4v_5, v_5v_6, v_4v_6, v_1v_4, v_2v_5\}$ . Suppose by way of contradiction that  $G$  has a proper 2-orientation  $D$ . Observe that the sets  $\{v_1, v_2, v_3\}$  and  $\{v_4, v_5, v_6\}$  are cliques in  $G$ . Thus Lemma 18 implies that the edges  $v_1v_4$  and  $v_2v_5$  must be oriented in both ways, a contradiction.  $\square$

## 4.2 $\vec{\chi}$ -bounded families of graphs

Gyárfás [7] introduced the concept of  $\chi$ -bounded graph classes. A class of graph  $\mathcal{G}$  is said to be  $\chi$ -bounded if there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $G \in \mathcal{G}$ . Similarly, one can define  $\vec{\chi}$ -bounded graph classes. A class of graph  $\mathcal{G}$  is said to be  $\vec{\chi}$ -bounded if there is a function  $f$  such that  $\vec{\chi}(G) \leq f(\omega(G))$  for every  $G \in \mathcal{G}$ . Because  $\chi \leq \vec{\chi}$ , a  $\vec{\chi}$ -bounded graph class is also  $\chi$ -bounded. Conversely, one might wonder which  $\chi$ -bounded graph classes are also  $\vec{\chi}$ -bounded.

The  $\chi$ -boundedness of graph classes defined by forbidden induced subgraphs have been particularly investigated. For a fixed graph  $F$ , let us denote by  $\text{Forb}(F)$  the class of graphs that do not contain  $F$  as an induced subgraph. Erdős [4] showed that there are graphs with arbitrarily high girth and chromatic number. This implies that if  $F$  contains a cycle, then  $\text{Forb}(F)$  is not  $\chi$ -bounded. Conversely, Gyárfás [6] and Sumner [9] independently made the following beautiful and difficult conjecture

**Conjecture 20** ([6] and [9]). *For every tree  $T$ , the class  $\text{Forb}(T)$  is  $\chi$ -bounded.*

It is natural to ask whether this conjecture generalizes to proper orientations.

**Problem 21.** *Is the class  $\text{Forb}(T)$   $\vec{\chi}$ -bounded for all tree  $T$  ?*

Gyárfás [7] establishes Conjecture 20 for stars by showing that a graph in  $\text{Forb}(K_{1,n})$  has maximum degree  $R(n, \omega(G))$ , where  $R(p, q)$  denotes the Ramsey number  $(p, q)$ . In particular, this shows that  $\text{Forb}(K_{1,n})$  is also  $\vec{\chi}$ -bounded.

In particular, if  $G$  is a planar claw-free graph (recall that the claw is the graph  $K_{1,3}$ ), Gyárfás result gives us that  $\vec{\chi}(G) \leq \Delta(G) \leq R(3, 4) = 9$ . This is also a partial answer to whether planar graphs have bounded proper orientation number. However, this bound is not tight, as we show next. In [8], Plummer showed that any claw-free 3-connected planar graph has maximum degree at most 6. His result can be extended to any claw-free planar graph.

**Theorem 22.** *If  $G$  is a claw-free planar graph, then  $\Delta(G) \leq 6$ .*

*Proof.* The proof is by induction on the number of vertices of  $G$ . If  $G$  is disconnected, then, by the induction hypothesis, each connected component of  $G$  has maximum degree at most 6 and so  $\Delta(G) \leq 6$ .

Assume that  $G$  has a cut-vertex  $u$ . As  $G$  is claw-free,  $G - u$  has exactly two components  $C_i$ ,  $i = 1, 2$ , and the neighborhood of  $u$  in each  $C_i$  is a clique  $N_i$ . Observe that  $N_i \cup \{u\}$  is a clique, which has size at most 4 because  $G$  is planar, so  $|N_i| \leq 3$ . Hence  $d(u) = |N_1| + |N_2| \leq 6$ . Now by the induction hypothesis applied to  $G[V(C_1) \cup \{u\}]$  and  $G[V(C_2) \cup \{u\}]$ , we obtain that every vertex distinct from  $u$  has degree at most 6. Therefore  $\Delta(G) \leq 6$ . Henceforth we may assume that  $G$  is 2-connected.

Assume that  $G$  has a 2-cut  $\{u, v\}$  (that is  $G - \{u, v\}$  is disconnected). The graph  $G' = G - v$  is connected with cut-vertex  $u$ . As above,  $G' - u$  has exactly two components,  $C_1$  and  $C_2$ , and  $N_i = N(u) \cap C_i$  is a clique, for  $i = 1, 2$  of size at most 3. We claim that  $d(u) \leq 6$ . If  $uv \notin E(G)$ , then  $d(u) = |N_1| + |N_2|$ , so  $d(u) \leq 6$ . If  $uv \in E(G)$ , then  $d(u) = |N_1| + |N_2| + 1$ .

But  $|N_1| + |N_2| \leq 5$  for otherwise there exist  $u_1 \in N_1$  and  $u_2 \in N_2$  non-adjacent to  $v$  (because  $G$  has no clique of size 5), so  $G[\{u, v, u_1, u_2\}]$  is a claw, a contradiction. Therefore  $d(u) \leq 6$ . Similarly, one proves  $d(v) \leq 6$ . Now by the induction hypothesis applied to  $G[V(C) \cup \{u, v\}]$  for each connected component of  $G - \{u, v\}$ , we obtain that every vertex distinct from  $u$  and  $v$  has degree at most 6; hence  $\Delta(G) \leq 6$ .

Henceforth, we may assume that  $G$  is 3-connected and the result follows by Plummer [8].  $\square$

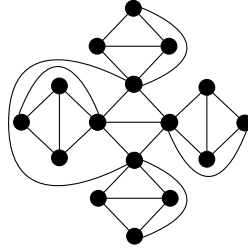


Figure 10: A planar claw-free graph  $G^*$  with maximum degree 6 and proper orientation number 6.

Theorem 22 is tight as shown by the graph  $G$  depicted in Figure 10 which is claw-free, planar and has maximum degree 6. Moreover, Theorem 22 implies that every planar claw-free graph has proper-orientation number at most 6. This is tight as shown by the following proposition.

**Proposition 23.** *The graph  $G^*$ , depicted in Figure 10, has proper orientation number equal to 6.*

*Proof.* The graph  $G^*$  is made of 5 blocks isomorphic to  $K_4$ . One of them (in the center of the figure), denoted by  $C$  intersects the four others. For every vertex  $v$  of  $C$ , let  $B(v)$  be the block intersecting  $C$  in  $v$ . Assume for a contradiction that  $G$  has a proper 5-orientation  $D$ . There are two vertices  $v_1$  and  $v_2$  in  $C$ , such that  $d_D^-(v_i) \in \{0, 1, 2, 3\}$ . Now the set of in-degrees of the other vertices of  $B(v_i)$  is exactly  $\{0, 1, 2, 3\} \setminus \{d_D^-(v_i)\}$ . Thus inside  $B(v)$  there are exactly  $6 - (0 + 1 + 2 + 3 - d_D^-(v_i)) = d_D^-(v_i)$  arcs towards  $v$ . Hence  $v_i$  dominates all other vertices of  $C$ . This is a contradiction, because the edge  $v_1v_2$  cannot be oriented both ways.  $\square$

## References

- [1] Arash Ahadi and Ali Dehghan. The complexity of the proper orientation number. *Information Processing Letters*, 113(19-21):799–803, 2013.
- [2] Julio Araujo, Nathann Cohen, Susanna F. de Rezende, Frédéric Havet, and Phablo F.S. Moura. On the proper orientation number of bipartite graphs. *Theoretical Computer Science*, 566(0):59 – 75, 2015.
- [3] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.
- [4] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.
- [5] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Co., New York, NY, USA, 1979.

- 
- [6] A. Gyárfás. On ramsey covering-numbers. In *Infinite and Finite Sets*, Coll. Math. Soc. János Bolyai, page 10. North Holland/American Elsevier, New York, 1975.
  - [7] A. Gyárfás. Problems from the world surrounding perfect graphs. *Zastosow Ania Matematyki Applicationes Mathematicae XIX*, 3-4:413–441, 1987.
  - [8] Michael D Plummer. Claw-free maximal planar graphs. Technical report, DTIC Document, 1989.
  - [9] D. P. Sumner. Subtrees of a graph and the chromatic number. In *The theory and applications of graphs (Kalamazoo, Mich., 1980)*, pages 557–576. Wiley, New York, 1981.



**RESEARCH CENTRE  
SOPHIA ANTIPOLIS – MÉDITERRANÉE**

2004 route des Lucioles - BP 93  
06902 Sophia Antipolis Cedex

Publisher  
Inria  
Domaine de Voluceau - Rocquencourt  
BP 105 - 78153 Le Chesnay Cedex  
[inria.fr](http://inria.fr)

ISSN 0249-6399