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# A strong Tauberian theorem for characteristic functions

R. Riedi<sup>†</sup> and P. Gonçalves<sup>‡</sup>

<sup>†</sup>Dept of Mathematics, EIA-Fribourg, Switzerland [rudolf.riedi@hefr.ch](mailto:rudolf.riedi@hefr.ch)

<sup>‡</sup>INRIA, ENS Lyon [paulo.goncalves@inria.fr](mailto:paulo.goncalves@inria.fr)

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## Abstract

Using wavelet analysis we show that if the characteristic function of a random variable  $X$  can be approximated at 0 by some polynomial of even degree  $2p$  then the moment of order  $2p$  of  $X$  exists. This strengthens a Tauberian-type result by Ramachandran and implies that the characteristic function is actually  $2p$  times differentiable at 0. This fact also provides the theoretical basis for a wavelet based non-parametric estimator of the tail index of a distribution.

## Introduction

When studying the existence of moments  $\beta_q = \mathbb{E}|X|^q$  of a random variable  $X$  it is essential to understand the behavior of its characteristic function  $\phi(u) = \mathbb{E}[\exp(iuX)]$  at 0, in particular its degree of differentiability, and more precisely its Hölder regularity.

A function  $g$  is called *Hölder-regular* of degree  $r \geq 0$  at  $u_0$ , denoted  $g \in \mathcal{H}^r(u_0)$ , if there exists a polynomial  $P$  of degree at most  $\lfloor r \rfloor := \max\{n \in \mathbb{N}_0 : n \leq r\}$  such that

$$|g(u) - P(u)| \leq C \cdot |u - u_0|^r \quad (1)$$

If  $g$  is actually  $\lfloor r \rfloor$  times differentiable, then  $P$  must be equal to the Taylor polynomial and we call  $g$  *Taylor regular* of degree  $r$ . Functions which obey (1) but are not  $\lfloor r \rfloor$  times differentiable are abundant. An example is  $g(u) = 1 + u + u^2 + u^{3.1} \sin(1/u)$  which is only once differentiable at  $u = 0$  but Hölder regular for any  $r \leq 3.1$ .

However, differentiability of a characteristic function at 0 is intimately related to the existence of moments: for any positive integer  $p$  we have  $\phi^{(2p)}(0) = (-1)^p \mathbb{E}[X^{2p}]$  whenever one of the two is defined. To make this point even stronger, Ramachandran shows: if  $\phi(u)$  is symmetric and Taylor regular of degree  $r = 2p + \delta$  ( $0 < \delta < 2$ ) at 0 then all moments  $\beta_q$  of degree  $q < r$  are finite (see [6, Thm. 3]).

Thereby, the assumption of  $\phi$  being symmetric is not required (see, e.g., Lukacs [5, Thm. 2.2.2.]). Indeed, the symmetric function  $\operatorname{Re} \phi(u)$  is the characteristic function of the random variable  $Y$  which is equal to  $X$  with probability 1/2, and equal to  $-X$  with probability 1/2. Since  $|Y| = |X|$  a.s. we have  $\mathbb{E}|Y|^q = \mathbb{E}|X|^q = \beta_q$  and it suffices to consider  $\operatorname{Re} \phi$ .

## The main result

We are able to strengthen Ramachandran's result considerably via the following:

**Theorem 1** *Let  $r > 0$  be positive and  $p$  integer such that  $2p \leq r < 2p + 2$ . If  $\operatorname{Re} \phi(u) \in \mathcal{H}^r(0)$  then  $\phi(u)$  is  $2p$ -times differentiable at 0.*

**Proof**

For  $p = 0$  there is nothing to show. For  $p \geq 1$  denote by  $\psi_p$  the function with real valued, non-negative Fourier transform

$$\Psi_p(\nu) = \nu^{2p} \exp\left(-\nu^2/4\right). \quad (2)$$

Up to a constant factor,  $\psi_p(u)$  is the  $2p$ -th derivative of  $\exp(-u^2)$ . Now consider the wavelet coefficients of  $\operatorname{Re} \phi$  of scale  $s$  at  $t$  which are given by

$$W_p(s, t) := \int \frac{1}{s} \psi_p\left(\frac{u-t}{s}\right) \operatorname{Re} \phi(u) du. \quad (3)$$

The Fourier transform of  $\frac{1}{s} \psi_p\left(\frac{u}{s}\right)$  being  $\Psi_p(s\nu)$ , Parseval's identity yields

$$W_p(s, 0) = \operatorname{Re} \int \frac{1}{s} \psi_p\left(\frac{u}{s}\right) \phi(u) du = \operatorname{Re} \int \Psi_p(s\nu) dF_X(\nu) = \mathbb{E}[\Psi_p(sX)]. \quad (4)$$

Monotone convergence gives

$$\sup_{s>0} \frac{W_p(s, 0)}{s^{2p}} = \sup_{s>0} \mathbb{E}[X^{2p} \exp(-s^2 X^2/4)] = \mathbb{E}[X^{2p}] \in \mathbb{R}_0^+ \cup \{\infty\}.$$

To control the decay of  $W_p(s, 0)$  at 0 note that  $\operatorname{Re} \phi$  is Hölder regular of degree  $2p$  which can be seen by absorbing higher order terms into  $C|u|^{2p}$  from (1) for  $|u| \leq 1$  and by using  $|\phi(u)| \leq 1$  for  $|u| > 1$ . Also,  $\operatorname{Re} \phi$  is even. Thus, there is an even polynomial  $Q$  of degree at most  $2p-2$  and a constant  $C_1$  such that  $|\operatorname{Re} \phi(u) - Q(u)| \leq C_1|u|^{2p}$ . By repeated integration by parts, we find that  $\int Q(u) \psi_p(u/s) du = 0$  for any  $s > 0$ . From this, we find

$$\begin{aligned} |W_p(s, 0)| &= \left| \int \frac{1}{s} \psi_p\left(\frac{u}{s}\right) (\operatorname{Re} \phi(u) - Q(u)) du \right| \leq C_1 \int \frac{1}{s} |\psi_p\left(\frac{u}{s}\right)| |u|^{2p} du \\ &= C_1 |s|^{2p} \int |\psi_p(v)| |v|^{2p} dv = C_2 |s|^{2p}. \end{aligned} \quad (5)$$

Note that  $C_2 < \infty$  due to the exponential tail of  $\psi_p$ . The claim follows.  $\diamond$

Theorem 1 strengthens Ramachandran [6, Thm. 3] as follows.

**Theorem 2** *Let  $r > 0$ . If  $\operatorname{Re} \phi \in \mathcal{H}^r(0)$  then  $\beta_q < \infty$  for  $0 \leq q < r$ .*

Remark 1: The stable laws with  $\phi(u) = \exp(-|u|^r)$  ( $0 < r < 2$ ) demonstrate that  $\beta_r$  may not be finite in general. However, if  $r$  is an even integer, we always have  $\beta_r < \infty$ .

**Proof**

In the light of Theorem 1 we may simply quote Ramachandran [6, Thm. 3] to complete the proof. However, just a little more work provides an alternative proof

to this classical result. In the notation of (1) and (2) we have  $\int P(u)\psi_{p+1}(u/s)du = 0$  ( $p \geq 0$ ). Similar as in (5) we get

$$\mathbb{E}[\Psi_{p+1}(sX)] \leq C \int \frac{1}{s} |\psi_{p+1}(\frac{u}{s})| |u|^r du = C_3 |s|^r. \quad (6)$$

Using  $\Psi_{p+1}(x) \geq e^{-1/4} > 1/2$  for  $1 \leq |x| \leq 2$  ( $p \geq 0$ ) we find

$$\int_{2^k \leq |x| \leq 2^{k+1}} dF_X(x) = \mathbb{E}[\mathbb{1}_{[-2,-1] \cup [1,2]}(2^{-k}X)] \leq 2\mathbb{E}[\Psi_{p+1}(2^{-k}X)] \leq 2C_3 \cdot 2^{-kr}.$$

It follows immediately that the moments  $\beta_q$  are finite for  $0 \leq q < r$ .  $\diamond$

#### Classical results

Theorem 1 is quite well known in the case  $r = 2$ :

**Theorem 3** *If  $|\phi(u) - 1| \leq C \cdot u^2$ , then the variance of the law is finite.*

#### Proof

Note that  $|\operatorname{Re} \phi(u) - 1| \leq C \cdot u^2$ . But  $1 - \operatorname{Re} \phi(u) = \mathbb{E}[2 \sin^2(uX/2)]$ . Since  $X^2 = \lim_{u \rightarrow 0} \sin^2(uX)/u^2$  a.s. it suffices then to apply Fatou's lemma.  $\diamond$

#### Alternative choice of wavelet:

Our choice of wavelet in the proofs of Theorem 1 and Theorem 2 (see (2)) is by far not the only possible. Here is one interesting alternative. Motivated by the classical argument of Theorem 3 we may replace  $\Psi_p$  by  $\sin^{2p}$  which leads to a somewhat more elementary computation akin to the proofs of Ramachandran and Lukacs.

Indeed, using  $(\sin sx)^{2p} = (2i)^{-2p} (\exp(isx) - \exp(-isx))^{2p}$  we obtain (cpre. (4))

$$\mathbb{E}[\sin^{2p}(sX)] = (-1/4)^p \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \phi(s(2p - 2k)) =: \Delta_p^s \phi(0).$$

Here, we introduced an order  $2p$  difference operator  $\Delta_p^s$  centered at 0 which is in essence the  $2p$ -fold convolution of the Haar wavelet. Being non-negative real  $\Delta_p^s \phi(0)$  equals  $\Delta_p^s \operatorname{Re} \phi(0)$ . Also,  $\Delta_p^s$  possesses the appropriate number of vanishing moments, meaning that  $\Delta_p^s Q = 0$  for  $Q$  from (5), since  $Q$  is of degree at most  $2p - 2$  and  $\sum_{k=0}^{2p} \binom{2p}{k} (-1)^k k^m = 0$  for any  $0 \leq m \leq 2p - 2$ . Setting  $C_1$  as in (5) we find

$$\mathbb{E}[\sin^{2p}(sX)] = |\Delta_p^s(\operatorname{Re} \phi - Q)| \leq 4^{-p} \sum_{k=0}^{2p} \binom{2p}{k} C_1 |2p - 2k|^{2p} |s|^{2p}.$$

Using Fatou as in Theorem 3 establishes finiteness of  $\beta_{2p}$  and, therefore, Theorem 1. To establish Theorem 2 in this setup, write  $\mathbb{E}[\sin^{2p+2}(sX)] = |\Delta_{p+1}^s(\operatorname{Re} \phi - P)| \leq C_4 |s|^r$  and use  $\sin \theta \geq \sqrt{3}/2$  for  $\pi/3 \leq \theta \leq 2\pi/3$  to establish finiteness of  $\beta_q$  for  $0 \leq q < r$  as in Theorem 2 above.  $\diamond$

Remark 2: Lukacs [5] uses essentially the same operator  $\Delta_p^s \phi(t)$ , centered at an arbitrary  $t$ ; he also uses the vanishing moments. But writing  $\Delta_p^s \phi(t) = \Delta_p^s P(t) + \Delta_p^s(\phi - P)(t)$  his estimates contain a term  $s^{2p}$  in addition to  $s^r$ . Lukacs assumes

differentiability of  $\phi$  in order to deal with the extra term  $s^{2p}$ . This crucial fact is obscured by a minor typing error in the statement of the theorem which makes it appear as if Lukacs assumed only Hölder regularity.

*Further regularity results*

The alternative wavelets mentioned above provide regularity conditions that do not require an approximating polynomial of the characteristic function:

**Corollary 4** *Let  $r$  be positive real and  $m$  and  $n$  positive integers. Then*

$$\begin{aligned} |\Delta_m^s \phi(0)| \leq C|s|^{2m} (s \rightarrow 0^+) &\Rightarrow \phi^{(2m)}(u) \text{ exists.} \\ |\Delta_n^s \phi(0)| \leq C|s|^r (s \rightarrow 0^+) &\Rightarrow \beta_q < \infty (0 \leq q < r). \end{aligned}$$

In the special case  $n = 1$  we are lead to the following notion: A function  $g$  is said to lie in the *Zygmund class*  $\dot{C}^1(u_0)$  if  $g$  satisfies

$$|g(u_0 + s) - 2g(u_0) + g(u_0 - s)| \leq C|s|^r \quad (7)$$

with  $r = 1$ . The condition (7) is equivalent to  $g \in \mathcal{H}^r(u_0)$  for  $0 < r < 1$ , but the inclusion  $\mathcal{H}^1(u_0) \subset \dot{C}^1(u_0)$  is strict. However, since  $1 - \text{Re } \phi(2s) = 2\Delta_1^s \phi(0)$  the real part of any characteristic function in Zygmund class  $\dot{C}^1(0)$  lies actually in  $\mathcal{H}^1(0)$ .

We study the derivatives of characteristic functions in terms of the following spaces: we set  $\dot{C}^r(u_0) = \mathcal{H}^r(u_0)$  for  $0 < r < 1$ ; for  $r > 1$  we denote by  $\dot{C}^r(u_0)$  the space of functions who's derivatives exist around  $u_0$  up to order  $m$  ( $m < r \leq m + 1$ ) and who's  $m$ -th derivative lies in  $\dot{C}^{r-m}(u_0)$ .

For general functions, control on the regularity of derivatives is not obvious (see the example of a function  $g \in \mathcal{H}^{3.1}(0)$  from the introduction). In order to gain such control we make use of the so-called *two-microlocalization*: with reference to (3) the function  $\text{Re } \phi(u)$  is said to belong to  $\mathcal{C}^{r,r'}(u_0)$  if

$$|W_{p+2}(s, t)| \leq C s^r \left(1 + \frac{|t - u_0|}{s}\right)^{-r'}. \quad (8)$$

Here, the wavelet  $\psi_{p+2}$  needs to be sufficiently regular, i.e.,  $2p + 4 > r + 2$ .

We will need the following results

$$\begin{aligned} \mathcal{H}^r(u_0) &\subset \mathcal{C}^{r,-r}(u_0) && (r > 0) \\ \mathcal{C}^{r,r'}(u_0) &\subset \mathcal{H}^r(u_0) && (r' > -r, r > 0 \text{ non-integer}) \\ \mathcal{C}^{1,0}(u_0) &\subset \dot{C}^1(u_0) \end{aligned} \quad (9)$$

The first two inclusions are found in [3, Prop. 1] as well as [4, Prop. 1.3]. For the third inclusion one proves the somewhat more general fact that  $g \in \mathcal{C}^{r,0}(u_0)$  ( $1 \leq r < 2$ ) implies (7) by applying the mean value theorem twice in [3, p. 292].

At this point, the wavelet-maxima-property of characteristic functions proves useful:

**Lemma 5** *The wavelet coefficients (3) of  $\phi$  and  $\text{Re } \phi$  are both maximal at 0:*

$$|W_p^{\text{Re } \phi}(s, t)| \leq |W_p^\phi(s, t)| \leq W_p(s, 0) = \mathbf{E}[\Psi_p(sX)].$$

**Proof**

This follows easily from Parseval's identity, i.e.,

$$W_p^\phi(s, t) = \int \frac{1}{s} \psi_p\left(\frac{u-t}{s}\right) \phi(u) du = \int \Psi_p(s\nu) \exp(-it\nu) dF_X(\nu), \quad (10)$$

combined with  $\Psi_p \geq 0$ , and (4).  $\diamond$

Note that  $\operatorname{Re} \phi \in \mathcal{H}^r(0)$  implies  $|W_{p+2}(s, 0)| \leq C|s|^r$  due to (9) and (8) with  $t = u_0$ . On the other hand, this bound implies with Lemma 5 that  $\phi \in \mathcal{C}^{r,0}(u_0)$  for any  $u_0$ . Notably, the same constant  $C$  can be used in the wavelet bound (8) for all  $u_0$ . This fact is indicated by writing  $\phi \in \mathcal{C}^{r,0}(\mathbb{R})$  and analogously for other regularity spaces. Now, (9) implies

**Corollary 6** *For non-integer  $r > 0$  we have*

$$\operatorname{Re} \phi \in \mathcal{H}^r(0) \Leftrightarrow |W_{p+2}(s, 0)| \leq C s^r \ (s > 0) \Leftrightarrow \phi \in \mathcal{H}^r(\mathbb{R}) \Leftrightarrow \operatorname{Re} \phi \in \mathcal{H}^r(\mathbb{R}).$$

Due to Theorem 1 and classical results,  $\operatorname{Re} \phi \in \mathcal{H}^r(0)$  implies that the derivatives of  $\phi$  exist and are uniformly continuous up to order  $m$  ( $m < r \leq m + 1$ ). It is known that [2, 4]

$$\operatorname{Re} \phi(u) \in \mathcal{C}^{r,r'}(u_0) \Rightarrow \frac{\partial^m}{\partial u^m} \operatorname{Re} \phi(u) \in \mathcal{C}^{r-m,r'}(u_0)$$

This precise control of the regularity of the derivatives via two-microlocalization can be easily verified and sharpened for even order derivatives of characteristic functions:

**Lemma 7** *Assume  $\beta_{2m} < \infty$  and  $p > m$ . Then (analogous for  $\operatorname{Re} \phi$ )*

$$W_{p-m}^{\phi^{(2m)}}(s, t) = (-1)^m s^{-2m} \cdot W_p^\phi(s, t).$$

**Proof**

Consider the probability distribution  $dF_m(x) = \frac{1}{\beta_{2m}} x^{2m} dF_X(x)$  with characteristic function

$$\phi_m(u) = \frac{1}{\beta_{2m}} \int e^{iux} x^{2m} dF_X(x) = \frac{1}{\beta_{2m}} \mathbb{E}[e^{iuX} X^{2m}] = \frac{(-1)^m}{\beta_{2m}} \phi^{(2m)}(u). \quad (11)$$

A direct computation using Parseval's identity (10) leads to the claimed relation.  $\diamond$

In particular,  $\operatorname{Re} \phi^{(2m)}$  and  $\phi^{(2m)}$  are, up to a factor, characteristic functions.

**Corollary 8** *For all  $r > 0$  we have:  $\operatorname{Re} \phi \in \mathcal{H}^r(0) \Rightarrow \phi \in \dot{\mathcal{C}}^r(\mathbb{R})$ .*

*For  $r = 2p$  even,  $\phi^{(2p)}$  exists and is uniformly continuous.*

*For  $r = 2p + 1$  odd,  $\phi^{(2p)} \in \dot{\mathcal{C}}^1(\mathbb{R})$  and  $\operatorname{Re} \phi^{(2p)} \in \mathcal{H}^1(\mathbb{R})$ .*

Remark 3: The Cauchy distribution with  $\phi(u) = \exp(-|u|)$  lies in  $\mathcal{H}^1(0)$  but is not differentiable. A condition for  $\phi^{(2p)}$  being differentiable at 0 and, thus, everywhere is provided in [2, Prop. 1] and [4, Prop 1.5].

An alternative approach to using wavelets is a Littlewood-Paley decomposition of  $\text{Re } \phi$  into functions  $\Delta_j \text{Re } \phi$ . To allow recycling of notation, consider  $\Psi_0(\nu) = h(\nu/2) - h(\nu)$  where  $h$  is a function of the Schwartz class such that  $0 \leq h \leq 1$ ,  $h(\nu) = 1$  for  $|\nu| \leq 1/2$  and  $h(\nu) = 0$  for  $|\nu| \geq 1$ . Then, by definition, the Fourier transform of  $\Delta_j \text{Re } \phi$  equals  $\Psi_0(2^{-j}\nu)$  times the Fourier transform of  $\text{Re } \phi$ . In the notation of (3) we get  $\Delta_j \text{Re } \phi(u) = W_0(2^{-j}, u)$ . Since  $\Psi_0$  is positive, Lemma 5 holds also in this context; with (4), (6) and  $\Psi_0 \leq K\Psi_p$  for some  $K > 0$  we see that  $\text{Re } \phi \in \mathcal{H}^r(0)$  implies

$$|\Delta_j \text{Re } \phi(u)| \leq \Delta_j \text{Re } \phi(0) = \mathbb{E}[\Psi_0(2^{-j}X)] \leq KC_3 2^{-jr}$$

These estimates can be used in a manner similar to the arguments provided earlier.

### Statistical estimation

As an application we mention a wavelet-based non-parametric estimator of the *tail-exponent*

$$\lambda := \sup\{q > 0 : \text{the moment } \beta_q \text{ is finite}\}$$

of the distribution of  $X$ . As is well known  $\beta_q < \infty$  ( $q > 0$ ) implies  $\text{Re } \phi \in \mathcal{H}^q(0)$  (see [5, Thm. 2.2.1.]). With Theorem 2 it follows that  $\lambda = \sup\{r > 0 : \text{Re } \phi \in \mathcal{H}^r(0)\}$ . With Corollary 6 we conclude

**Corollary 9** *Provided  $2n > \lambda + 2$ , the tail exponent  $\lambda$  satisfies*

$$\lambda = \sup\{r > 0 : \mathbb{E}[\Psi_n(sX)] \cdot s^{-r} \text{ is bounded for } s \rightarrow 0^+\}$$

In conclusion, the estimation of  $\lambda$  can be based on estimating  $\mathbb{E}[\Psi_n(sX)]$  for various values of  $s > 0$  followed by a least square linear regression in log-log towards fitting  $\mathbb{E}[\Psi_n(sX)] \simeq s^\lambda$ . Various values of  $n$  should be employed to ensure  $2n > \lambda$ . The exact setup and performance of this estimation procedure is detailed in [1].

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