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A strong Tauberian theorem for characteristic functions

R. Riedi† and P. Gonçalves‡
†Dept of Mathematics, EIA-Fribourg, Switzerland rudolf.riedi@hefr.ch
‡INRIA, ENS Lyon paulo.goncalves@inria.fr

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Abstract

Using wavelet analysis we show that if the characteristic function of a random variable \( X \) can be approximated at 0 by some polynomial of even degree \( 2p \) then the moment of order \( 2p \) of \( X \) exists. This strengthens a Tauberian-type result by Ramachandran and implies that the characteristic function is actually \( 2p \) times differentiable at 0. This fact also provides the theoretical basis for a wavelet based non-parametric estimator of the tail index of a distribution.

Introduction

When studying the existence of moments \( \beta_q = \mathbb{E}|X|^q \) of a random variable \( X \) it is essential to understand the behavior of its characteristic function \( \phi(u) = \mathbb{E}[\exp(iuX)] \) at 0, in particular its degree of differentiability, and more precisely its Hölder regularity.

A function \( g \) is called Hölder-regular of degree \( r \geq 0 \) at \( u_0 \), denoted \( g \in \mathcal{H}^r(u_0) \), if there exists a polynomial \( P \) of degree at most \( \lfloor r \rfloor := \max\{n \in \mathbb{N}_0 : n \leq r\} \) such that

\[
|g(u) - P(u)| \leq C \cdot |u - u_0|^r
\]

If \( g \) is actually \( |r| \) times differentiable, then \( P \) must be equal to the Taylor polynomial and we call \( g \) Taylor regular of degree \( r \). Functions which obey (1) but are not \( |r| \) times differentiable are abundant. An example is \( g(u) = 1 + u + u^2 + u^{3.1}\sin(1/u) \) which is only once differentiable at \( u = 0 \) but Hölder regular for any \( r \leq 3.1 \).

However, differentiability of a characteristic function at 0 is intimately related to the existence of moments: for any positive integer \( p \) we have \( \phi^{(2p)}(0) = (-1)^p \mathbb{E}|X|^{2p} \) whenever one of the two is defined. To make this point even stronger, Ramachandran shows: if \( \phi(u) \) is symmetric and Taylor regular of degree \( r = 2p + \delta \) (\( 0 < \delta < 2 \)) at 0 then all moments \( \beta_q \) of degree \( q < r \) are finite (see [6, Thm. 3]).

Thereby, the assumption of \( \phi \) being symmetric is not required (see, e.g., Lukacs [5, Thm. 2.2.2.]). Indeed, the symmetric function \( \text{Re} \phi(u) \) is the characteristic function of the random variable \( Y \) which is equal to \( X \) with probability \( 1/2 \), and equal to \(-X\) with probability \( 1/2 \). Since \( |Y| = |X| \) a.s. we have \( \mathbb{E}|Y|^q = \mathbb{E}|X|^q = \beta_q \) and it suffices to consider \( \text{Re} \phi \).

The main result

We are able to strengthen Ramachandran’s result considerably via the following:
Theorem 1 Let $r > 0$ be positive and $p$ integer such that $2p \leq r < 2p + 2$. If $\Re \phi(u) \in \mathcal{H}^r(0)$ then $\phi(u)$ is $2p$-times differentiable at 0.

Proof

For $p = 0$ there is nothing to show. For $p \geq 1$ denote by $\psi_p$ the function with real valued, non-negative Fourier transform

$$
\Psi_p(\nu) = \nu^{2p} \exp\left( -\frac{\nu^2}{4} \right).
$$

(2)

Up to a constant factor, $\psi_p(u)$ is the $2p$-th derivative of $\exp(-u^2)$. Now consider the wavelet coefficients of $\Re \phi$ of scale $s$ at $t$ which are given by

$$
W_p(s, t) := \int \frac{1}{s} \psi_p\left( \frac{u - t}{s} \right) \Re \phi(u) du.
$$

(3)

The Fourier transform of $\frac{1}{s} \psi_p\left( \frac{u}{s} \right)$ being $\psi_p(s \nu)$, Parseval’s identity yields

$$
W_p(s, 0) = \Re \int \frac{1}{s} \psi_p\left( \frac{u}{s} \right) \phi(u) du = \Re \int \Psi_p(sv)dF_X(\nu) = \mathbf{E}[\Psi_p(sX)].
$$

(4)

Monotone convergence gives

$$
\sup_{s > 0} \frac{W_p(s, 0)}{s^{2p}} = \sup_{s > 0} \mathbf{E}[X^{2p} \exp(-s^2 X^2/4)] = \mathbf{E}[X^{2p}] \in \mathbb{R}_0^+ \cup \{\infty\}.
$$

To control the decay of $W_p(s, 0)$ at 0 note that $\Re \phi$ is Hölder regular of degree $2p$ which can be seen by absorbing higher order terms into $C|u|^{2p}$ from (1) for $|u| \leq 1$ and by using $|\phi(u)| \leq 1$ for $|u| > 1$. Also, $\Re \phi$ is even. Thus, there is an even polynomial $Q$ of degree at most $2p - 2$ and a constant $C_1$ such that $|\Re \phi(u) - Q(u)| \leq C_1 |u|^{2p}$. By repeated integration by parts, we find that $\int Q(u) \psi_p(u/s) du = 0$ for any $s > 0$. From this, we find

$$
|W_p(s, 0)| = \left| \int \frac{1}{s} \psi_p\left( \frac{u}{s} \right) (\Re \phi(u) - Q(u)) du \right| \leq C_1 \int \frac{1}{s} |\psi_p\left( \frac{u}{s} \right)| |u|^{2p} du
$$

$$
= C_1 |s|^{2p} \int |\psi_p(v)| |v|^{2p} dv = C_2 |s|^{2p}.
$$

(5)

Note that $C_2 < \infty$ due to the exponential tail of $\psi_p$. The claim follows.

Theorem 1 strengthens Ramachandran [6, Thm. 3] as follows.

Theorem 2 Let $r > 0$. If $\Re \phi \in \mathcal{H}^r(0)$ then $\beta_q < \infty$ for $0 \leq q < r$.

Remark 1: The stable laws with $\phi(u) = \exp(-|u|^r)$ ($0 < r < 2$) demonstrate that $\beta_r$ may not be finite in general. However, if $r$ is an even integer, we always have $\beta_r < \infty$.

Proof

In the light of Theorem 1 we may simply quote Ramachandran [6, Thm. 3] to complete the proof. However, just a little more work provides an alternative proof.
Remark 2: Lukacs [5] uses essentially the same operator \( \Psi_{p+1}(x) \) to establish Theorem 2 in this setup, write \( I \in [\sin \theta] \). Using Fatou as in Theorem 3 establishes finiteness of the moments \( \beta_q \). Here, we introduced an order 2\( p \) difference operator \( \Delta_p^* \), centered at 0 which is in essence the 2\( p \)-fold convolution of the Haar wavelet. Being non-negative real \( \Delta_p^* \) equals \( \Delta_p^* \) centered at 0 which is in essence the 2\( p \)-fold convolution of the Haar wavelet. Being non-negative real \( \Delta_p^* \) equals \( \Delta_p^* \). Also, \( \Delta_p^* \) possesses the appropriate number of vanishing moments, meaning that \( \Delta_p^* Q = 0 \) for \( Q \) from (5), since \( Q \) is of degree at most 2\( p \) and \( \sum_{k=0}^{2\( p \)} (2k)^{-1} k^m = 0 \) for any \( 0 \leq m \leq 2\( p \) - 2. Setting \( C_1 \) as in (5) we find

\[
\mathbb{E}[\sin^{2\( p \)}(sX)] = |\Delta_p^*(\Re \phi - Q)| \leq 4^{-\( p \)} \sum_{k=0}^{2\( p \)} \binom{2\( p \)}{k} C_1 |2\( p \) - 2k|^{2\( p \)} |s|^{2\( p \)}.
\]

Using Fatou as in Theorem 3 establishes finiteness of \( \beta_{2\( p \)} \) and, therefore, Theorem 1. To establish Theorem 2 in this setup, write \( \mathbb{E}[\sin^{2\( p \)+2}(sX)] = |\Delta_p^*(\Re \phi - P)| \leq C_4|s|^r \) and use \( \sin \theta \geq \sqrt{3}/2 \) for \( \pi/3 \leq \theta \leq 2\pi/3 \) to establish finiteness of \( \beta_q \) for \( 0 \leq q < r \) as in Theorem 2 above.

Remark 2: Lukacs [5] uses essentially the same operator \( \Delta_p^* \phi(t) \), centered at an arbitrary \( t \); he also uses the vanishing moments. But writing \( \Delta_p^* \phi(t) = \Delta_p^* P(t) + \Delta_p^*(\phi - P)(t) \) his estimates contain a term \( s^{2\( p \)} \) in addition to \( s^r \). Lukacs assumes
differentiability of \( \phi \) in order to deal with the extra term \( s^{2p} \). This crucial fact is
obscured by a minor typing error in the statement of the theorem which makes it
appear as if Lukacs assumed only Hölder regularity.

**Further regularity results**

The alternative wavelets mentioned above provide regularity conditions that do not
require an approximating polynomial of the characteristic function:

**Corollary 4** Let \( r \) be positive real and \( m \) and \( n \) positive integers. Then

\[
|\Delta_s^m \phi(0)| \leq C|s|^{2m}(s \to 0^+) \Rightarrow \phi^{(2m)}(u) \text{ exists.}
\]

\[
|\Delta_n^s \phi(0)| \leq C|s|^r \quad (s \to 0^+) \Rightarrow \beta_q < \infty \quad (0 \leq q < r).
\]

In the special case \( n = 1 \) we are lead to the following notion: A function \( g \) is said to
lie in the Zygmund class \( C^1(u_0) \) if \( g \) satisfies

\[
|g(u_0 + s) - 2g(u_0) + g(u_0 - s)| \leq C|s|^r
\]

with \( r = 1 \). The condition (7) is equivalent to \( g \in H^r(u_0) \) for \( 0 < r < 1 \), but the
inclusion \( H^1(u_0) \subset C^1(u_0) \) is strict. However, since \( 1 - \Re \phi(2s) = 2\Delta_s \phi(0) \) the real
part of any characteristic function in Zygmund class \( C^1(0) \) lies actually in \( H^1(0) \).

We study the derivatives of characteristic functions in terms of the following spaces:
we set \( \mathcal{C}^r(u_0) = H^r(u_0) \) for \( 0 < r < 1 \); for \( r > 1 \) we denote by \( \mathcal{C}^r(u_0) \) the space of
functions who’s derivatives exist around \( u_0 \) up to order \( m \) \( (m < r < m + 1) \) and
who’s \( m \)-th derivative lies in \( \mathcal{C}^{r-m}(u_0) \).

For general functions, control on the regularity of derivatives is not obvious (see the
example of a function \( g \in H^{3,1}(0) \) from the introduction). In order to gain such
control we make use of the so-called **two-microlocalization**: with reference to (3) the
function \( \Re \phi(u) \) is said to belong to \( \mathcal{C}^{r,r'}(u_0) \) if

\[
|W_{p+2}(s,t)| \leq Cs^r \left(1 + \frac{|t-u_0|}{s}\right)^{-r'}.
\]

Here, the wavelet \( \psi_{p+2} \) needs to be sufficiently regular, i.e., \( 2p + 4 > r + 2 \).

We will need the following results

\[
\begin{align*}
H^r(u_0) & \subset C^{r-r}(u_0) \quad (r > 0) \\
C^{r,r'}(u_0) & \subset H^r(u_0) \quad (r' > -r, \ r > 0 \ \text{non-integer}) \\
C^{1,0}(u_0) & \subset C^1(u_0)
\end{align*}
\]

The first two inclusions are found in [3, Prop. 1] as well as [4, Prop. 1.3]. For the
third inclusion one proves the somewhat more general fact that \( g \in C^{r,0}(u_0) \)
\( (1 \leq r < 2) \) implies (7) by applying the mean value theorem twice in [3, p. 292].

At this point, the wavelet-maxima-property of characteristic functions proves useful:

**Lemma 5** The wavelet coefficients (3) of \( \phi \) and \( \Re \phi \) are both maximal at 0:

\[
|W_p^{Re \phi}(s,t)| \leq |W_p^{\phi}(s,t)| \leq W_p(s,0) = \mathbb{E}[\Psi_p(sX)].
\]
Proof
This follows easily from Parseval’s identity, i.e.,
\[ W_p^*(s, t) = \int \frac{1}{s} \psi_p \left( \frac{s-t}{s} \right) \phi(u) du = \int \Psi_p(s \nu) \exp(-it \nu) dF_X(\nu), \]  
(10)
combined with \( \Psi_p \geq 0 \), and (4).
\[ \square \]

Note that \( \text{Re} \phi \in \mathcal{H}^r(0) \) implies \(|W_{p+2}(s, 0)| \leq C|s|^r \) due to (9) and (8) with \( t = u_0 \). On the other hand, this bound implies with Lemma 5 that \( \phi \in C^{r,0}(u_0) \) for any \( u_0 \). Notably, the same constant \( C \) can be used in the wavelet bound (8) for all \( u_0 \).

Now, (9) implies
Corollary 6 For non-integer \( r > 0 \) we have
\[ \text{Re} \phi \in \mathcal{H}^r(0) \iff |W_{p+2}(s, 0)| \leq C|s|^r (s > 0) \iff \phi \in \mathcal{H}^r(\mathbb{R}) \iff \text{Re} \phi \in \mathcal{H}^r(\mathbb{R}). \]
Due to Theorem 1 and classical results, \( \text{Re} \phi \in \mathcal{H}^r(0) \) implies that the derivatives of \( \phi \) exist and are uniformly continuous up to order \( m \) \((m < r \leq m + 1)\). It is known that \[ 2 \]
\[ \text{Re} \phi(u) \in C^{r,r'}(u_0) \Rightarrow \frac{\partial^m}{\partial u^m} \text{Re} \phi(u) \in C^{r-m,r'}(u_0) \]
This precise control of the regularity of the derivatives via two-microlocalization can be easily verified and sharpened for even order derivatives of characteristic functions:

Lemma 7 Assume \( \beta_{2m} < \infty \) and \( p > m \). Then (analogous for \( \text{Re} \phi \))
\[ W^\phi_{p-m}(s, t) = (-1)^m s^{-2m} \cdot W^\phi_p(s, t). \]

Proof
Consider the probability distribution \( dF_m(x) = \frac{1}{\beta_{2m}} x^{2m} dF_X(x) \) with characteristic function
\[ \phi_m(u) = \frac{1}{\beta_{2m}} \int e^{iux} x^{2m} dF_X(x) = \frac{1}{\beta_{2m}} \mathbb{E}[e^{iux} X^{2m}] = \frac{(-1)^m}{\beta_{2m}} \phi^{(2m)}(u). \]
(11)
A direct computation using Parseval’s identity (10) leads to the claimed relation.
\[ \square \]

In particular, \( \text{Re} \phi^{(2m)} \) and \( \phi^{(2m)} \) are, up to a factor, characteristic functions.

Corollary 8 For all \( r > 0 \) we have: \( \text{Re} \phi \in \mathcal{H}^r(0) \Rightarrow \phi \in \dot{C}^r(\mathbb{R}). \)
For \( r = 2p \) even, \( \phi^{(2p)} \) exists and is uniformly continuous.
For \( r = 2p + 1 \) odd, \( \phi^{(2p)} \in \dot{C}^1(\mathbb{R}) \) and \( \text{Re} \phi^{(2p)} \in \dot{H}^1(\mathbb{R}). \)

Remark 3: The Cauchy distribution with \( \phi(u) = \exp(-|u|) \) lies in \( \mathcal{H}^1(0) \) but is not differentiable. A condition for \( \phi^{(2p)} \) being differentiable at 0 and, thus, everywhere is provided in [2, Prop. 1] and [4, Prop 1.5].
An alternative approach to using wavelets is a Littlewood-Paley decomposition of $\Re \phi$ into functions $\Delta_j \Re \phi$. To allow recycling of notation, consider $\Psi_0(\nu) = h(\nu/2) - h(\nu)$ where $h$ is a function of the Schwartz class such that $0 \leq h \leq 1$, $h(\nu) = 1$ for $|\nu| \leq 1/2$ and $h(\nu) = 0$ for $|\nu| \geq 1$. Then, by definition, the Fourier transform of $\Delta_j \Re \phi$ equals $\Psi_0(2^{-j}\nu)$ times the Fourier transform of $\Re \phi$.

In the notation of (3) we get $\Delta_j \Re \phi(u) = W_0(2^{-j}, u)$. Since $\Psi_0$ is positive, Lemma 5 holds also in this context; with (4), (6) and $\Psi_0 \leq K \Psi_p$ for some $K > 0$ we see that $\Re \phi \in \mathcal{H}^r(0)$ implies

$$|\Delta_j \Re \phi(u)| \leq \Delta_j \Re \phi(0) = \mathbf{E}[\Psi_0(2^{-j}X)] \leq KC_3 2^{-jr}$$

These estimates can be used in a manner similar to the arguments provided earlier.

**Statistical estimation**

As an application we mention a wavelet-based non-parametric estimator of the tail-exponent

$$\lambda := \sup\{q > 0 : \text{the moment } \beta_q \text{ is finite}\}$$

of the distribution of $X$. As is well known $\beta_q < \infty$ ($q > 0$) implies $\Re \phi \in \mathcal{H}^r(0)$ (see [5, Thm. 2.2.1.]). With Theorem 2 it follows that $\lambda = \sup\{r > 0 : \Re \phi \in \mathcal{H}^r(0)\}$. With Corollary 6 we conclude

**Corollary 9** Provided $2n > \lambda + 2$, the tail exponent $\lambda$ satisfies

$$\lambda = \sup\{r > 0 : \mathbf{E}[\Psi_n(sX)] \cdot s^{-r} \text{ is bounded for } s \to 0^+\}$$

In conclusion, the estimation of $\lambda$ can be based on estimating $\mathbf{E}[\Psi_n(sX)]$ for various values of $s > 0$ followed by a least square linear regression in log-log towards fitting $\mathbf{E}[\Psi_n(sX)] \approx s^\lambda$. Various values of $n$ should be employed to ensure $2n > \lambda$. The exact setup and performance of this estimation procedure is detailed in [1].

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**References**


