



**HAL**  
open science

# A New Approach for Volume Reconstruction in TomoPIV with the Alternating Directions Method of Multipliers

Ioana Barbu, Cédric Herzet

► **To cite this version:**

Ioana Barbu, Cédric Herzet. A New Approach for Volume Reconstruction in TomoPIV with the Alternating Directions Method of Multipliers. The 11th International Symposium on Particle Image Velocimetry, Sep 2015, Santa Barbara, United States. hal-01245009v1

**HAL Id: hal-01245009**

**<https://inria.hal.science/hal-01245009v1>**

Submitted on 16 Dec 2015 (v1), last revised 19 Feb 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A New Approach for Volume Reconstruction in TomoPIV with the Alternating Directions Method of Multipliers

Ioana Barbu<sup>1</sup> and Cédric Herzet<sup>2</sup>

<sup>1</sup>Irstea, UR TERE, F-35044, France

<sup>2</sup>Fluminance, INRIA Centre Rennes - Bretagne Atlantique, Rennes, France

<sup>1</sup>ioana.barbu@irstea.fr, <sup>2</sup>cedric.herzet@inria.fr

## ABSTRACT

We adapt and import into the TomoPIV scenery a fast algorithm for solving the volume reconstruction problem. Our approach is based on the reformulation of the volume reconstruction task as a constrained optimization problem and the resort to the “Alternating Directions Method of Multipliers” (ADMM). This approach is shown to be an interesting alternative to the “row-action” methods, classically used by the Tomo-PIV community. In particular, our problem formulation allows to: *i*) take explicitly into account the level of the noise affecting the data; *ii*) account for both the nonnegativity and the sparsity of the solution. Experiments on a numerical TomoPIV benchmark show that the proposed framework is a serious contender for the state-of-the-art.

## 1. Introduction

The Tomographic PIV (TomoPIV) has been considered, since its debut by Elsinga *et al.* [10], the experimental solution to the understanding of turbulent fluid velocity. Its methodology deals with inferring the tridimensional (3D) motion from the observation of a multiview set of images captured at each time frame. A crucial step in this development is the volumetric reconstruction of the particles’ distribution at each time frame and this problem has thus been studied in many works of literature, see [2, Chapter 3] for a review. The most popular of these contributions belong to the class of the so-called “row-action method” [7]. These procedures accomplish the volume reconstruction task with low computational and storage requirements; unfortunately, they also often suffer from a low convergence rate to the desired solution. As a result, while a few iterations of these procedures may lead to pretty poor reconstruction performance, letting them running up to convergence may lead to an unacceptable computational burden.

New enhanced tools have thus to be employed to deal with this conundrum. We exhibit, in our companion paper [3], ways to accelerate convergence rates of standard methods with no inflation of the computational cost. In the current work, we focus our attention on an alternative approach to the volume reconstruction problem by resorting to the powerful “Alternating Direction Method of Multipliers” (ADMM) [9], an optimization procedure for large-scale problems which has recently become popular in the machine-learning community. Our motivation is backed up by the numerous advantages of the method, namely *i*) accounting for both the nonnegativity and the sparsity of the solution *ii*) explicitly handling the noise levels affecting the observations *iii*) possessing – in certain variants of the proposed paradigm – the same complexity per iteration as the “row action” methods, but at faster convergence rates than the latter.

The paper is organized as follows. Section 2 briefly presents the model underlying the volume reconstruction problem. In Section 3, we formalize the reconstruction task as a convex optimization problem and apply the ADMM technique to this particular setup. Section 4 reports convincing numerical results and Section 5 concludes with a few remarks.

## 2. Problem Formulation

The estimation of the 3D positions of the particles from the set of the collected images consists in inverting a model of the form

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \text{noise}, \quad (1)$$

where  $\mathbf{y} \in \mathbb{R}^m$  is a vector collecting the set of image pixels;  $\mathbf{x} \in \mathbb{R}^n$  is a vector such that each element  $x_j$  is associated to a discretized particle position (*i.e.*, a voxel) in the visualized volume:  $x_j = 0$  if and only if there is no particle at the corresponding location; finally,  $\mathbf{A} \in \mathbb{R}^{m \times n}$  is the “projection matrix” which relates the collected images to some particle configuration  $\mathbf{x}$ . We refer the reader to our companion paper [3] for a flavour of the reasoning justifying Equation (1) and to [2, Chapter 2] for a detailed description of the model construction.

When  $n > m$  and  $\mathbf{A}$  is full rank, (1) has an infinity of solutions. In order to distinguish between them, we can exploit some information available on the TomoPIV system: *i*) the elements of  $\mathbf{x}$  corresponds to intensity and should therefore be nonnegative; *ii*) the sought vector  $\mathbf{x}$  is typically sparse, that is, it contains much more zero than nonzero elements. Constraining the solution to be nonnegative is often an easy

task. Accounting for sparsity is usually more tricky and many works have focussed on this problem in the last decades [4, 8, 11]. A standard technique proposed in this literature to enforce sparsity consists in looking for the solution of (1) with the minimal  $\ell_1$  norm. We adopt this approach hereafter. Our volume reconstruction problem then takes the form of the following optimization problem

$$\min_{\mathbf{x} \geq 0} \|\mathbf{x}\|_1 \text{ such that } \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \varepsilon, \quad (2)$$

where  $\varepsilon > 0$  is a parameter accounting for the level of the noise corrupting the collected data. The resulting optimization problem is convex, but nonsmooth. This, on top of the high dimensionality of the system (typically,  $m \sim 10^6$ ,  $n \sim 10^9$ ), incites us to consider the volume reconstruction problem within the ADMM framework.

### 3. TomoPIV meets ADMM

In this section, we expose our methodology based on ADMM to solve the volume reconstruction problem. We start by introducing some useful notations. Then, we depict the general ADMM framework. Finally, we aim attention at the ADMM iterates to address the problem (2) and show its pertinence within our context.

#### 3.1 Notations and Useful Computations

We define the indicator function  $\mathbb{I}_{\mathcal{X}}(\mathbf{x})$  of a set  $\mathcal{X}$  as

$$\mathbb{I}_{\mathcal{X}}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{X}, \\ +\infty & \text{otherwise.} \end{cases}$$

The projection operator of a point  $\mathbf{v}$  on a convex set  $\mathcal{X}$  is defined as

$$\Pi_{\mathcal{X}}(\mathbf{v}) = \arg \min_{\mathbf{x} \in \mathcal{X}} \|\mathbf{x} - \mathbf{v}\|_2.$$

The proximal operator of a convex proper function  $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \infty$  is defined as

$$\text{prox}_f(\mathbf{v}) = \arg \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) + \frac{1}{2} \|\mathbf{x} - \mathbf{v}\|_2^2.$$

#### 3.2 Quick Tutorial on ADMM

ADMM focusses on the following type of optimization problems

$$\begin{aligned} & \min_{\mathbf{x}, \mathbf{z}} f(\mathbf{x}) + g(\mathbf{z}) \\ & \text{such that } \mathbf{D}\mathbf{x} + \mathbf{z} = \mathbf{0}, \end{aligned} \quad (3)$$

where  $f: \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  are closed, proper and convex functions. The ADMM is an iterative procedure searching for a minimizer of (3) via the following recursion

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \arg \min_{\mathbf{x}} f(\mathbf{x}) + \frac{\rho}{2} \|\mathbf{D}\mathbf{x} + \mathbf{z}^{(k)} + \mathbf{u}^{(k)}\|_2^2 \\ \mathbf{z}^{(k+1)} &= \arg \min_{\mathbf{z}} g(\mathbf{z}) + \frac{\rho}{2} \|\mathbf{D}\mathbf{x}^{(k+1)} + \mathbf{z} + \mathbf{u}^{(k)}\|_2^2 \\ \mathbf{u}^{(k+1)} &= \mathbf{u}^{(k)} + \mathbf{D}\mathbf{x}^{(k+1)} + \mathbf{z}^{(k+1)}, \end{aligned} \quad (4)$$

where  $\rho > 0$ . We refer the reader to the very good tutorial on ADMM [5] for an explanation of the rationale behind this type of methodology.

ADMM has recently caught the attention of the signal-processing community for several reasons. First, the conditions on  $f$  and  $g$  in (3) are mild (they are not required to be differentiable and can take on infinite values); therefore, problem (3) encompasses a large number of optimization problems as particular cases. Second, the ADMM recursion (4) converges to a solution of (3) under very general conditions, see [5, section 3.2]. Third, the algorithmic scheme has been shown empirically to converge to modest accuracy to a solution of (3) in only a few tens of iterations. Finally, the optimization problems involved in the updates of  $\mathbf{x}^{(k+1)}$  and  $\mathbf{z}^{(k+1)}$  admit very fast solution or closed-form expression in many setups. These two last features make ADMM very appealing in **large-scale problems** where modest accuracy is often sufficient but complexity load is of utmost importance.

### 3.3 ADMM Applied to the TomoPIV Problem

Our ensuing developments are inspired from the ‘‘C-SALSA’’ algorithm introduced in [1]. In order to take the algorithmic cue from the upper-evoked scheme, we express the problem (2) along the lines of that expressed by Equation (3):

$$\min_{\mathbf{x}, \mathbf{z}_1, \mathbf{z}_2} \mathbb{I}_{\mathbb{R}_+^n}(\mathbf{z}_1) + \|\mathbf{z}_1\|_1 + \mathbb{I}_{\mathcal{B}(\mathbf{y}, \varepsilon)}(\mathbf{z}_2) \text{ such that } \begin{bmatrix} \mathbf{I}_n \\ \mathbf{A} \end{bmatrix} \mathbf{x} - \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = 0, \quad (5)$$

where  $\mathcal{B}(\mathbf{y}, \varepsilon) = \{\mathbf{v} \in \mathbb{R}^m \mid \|\mathbf{y} - \mathbf{v}\|_2 \leq \varepsilon\}$  is the  $\ell_2$  ball of radius  $\varepsilon$  centered on  $\mathbf{y}$ .

Particularizing the ADMM update rules (4) to the targeted problem (5), we obtain the following recursions:

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \arg \min_{\mathbf{x}} \|\mathbf{x} - \mathbf{z}_1^{(k)} - \mathbf{u}_1^{(k)}\|_2^2 + \frac{\rho}{2} \|\mathbf{A}\mathbf{x} - \mathbf{z}_2^{(k)} - \mathbf{u}_2^{(k)}\|_2^2 \\ \begin{cases} \mathbf{z}_1^{(k+1)} &= \text{prox}_{\rho^{-1}\|\mathbf{z}_1\|_1 + \mathbb{I}_{\mathbb{R}_+^n}}(\mathbf{x}^{(k+1)} - \mathbf{u}_1^{(k)}) \\ \mathbf{z}_2^{(k+1)} &= \Pi_{\mathcal{B}(\mathbf{y}, \varepsilon)}(\mathbf{A}\mathbf{x}^{(k+1)} - \mathbf{u}_2^{(k)}) \end{cases} \\ \begin{cases} \mathbf{u}_1^{(k+1)} &= \mathbf{u}_1^{(k)} - \mathbf{x}^{(k+1)} + \mathbf{z}_1^{(k+1)} \\ \mathbf{u}_2^{(k+1)} &= \mathbf{u}_2^{(k)} - \mathbf{A}\mathbf{x}^{(k+1)} + \mathbf{z}_2^{(k+1)}. \end{cases} \end{aligned}$$

We can make the following remarks on the complexity of these update rules. The ‘‘ $\mathbf{x}$ -update’’ is tantamount to solving a least-square problem. Several options are possible to carry out this task. First, one can rely on standard optimality conditions to characterize the solution. The identification of the latter requires to solve a linear system leading to a complexity scaling (at best) as  $O(mn + n^2)$ . Another route to solve the least-square problem consists in resorting to iterative search techniques such as, for example, the **conjugate-gradient method (CGM)**. This approach leads to a complexity scaling as  $O(mn)$  per iteration.

The  $\mathbf{z}$ -update requires to apply some simple proximal and projection operators. The update of  $\mathbf{z}_1$  relies on the proximal operator of the  $\ell_1$ -norm defined by

$$\left(\text{prox}_{\rho^{-1}\|\mathbf{z}_1\|_1 + \mathbb{I}_{\mathbb{R}_+^n}}(\mathbf{v})\right)_i = \begin{cases} v_i - \rho^{-1} & \text{if } v_i \geq \rho^{-1} \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

which is a simple element-wise thresholding operation. The complexity of (6) thus scales as  $O(n)$ . The update of  $\mathbf{z}_2$  requires the projection of  $\mathbf{A}\mathbf{x}^{(k+1)} - \mathbf{u}_2^{(k)}$  onto the  $\ell_2$  ball  $\mathcal{B}(\mathbf{y}, \varepsilon)$  and has a simple closed-form expression, see [1, section 3]:

$$\Pi_{\mathcal{B}(\mathbf{y}, \varepsilon)}(\mathbf{v}) = \mathbf{y} + \begin{cases} \varepsilon \frac{\mathbf{v} - \mathbf{y}}{\|\mathbf{v} - \mathbf{y}\|_2} & \text{if } \|\mathbf{v} - \mathbf{y}\|_2 > \varepsilon \\ \mathbf{v} - \mathbf{y} & \text{if } \|\mathbf{v} - \mathbf{y}\|_2 \leq \varepsilon \end{cases} \quad (7)$$

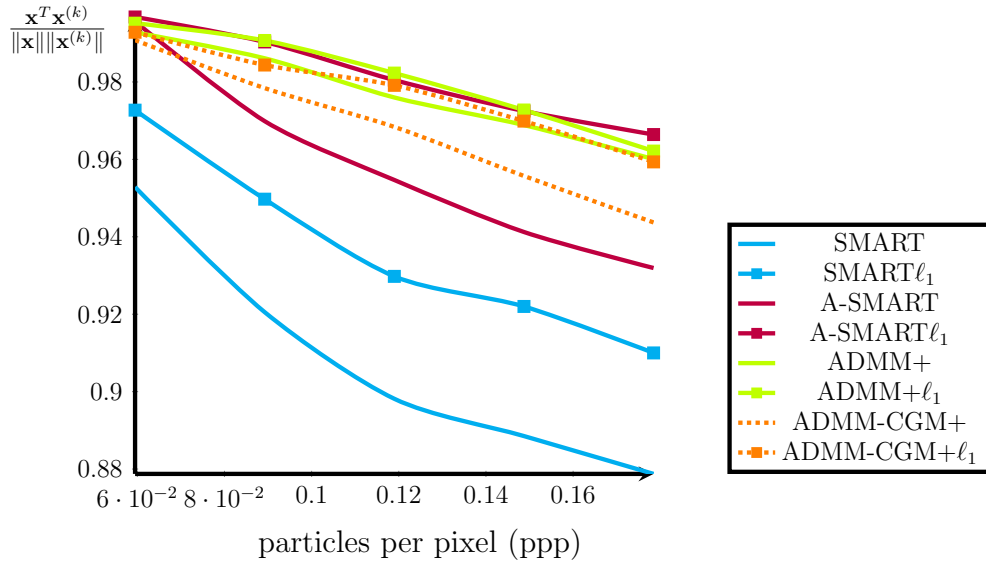
This operation can thus be performed with a complexity scaling as  $O(n)$ . The updates of  $\mathbf{u}_1$  and  $\mathbf{u}_2$  only involve simple vector additions.

As a consequence, the overall complexity of the ADMM recursion is dominated by the  $\mathbf{x}$ -update, that we can solve both analytically and iteratively. The two approaches will be considered in our simulation in the next section. To rehash some of the information provided earlier, the analytical resolution of the  $\mathbf{x}$ -update scales (at best) as  $O(mn + n^2)$ , whereas its iterative counterpart has a complexity scaling as  $O(mn)$  per iteration. We note that a complexity scaling as  $O(mn)$  is similar to the row-action methods, widely used in the TomoPIV community. On the other hand, the complexity  $O(mn + n^2)$  is prohibitive for typical dimensions of the TomoPIV systems. However, this approach will serve as a point of comparison in our simulations.

## 4. Numerical Validation

Figure 1 depicts comparative numerical assessment of our newly advanced methods against state-of-the-art SMART [6] and its accelerated counterpart – that we have coined A-SMART – introduced in our companion paper [3]. Nomenclature-wise, positivity-enforcement will be suggested by appending ‘‘+’’ to algorithms’ labels – (A-)SMARTs excluded, which implicitly constraints non-negativity –, while the ‘‘ $\ell_1$ ’’ hallmark will depict the sparsity constraint. The approximated variant of ADMM will be referred to as ADMM-CGM and will follow the similar paradigm concerning the appended labels.

We simulate a medium-scale perturbed setting (*i.e.*,  $m = 6724$ ,  $n = 99944$ ), where  $y_i$  is affected by a Gaussian noise of zero mean and standard deviation equal to  $0.1y_i$ . The results are averaged on 10 experiments and the number of iterations counts 50. We notice in particular that state-of-the-art SMART is outperformed by the newly generalized (A-)SMART( $\ell_1$ ) schemes – see [3] – and by the here-introduced ADMM procedures and their approximated variants ADMM-CGM. This reinforces our statement that the latter should be resolutely invited – along accelerated versions of SMART – into the experimental TomoPIV scene.



**Figure 1:** Numerical assessment of the reconstruction quality of  $\mathbf{x}^{(80)}$  in terms of the normalized correlation against the “particles per pixel (ppp)” ratio.

## 5. Conclusion

This paper advocates a new class of algorithms to undertake the TomoPIV Volume Reconstruction Problem. The latter make for an interesting alternative to the classical schemes employed for the TomoPIV Reconstruction Problem in terms of quality of reconstruction versus complexity and speed.

## References

- [1] M.V. Alfonso, J.M. Bioucas-Dias, and M.A. Figueiredo. An Augmented Lagrangian Approach to the Constrained Optimization Formulation of Imaging Inverse Problems. *IEEE Trans Image Process*, 20(3), 2011.
- [2] I. Barbu. *Tridimensional Estimation of Turbulent Fluid Velocity*. PhD thesis, Université Rennes 1, 2014.
- [3] I. Barbu and C. Herzet. Accelerated, Sparsity-Aware Generalizations of Classical Algorithms for TomoPIV. In *Proc. PIV*, 2015.
- [4] I. Barbu, C. Herzet, and E. Mémin. Sparse models and pursuit algorithms for PIV tomography. In *FVR*, 2011.
- [5] S.P. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers. *Found. Trends Mach. Learning*, 3(1), 2011.
- [6] C.L. Byrne. Iterative Image Reconstruction Algorithms Based on Cross-Entropy Minimization. *IEEE Trans. Image Process.*, 2(1), 1993.
- [7] Y. Censor. Row-Action Methods for Huge and Sparse Systems and Their Applications. *SIAM Rev.*, 23(4), 1981.
- [8] P. Cornic, F. Champagnat, A. Cheminet, B. Leclaire, and G. Besnerais. Computationally efficient sparse algorithms for tomographic PIV Reconstruction. In *PIV*, 2013.
- [9] J. Eckstein and D.P. Bertsekas. On the Douglas-Rachford Splitting Method and the Proximal Point Algorithm for Maximal Monotone Operators. *Math. Prog.*, 55(3), 1992.
- [10] G. Elsinga, F. Scarano, B. Wieneke, and B. van Oudheusden. Tomographic particle image velocimetry. *Exp. Fluids*, 41(6), 2006.
- [11] S. Petra and Schnörr, C. TomoPIV meets Compressed Sensing. *Pure Math. Appl.*, 2009.