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# Auctions for online ad space among advertisers sensitive to both views and clicks

Patrick Maillé and Bruno Tuffin

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## Abstract

Advertisement in dedicated webpage spaces or in search engines sponsored slots is usually sold using auctions, with a payment rule that is either per view or per click. But advertisers can be both sensitive to being viewed (brand awareness effect) and being clicked (conversion into sales).

In this paper, we generalize the auction mechanism by including both pricing components: the displayed advertisers are charged when their ad is displayed, and pay an additional price if the ad is clicked. Applying the results for Vickrey-Clarke-Groves (VCG) auctions, we show how to compute payments to ensure incentive compatibility from advertisers as well as maximize the total value of the advertisement slot(s). We provide tight upper bounds for the loss of efficiency due to applying only pay-per-click (or pay-per-view) pricing instead of our scheme. Those bounds depend on the joint distribution of advertisement visibility and population likelihood to click on ads, and can help identify situations where our mechanism yields significant improvements. We also describe how the commonly used generalized second price (GSP) auction can be extended to this context.

## 1 Introduction

An increasing number of online services (search engines, news, e-mail, video, maps, ...) are provided free of charge, with a business model based on advertising: upon accessing the service, the user is presented with advertisements, usually next to (or before, for videos) the data of initial interest. The available advertisement space is limited (among others, to maintain the service attractiveness for users), and is generally allocated among advertisers using auction mechanisms: advertisers submit *bids*, in the form of prices, and advertisement slots are decided based on those bids and possibly other factors such as the advertisers' likelihood of being of interest (clicked) to users. The price paid by each advertiser may depend on all submitted bids, and would ideally satisfy a property called *incentive compatibility* or *truthfulness*, meaning that even strate-

gic bidders cannot do better than truthfully declare the price they are willing to pay to get a slot.

In practice, two types of payment mechanisms are applied: either a price is paid each time the ad is displayed (*pay-per-impression* scheme), or each time the ad is clicked by a user (*pay-per-click* scheme). The former mechanism seems appropriate for advertisers aiming at building a “brand awareness effect” (getting the name of a brand or product known), while the latter rather corresponds to advertisers aiming at converting the ad into a more tangible outcome such as a product sale or a registration. Typically, Coca-Cola should be more interested in views, since clicks will hardly directly result in sales. Pay-per-click on the other hand more likely corresponds to online shops displaying their products. But many advertisers could be interested in both options.

For both of those payment types, appropriate pricing rules exist, and have been the subject of extensive research (see for example [10] and references therein). Remark that some services use a fixed payment scheme (pay-per-click for most search engines like Google or Bing) while others (e.g., Facebook) leave the choice to the advertiser. However, to the best of our knowledge no platform proposes hybrid schemes, where a price would be paid per view and an additional charge would be applied if the ad is clicked. Actually the two models are mathematically equivalent when there are known values of the the click-through-rate (CTR) and click-to-sale-conversion probabilities because a view or a click can then be translated into a financial gain [6], even if it might actually be difficult to quantify the money gain associated to a view into in the case of brand awareness.

A hybrid mechanism similar to what we suggest has been defined and studied in [6, 7], under the name *impression plus click* pricing: the equivalence result is not satisfied there because the beliefs about the CTR are considered potentially different between the publisher and the advertiser, hence the need for a richer mechanism. An interesting feature of the proposed auction scheme is that despite the relaxation of this common knowledge assumption, the mechanism is still incentive-compatible. On the other hand, it requires strong assumptions on bounded valuations that we do not need here.

In this paper, we do not consider such discrepancies in CTR estimation—we assume it is the same for the advertiser and the publisher—but we rather go in a different direction in terms of assumption relaxation. We indeed investigate the situation where a common bid is used for a (possibly large) *set of webpages*, with different ad visibilities but also different likelihoods that an ad is clicked: typically, a page about fashion is more likely to bring clicks and then sales than a page about news; similarly, for a given page the click likelihood can depend on the population viewing it, which can be determined using different means: knowing how the user landed on this page, having users connect using a social network account, etc. In this paper we will treat that case as distinct pages, without loss of generality.

Assuming such variability in ad visibility and click likelihood also implies that the equivalence result is not satisfied: we cannot simply and directly relate the valuation per impression and the valuation per click, since the click/impression

ratio differs between pages. An advantage is that instead of requiring an estimation of the CTR for each page, the bids have to be provided once and for all independently of the web page (but under a separability assumption).

In such a context, it appears necessary to have advertisers declare *two* values, namely their willingness-to-pay for being clicked, and for being viewed. We show on simple examples that using one-dimensional bids leads to significant losses of efficiency (in the sense of the value obtained from the ad slots). On the other hand, with a Vickrey-Clarke-Groves (VCG)-based scheme applied for each webpage to 2-dimensional bids, we ensure truthful bidding (as a dominant bidding strategy) and maximized efficiency. We present bounds on the relative performance of one-dimensional bids with respect to our proposed scheme, and also explain how the commonly-used generalized second price (GSP) auctions can be extended to this context, with their pros and cons.

The remainder of this paper is organized as follows. Section 2 introduces our mathematical model, specifying our assumptions for ad visibility and click attractivity. Section 3 then treats the case when each page only has one advertisement slot, describing the scheme and its social surplus performance bounds. Some of those results are generalized to the multiple-slot case in Section 4. Section 5 discusses our findings and proposes directions for future work.

## 2 Mathematical model

Consider advertisement slots to be sold on a web page or on a search engine. Those slots are sold through auctions; for more about such auctions, the reader is advised to look at [10]. This section specifies our assumptions in terms of ad slot (random) visibility and user (random) click behavior, and advertiser preferences.

### 2.1 Advertisement visibility and click attractivity effectiveness

Inspired by the literature [5, 9] considering separable click-through rates, we distinguish three effects in terms of the impact of a given ad.

- The **ad visibility** depends on the type of page visited, the size and location of the advertising slot, etc. It can be interpreted as the *probability* that the user actually *sees* the ad (although this may be difficult to measure), in particular we assume it is independent of the ads possibly displayed in the other ad slots. We denote by  $\alpha_s$  the visibility of the ad displayed in the  $s^{\text{th}}$  ad slot of a given page: we model it as a random variable to take into account the variability in pages, sizes and locations. But for any given visited page, the vector  $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$  is assumed known to the auctioneer, where  $m$  is the (max) number of slots available on each page. Also, without loss of generality we order ad slots on each page so that  $\alpha_1 \geq \alpha_2 \geq \dots$ , i.e., ad slot 1 being more visible than ad slot 2, etc. Note

that the case when some pages have less than  $m$  ad slots is also covered: for such spaces one just has to set  $\alpha_s = 0$  if there is no ad slot  $s$ .

- Once an ad is viewed, there is a probability that it is clicked by the user. We assume this probability is multiplicatively separable into
  - a **page effect**, which reflects the fact that the likelihood of a click depends on the type of page visited. For example, e-commerce webpages may lead to more ad-clicks (to products related to the page) than news webpages; or similarly, general knowledge about the audience of a webpage (age, gender, location) can give information about the likelihood to get a click. We represent that effect through another random variable  $\beta$ .
  - an **ad effect**, which represents the intrinsic attractiveness of the ad, and which we denote by  $\text{CTR}_j$  for advertiser  $j$ .

To summarize, in our model each webpage is characterized by a pair  $(\vec{\alpha}, \beta)$ , with  $\vec{\alpha}$  a vector giving the visibility value  $\alpha_s$  for each slot  $s$  in the page. When displayed in slot  $s$  on such a page, the ad from an advertiser  $j$  is:

- viewed with probability  $\alpha_s$ ;
- clicked with probability  $\alpha_s \beta \text{CTR}_j$ .

## 2.2 Advertiser preferences

Consider a set  $\mathcal{J}$  of advertisers. Each advertiser is potentially interested in both views and clicks. More exactly, Advertiser  $j \in \mathcal{J}$  is characterized by a pair

$$(b_j^v, b_j^c)$$

where  $b_j^v$  is its valuation for a view (the brand awareness effect) and  $b_j^c$  its valuation for a click (related to potential sales). Due to the large number of page visits, it is reasonable to assume that advertisers are *risk-neutral*, i.e., they are interested in the value they can extract over the whole set of webpages and of visits.

Under the assumptions of the previous subsection, for an Advertiser  $j$  characterized by valuations  $(b_j^v, b_j^c)$ , the expected value of ad slot  $s$  at web page  $(\vec{\alpha}, \beta)$  is

$$U_j(\vec{\alpha}, \beta) = \alpha_s (b_j^v + \beta \text{CTR}_j b_j^c). \quad (1)$$

Advertisers being sensitive to the *aggregated* effects of their ad behavior, each advertiser  $j \in \mathcal{J}$  is assumed to behave (here, bid) in order to maximize its expected *net utility*, that is, the expectation

$$\mathbb{E}_{(\vec{\alpha}, \beta)} [U_j(\vec{\alpha}, \beta) - p_j(\vec{\alpha}, \beta)] = \mathbb{E}_{(\vec{\alpha}, \beta)} [\alpha_s (b_j^v + \beta \text{CTR}_j b_j^c) - p_j(\vec{\alpha}, \beta)],$$

where  $p_j(\vec{\alpha}, \beta)$  is the (expected) amount that Advertiser  $j$  is charged when page  $(\vec{\alpha}, \beta)$  is visited.

### 2.3 Auctioning for ad space over a set of webpages

In this paper, we suggest that publishers offer a simple interface to advertisers: advertisers are asked to declare how much they are willing to pay to be viewed, and—when displayed—what *additional* price they would be willing to pay to have their ad clicked. Those bids are then used for a possibly large set of webpages which has been agreed upon beforehand between the auctioneer and each particular advertiser. Note that the set of webpages can differ among advertisers: each advertiser  $j$  may define a set of webpages  $W_j$  it is interested in, although we do not develop that aspect in this paper because it is not its purpose to discuss the strategies on the slots to target. The difference for our analysis would be a decomposition of the set of ads into  $2^{|\mathcal{J}|} - 1$  subsets, corresponding to all potential combinations of bidders, and the surpluses and expected revenues would just be a sum over all those subsets. We therefore simplify the notations by assuming without much loss of generality that the advertisers submit their bid on all slots.

The auctioneer allocates the ad slot(s) of a page in order to maximize the generated value, based on bids, i.e., on declared values  $(\tilde{b}_j^v)_{j \in \mathcal{J}}$  and  $(\tilde{b}_j^c)_{j \in \mathcal{J}}$ . Focusing on a single slot, it should be given to the advertiser with the highest value for  $\tilde{b}_j^v + \beta \text{CTR}_j \tilde{b}_j^c$ . Intuitively, if a webpage is not likely to attract clicks ( $\beta$  small), its ad slot will rather be allocated to advertisers focusing on building a brand effect; on the contrary if  $\beta$  is large then the slot should be given to an advertiser with a high valuation for clicks.

### 2.4 Performance metrics

As is classical in mechanism design, a key property will be *dominant-strategy incentive compatibility* (or *incentive compatibility* in short), meaning that for any bid profile of one’s competitors, one’s best bidding strategy is to reveal one’s willingness-to-pay for the ad slot resource. In this paper we will directly apply some well-known results on second-price auctions and their generalizations, which ensure incentive compatibility.

The other performance metric is relative to the *efficiency* of the resulting allocation, and quantifies the overall value that is extracted from the ad slots. This value takes the form of utility for advertisers (for which we consider their economic counterpart) and of direct revenues for the auctioneer. Their sum equals the total willingness-to-pay of advertisers for the slots they obtain. Such a measure is often called *social welfare*, here we rather call it *surplus* to stress the fact that users are not included in the considered “society”.

## 3 The one-slot case

For simplicity of exposition, we first treat here the case when only one advertisement slot is available on each webpage. In that case, the vector  $\vec{\alpha}$  is just a scalar  $\alpha \geq 0$ .

### 3.1 The *impression plus click* second-price auction scheme

We suggest that *on each webpage*  $w$  (with given values for  $(\alpha, \beta)$ ), the publisher performs a second-price auction [14] to allocate the advertising slot. The multiple-slot case will be discussed in the next section.

#### 3.1.1 2D-auction rules

**Definition 1** (2D-auction scheme). *Based on the submitted two-dimensional bids  $(\tilde{b}_j^v, \tilde{b}_j^c)_{j \in \mathcal{J}}$ , on a webpage with ad visibility  $\alpha$  and click-likelihood  $\beta$ ,*

- *the slot is given to an advertiser  $j_1$  with the highest declared expected value*

$$j_1 \in \arg \max_{j \in \mathcal{J}} \underbrace{\tilde{b}_j^v + \beta CTR_j \tilde{b}_j^c}_{:=m_j(\beta)}$$

- *that advertiser is charged  $\alpha p$  for each impression (or equivalently,  $p$  per expected view) where  $p$  is the second-highest declared valuation*

$$p := \max_{j \in \mathcal{J} \setminus j_1} \{\tilde{b}_j^v + \beta CTR_j \tilde{b}_j^c\}. \quad (2)$$

Remark from the definition that the winner identity depends on  $\beta$ , and the price charged depends on  $\beta$  and  $\alpha$ .

#### 3.1.2 2D-auction incentive and surplus properties

As a second-price auction, the 2D-auction has good incentive properties, as stated below.

**Proposition 1.** *The 2D-auction scheme of Definition 1 satisfies the truthfulness property: for each Advertiser  $j$ , submitting its true pair of valuations  $(\tilde{b}_j^v, \tilde{b}_j^c) = (b_j^v, b_j^c)$  is a dominant strategy.*

*Moreover, the average social surplus per visited page is*

$$S_{2D} = \mathbb{E} \left[ \max_{j \in \mathcal{J}} \alpha (b_j^v + \beta CTR_j b_j^c) \right]. \quad (3)$$

*Proof.* On any page  $(\alpha, \beta)$ , the ad slot is allocated to the advertiser who values it most in the sense of the declared valuation  $\tilde{U}_j := \alpha (\tilde{b}_j^v + \beta CTR_j \tilde{b}_j^c)$ , which corresponds to the real valuation when advertisers bid truthfully. The price paid then corresponds to the so-called *social opportunity cost* of the winner's presence, that is here simply the highest valuation among losing advertisers. We therefore exactly have a Vickrey auction using the declared values  $(\tilde{U}_j)_{j \in \mathcal{J}}$ , which is known to be incentive compatible [11]: for each webpage the advertisers cannot do better than being truthful about their valuation  $U_j$ . Since the value of  $\beta$  is unknown to advertisers, the only way to ensure that the value considered by the auctioneer is the true one for each page is for advertisers to truthfully declare their value-per-view and value-per-click.  $\square$

Remark that another implementation of the 2D pricing scheme can be in the form of impression+click pricing, as suggested in [7], instead of the per-impression price of (2) charged each time: Advertiser  $j_1$  would be charged only  $\alpha \tilde{b}_{j_2}^v$  when displayed, and an additional  $\frac{\text{CTR}_{j_2}}{\text{CTR}_{j_1}} \tilde{b}_{j_2}^c$  if the ad is clicked. That implementation gives the same expected payment  $\alpha p$  with  $p$  given in (2), and therefore will be considered equivalent in this paper. Nevertheless, it may be preferred by advertisers since being closer to the actual user behavior (the extra charge for a click being applied only when a click occurs instead of being averaged for each impression). But note on the other hand that it induces more variance in prices.

Given that the parameter  $\beta$  differs among pages (a difference with [6, 7]), there is no direct equivalence between a price per view and a price per click when submitting bids. Hence, as we illustrate later, using one-dimensional bids (interpreted as per impression, or per click) leads to an inevitable loss of efficiency: the ad slots cannot always be allocated to the advertiser valuing it most. At least a two-dimensional bid is needed, and we argue it is sufficient to ensure optimality.

Of course, separate auctions could be run independently on each webpage. The advantage of our method is that the bidding process is used once, for a possibly very large set of webpages, and remains very simple (only a two-dimensional bid is sent by each advertiser at the beginning of its campaign) while still permitting advertisers to play their dominant-strategy bid on *each* page. Said differently, even if advertisers were allowed to submit a different bid on each page, they could not get a higher utility.

## 3.2 A 1D-auction scheme

For comparison purposes, we define the (classical) one-dimensional bidding auction: even if advertiser utilities are bi-dimensional, they are only allowed to submit a one-dimensional bid, representing either their valuation for being clicked or for being displayed.

### 3.2.1 1D-auction rules

**Definition 2** (1D-auction scheme). *Based on values  $(\tilde{v}_j)_{j \in \mathcal{J}}$  representing the average value of being displayed, and computed from one-dimensional bids:*

- *the slot is given to an advertiser  $j$  with maximum value  $\tilde{v}$ :*

$$j \in \arg \max_{i \in \mathcal{J}} \tilde{v}_i;$$

- *the (average) price paid per impression is the second-highest value  $p_j = \max_{i \in \mathcal{J} \setminus \{j\}} \tilde{v}_i$ .*

This definition is independent of the bid interpretation:



- if a bid represents the value-per-impression, then  $\tilde{v}_j$  is directly the submitted bid of Advertiser  $j$ ;
- if bids are interpreted as a value-per-click, then  $\tilde{v}_j$  is computed from bid  $b_j$  as:  $\tilde{v}_j = b_j \cdot \mathbb{E}[\alpha\beta]CTR_j$ .

Also, both price implementations are covered:

- in a pay-per-impression scheme, the displayed Advertiser  $j$  is charged  $p_j$  at each impression;
- in a pay-per-click scheme, Advertiser  $j$  is charged  $\frac{p_j}{\mathbb{E}[\alpha\beta]CTR_j}$  when clicked, hence on average  $p_j$  per impression.

### 3.2.2 1D-auction properties

Let us formalize the equivalence between pay-per-click and pay-per-impression in that context, as well as incentive and properties.

**Lemma 1.** *For the 1D-auction scheme of Definition 2, both pay-per-impression and pay-per-click implementations are equivalent in average, in the following sense:*

1. *for any interpretation of the bid (as the willingness-to-pay for an impression, or for a click), truthfully declaring one's (corresponding) expected value is the only dominant strategy;*
2. *the same advertiser is displayed in the ad slot of all webpages, and is an advertiser  $j_0$  with maximum expected value for being displayed*

$$j_0 \in \arg \max_{j \in \mathcal{J}} \mathbb{E} [\alpha(b_j^y + \beta CTR_j b_j^c)]$$

3. *the resulting average surplus (per visited webpage) is the maximum expected value*

$$S_{1D} = \max_{j \in \mathcal{J}} \mathbb{E} [\alpha(b_j^y + \beta CTR_j b_j^c)] \quad (4)$$

4. *the average revenue per visited webpage is the second-highest maximum expected value*

$$R_{1D} = \max_{j \in \mathcal{J} \setminus j_0} \mathbb{E} [\alpha(b_j^y + \beta CTR_j b_j^c)] \quad (5)$$

*Proof.* The 1D-auction actually allocates only one indivisible good (the right to be displayed in the ad slot of *all* webpages), and allocates it using a classical Vickrey auction: bidders are asked (directly or indirectly) to declare how much they are willing to pay to obtain that good, which is then allocated to the highest bidder who is charged the second-highest declared value. It is well-known and easy to check that bidding truthfully is the only dominant strategy.

In our setting, this means that an advertiser  $j$  characterized by its valuations  $(b_j^c, b_j^y)$  should bid so that the value considered by the auction equals its truthful

(expected) valuation for being displayed, that is  $\mathbb{E}[\alpha(b_j^y + \beta \text{CTR}_j b_j^c)] = \bar{\alpha} b_j^y + \overline{\alpha\beta} \text{CTR}_j b_j^c$ . If bids are interpreted as values per impression, that bidder should then directly declare that value, while it should divide it by the click probability  $\overline{\alpha\beta} \text{CTR}_j$  if bids are interpreted as values per click.

The two other points of the lemma are then direct consequences of the truthfulness property: the ad slot goes to an advertiser with maximal expected valuation, and who is charged on average the second-highest expected valuation for each display.  $\square$

### 3.3 Relative performance of 1D and 2D auctions

The following proposition compares the performance of one-dimensional bidding (pay-per-click or pay-per-impression) with our two-dimensional bidding scheme.

**Proposition 2.** *Recall that  $S_{1D}$  (resp.  $S_{2D}$ ) are for the expected surplus over the set of webpages over which the auction is performed for the 1D (resp. 2D) auction. Then if advertisers are assumed to play their (unique) dominant strategy, the respective surpluses raised by both auction schemes verify:*

$$1 \leq \frac{S_{2D}}{S_{1D}} \leq \mathbb{E} \left[ \alpha \max \left( \frac{1}{\bar{\alpha}}, \frac{\beta}{\overline{\alpha\beta}} \right) \right] \quad (6)$$

with  $\bar{\alpha} := \mathbb{E}[\alpha]$  and  $\overline{\alpha\beta} := \mathbb{E}[\alpha\beta]$ .

Moreover, those bounds are tight, i.e., for any distribution of  $(\alpha, \beta)$  there exists a valuation profile such that  $S_{2D} = S_{1D}$  and a valuation profile such that  $S_{2D} = S_{1D} \mathbb{E} \left[ \alpha \max \left( \frac{1}{\bar{\alpha}}, \frac{\beta}{\overline{\alpha\beta}} \right) \right]$ .

*Proof.* To simplify notations, define for each  $i \in \mathcal{J}$

$$a_i := \text{CTR}_i b_i^c.$$

The expressions for  $S_{1D}$  and  $S_{2D}$  are then:

$$\begin{aligned} S_{1D} &= \max_{i \in \mathcal{J}} \mathbb{E} [\alpha \{b_j^y + \beta a_j\}] = \max_{i \in \mathcal{J}} \{\bar{\alpha} b_j^y + \overline{\alpha\beta} a_j\}, \\ S_{2D} &= \mathbb{E} \left[ \alpha \max_{i \in \mathcal{J}} \{b_j^y + \beta a_j\} \right]. \end{aligned}$$

The inequality  $S_{2D} \geq S_{1D}$  is trivial, as well as the fact it is tight (just assume all bidders in  $\mathcal{J}$  have the same values for  $a_j$  and  $b_j^y$ ).

Now, notice that for all  $j \in \mathcal{J}$ ,  $\bar{\alpha} b_j^y + \overline{\alpha\beta} a_j \leq S_{1D}$ , hence  $a_j \leq \frac{S_{1D} - \bar{\alpha} b_j^y}{\overline{\alpha\beta}}$ , and thus for any page  $(\alpha, \beta)$  we have

$$b_j^y + \beta a_j \leq \frac{\beta}{\overline{\alpha\beta}} S_{1D} + b_j^y \left( 1 - \frac{\beta \bar{\alpha}}{\overline{\alpha\beta}} \right).$$

Therefore, since  $b_j^y \geq 0$  we have when  $\beta > \frac{\overline{\alpha\beta}}{\bar{\alpha}}$

$$b_j^y + \beta a_j \leq \frac{\beta}{\overline{\alpha\beta}} S_{1D} \quad \text{for all } j \in \mathcal{J}.$$

Consequently,

$$\begin{aligned}
S_{2D} &= \mathbb{E} \left[ \mathbb{1}_{\{\beta \leq \frac{\alpha\beta}{\alpha}\}} \left( \alpha \max_{j \in \mathcal{J}} \{b_j^v + \beta a_j\} \right) + \mathbb{1}_{\{\beta > \frac{\alpha\beta}{\alpha}\}} \left( \alpha \max_{j \in \mathcal{J}} \{b_j^v + \beta a_j\} \right) \right] \\
&\leq \mathbb{E} \left[ \mathbb{1}_{\{\beta \leq \frac{\alpha\beta}{\alpha}\}} \underbrace{\left( \alpha \max_{j \in \mathcal{J}} \left\{ b_j^v + \frac{\alpha\beta}{\alpha} a_j \right\} \right)}_{= \frac{\alpha}{\alpha} S_{1D}} + \mathbb{1}_{\{\beta > \frac{\alpha\beta}{\alpha}\}} \frac{\alpha\beta}{\alpha\beta} S_{1D} \right] \\
&= S_{1D} \mathbb{E} \left[ \alpha \max \left( \frac{1}{\alpha}, \frac{\beta}{\alpha\beta} \right) \right],
\end{aligned}$$

which gives the right-hand side of (6).

There remains to show that this last inequality is tight: let us fix a joint distribution for  $(\alpha, \beta)$  and consider a setting with two advertisers, both having the same expected value  $\bar{\alpha}b_1^v + \alpha\beta a_1 = \bar{\alpha}b_2^v + \alpha\beta a_2$  per visit for the slot (hence that common value is  $S_{1D}$ ), but where:

- Advertiser 1 is only interested in clicks, i.e.,  $b_1^v = 0$  and thus  $a_1 = \frac{S_{1D}}{\alpha\beta}$ ;
- Advertiser 2 is only interested in views, i.e.,  $a_2 = 0$  and  $b_2^v = \frac{S_{1D}}{\alpha}$ .

Then with 2D-bidding, the slot goes to Advertiser 1 if and only if  $\beta > \frac{\alpha\beta}{\alpha}$ , and

$$\begin{aligned}
S_{2D} &= \mathbb{E} \left[ \mathbb{1}_{\{\beta \leq \frac{\alpha\beta}{\alpha}\}} \underbrace{\alpha \left( \max_{i \in \mathcal{J}} \{b_i^v + \beta a_i\} \right)}_{= b_2^v + \beta a_2 = S_{1D}/\alpha} + \mathbb{1}_{\{\beta > \frac{\alpha\beta}{\alpha}\}} \underbrace{\alpha \left( \max_{i \in \mathcal{J}} b_i^v + \beta a_i \right)}_{= b_1^v + \beta a_1 = \frac{\beta}{\alpha\beta} S_{1D}} \right] \\
&= S_{1D} \mathbb{E} \left[ \mathbb{1}_{\{\beta \leq \frac{\alpha\beta}{\alpha}\}} \frac{\alpha}{\alpha} + \mathbb{1}_{\{\beta > \frac{\alpha\beta}{\alpha}\}} \frac{\alpha\beta}{\alpha\beta} \right] \\
&= S_{1D} \mathbb{E} \left[ \alpha \max \left( \frac{1}{\alpha}, \frac{\beta}{\alpha\beta} \right) \right].
\end{aligned}$$

□

### 3.4 An illustrative example

A particular case is when  $\alpha$  and  $\beta$  are independent. For example, if in that case  $\beta$  is uniformly distributed over an interval  $[0, \beta_{\max}]$ , then from Proposition 2 we have

$$1 \leq \frac{S_{2D}}{S_{1D}} \leq \frac{5}{4},$$

i.e., using 2D-bidding instead of 1D-bidding can yield a 25% surplus improvement.

Let us for example assume that ad slots on all webpages have the same visibility  $\alpha$ , and take  $\alpha = 1$  without loss of generality. Hence webpages differ

only in the likelihood  $\beta$  of ads attracting clicks; in this example, we consider that  $\beta$  is uniformly distributed over the interval  $[0, 1]$ .

Consider two advertisers with the same CTR, with Advertiser 1 only interested in clicks, i.e., characterized by valuations  $(0, b_1^c)$ , and Advertiser 2 only interested in views, i.e., characterized by valuations  $(b_2^v, 0)$ .

### Behavior of our two-dimensional auction

Under our impression+click (2D-auction) scheme, from Proposition 1, advertisers submit truthful bids. Hence,

- if  $\beta > \frac{b_2^v}{b_1^c \text{CTR}}$  the slot goes to Advertiser 1, who is charged  $b_2^v$  per impression;
- otherwise the slot goes to Advertiser 2, who is charged  $b_1^c \beta \text{CTR}$  on average per impression (either as a price per impression, or as a price  $b_1^c$  per click).

Then, assuming  $b_2^v < b_1^c \text{CTR}$ , the expected value (over the random  $\beta$ ) from the slot for each impression is

$$\begin{aligned} \mathbb{E}[\max(b_1^c \beta \text{CTR}, b_2^v)] &= b_2^v \frac{b_2^v}{b_1^c \text{CTR}} + \int_{x=\frac{b_2^v}{b_1^c \text{CTR}}}^1 b_1^c x \text{CTR} dx \\ &= \frac{(b_2^v)^2}{b_1^c \text{CTR}} + \frac{b_1^c \text{CTR}}{2} \left(1 - \left(\frac{b_2^v}{b_1^c \text{CTR}}\right)^2\right) \\ &= \frac{(b_2^v)^2}{2b_1^c \text{CTR}} + \frac{b_1^c \text{CTR}}{2}. \end{aligned} \quad (7)$$

When  $b_2^v \geq b_1^c \text{CTR}$  the slot always goes to Advertiser 2 so that the expected slot value is simply  $b_2^v$ .

### Behavior of a one-dimensional auction

On the other hand, let us investigate what would happen under a 1D auction, where each advertiser could only submit a one-dimensional bid, for example indicating its willingness-to-pay for a click (from Lemma 1, a pay-per-impression scheme would be equivalent).

Denoting by  $b_i$  the (one-dimensional) bid of Advertiser  $i$  for  $i = 1, 2$ , the auctioneer would for each page:

- sort bids in terms of their product *click-through-rate*  $\times$  *bid*—here since we assumed both advertisers have the same CTR, the ranking is just based on bids—;
- give the slot to the highest bidder (in terms of this product);
- charge (per impression, on average) the second-highest price (still in terms of the *click-through-rate*  $\times$  *bid* product), hence for example  $b_2 \frac{\text{CTR}_2}{\text{CTR}_1}$  per click if Advertiser 1 obtains the slot [10]. Here, with both advertisers having the same CTR, the price per click is just the second-highest bid.

The problem here is that the click likelihood is a random variable, so that as evoked before there is no predictable equivalence between the value for being displayed and the value for being clicked. While Advertiser 1 can declare its true valuation  $b_1 = b_1^c$ , bidding in such a scheme is more tricky for the view-sensitive Advertiser 2, since it cannot bid so that its truthful valuation be taken into account for each page. Since bidders are assumed risk-neutral, and as stated in Lemma 1, Advertiser 2 will declare its truthful *expected* valuation for the slot (in terms of value per click): since winning the auction will bring some (random) number  $N$  of views, and a corresponding expected number of clicks  $N\beta$ , that bidder should declare it is willing to pay  $b_2^y N$  for those views, and hence  $b_2 = b_2^y/\beta$  per click.

For our example, if  $b_2 < b_1$  then the slot always goes to Advertiser 1, and hence has an expected value per display

$$\mathbb{E}[\beta \text{CTR} b_1^c] = \frac{\text{CTR} b_1^c}{2}.$$

On the other hand, if  $b_1 < b_2$  the slot goes to Advertiser 2, for which it has an expected value per display  $b_2^y$ .

As a result, if advertisers bid truthfully, the expected value per impression is

$$\max\left(b_2^y, \frac{\text{CTR} b_1^c}{2}\right). \quad (8)$$

### Behavior of a pay-per-impression auction

Similarly, a pay-per-impression auction would ask for a one-dimensional bid and always give the slot to the same bidder, yielding the same maximum expected value as the one in (8).

Figure 1 shows that expected value per ad impression for our 2D-bid scheme and for the two one-dimensional bidding schemes (click-based or impression-based) when  $b_2^y = 0.01$ ,  $\text{CTR} = 0.01$  and  $b_1^c$  varies. The second graph displays the ratio of allocated slot values. Our 2D-auction reaching maximum efficiency, a value 1 means an efficient outcome. A loss of efficiency can be observed for some per-click valuations  $b_1^c$  of Advertiser 1. Interestingly, this (relative) loss of efficiency is not monotonous as  $b_1^c$  increases. We also observe that the bound obtained in Proposition 2 is attained: when  $b_1^c = \frac{b_2^y}{\beta \text{CTR}}$  we have  $\frac{S_{2D}}{S_{1D}} = \frac{5}{4}$ , or equivalently an 80% efficiency.

## 4 Extension to multi-slot pages

Our 2D-auction mechanism can quite easily be extended to the case where each page has a number  $m$  of ad slots. Let us index slots with the subscript  $s$ ,  $1 \leq s \leq m$ . Recall that the slot visibilities are  $(\alpha_s)_{1 \leq s \leq m}$  where  $\alpha_1 \geq \dots \geq \alpha_m$ , and the page effect  $\beta$  is the same for all slots of a given page. As before we also have the ad effect on clicks, that we still denote by  $\text{CTR}_j$  for advertiser

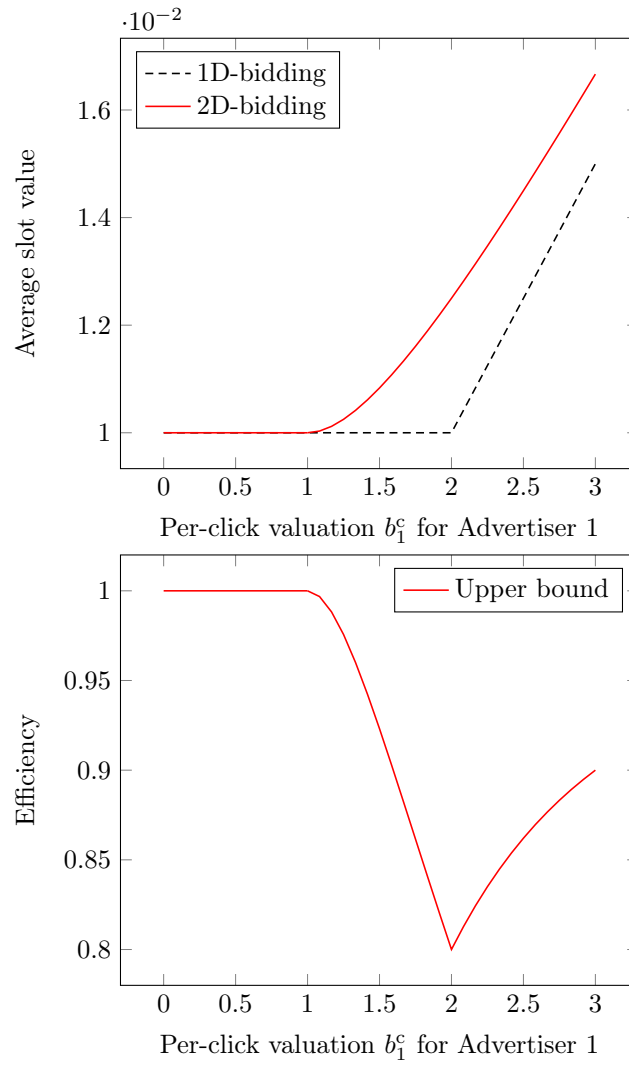


Figure 1: Efficiency of a 1D-bidding scheme

*j.* Remark that, since  $\beta$  is the same for all slots, the preference ranking of all advertisers is the same: they all prefer Slot 1 over Slot 2, etc.

As pointed out in Section 2, assuming  $m$  to be the same over all pages is without loss of generality.

Similar to what is suggested in [6], our auction scheme can be extended in two different ways.

- A first possibility is to apply the VCG principle to allocate and price ad slots on each visited page (see [14, 2, 8] or [11, 12] for a definition of VCG auctions). More precisely, for each page one would allocate slots to ads in order to maximize the “total declared value” of that allocation; and each advertiser would be charged the cost its presence imposes on the others’ total declared value. This approach is detailed in Subsection 4.1.
- The second approach is inspired by the Generalized Second Price mechanism, which is extensively applied in sponsored search auctions [10, 11]: as for VCG, the allocation is the most efficient one (in the sense of declared values); but the price each displayed advertiser pays is here the minimum one she could have declared and would have earned her the same slot. This approach is detailed in Subsection 4.2.

#### 4.1 Applying VCG on each page

In VCG auctions, each bidder should declare how much she is willing to pay for all the possible outcomes. In our setting, each bidder is sensitive only to her allocated slot (if any) since our model assumes no interdependency among the view/click probabilities of jointly displayed ads.

Hence, the mechanism needs to know the willingness-to-pay of each bidder for all slots, which depend on the page characteristics  $(\vec{\alpha}, \beta)$ . As in the one-slot case, it is sufficient that each bidder  $j$  declares her per-view valuation  $b_j^v$  and per-click valuation  $b_j^c$ . Therefore the mechanism can simply ask each bidder  $j$  to declare a per-view bid  $\tilde{b}_j^v$  and a per-valuation bid  $\tilde{b}_j^c$ .

Vickrey-Clarke-Groves mechanisms are well-known for having three desirable properties:

- *Incentive Compatibility*, meaning that bidding truthfully is a dominant strategy (i.e., the best each bidder could do, whatever the others’ actions);
- *Efficiency*, meaning that the equilibrium (in dominant strategies) of bidding strategies leads to a value-maximizing allocation;
- *Individual Rationality*, meaning that each bidder gets a non-negative utility (i.e., one never loses by entering the game).

In practice, once the bid values  $(\tilde{b}_j^v, \tilde{b}_j^c)$  are collected from all bidders  $j$ , the mechanism follows the same method, independently, on each webpage, as described below.

1. The slots  $1, \dots, m$  are allocated in order to maximize the declared valuation. More specifically, the advertisers  $(\pi(s))_{1 \leq i \leq m}$ , where  $\pi(i)$  is the advertiser whose ad is displayed in slot  $i$ , are a solution of

$$\arg \max_{\pi(1), \dots, \pi(m)} \sum_{i=1}^m \alpha_i \left( \tilde{b}_{\pi(i)}^v + \beta \text{CTR}_{\pi(i)} \tilde{b}_{\pi(i)}^c \right) \quad (9)$$

s.t.  $\pi(i) \neq \pi(i') \quad \forall i \neq i'$ .

This is not computationally hard: since slots are such that  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ , advertisers just need to be sorted by decreasing values of  $\tilde{b}_j^v + \beta \text{CTR}_j \tilde{b}_j^c$ , the  $i^{\text{th}}$  advertiser of that ranking being then displayed in slot  $i$  for  $i = 1, \dots, m$ . Remark that, as in the one-slot case, the ranking depends on the page only through the click-likelihood  $\beta$ .

2. VCG payments are so that each advertiser pays the loss of (declared) value her presence imposes on others: here only providers ranked lower than Advertiser  $j$  (and in the first  $m + 1$  positions) are affected by Advertiser  $j$ 's presence. Had Advertiser  $j$  not been here, each of them would be one slot ahead, hence the price paid by Advertiser  $j$  is

$$p_j^{\text{VCG}} = \sum_{k=s(j)+1}^{m+1} (\alpha_k - \alpha_{k+1}) \left( \tilde{b}_{j(k)}^v + \beta \text{CTR}_{j(k)} \tilde{b}_{j(k)}^c \right),$$

where  $s(j)$  is the slot allocated to Advertiser  $j$  on this page.

## 4.2 Using Generalized Second Price

In GSP, the allocation rule is the same as in VCG—hence, determined by (9)—, but the pricing rule is different, following a “next-price” principle [1]: Advertiser  $j$  pays the minimum bid (actually, the declared valuation) she could have bid while keeping the same ad slot, i.e., her price depends on the bid of the advertiser ranked just after her. Denoting by  $s(j)$  the slot of Advertiser  $j$  and by  $\pi(s(j)+1)$  the advertiser ranked just below her, we have an average price per display

$$p_j^{\text{GSP}} = \alpha_{s(j)} \left( b_{\pi(s(j)+1)}^v + \beta \text{CTR}_{\pi(s(j)+1)} b_{\pi(s(j)+1)}^c \right),$$

which can be decomposed in an *impression plus click* manner, with a price per impression

$$p_j^{\text{GSP}^{\text{imp}}} = \alpha_{s(j)} b_{\pi(s(j)+1)}^v$$

and an additional price per click

$$p_j^{\text{GSP}^{\text{click}}} = \frac{\text{CTR}_{\pi(s(j)+1)}}{\text{CTR}_j} b_{\pi(s(j)+1)}^c.$$



Indeed, on a visited webpage  $(\alpha, \beta)$  such a pricing yields an average (per visit) payment due to clicks

$$\begin{aligned} \mathbb{E}[p_j^{\text{GSPclick}} \mathbb{1}_{\{\text{ad clicked}\}}] &= \alpha_{s(j)} \beta \text{CTR}_j p_j^{\text{GSPclick}} \\ &= \alpha_{s(j)} \beta \text{CTR}_{\pi(s(j)+1)} b_{\pi(s(j)+1)}^c. \end{aligned}$$

Despite not incentive compatible (see below for an illustration), this pricing scheme is more likely to be used than VCG for two reasons, as in traditional sponsored search auctions:

- it is claimed to be simpler to understand for advertisers (instead of depending on many bids, the price paid only depends on the next-high bid);
- it yields a larger revenue (still, with respect to VCG) to the slots owner [11].

### 4.3 A numerical example

Let us consider an example with  $m$  slots, for which  $N^c + 1$  advertisers compete: we assume  $N^c$  advertisers are interested in clicks, while one advertiser (Advertiser 0) is interested in views. For simplicity, we assume that

- all click-interested advertisers have the same click-through-rate
- the willingness-to-pay values for a click are linearly distributed among click-interested advertisers, between  $b_{\min}^c$  and  $b_{\max}^c > b_{\min}^c$ : Advertiser  $k$  is willing to pay a price  $b_{\min}^c + \frac{k-1}{N^c-1}(b_{\max}^c - b_{\min}^c)$
- the view probabilities are the same on all pages. More precisely, at each page the probability that an ad displayed in slot  $s$  is viewed is given in Table 1. Those numbers are taken from [3], and indicate the relative numbers of clicks on ranked results in a search-engine result page: even if the context is different here, this gives an idea of the influence of the position on the click likelihood.
- the page effect on the click likelihood is such that  $\beta$  is uniformly distributed on an interval of the form  $[0, \beta_{\max}]$ .

$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\alpha_5$	$\alpha_6$	$\alpha_7$	$\alpha_8$	$\alpha_9$	$\alpha_{10}$
0.364	0.125	0.095	0.079	0.061	0.041	0.038	0.035	0.03	0.022

Table 1: Values for  $\alpha_s$  used in the example.

Let us investigate a specific example, for which the parameter values are given in Table 2.

We compare in Figures 2-4 our 2D-auction scheme—with VCG or GSP pricing—to a 1D-auction scheme, which can be seen as the currently used pay-per-click

$m$	$N^c$	$\text{CTR}_k, k > 0$	$\text{CTR}_0$	$b_{\min}^c$	$b_{\max}^c$	$b_0^v$	$\beta$
5	6	0.1	0.05	10	100	1	$\mathcal{U}[0, 0.3]$

Table 2: Parameter values for the  $m$ -slot scenario considered.

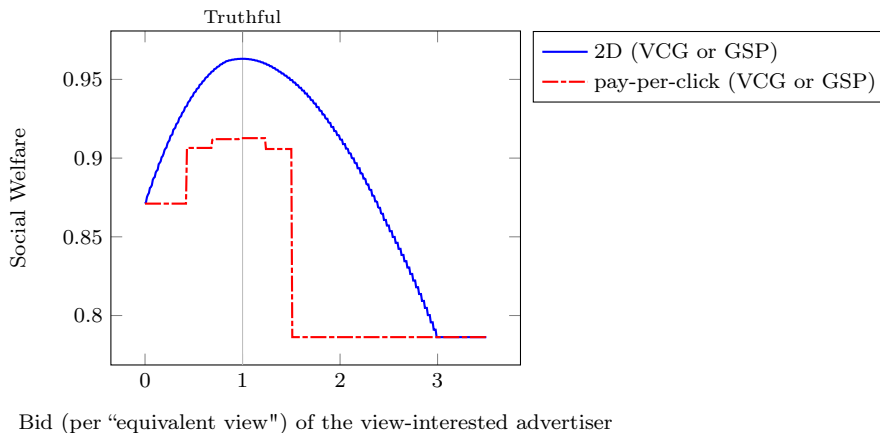


Figure 2: Average value extracted from the ad slots

(PPC) scheme—implementing also the VCG and GSP pricing rules. We vary the bid (per click) of the view-interested advertiser, assuming the other advertisers (which are all click-interested) truthfully declare their willingness-to-pay for a click. Figures 2 and 3 respectively show the per-display expected surplus (that is, the total expected value that slots have for advertisers) and the view-interested valuation for the resulting allocation. The allocation rules being the same for VCG and GSP, the social welfare curves in Figure 2 coincide for both pricing rules. Figure 3 on the other hand distinguishes both schemes, and illustrates the incentive-compatibility property of the VCG pricing rule (both for 1D and 2D auctions): bidding truthfully maximizes the view-interested advertiser utility. Also, bidding truthfully ensures a non-negative utility (individual rationality property of the VCG scheme). Finally, both for 1D and 2D auction schemes, the reachable social welfare is maximized when bidders bid truthfully.

On the other hand, with GSP pricing truthfulness is not ensured: here Advertiser 0 (the view-interested advertiser) would be better off bidding below its actual (average) per-click valuation. However, determining the best bidding strategies with GSP pricing is a complex problem, that is beyond the scope of this paper: we rather refer to [5, 10, 13] for works related to this issue of optimal bidding when GSP is used.

Finally, Figure 4 shows the corresponding revenues. As stated before, if bidders bid truthfully, the GSP pricing rule yields a higher revenue than the VCG one. Somewhat counterintuitively, in this numerical experiment the 2D-

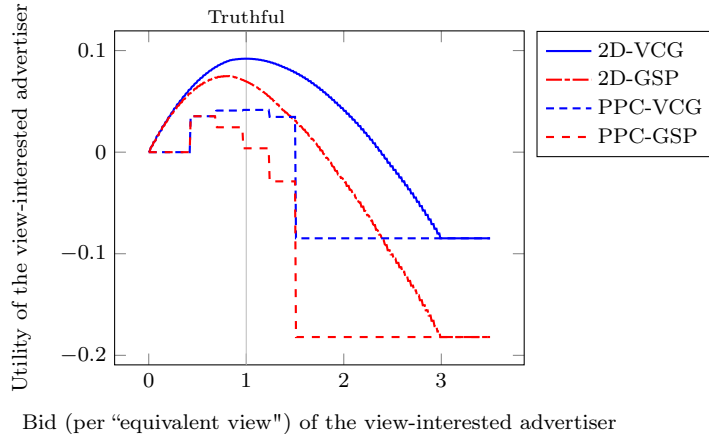


Figure 3: Average utility (value minus price) per visit for the view-interested advertiser

auction revenues are lower than the 1D-auction revenues: even if more value (the social welfare) is extracted from the ad slots in the 2D-case, GSP pricing favors the auctioneer to an extent that exceeds this difference since the auctioneer extracts more revenue in a GSP-1D auction than a VCG-2D auction. A direct conclusion is that in terms of utility (valuation minus prices), advertisers suffer twice from the current most applied scheme: they pay more (due to the GSP rule) and get less from being displayed (from the 1D-auction) than if a 2D-auction with VCG payments were applied.

## 5 Conclusions

We proposed in this paper an *impression plus click* pricing scheme for online advertisement slots and sponsored searches, which allows to optimally allocate the slots among view- and click-sensitive advertisers. Our scheme presents the advantage of being simple to implement, and efficient. We obtained bounds on the relative efficiency with respect to bids based on clicks or impressions only. In the multiple-slot case, we discussed the VCG implementation of the scheme, but also the GSP one, more likely to be considered since GSP is extensively used for online advertising even if not incentive compatible.

Future works could be in several directions. First, the case where advertisers focus on different (but overlapping) subsets of pages or searches, as highlighted in Section 2.3, is worth studying more deeply, by including it in the decision variables of advertisers. Second, we would like to extend our scheme to the case where CTRs are estimated, in a way to also generalize the work in [6, 7]. Finally, the case of budget limits for advertisers in ad campaigns is important to consider, but yields other difficulties since all page visits cannot be treated

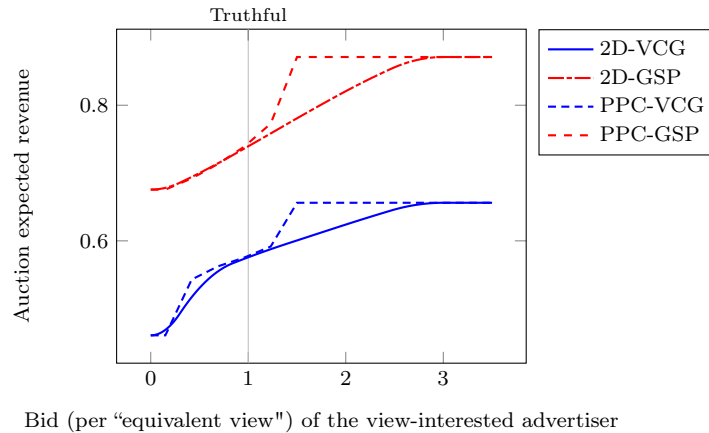


Figure 4: Auctioneer revenue for the different mechanisms

independently (the budget limits can be reached) [4].

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