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On nontraditional quasi-geostrophic equations

Carine Lucas
MAPMO UMR CNRS 7349
Fédération Denis Poisson FR CNRS 2964
Université d'Orléans, 45067 Orléans cedex 2, France
(Carine.Lucas@univ-orleans.fr)

James C. McWilliams
Dept. of Atmospheric and Oceanic Sciences
University of California, Los Angeles (UCLA)
Mathematical Sciences Building, Room 7983
Los Angeles, CA 90095-1565
(jcm@atmos.ucla.edu)

Antoine Rousseau
Inria and IMAG UMR CNRS 5159
Team LEM0N, 860 rue Saint-Priest, 34095 Montpellier Cedex 5, France
(Antoine.Rousseau@inria.fr)

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Abstract

In this article, we work on nontraditional models where the so-called *traditional* approximation on the Coriolis force is removed. In the derivation of the quasi-geostrophic equations, we obtain new terms in δ/ε, where δ (aspect ratio) and ε (Rossby number) are both small numbers. We provide here some rigorous crossed-asymptotics with regards to these parameters, prove some mathematical and physical results on the nontraditional models, and situate them among traditional ones.

Keywords: Ocean modeling, Coriolis force, traditional approximation, tilted quasi-geostrophic equations, slanted rotation.

1 Introduction

Equations used to model oceans rely on several hypotheses, depending on the underlying accuracy requirements in numerical simulations. One of these hypotheses is the so-called *traditional approximation*, which consists in neglecting the cosine part of the Coriolis force, modifying the zonal and vertical components of the momentum equations (see f^* terms in Equations (1a) and (1c)). This approximation has been widely discussed in the literature (see [7] and the correspondence in [17, 23, 18, 25]). Following for example [16, 27], the objective of this article is to study the relaxation of this assumption.

Discussions on the traditional approximation usually appear when studying oceans near the equator. Indeed, the Coriolis force expresses as $\overrightarrow{\Omega} \times U$ where U is the fluid velocity, and $\overrightarrow{\Omega} = {}^t(0, \cos\theta, \sin\theta)$, θ being the latitude. When modeling ocean in the equatorial zone, the latitude becomes smaller and the cosine part of the Coriolis force is not negligible compared to the sine part. This is the approach developed for example in [5], based on [24], or in [19], where the author refers to [28] to assert that the $\cos\theta$ Coriolis terms cannot be omitted.

In more general cases, taking into account this cosine effect leads to a second order correction in the equations, as already studied by the authors for the Shallow-Water equations in [14]. It can have a significant effect on the results depending on the time scale and the configuration of the experiment.

In this paper, we consider another situation. We are interested in the quasi-geostrophic equations, that can be obtained from the 3D primitive (hydrostatic) equations thanks to an asymptotic expansion with respect to the Rossby number. In this derivation, the ratio between the aspect ratio (usually small), and the Rossby number (also usually small) appears as the coefficient of the cosine terms. Consequently, a precise analysis is required to write a correct derivation of the nontraditional equations. The model we get is the same as Equation (4.11) obtained by a fast wave averaging approach in [8], known as the quasi-geostrophic equations for slanted rotation, and the tilted quasi-geostrophic equations cited by [12]. An existence result is proved and some physical properties are given, underlying the similitude with the traditional equations through a tilted vertical coordinate.

After this study, we set up a hierarchy of models summarized in Figure 1, showing that these so-called tilted quasi-geostrophic equations can be seen as an intermediate model between the non-hydrostatic equations and the usual three dimensional quasi-geostrophic equations. In particular, we prove that a solution to the nontraditional quasi-geostrophic equations converge, when the aspect ratio tends to zero, to the traditional quasi-geostrophic equations.

2 Quasi-geostrophic equations

The quasi-geostrophic equations are very familiar to oceanographers and meteorologists, they have been extensively used for modeling oceanic and atmospheric circulations [3, 2]. A way to obtain these equations is to perform an asymptotic expansion, with respect to the Rossby number, in the 3D primitive (hydrostatic) equations. The model is also very familiar to applied mathematicians, and several studies establishing the well-posedness of the corresponding boundary-value problem have been published (see *e.g.*, [1, 10, 15]). As we do not perform the traditional approximation of the Coriolis force, the primitive equations we start with are the quasi-hydrostatic ones (see [13] for a mathematical study).
Since the QG equations are obtained in the zero-limit of the Rossby number (large rotational effects), one could think of retaining all the rotating terms in the original equations, before performing the asymptotic analysis. This is the main purpose of this paper. We will see in (16) that the modified QG equations, that we will call quasi-hydrostatic quasi-geostrophic (QHQG) equations, are very similar to the traditional QG ones, except that they raise a new vertical direction (denoted Z hereafter), which differs from the traditional vertical

direction z, see also [12]. The tilt between z and Z is proportional to the nondimensional parameter λ introduced in (2.1.1), which mesures the ratio between traditional and nontraditional Coriolis terms. Experimental and numerical evidences of this tilted vertical direction can be found in [20, 26, 11, 21].

2.1 Derivation of the QHQG model

In this section we present the derivation of the quasi-hydrostatic quasi-geostrophic (QHQG) equations. The derivation follows classical principles (as in [1]): scaling, asymptotic expansion with respect to a small parameter, equations satisfied at order zero and one. Here, the small parameter (denoted ε in the sequel) is the Rossby number, so that we underline the effect of rotating terms. In order to account for the complete Coriolis force (see *e.g.*, [13] and references therein), we retain all the rotating terms in the original equations, including the terms that are usually neglected in the so-called *traditionnal approximation*. This denomination was introduced by Carl Eckart [7].

2.1.1 Scaling Parameters and Scaled Equations

We consider a three-dimensional domain with periodic boundary conditions in the horizontal directions, rigid lid and flat bottom in the vertical. The governing non-hydrostatic equations, including the complete Coriolis force, read:

$$\frac{Du}{Dt} - fv + f^* w = -\frac{\partial \varphi}{\partial x}, \tag{1a}$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \varphi}{\partial y}, \tag{1b}$$

$$\frac{Dw}{Dt} - f^* u + \frac{g\,\varrho}{\varrho_0} = -\frac{\partial \varphi}{\partial z}, \tag{1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1d}$$

$$\frac{D\varrho}{Dt} = 0. \tag{1e}$$

Here (u, v, w) and ϱ are respectively the three-dimensional velocity and density of the fluid, ϱ_0 stands for the averaged density of the fluid, and φ is the renormalized pressure, $\varphi = p/\varrho_0$. The scalars $f = 2\Omega \sin(\theta)$ and $f^* = 2\Omega \cos(\theta)$ are the Coriolis parameters where Ω stands for the angular velocity of the earth and θ is the latitude; g is the universal gravity constant.
Equations (1a) to (1c) describe the conservation of momentum, where D/Dt is the material derivative $D/Dt = \partial/\partial_t + u\partial_x + v\partial_y + w\partial_z$, and Equation (1d) corresponds to the conservation of mass. Finally, Equation (1e) describes the advection of tracers (here the density ϱ). The density and the pressure may be classically decomposed as

$$\varrho(x, y, z, t) = \overline{\rho}(z) + \rho(x, y, z, t)$$

and

$$\varphi(x, y, z, t) = \overline{\varphi}(z) + \phi(x, y, z, t),$$

where $\overline{\rho}$ and $\overline{\varphi}$ are the (known) background density and potential, depending only on the vertical variable. The functions $\overline{\rho}$ and $\overline{\varphi}$ are in hydrostatic balance.

We also denote by $N^2(z) = -\frac{d\overline{\rho}}{dz}(z)$ the buoyancy frequency, assuming that $\frac{d\overline{\rho}}{dz}$ is bounded away from zero.

Before going further in the derivation of the corresponding QG model, let us insist on the fact that we keep in Equations (1a) and (1c) the Coriolis terms f^*w and f^*u. We will finally lead to a slightly modified QG model (see Equation (14)). We think that it is a relevant modification, since the QG approximation aims at underlying the earth's rotation effects: one should thus include every rotation term in the primitive equations prior to an asymptotic expansion with respect to the Rossby number.
In the context of the β-plane approximation (see [6] for example), as suggested in [11] we have, with θ_0 the average latitude and R_e the mean radius of the Earth:

$$f = 2\Omega \sin\theta_0 + \frac{1}{R_e} 2\Omega \cos\theta_0 y,$$
$$f^* = f_0^* = 2\Omega \cos\theta_0.$$

We now introduce the following dimensionless variables, as it is classically done in QG modeling:

$$(x, y) = L\,(x', y'), \qquad z = H\,z', \qquad t = \frac{L}{U} t',$$
$$u = U\,u', \qquad v = U\,v', \qquad w = \frac{UH}{L} w',$$
$$\overline{\rho} = P\overline{\rho}', \qquad \rho = \frac{\varrho_0 f_0 U L}{gH}\rho', \qquad \phi = f_0 U L \phi'.$$

The Rossby number $\varepsilon = U/f_0 L$ is the fundamental ordering parameter in the following asymptotic expansion. A secondary ordering parameter is the scale ratio of ρ to ϱ: this ratio is assumed to be ε, that is[1], we assume:

$$\frac{\varrho_0 f_0 U L}{gH} = P\varepsilon.$$

Finally, the density may be expressed in terms of nondimensionnal quantities:

$$\varrho = P\Big(\overline{\rho}'(z) + \varepsilon\rho'\Big),$$

as well as the Coriolis parameter:

$$f = f_0(1 + \varepsilon\beta_0 y') \quad \text{with } f_0 = 2\Omega\sin\theta_0, \text{ and } \beta_0 = \frac{2\Omega L^2 \cos\theta_0}{U R_e}.$$

Another usual nondimentional number is the aspect ratio $\delta = H/L$; δ also appears in the ratio between the two Coriolis terms in the zonal momentum equation (1a), which scales as

$$\lambda = \delta \cot\theta_0.$$

When we considered the scaling numbers introduced above, we have implicitely assumed that the leading term at the left-hand-side of Equation (1a) was fv, which means that $\delta\cot\theta_0$ should not be too large:

$$\lambda \lesssim 1. \tag{2}$$

[1]This assumption correspond to a small Froude number, of order ε.

Fortunately, because the aspect ratio $\delta = H/L$ is rather small in large ocean models, the condition (2) is easily satisfied.

We end this section with the non-hydrostatic scaled equations (we naturally drop the primes):

$$\frac{Du}{Dt} - \left(\frac{1}{\varepsilon} + \beta_0 y\right) v + \frac{\lambda}{\varepsilon} w = -\frac{1}{\varepsilon}\frac{\partial \phi}{\partial x}, \tag{3a}$$

$$\frac{Dv}{Dt} + \left(\frac{1}{\varepsilon} + \beta_0 y\right) u = -\frac{1}{\varepsilon}\frac{\partial \phi}{\partial y}, \tag{3b}$$

$$\delta^2 \frac{Dw}{Dt} - \frac{\lambda}{\varepsilon} u + \frac{\rho}{\varepsilon} = -\frac{1}{\varepsilon}\frac{\partial \phi}{\partial z}, \tag{3c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3d}$$

$$\frac{D\rho}{Dt} + \frac{w}{\varepsilon}\overline{\rho}_z = 0, \tag{3e}$$

where we recall that ε is the Rossby number (meant to go to zero), $\delta = H/L$ is the domain aspect ratio, and $\lambda = \delta \cot \theta_0$.

Remark 1. *Actually, the term $\delta^2 Dw/Dt$ could be set to zero in* (3c) *above with no modification in the sequel: indeed the reader will see below that $w^{(0)} = 0$, hence $\delta^2 Dw/Dt = O(\varepsilon)$ can be neglected prior to the QG approximation.*
The final model is thus called QHQG since the differences between the new QG model and the traditional one rely only on the terms related to the Coriolis force.

The objective of the present work is to draw the reader's attention on the fact that the ratio $\lambda/\varepsilon = \cot\theta_0\ \delta/\varepsilon$ is not *necessarily* small compared to $1/\varepsilon$, and that it may have some physical repercussions (see Section 2.3).

2.1.2 Geostrophic Balance

We now consider an asymptotic expansion of all variables with respect to the Rossby number: for every unknown function γ, we write the formal asymptotic expansion

$$\gamma = \gamma^{(0)} + \varepsilon\gamma^{(1)} + \varepsilon^2\gamma^{(2)} + ...$$

where $(\gamma^{(j)})_{j\geq 0}$ behave as $O(1)$ as ε goes to zero. Equations (3a)-(3c) give, keeping only the leading order terms in ε:

$$-v^{(0)} + \lambda w^{(0)} = -\phi_x^{(0)}, \tag{4a}$$

$$u^{(0)} = -\phi_y^{(0)}, \tag{4b}$$

$$-\lambda u^{(0)} + \rho^{(0)} = -\phi_z^{(0)}. \tag{4c}$$

The incompressibility condition reads $w_z^{(0)} = -u_x^{(0)} - v_y^{(0)}$ and this *traditionally* leads to $w^{(0)} = 0$, thanks to Equations (4a), (4b) and boundary conditions on w (see [1]). Here, the incompressibility condition does not provide $w_z^{(0)} = 0$, but

we have, denoting $\partial_Z = \partial_z + \lambda \partial_y$:

$$\begin{aligned} w_Z^{(0)} &= w_z^{(0)} + \lambda w_y^{(0)} \\ &= -u_x^{(0)} - v_y^{(0)} + \lambda w_y^{(0)} \\ &= \mathrm{curl}(\phi_y, \phi_x) \\ w_Z^{(0)} &= 0. \end{aligned} \tag{5}$$

Thanks to (5) and to homogeneous boundary conditions on $w^{(0)}$, we finally obtain that $w^{(0)} = 0$. Alternatively, we have Equation (3e) which (written to the order zero and since $\overline{\rho}_z$ never vanishes) leads to $w^{(0)} = 0$.

The geostrophic equations read:

$$-v^{(0)} = -\phi_x^{(0)}, \tag{6a}$$
$$u^{(0)} = -\phi_y^{(0)}, \tag{6b}$$
$$\rho^{(0)} = -\phi_z^{(0)} - \lambda \phi_y^{(0)} = -\phi_Z^0, \tag{6c}$$
$$w^{(0)} = 0. \tag{6d}$$

2.1.3 Quasi-Geostrophic Equations

Now we need the first order equations in order to determine the evolution of $\phi^{(0)}$. We denote by d_g the zero-order material derivative:

$$d_g = \partial_t + u^{(0)} \partial_x + v^{(0)} \partial_y.$$

The first order equations are:

$$d_g u^{(0)} - \beta_0 y v^{(0)} - v^{(1)} + \lambda w^{(1)} = -\phi_x^{(1)}, \tag{7a}$$
$$d_g v^{(0)} + \beta_0 y u^{(0)} + u^{(1)} = -\phi_y^{(1)}, \tag{7b}$$
$$-\lambda u^{(1)} + \rho^{(1)} = -\phi_z^{(1)}, \tag{7c}$$
$$u_x^{(1)} + v_y^{(1)} + w_z^{(1)} = 0, \tag{7d}$$
$$d_g \rho^{(0)} + w^{(1)} \overline{\rho}_z = 0. \tag{7e}$$

We now take the curl of Equations (7a)-(7b) to obtain, thanks to Equation (7d)

$$d_g(v_x^{(0)} - u_y^{(0)}) - w_z^{(1)} - \lambda w_y^{(1)} + \beta_0 v^{(0)} = 0. \tag{8}$$

We notice, as for the traditional QG equations, that $\beta_0 v^{(0)} = d_g(\beta_0 y)$. We thus try to express $-w_z^{(1)} - \lambda w_y^{(1)} = -w_Z^{(1)}$ as $d_g(\Gamma)$ where Γ is a function to be defined. To this aim, we will extensively make use of Equation (7e) that we reformulate:

$$w^{(1)} = N^{-2} d_g \rho^{(0)} = d_g \left(N^{-2} \rho^{(0)} \right). \tag{9}$$

Given (9), we may compute the required quantity

$$w_z^{(1)} + \lambda w_y^{(1)} = \left(d_g \left(N^{-2} \rho^{(0)} \right) \right)_z + \lambda \left(d_g \left(N^{-2} \rho^{(0)} \right) \right)_y. \tag{10}$$

We remark that for any function γ and any variable $*$ we have the identity

$$(d_g(\gamma))_* = d_g(\gamma_*) + u_*^{(0)}\partial_x\gamma + v_*^{(0)}\partial_y\gamma,$$

so that we can write Equation (10)

$$w_z^{(1)} + \lambda w_y^{(1)} = d_g\left(\left(N^{-2}\rho^{(0)}\right)_z\right) + \lambda d_g\left(\left(N^{-2}\rho^{(0)}\right)_y\right) + R, \tag{11}$$

where the remainder R, according to the remark above, is

$$\begin{aligned} R &= u_z^{(0)}(N^{-2}\rho^{(0)})_x + v_z^{(0)}(N^{-2}\rho^{(0)})_y \\ &\quad + \lambda u_y^{(0)}(N^{-2}\rho^{(0)})_x + \lambda v_y^{(0)}(N^{-2}\rho^{(0)})_y \\ &= (N^{-2}\rho^{(0)})_x\,(u_z^{(0)} + \lambda u_y^{(0)}) + (N^{-2}\rho^{(0)})_y\,(v_z^{(0)} + \lambda v_y^{(0)}). \end{aligned}$$

Using (6a)-(6c) again, we have

$$R = N^{-2}\rho_x^{(0)}\rho_y^{(0)} - N^{-2}\rho_y^{(0)}\rho_x^{(0)} = 0,$$

which simplifies Equation (11) as follows:

$$w_Z^{(1)} = w_z^{(1)} + \lambda w_y^{(1)} = d_g\left(\left(N^{-2}\rho^{(0)}\right)_z + \lambda\left(N^{-2}\rho^{(0)}\right)_y\right). \tag{12}$$

Back to Equation (8), we obtain the quasi-hydrostatic quasi-geostrophic equation:

$$d_g\Big(v_x^{(0)} - u_y^{(0)} - (N^{-2}\rho^{(0)})_z - \lambda(N^{-2}\rho^{(0)})_y + \beta_0 y\Big) = 0. \tag{13}$$

The potential vorticity

$$v_x^{(0)} - u_y^{(0)} - (N^{-2}\rho^{(0)})_z - \lambda(N^{-2}\rho^{(0)})_y + \beta_0 y$$

is thus conserved along material paths.

Let us now rewrite Equation (13), expressing everything in terms of $\phi^{(0)}$. We have

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big)\zeta = 0, \tag{14}$$

$$\zeta = \Delta\phi^{(0)} + N^{-2}(\partial_z + \lambda\partial_y)^2\phi^{(0)} + N_z^{-2}(\partial_z + \lambda\partial_y)\phi^{(0)} + \beta_0 y, \tag{15}$$

where Δ is horizontal Laplacian operator. We could write this more compactly with $\partial_Z = \partial_z + \lambda\partial_y$,

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big)\Big(\Delta\phi^{(0)} + (N^{-2}\phi_Z^{(0)})_Z + \beta_0 y\Big) = 0. \tag{16}$$

One can thus easily recognize the traditional QG equation (see Equation (2.23) in [1]), except that the differential operator ∂_z is replaced by the tilted one $\partial_Z = \partial_z + \lambda\partial_y$, see [12]. We recall here that $\lambda = \delta\cot\theta_0$ is proportional to the domain aspect ratio. In particular, we recover the traditional QG equation when setting $\delta = 0$ in Equations (13), (14) or (16). These equations are also called quasi-geostrophic equations for slanted rotation in [8].

2.2 Existence of solutions

In this section, we are interested in existence results for the quasi-hydrostatic quasi-geostrophic equation (14). We consider a periodic domain in the horizontal variables with rigid boundaries at the top and bottom. More precisely, we denote by $B = \Sigma \times (0,h) = \left(-\frac{1}{2},\frac{1}{2}\right) \times \left(-\frac{1}{2},\frac{1}{2}\right) \times (0,h)$ the spatial domain where we establish existence of solutions, with $t \in [0,T]$ (where $T > 0$) the time interval.

We assume the solutions to be horizontally periodic with period 1. The pressure potential $\phi^{(0)}$ is defined by Equation (16) up to a constant at each time; then we add the following condition: $\int_B \phi^{(0)} = 0$. We define $\mathbf{u} = (u^{(0)}, v^{(0)})$ the horizontal velocity at the leading order, we omit the (0) superscripts and rewrite Equation (14) as:

$$\begin{aligned} v &= \phi_x, && (17)\\ u &= -\phi_y, && (18)\\ \rho &= -\phi_z - \lambda\phi_y, && (19)\\ \omega &:= v_x - u_y - \left(N^{-2}\rho\right)_z - \lambda\left(N^{-2}\rho\right)_y \quad \text{in } B, && (20)\\ \omega_t + \mathbf{u}\cdot\nabla\omega &= -\beta_0 v \quad \text{in } B\times[0,T], && (21) \end{aligned}$$

with the initial condition

$$\omega(x,y,z,0) = \omega_0(x,y,z), \tag{22}$$

and the boundary conditions

$$\rho = 0 \quad \text{at } z = 0 \text{ and } z = h. \tag{23}$$

Existence results are based on the following property:

Lemma 1. *The partial differential operator $\mathcal{L}$ defined by*

$$\mathcal{L}\phi := \Delta\phi + \left(N^{-2}\left(\phi_z + \lambda\phi_y\right)\right)_z + \lambda\left(N^{-2}\left(\phi_z + \lambda\phi_y\right)\right)_y \ \textit{in } B$$

is uniformly elliptic.

Proof. The function $N^{-2}(z)$ is assumed to be bounded away from 0 for $z \in [0,h]$. We expand $\mathcal{L}$ as:

$$\begin{aligned} \mathcal{L}\phi &= \phi_{xx} + \left(1+\lambda^2N^{-2}\right)\phi_{yy} + N^{-2}\phi_{zz} + 2\lambda N^{-2}\phi_{yz} + \lambda\left(N^{-2}\right)_z\phi_y + \left(N^{-2}\right)_z\phi_z\\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\partial^2_{i,j}\phi + \sum_{i=1}^{3} b_i\partial_i\phi, \end{aligned}$$

where:

$$A = (a_{ij})_{i,j=1\ldots3} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1+\lambda^2N^{-2} & \lambda N^{-2}\\ 0 & \lambda N^{-2} & N^{-2} \end{pmatrix}.$$

For all $\xi = (\xi_1, \xi_2, \xi_3)^\intercal$, we have, for almost every z in $[0, h]$:

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\xi_i\xi_j &= \xi_1^2 + \left(1 + \lambda^2 N^{-2}\right)\xi_2^2 + N^{-2}\xi_3^2 + 2\lambda N^{-2}\xi_2\xi_3 \\
&\geq \xi_1^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2\lambda^2 + N^2}\xi_3^2 \quad \text{(thanks to Young inequality)} \\
&\geq \theta|\xi| \quad \text{with } \theta := \min\left(\frac{1}{2}, \frac{1}{2\lambda^2 + N^2}\right) > 0.
\end{aligned}$$

□

The condition $\theta > 0$ is satisfied as soon as $\lambda > 0$. In the case where $\lambda = 0$ (traditional equation), one needs to assume that N^{-2} is bounded away from zero (see [1]).
Lemma 1 generalizes the ellipticity result of [1] for any $\lambda \in \mathbb{R}$ and we can now obtain the same existence results. Indeed, System (17)–(21) only differs from the corresponding system in [1] in Equations (19) and (20).

We first obtain the local existence of solutions:

Theorem 2.1 (Short-time existence of solutions to the QHQG model)**.** *If the initial vorticity ω_0 is in $H^s(B)$ for some $s \geq 3$ with $|\omega_0|_s \leq M$, then there exists a time $T^* > 0$ and a solution ω in $\mathcal{C}\left([0, T^*], H^s(B)\right)$ to the QHQG model, where T^* depends only on M, B, λ and β_0. The vorticity ω satisfies the estimate $\|\omega\|_{s,T^*} \leq 2M$.*

Proof. The proof follows the lines of [1], considering an iterative process:

- starting from $\xi^0(x, y, z, t) = \omega_0(x, y, z)$ given,
- for $k \geq 0$ compute ϕ^k thanks to the relation:

$$\begin{cases}
\phi^k_{xx} + \phi^k_{yy} + \left(\dfrac{\phi^k_z + \lambda\phi^k_y}{N^2}\right)_z + \dfrac{\lambda}{N^2}\left(\phi^k_z + \lambda\phi^k_y\right)_y = \xi^k \text{ in } B, \\
\phi^k_z + \lambda\phi^k_y = 0 \quad \text{for } z = 0 \text{ and } z = h, \\
\displaystyle\int_B \phi^k = 0,
\end{cases} \tag{24}$$

- then set $u^k = -\phi^k_y$ and $v^k = \phi^k_x$,
- for every $z \in [0, h]$, compute ξ^{k+1} solution of

$$\xi^{k+1}_t + \left(u^k, v^k\right)^\intercal \cdot \nabla\xi^{k+1} = -\beta_0 v^k,$$

  with $\xi^{k+1}(t = 0) = \omega_0$, with periodic boundary conditions in x and y.

The elliptic result (Lemma 1) still gives the estimate:

$$|\phi^k|_{s+2} \leq C_0|\xi^k|_s \quad \text{with } C_0 = C_0(B, N(z), \lambda) > 0$$

such that, Equations (17), (18) and (21) being unchanged, the short-time existence of the k-th vorticity iterate ξ^k follows, as well as the upper bound on its H^s norm.

The end of the proof, for the convergence of the iterates $\{\xi^k(t)\}_{k\geq 0}$, is not modified by the λ coefficient. □

We are also able to adapt the proof of global solutions for the QG model by [1] to get:

Theorem 2.2 (Global existence of solutions to the QHQG model)**.** *If ω_0 is in $H^s(B)$ for some $s \geq 3$, then given any time $T > 0$, there exists a solution ω in $\mathcal{C}([0,T], H^s(B))$ to the QHQG model.*

Proof. From Lemma 1, relations (17), (18), (19) and from

$$\begin{cases} \phi_{xx} + \phi_{yy} + \left(\dfrac{\phi_z + \lambda\phi_y}{N^2}\right)_z + \dfrac{\lambda}{N^2}\left(\phi_z + \lambda\phi_y\right)_y = \omega \text{ in } B, \\ \phi_z + \lambda\phi_y = 0 \quad \text{for } z = 0 \text{ and } z = h, \\ \displaystyle\int_B \phi = 0, \end{cases}$$

we have the elliptic estimate:

$$|\mathbf{u}|_{s+1} + |\rho|_{s+1} \leq C|\omega|_s,$$

such that we also have a global estimate on the QHQG vorticity ω of the form

$$|\omega(t)|_s \leq K(t) \quad \forall t \in [0,T],$$

where the function K only depends on the H^s norm of the initial condition ω_0. Thanks to the short-time existence, the global existence can be proved, using the global bound

$$|\omega(t)|_s \leq \max_{t\in[0,T]} K(t) \quad \forall t \in [0,T]$$

to reach any time T thanks to an iterative process. □

2.3 Simple physical properties of the QHQG model

We detail here some basic physical properties that can be obtain for the QHQG model and we perform comparisons with the well-known QG system.

We reformulate the QHQG equation (16) as follows:

$$D[q + \beta y] = 0, \tag{25}$$

$$\Big(\Delta + \partial_Z[\frac{1}{N^2}\partial_Z]\Big)\phi = q, \tag{26}$$

where $D = \partial_t + J[\phi,\ ]$, $J[a,b] = a_x b_y - a_y b_x$, $\Delta = \partial_x^2 + \partial_y^2$, $\partial_Z = \partial_z + \lambda\partial_y$, and $\lambda = \dfrac{H}{L}\cot\theta_0$.

2.3.1 Coordinate transformation

We define a coordinate transform from $(\mathbf{x}, t) = (x, y, z, t)$ to $(\mathbf{X}, T) = (X, Y, Z, T)$ by

$$X = x, \quad Y = y - \lambda z, \quad Z = z, \quad T = t. \tag{27}$$

The transformed QHQG system is isomorphic to (26) because $D = \partial_T + J[\phi,\ ]$, $\Delta = \partial_X^2 + \partial_Y^2$, and $J[a,b] = a_X b_Y - a_Y b_X$ have identical functional forms in the transformed coordinates; $N^2(z) = N^2(Z)$; and $\beta y = \beta(Y + \lambda Z)$ can be replaced by βY because $D[\lambda Z] = 0$. Thus, all QG solutions $\phi(\mathbf{x}, t)$ are also QHQG solutions $\phi(\mathbf{X}, T)$ if the initial and boundary conditions are consistent. Initial conditions are equivalently specified at $t = 0$ or $T = 0$. Planar vertical boundary surfaces at $z = z_0$ are also planar at $Z = z_0$. The same is true in x and X. Only in y is a planar surface in y no longer planar in Y; therefore, the simple equivalence is for solutions that have an unbounded y domain, or are y-periodic, or decay away in y before the boundary.

2.3.2 Separable solutions and vertical modes

QHQG has vertically separable solutions when linearized by neglecting $J[\phi, q]$, even for general $N(z)$, $\phi = \Phi(X, Y, T)F(Z)$. (QG has an analogous property.) Such QHQG solutions have a vertically upward phase tilt in the (y, z) plane with a slope of $dz/dy = \lambda^{-1}$ relative to the z axis. This direction is aligned with the full rotation vector. Many phenomena influenced by the non-traditional approximation (*e.g.*, convection and centrifugal instability as reviewed in [11]) are known to exhibit this type of phase tilt. If we assume constant density at the vertical boundaries, F is determined by the 1D eigenvalue problem,

$$(N^{-2}F_Z)_Z + \mathcal{R}^{-2}F = 0\,, \qquad F_Z = 0 \text{ at } Z = Z_0, Z_1\,. \tag{28}$$

The eigenvalue $\mathcal{R}$ is the horizontal radius of deformation. With N constant, the eigenmodes F are cosine functions in Z.

2.3.3 Thermal wind balance

The geostrophic equations (6a)-(6c) have a steady solution for a zonal flow. This can be expressed as $u = U(Y, Z)$, $\varrho = -\int^Z N^2(Z')\,dZ' + B(Y, Z)$ with $U_Z = -B_Y$, which is thermal wind balance.

2.3.4 Rossby wave modes

One class of simple solutions is Rossby wave modes. They satisfy a linearized PDE usually justified by an assumption of small amplitude flow,

$$\frac{\partial q}{\partial t} + \beta\frac{\partial \phi}{\partial x} = 0. \tag{29}$$

With constant N in either a vertically bounded or unbounded domain, eigenmodes are

$$\phi \propto e^{i(kx+ly+mz-\omega t)}, \tag{30}$$

with a dispersion relation,

$$\omega = \frac{-\beta k}{K^2}, \quad K^2 = k^2 + l^2 + \frac{1}{N^2}(m + \lambda l)^2. \tag{31}$$

We can equivalently write this as

$$\phi \propto e^{i(kX+lY+MZ-\omega T)}, \tag{32}$$

with $M = m + \lambda l$.

Thus, $\lambda \neq 0$ implies lower frequency and slower phase speed compared to QG Rossby waves with the same wavenumber (k, l, m). We also consider the group velocity for wave energy propagation, $c_g = \partial_k \omega$. Its components are

$$\begin{aligned} \partial_k \omega &= \frac{\beta(2k^2 - K^2)}{K^4}, \\ \partial_l \omega &= \frac{2\beta k(l + \lambda(m + \lambda l)/N^2)}{K^4}, \\ \partial_m \omega &= \frac{2\beta k(m + \lambda l)/N^2}{K^4}. \end{aligned} \tag{33}$$

This usually has a more westward zonal propagation due to $\lambda \neq 0$, and its (y, z) propagation is altered as well. These effects also occur for the barotropic mode $(m = 0)$ where the vertical propagation is nonzero because of the QH correction.

2.3.5 Vortex solutions

A simple vortex solution is a nonlinear stationary state when $\beta = 0$ and N is constant. In QG this occurs for any axisymmetric profile, $\phi(r, z)$, where $r = \sqrt{x^2 + y^2}$ is the radial coordinate, which is sufficient to make $J[\phi, q] = 0$ even when the arguments have large amplitude. Physical interest usually lies in profiles that are localized in r, *e.g.*, a monopole with $\phi(r)$ decaying away from a central extremum. In QHQG nonlinear stationary solutions exist for any profile $\phi(R, Z)$, where $R^2 = X^2 + Y^2$. Thus, QH vortices are meridionally tilted rather than verically aligned. This tilted structure was previously proposed for Meddy vortices by [22].

2.3.6 Fourier-space estimates for non-traditional effects

Assuming a wavenumber-space characterization of the solution, as commonly done for turbulent flows, in terms of vertical and horizontal wavenumbers, kv and kh, or equivalently local values of H and L, we can ask when the λ value is not small.

In QG the common view (sometimes called Charney's stretched isotropy) is that the Burger number, $Bu = NH/2\Omega L = kh/(2\Omega kv/N)$, is order one while $2\Omega/N$ is small. If $H/L \sim 2\Omega/N \ll 1$, then $\lambda \sim (2\Omega/N)\cot\theta_0$, which will be small except when θ_0 is very close to the Equator.

Alternatively, if we consider flow patterns with $2\Omega kv/Nkh \sim r \ll 1$, then $\lambda \sim (2\Omega/Nr)\cot\theta_0$. For small enough r, λ need not be small, and the QH correction to QG will be important. This happens when the aspect ratio H/L is large, the stratification is weak, and/or θ_0 is small. Large aspect ratio can be described from the QG perspective as atypically "tall" flows.

3 The QHQG model

The objective of this section is to show that the quasi-hydrostatic quasi-geostrophic equation (14) is an intermediate model between the non-hydrostatic equations and the usual (traditional) quasi-geostrophic equation, namely Equation (14) with $\lambda = 0$.

3.1 Hierarchy of models

We can represent these models on the diagram of Figure 1.

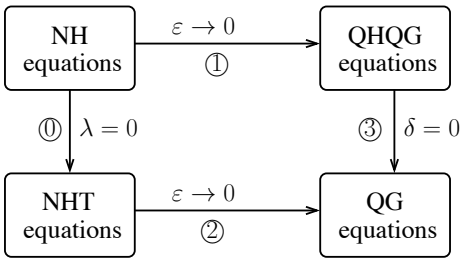

Figure 1: Non-hydrostatic equations and quasi-geostrophic equations: models with and without nontraditional terms. NH: non-hydrostatic equations, see (1); NHT: non-hydrostatic traditional equations, see (1) with $\lambda = 0$; QHQG: quasi-hydrostatic quasi-geostrophic equation, see (14); QG: (traditional) quasi-geostrophic equation, see (14) with $\lambda(= \delta \cot \theta_0) = 0$.

The convergence of NH towards QHQG (see ① in Figure 1) is proved in [8]: based on techniques developed in [9], the authors prove that the solution of NH weakly converges, when ε tends to zero, to a limit that satisfies QHQG. The convergence cannot be in the strong sense due to fast oscillations that can appear in NH and that disappear in QHQG.

The convergence of NHT (that are exactly NH with $\lambda = 0$) to QG (that is ② in Figure 1) have been studied in several papers; let us mention [1] in a periodic domain in the horizontal variables, with regular initial data, or in [4] in a three dimensional periodic domain, for ill prepared data.

In the following of this section, we are interested in the right part ③ of the diagram 1: we prove that a solution of QHQG converges to a solution of QG when δ (and consequently λ) tends to zero.

3.2 Convergence as δ (or λ) tends to zero

We are interested in the limit when the aspect ratio δ tends to zero, which means that λ also tends to zero. As we will prove this convergence in the case of a viscous fluid, we start with the formulation of the viscous QHQG equations.

3.2.1 Viscous QHQG

Starting from the viscous non-hydrostatic equations:

$$\begin{aligned}
\varepsilon\frac{Du}{Dt} - (1+\varepsilon\beta_0 y)v + \lambda w &= -\frac{\partial\phi}{\partial x} + \varepsilon\nu_1\Delta u,\\
\varepsilon\frac{Dv}{Dt} + (1+\varepsilon\beta_0 y)u &= -\frac{\partial\phi}{\partial y} + \varepsilon\nu_1\Delta v,\\
\varepsilon\delta^2\frac{Dw}{Dt} - \lambda u + \rho &= -\frac{\partial\phi}{\partial z} + \varepsilon\nu_2\Delta w,\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,\\
\varepsilon\frac{D\rho}{Dt} + w\overline{\rho}_z &= \varepsilon\nu_3\Delta\rho,
\end{aligned}$$

we perform exactly the same derivation as in Section 2.1, with the four additional viscous terms. We obtain:

$$\left(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\right)\zeta = \nu_1\Delta_h\Delta\phi^{(0)} + \nu_3\left(\partial_z + \lambda\partial_y\right)\left(N^{-2}\Delta\left(\phi_z^{(0)} + \lambda\phi_y^{(0)}\right)\right),$$

with

$$\zeta = \Delta_h\phi^{(0)} + N^{-2}(\partial_z + \lambda\partial_y)^2\phi^{(0)} + N_z^{-2}(\partial_z + \lambda\partial_y)\phi^{(0)} + \beta_0 y,$$

where Δ_h is horizontal Laplacian operator. This equation rewrites:

$$\begin{aligned}
&\left(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\right)\left(\Delta_h\phi^{(0)} + (N^{-2}\phi_Z^{(0)})_Z + \beta_0 y\right)\\
&\qquad = \nu_1\Delta_h\Delta\phi^{(0)} + \nu_3\left(N^{-2}\Delta\phi_Z^{(0)}\right)_Z, \quad (35)
\end{aligned}$$

with $\partial_Z = \partial_z + \lambda\partial_y$. We still supplement this equation with a given initial condition $\phi_0^{(0)}$ and with the following boundary conditions:

$$\phi_Z^{(0)} = 0 \text{ on the top and bottom,}$$

and we will consider periodic conditions in x and y.

3.2.2 Convergence of QHQG to QG

In this part, we prove that the quasi-hydrostatic quasi-geostrophic model converges to its corresponding hydrostatic model when the aspect ratio δ tends to 0, that is, as $\lambda = \cot\theta_0\delta$, when the nontraditional Coriolis parameter λ tends to 0.

Let us denote by ϕ^λ the solution of the nontraditional (quasi-hydrostatic) quasi-geostrophic model (35) for a fixed λ:

$$\begin{aligned}
&\left(\partial_t - \phi_y^\lambda\partial_x + \phi^\lambda x\partial_y\right)\left(\Delta_h\phi^\lambda\right.\\
&\qquad \left. + N^{-2}(\partial_z + \lambda\partial_y)^2\phi^\lambda + N_z^{-2}(\partial_z + \lambda\partial_y)\phi^\lambda + \beta_0 y\right)\\
&\qquad\qquad = \nu_1\Delta_h\Delta\phi^\lambda + \nu_3\left(N^{-2}\Delta\phi_Z^\lambda\right)_Z, \quad (36)
\end{aligned}$$

in the domain Ω, with periodic conditions in x and y, and verifying $\partial_Z \phi^\lambda = 0$ on the top ($z = h$) and bottom ($z = 0$). At $t = 0$, the function is given by ϕ_0^λ. We assume the data to be well-prepared, that is $\phi_0^\lambda \to \phi_0$ when $\lambda \to 0$.

We want to prove that ϕ^λ converges when $\lambda \to 0$ towards a function ϕ, solution of the following traditional (hydrostatic) quasi-geostrophic model:

$$\left(\partial_t - \phi_y^{(G)}\partial_x + \phi_x^{(G)}\partial_y\right)\left(\Delta_h \phi^{(G)} + N^{-2}\partial_z^2\phi^{(G)} + N_z^{-2}\partial_z\phi^{(G)} + \beta_0 y\right) = \nu_1 \Delta_h \Delta\phi^{(G)} + \nu_3\left(N^{-2}\Delta\phi_z^{(G)}\right)_z, \quad (37)$$

with $\partial_z \phi^{(G)} = 0$ on the top and bottom, and $\phi^{(G)}(t = 0) = \phi_0$.

***A priori* estimates for ϕ^λ.** The function ϕ^λ is solution of

$$\left(\partial_t - \phi_y^\lambda\partial_x + \phi_x^\lambda\partial_y\right)\left(\Delta_h\phi^\lambda + \left(N^{-2}\partial_Z\phi^\lambda\right)_Z + \beta_0 y\right) = \nu_1\Delta_h\Delta\phi^\lambda + \nu_3\left(N^{-2}\Delta\phi_Z^\lambda\right)_Z. \quad (38)$$

Let us multiply equation (38) by ϕ^λ and integrate over the periodic domain (in x and y) Ω:

$$\int_\Omega \partial_t\left(\Delta_h\phi^\lambda + \left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right)\phi^\lambda = \nu_1\int_\Omega \Delta_h\Delta\phi^\lambda\phi^\lambda + \nu_3\int_\Omega\left(N^{-2}\Delta\phi_Z^\lambda\right)_Z\phi^\lambda$$

(due to the properties of the jacobian), which leads to:

$$-\frac{1}{2}\frac{d}{dt}\left\|\nabla_h\phi^\lambda\right\|_2^2 - \frac{1}{2}\frac{d}{dt}\left\|N^{-1}\partial_Z\phi^\lambda\right\|_2^2 = \nu_1\left\|\Delta_h\phi^\lambda\right\|_2^2 + \nu_1\left\|\nabla_h\partial_z\phi^\lambda\right\|_2^2 + \nu_3\left\|N^{-1}\nabla\phi_Z^\lambda\right\|_2^2,$$

that reads:

$\nabla_h\phi^\lambda$ is bounded in $L^\infty(0,T,L^2(\Omega))$ if, $\forall\lambda > 0$, $\left(\nabla_h\phi^\lambda\right)(t = 0)$ is in $L^2(\Omega)$,

$\partial_Z\phi^\lambda$ is bounded in $L^\infty(0,T,L^2(\Omega))$ if, $\forall\lambda > 0$, $\left(\partial_Z\phi^\lambda\right)(t = 0)$ is in $L^2(\Omega)$,

and

$$\begin{aligned}\Delta_h\phi^\lambda &\text{ is bounded in } L^2(0,T,L^2(\Omega)),\\ \nabla_h\phi_z^\lambda &\text{ is bounded in } L^2(0,T,L^2(\Omega)),\\ \nabla\phi_Z^\lambda &\text{ is bounded in } L^2(0,T,L^2(\Omega)),\end{aligned}$$

as the Brunt-Väisälä frequency $N(z)$ is bounded (from above and away from zero).

Let us also the multiply equation (38) by $\Delta_h\phi^\lambda + \left(N^{-2}\partial_Z\phi^\lambda\right)_Z$ and integrate over Ω. We obtain:

$$\int_\Omega\left(\partial_t\left(\Delta_h\phi^\lambda + \left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right) + \beta_0\phi_x^\lambda\right)\left(\Delta_h\phi^\lambda + \left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right)$$
$$= \nu_1\int_\Omega\Delta_h\Delta\phi^\lambda\left(\Delta_h\phi^\lambda + \left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right) + \nu_3\int_\Omega\left(N^{-2}\Delta\phi_Z^\lambda\right)_Z\left(\Delta_h\phi^\lambda + \left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right),$$

that gives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\Delta_h\phi^\lambda+\left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right\|_2^2+\nu_1\left\|\nabla\Delta_h\phi^\lambda\right\|_2^2+(\nu_1+\nu_3)\left\|N^{-1}\Delta_h\phi_Z^\lambda\right\|_2^2\\ +(\nu_1+\nu_3)\left\|N^{-1}\nabla_h\partial_z\phi_Z^\lambda\right\|_2^2+\nu_3\left\|(N^{-2}\nabla\phi_Z^\lambda)_Z\right\|_2^2\\ =-\beta_0\int_\Omega\phi_x^\lambda\left(\Delta_h\phi^\lambda+\left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right)+(\nu_1+\nu_3)\int_\Omega\partial_z N^{-2}\partial_z\phi_Z^\lambda\Delta_h\phi_Z^\lambda.\end{aligned}$$

The r.h.s. can be bounded by $C_1+C_2\left\|\Delta_h\phi^\lambda+\left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right\|_2^2+C_3\left\|\Delta_h\phi_Z^\lambda\right\|_2^2$ using the previous results, where the C_i are constants, leading to:

$\Delta_h\phi^\lambda$ is bounded in $L^\infty(0,T,L^2(\Omega))$ if, $\forall\lambda>0$, $\left(\Delta_h\phi^\lambda\right)(t=0)$ is in $L^2(\Omega)$,

$\left(N^{-2}\partial_Z\phi^\lambda\right)_Z$ is bounded in $L^\infty(0,T,L^2(\Omega))$ if, $\forall\lambda>0$, $\left(\left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right)(t=0)$ is in $L^2(\Omega)$,

$N^{-1}\partial_Z\nabla_h\phi^\lambda$ is bounded in $L^\infty(0,T,L^2(\Omega))$ if, $\forall\lambda>0$, $\left(N^{-1}\partial_Z\nabla_h\phi^\lambda\right)(t=0)$ is in $L^2(\Omega)$,

and

$$\begin{gathered}\nabla\Delta_h\phi^\lambda \text{ is bounded in } L^2(0,T,L^2(\Omega)),\\ \partial_Z\Delta_h\phi^\lambda \text{ is bounded in } L^2(0,T,L^2(\Omega)),\\ \left(N^{-2}\nabla\phi_Z^\lambda\right)_Z \text{ is bounded in } L^2(0,T,L^2(\Omega)),\\ \partial_{zZ}^2\nabla_h\phi^\lambda \text{ is bounded in } L^2(0,T,L^2(\Omega)).\end{gathered}$$

From equality (38), thanks to the previous properties, we also have:

$$\partial_t\left(\Delta_h\phi^\lambda+\left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right) \text{ is bounded in } L^2(0,T,H^{-1}(\Omega)).$$

Convergence when $\lambda\to 0$. From the previous results, the gradient of ϕ_λ is in $L^\infty(0,T,L^2(\Omega))$, so with Sobolev embeddings, ϕ^λ strongly converges to a limit denoted ϕ in $L^\infty(0,T,L^p(\Omega))$, with $p<6$. Let us prove that ϕ satisfies the traditional quasi-geostrophic equation (37).

We proved that $\partial_t\left(\Delta_h\phi^\lambda+\left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right)$ is bounded in $L^2(0,T,H^{-1}(\Omega))$. We also have $\nabla\left(\Delta_h\phi^\lambda+\left(N^{-2}\partial_Z\phi^\lambda\right)_Z\right)$ in $L^2(0,T,L^2(\Omega))$: thanks to Aubin-Lions theorem $\Delta_h\phi^\lambda+\left(N^{-2}\partial_Z\phi^\lambda\right)_Z$ strongly converges towards $\Delta_h\phi+\left(N^{-2}\partial_z\phi\right)_z$ in $\mathcal{C}^0(0,T,L^2(\Omega))$ (we recall that $\partial_Z=\partial_z+\lambda\partial_y$). So we can pass to the limit in the jacobian thanks to the strong convergence, the other terms do not bring any difficulty.

Theorem 3.1. *Let ϕ^λ be a sequence of weak solutions of the nontraditional quasi-geostrophic equation* (38) *with initial data $\phi^\lambda(t=0)=\phi_0^\lambda$ satisfying $\phi_0^\lambda\to\phi_0$ in $L^1(\Omega)$ with, for all $\lambda>0$,*

$$\nabla\phi_0^\lambda\in L^2(\Omega),\quad \Delta\phi_0^\lambda\in L^2(\Omega),\ \textit{and}\ \phi_{0\,Z}^\lambda\in L^2(\Omega).$$

Then, up to a subsequence, ϕ^λ strongly converges in $L^\infty(0,T,L^p(\Omega))$, with $p<6$ towards ϕ, weak solution of the quasi-geostrophic equation (37).

4 Conclusion

The quasi-hydrostatic equations have been used to derive the quasi-geostrophic model for the simulation of geophysical fluid dynamics. This QHQG model, also known as tilted QG model, is derived in Section 2.1, in the same spirit than the usual derivation of the traditional QG model, such as in [1]. The resulting QHQG model (16) is slightly different from the traditional one, in the sense that the traditional operator ∂z is changed in $\partial Z = \partial z + \lambda \partial y$ where λ measures the effect of nontraditional terms of the Coriolis force, $\lambda = (H/L)\cot\theta_0$. Mathematical properties such as local or global existence of solutions have been proved, in Section 2.2, in the same way as for the traditional QG equation.
In Section 2.3, we discuss some physical properties of the QHQG model, and point the similarities / differences with the traditional QG model.
Finally, in Section 3, we proved that QHQG equations are an intermediate model between the complete non-hydrostatic equations and the usual QG equations.

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