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Abstract: Stochastic hybrid systems have been largely studied in the literature in the framework of Markov mode transitions and linear Gaussian state space mode models. In order to generalize such results to hybrid systems involving phenomena characterized by differential-algebraic equations, this paper studies the problem of state estimation for stochastic hybrid systems with modes described by descriptor equations. The proposed state estimation algorithm follows the interacting multiple model (IMM for short) approach, but the classical state space system Kalman filters are replaced by descriptor system Kalman filters. Because of the difficulty for computing the innovation of each descriptor system Kalman filter, a new method for likelihood evaluation is proposed, as one important step of the new IMM algorithm. Numerical examples are presented to illustrate the performance of the proposed algorithm.

Keywords: Hybrid system, descriptor system, IMM estimator, descriptor system Kalman filter.

1. INTRODUCTION

Differential equations have been widely used in the study of dynamic systems, in particular, finite dimensional state space equations of the form

$$\dot{x}(t) = f(x(t), u(t)) \qquad (1)$$

are often used for modeling engineering systems, with $x(t)$ and $u(t)$ denoting respectively the state vector and the input vector of the considered system, $\dot{x}(t) = dx(t)/dt$, and f is a function characterizing the dynamic behavior of the system. Despite the large success of such theories in engineering practice, some complex systems cannot be appropriately described in this framework. Such exceptions include differential-algebraic systems and hybrid systems.

Algebraic constraints in engineering systems typically result from singularities of differential equations. For example, a train following a turning track is constrained by the geometrical form of the tack if the mass of the earth is considered infinitely large compared to the mass of the train. Such a system can be described by differential-algebraic equations (DAE) of the form

$$g(\dot{x}(t), x(t), u(t))$$

which can represent a wider class of systems than the classical state space systems of the form (1). After linearization and discretization in time, such equations can be approximated by implicit discrete time state space equations of the form

$$E_{k+1}x(k+1) = A_k x(k) + B_k u(k) + v(k), \qquad (2)$$

where $x(k), u(k)$ and $v(k)$ are respectively the discrete time state, input and the modeling errors indexed by $k = 1, 2 \ldots$, and E_{k+1}, A_k, B_k are time varying matrices of appropriate sizes. With a possibly rank deficient matrix E_{k+1}, refer to (2) which is known as a *descriptor* equation [Nikoukhah et al. (1992)].

On the other hand, some complex systems have different working modes, for example, the starting mode, the normal working mode, or some reduced regime mode in case of component failures. If in each of these modes the system is described by some differential equations, the over-all functioning of the system, including the mode switching mechanism and the behavior within each mode, is usually modeled as a *hybrid* system [Bar-Shalom et al. (2001)].

The purpose of the present paper is to study state estimation for *hybrid* systems with working modes described by *descriptor* equations of the form (2).

The study of descriptor systems, notably about state estimation, has a rich literature [Yeu et al. (2001); Nikoukhah et al. (1992); Marx et al. (2004); Koenig et al. (2002); Gao et al. (2006)] (for more information see references therein). Besides this topic, state estimation for hybrid systems has been largely studied with various applications such as target tracking [Bar-Shalom et al. (1988); Blom et al. (1988)], signal processing [Doucet et al. (2001)], and fault diagnosis [Hanlon et al. (2000); Koutsoukos et al. (2002)]. These reported results concern hybrid systems involving modes described by state space equations. To our knowledge, no study has been reported about hybrid systems with working modes described by *descriptor* equations.

In the case of stochastic hybrid systems with working modes described by classical linear Gaussian state space equations, it is known that the complexity of the optimal state estimator increases exponentially with time. In practice, heuristic algorithms of reduced complexity are used, without convergence proof. Among such algorithms, is the *interacting multiple model* (IMM for short) estimator [Bar-Shalom et al. (2001), C. E. Seah et al. (2009)].

It is based on multiple Kalman filters, each assuming a particular working mode. At each iteration, in addition to the classical prediction and update steps of the Kalman filter, all the Kalman filters are modified based on the knowledge about the mode transition mechanism modeled as a Markov chain.

In the present paper, the algorithm proposed for hybrid descriptor system state estimation follows also the IMM approach. The main novelties in this algorithm reside in the replacement of the classical state space system Kalman filter by the descriptor system Kalman filter, and in a new method for the evaluation of the likelihood of each Kalman filter at each iteration. Of course, like the classical IMM estimator for state-space hybrid systems, no convergence proof of the algorithm is known.

This paper is organized as follows: In Section 2 we present a formal formulation of the estimation problem for a stochastic linear hybrid descriptor system. In Section 3, we make a comparison between the Kalman filter for classical situation and the descriptor one. We state the IMM algorithm in the framework of descriptor system in 4. In Section 5, we provide numerical examples. Finally, conclusions are presented in Section 6.

2. PROBLEM FORMULATION

The stochastic hybrid systems considered in this paper are modeled at two levels. At the top level, each system has a finite number of working modes. At each time instant, one of the modes is active, and random transitions between different modes are characterized by a Markov model. At the bottom level, each mode of the system is described by a stochastic descriptor system model.

2.1 Top level Markov Transition Model

Let $\{M_1, M_2, \ldots, M_r\}$ be the set of r possible modes of a hybrid system, and $m(k) \in \{1, 2, \ldots, r\}$ denote the index of the mode at time instant $k = 0, 1, 2, \ldots$. The mode evolution, corresponding to the sequence $m(0), m(1), \cdots$, is a Markov chain described by the mode transition matrix
$$\Pi = \{p_{ij}\}_{i,j=1,\cdots,r},$$
where
$$p_{i,j} = P\left[m(k+1) = j \mid m(k) = i\right],$$
with known transition probabilities $p_{i,j}$ independent of k and satisfying
$$\sum_{j=1}^{r} p_{i,j} = 1, \qquad i = 1, 2, \ldots, r.$$
For the initial time instant $k = 0$, the *prior probability* that M_j is active is
$$P[m(0) = j] = \mu_j(0)$$
with known probabilities $\mu_1(0), \mu_2(0), \ldots, \mu_r(0)$ satisfying
$$\sum_{j=1}^{r} \mu_j(0) = 1.$$

2.2 Bottom level Stochastic Descriptor System Model

At the bottom level, the state $x(k) \in \mathbb{R}^n$, the input $u(k) \in \mathbb{R}^q$ and the output $y(k) \in \mathbb{R}^p$ of the hybrid system satisfy the stochastic descriptor equations

$$E_{M_j} x(k+1) = A_{M_j} x(k)$$
$$+ B_{M_j} u(k) + W_{M_j}(k), \qquad (3)$$
$$y(k) = C_{M_j} x(k)$$
$$+ D_{M_j} u(k) + V_{M_j}(k), \qquad (4)$$

where the time index $k = 0, 1, 2, \cdots$, the descriptor system matrices $E_{M_j} \in \mathbb{R}^{l \times n}$, $A_{M_j} \in \mathbb{R}^{l \times n}$, $B_{M_j} \in \mathbb{R}^{l \times q}$, $C_{M_j} \in \mathbb{R}^{p \times n}$, $D_{M_j} \in \mathbb{R}^{p \times q}$, and the state and output white noises $W_{M_j}(k) \in \mathbb{R}^l$, $V_{M_j}(k) \in \mathbb{R}^p$, with $W_{M_j}(k) \sim \mathcal{N}(0, Q_{M_j})$, $V_{M_j}(k) \sim \mathcal{N}(0, S_{M_j})$ which are mutually independent to each other.

When the mode of the hybrid system stays unchanged, say at M_j, the mode index $m(k) = j$, then the system described by equation (3) behaves like a stochastic descriptor system characterized by the matrices A_j, B_j, C_j, D_j, E_j, Q_j, S_j. Within each mode, it is assumed that descriptor system (3)-(4) is observable and controllable by the state noise. See [Nikoukhah et al. (1992)] for the definitions of descriptor system observability and controllability.

Other other hand, the mode M_j may evolve, so that the mode index sequence $m(0), m(1), \cdots$ forms a Markov chain, as described at the top level. In particular, during every mode change, say from $M_{m(k)} = M_j$ to $M_{m(k+1)} = M_i$, the evolution of the state $x(k+1)$ and the output $y(k+1)$ are also described by equation (4), given a realization of the mode sequence Markov chain.

Let $Z^k = [u(0), y(0), u(1), y(1), \ldots, u(k), y(k)]$ denote the measurements up to time k.

The following assumptions are made in this paper.

(A1): Assume that $x(0) \sim \mathcal{N}(0, P_0)$ is independent of $W_{M_j}(k)$ and $V_{M_j}(k)$;

(A2): For $j \in \{1, 2, \ldots, r\}$, $H_{M_j} = \begin{bmatrix} E_{M_j} \\ C_{M_j} \end{bmatrix}$ is full column rank;

(A3): For $j \in \{1, 2, \ldots, r\}$, the matrices Q_{M_j} and S_{M_j} are symmetric positive definite.

3. KALMAN FILTER FOR DESCRIPTOR SYSTEMS

In this section, let us consider the single-mode descriptor system

$$E_{k+1} x(k+1) = A_k x(k) + B_k u(k) + W(k), \qquad (5)$$
$$y(k+1) = C_{k+1} x(k+1) + D_{k+1} u(k+1)$$
$$+ V(k+1). \qquad (6)$$

First recall the classical Kalman filter for state space systems corresponding to the case $l = n$ and $E_{k+1} = I_{n \times n}$.

$$\begin{cases} \hat{x}(0|0) = \mathbb{E}(x_0), \ P(0,0) = Var(x_0), \\ P(k+1|k) = A(k)P(k|k)A^T(k) + Q(k), \\ G(k+1) = P(k+1|k)C^T(k+1) \\ \quad \cdot (C(k+1)P(k+1|k)C^T(k+1) + S(k+1))^{-1}, \\ \hat{x}(k+1|k) = A(k)\hat{x}(k|k) + B(k)u(k), \\ \hat{x}(k+1|k+1) = \hat{x}(k+1|k) + G(k+1)(y(k+1) \\ \quad - D(k+1)u(k+1) - C(k+1)\hat{x}(k+1|k)), \end{cases}$$

where the innovation process, also known as the prediction error, $v(k+1) = y(k+1) - \hat{y}(k+1|k)$ with
$$\hat{y}(k+1|k) = C(k+1)\hat{x}(k+1|k) + D(k+1)u(k+1),$$

is a white Gaussian sequence. The associated likelihood function can be computed by using the innovation $v(k+1)$.

Now let us consider the more general descriptor system (5)-(6). Its Kalman filter [Nikoukhah et al. (1992)] writes (d in subscript short for descriptor systems),

$$
\begin{cases}
\hat{x}_d(0|0) = \mathbb{E}(x_0), \\
\Sigma_d(k) = A(k)P_d(k|k)A^T(k) + Q_d(k), \\
G_d(k+1) \\
= (E^T(k+1)\Sigma_d^{-1}(k)E(k+1))^{-1}C^T(k+1) \\
\quad \cdot (C(k+1)(E^T(k+1)\Sigma_d^{-1}(k)E(k+1))^{-1} \\
\quad \cdot C^T(k+1) + S_d(k))^{-1}, \\
L_d(k+1) = (E^T(k+1)\Sigma_d^{-1}(k)E(k+1))^{-1} \\
\quad \cdot E^T(k+1)\Sigma_d^{-1}(k) - G_d(k+1)C(k+1) \\
\quad \cdot (E^T(k+1)\Sigma_d^{-1}(k)E(k+1))^{-1} \\
\quad \cdot E^T(k+1)\Sigma_d^{-1}(k), \\
\hat{x}_d(k+1|k+1) = L_d(k+1)A(k)\hat{x}_d(k|k) \\
\quad + L_d(k+1)B(k)u(k) \\
\quad - G_d(k+1)D(k+1)u(k+1) \\
\quad + G_d(k+1)y_d(k+1), \\
P_d(k+1|k+1) = L_d(k+1)A(k)P_d(k|k)A^T(k) \\
\quad \cdot L_d^T(k+1) + L_d(k+1)Q_d(k)L_d^T(k+1) \\
\quad + G_d(k+1)S_d(k)G_d^T(k+1).
\end{cases}
\tag{7}
$$

In [Nikoukhah et al. (1992); Ali et al. (2014)] the gain matrices $G_d(k+1)$ and $L_d(k+1)$ were given in implicit forms. Their explicit forms presented here are proved in Appendix A. Unlike the classical state space system Kalman filter which was formulated in two steps known as prediction and update, computing respectively $\hat{x}(k+1|k)$ and $\hat{x}(k+1|k+1)$. Here for descriptor systems the filter is written in a single step. It is thus not obvious if the computed state estimate $\hat{x}_d(k+1|k+1)$ in every iteration is the predicted state or the filtered state. In Appendix A it is shown that $\hat{x}_d(k+1|k+1)$ is indeed the filtered state. Consequently, the output estimation error $\bar{v}_d(k+1) = y_d(k+1) - \hat{y}_d(k+1|k+1)$ where
$\hat{y}_d(k+1|k+1) = C(k+1)\hat{x}_d(k+1|k+1) + D(k+1)u(k+1)$.
is not the innovation sequence in the sense of prediction error as defined in the classical Kalman filter. Usually the innovation sequence is used for likelihood evaluation in the IMM approach to hybrid system estimation. Fortunately, like the innovation sequence, this estimation error $v_d(k+1)$ is a white Gaussain sequence in [Ali et al. (2014)]. This result is important for the new IMM estimator presented in this paper for hybrid systems, at the step of likelihood computation.

4. HYBRID DESCRIPTOR SYSTEM IMM ESTIMATOR

The algorithm presented below for hybrid descriptor systems is quite similar to the IMM estimator for classical hybrid state space systems [Bar-Shalom et al. (2001)], the main differences reside in the Kalman filter for descriptor systems and in the evaluation of the likelihood at each iteration.

The key feature of IMM we point out is that it consists of r interacting filters operating in parallel.

The algorithm consists of the following steps.

1. **Calculation of the mixing probabilities** (i, $j = 1, \ldots, r$)**.** The probability that mode M_i was in effect at k given that M_j is in effect at $k+1$ conditioned on Z^{k+1} is

$$
\begin{aligned}
\mu_{i|j}(k|k+1) &= P\{M_i(k) \,|\, M_j(k+1), Z^{k+1}\} \\
&= \frac{1}{\bar{c}_j}p_{i,j}\mu_j(k),
\end{aligned}
$$

where

$$
\bar{c}_j = \sum_{i=1}^{r} p_{i,j}\mu_i(k).
$$

2. **Intermediate results mixing** ($j = 1, \ldots, r$)**.**

During the last iteration, r descriptor system Kalman filters were run in parallel, each assuming a different active mode at instant k, yielding r state estimates $\hat{x}(k|k)$ and r covariance matrices $P^i(k|k)$. Based on these results, the mixed state estimates and covariance matrices are

$$
\hat{x}^{0,j}(k|k) = \sum_{i=1}^{r} \hat{x}(k|k)\mu_{i|j}(k|k),
$$

$$
\begin{aligned}
P^{0,j}(k|k) = \sum_{i=1}^{r} \mu_{i|j}(k|k)\{ &P^i(k|k) \\
&+ [\hat{x}(k|k) - \hat{x}^{0,j}(k|k)] \\
&\cdot [\hat{x}(k|k) - \hat{x}^{0,j}(k|k)]^T \}.
\end{aligned}
$$

3. **Mode-matched filtering** ($j = 1, \ldots, r$)**.**

For each of the r assumed active modes at instant $k+1$, say M_j, a descriptor system Kalman filter delivers a state estimate $\hat{x}^j(k+1|k+1)$ as follows:

$$
\begin{aligned}
\hat{x}^j(0|0) &= \mathbb{E}(x_0), \\
\Sigma^j(k) &= A_{M_j}P^{0,j}(k|k)A_{M_j}^T + Q_{M_j}, \\
G_{M_j}(k+1) &= (E_{M_j}^T(\Sigma^j(k))^{-1}E_{M_j})^{-1}C_{M_j}^T \\
&\quad \cdot (C_{M_j}(E_{M_j}^T(\Sigma^j(k))^{-1}E_{M_j})^{-1}C_{M_j}^T \\
&\quad + S_{M_j})^{-1}, \\
L_{M_j}(k+1) &= (E_{M_j}^T(\Sigma^j(k))^{-1}E_{M_j})^{-1} \\
&\quad \cdot E_{M_j}^T(\Sigma^j(k))^{-1} - G_{M_j}(k+1)C_{M_j} \\
&\quad \cdot (E_{M_j}^T(\Sigma^j(k))^{-1}E_{M_j})^{-1}E_{M_j}^T(\Sigma^j(k))^{-1}, \\
\hat{x}^j(k+1|k+1) &= L_{M_j}(k+1)A_{M_j}\hat{x}^{0,j}(k|k) \\
&\quad + L_{M_j}(k+1)B_{M_j}u(k) \\
&\quad - G_{M_j}(k+1)D_{M_j}u(k+1) \\
&\quad + G_{M_j}(k+1)y(k+1), \\
P^j(k+1|k+1) &= L_{M_j}(k+1)A_{M_j}P^{0,j}(k|k)A_{M_j}^T \\
&\quad \cdot L_{M_j}^T(k+1) \\
&\quad + L_{M_j}(k+1)Q_{M_j}L_{M_j}^T(k+1) \\
&\quad + G_{M_j}(k+1)S_{M_j}G_{M_j}^T(k+1).
\end{aligned}
$$

The likelihood of the mode M_j, given the input-output data up to instant $k+1$, is evaluated through the "innovation"

$$\bar{v}^j(k+1) = y(k+1) - \hat{y}^j(k+1|k+1)$$
$$= y(k+1) - D_{M_j}u(k+1)$$
$$-C_{M_j}\hat{x}^j(k+1|k+1),$$
$$\Lambda_j(k+1) = \frac{1}{\sqrt{(2\pi)^p \det(\Sigma^j(k+1))}}$$
$$\cdot \exp(\Upsilon^j(k+1)),$$

where

$$\Upsilon^j(k+1) = -\frac{1}{2}\left(\bar{v}^j(k+1)\right)^T \left(\Sigma^j(k+1)\right)^{-1}\bar{v}^j(k+1).$$

4. **Mode probability update.**

The mode probabilities are updated as

$$\mu_j(k+1) = \frac{1}{c}\Lambda_j(k+1)\bar{c}_j$$

with

$$\bar{c}_j = \sum_{i=1}^{r} p_{i,j}\mu_i(k), \qquad c = \sum_{j=1}^{r}\Lambda_j(k+1)\bar{c}_j.$$

5. **Estimate and covariance combination.**

Combination of the model-conditioned estimates and covariances is completed according to the mixture equations

$$\hat{x}(k+1|k+1) = \sum_{j=1}^{r}\hat{x}^j(k+1|k+1)\mu_j(k+1)$$

and

$$P(k+1|k+1)$$
$$= \sum_{j=1}^{r}\mu_j(k+1)\Big\{P^j(k+1|k+1)$$
$$+ \left[\hat{x}^j(k+1|k+1) - \hat{x}(k+1|k+1)\right]$$
$$\cdot \left[\hat{x}^j(k+1|k+1) - \hat{x}(k+1|k+1)\right]^T\Big\}.$$

5. NUMERICAL EXAMPLES

Consider a hybrid descriptor stochastic system as formulated in Section 2, with (3) and (4), where $r = 3$, the mode dependent system matrices A_i, B_i, C_i, D_i, E_i, Q_i, S_i, Π and the control u are as follows:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 1 \\ 0 & 0 & -0.1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0 & 0.6 & 0 \\ 0 & 0.3 & 0.9 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.05 & -0.1 & 1 \end{bmatrix}, \ B_i = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ D_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \ C_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix};$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$Q_i = 0.001 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ S_i = 10 \cdot \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \ i = 1,2,3;$$
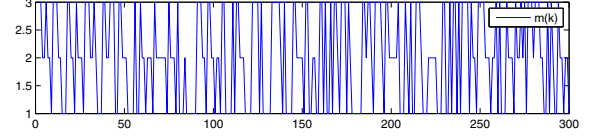

Fig. 1. Markov mode sequence.

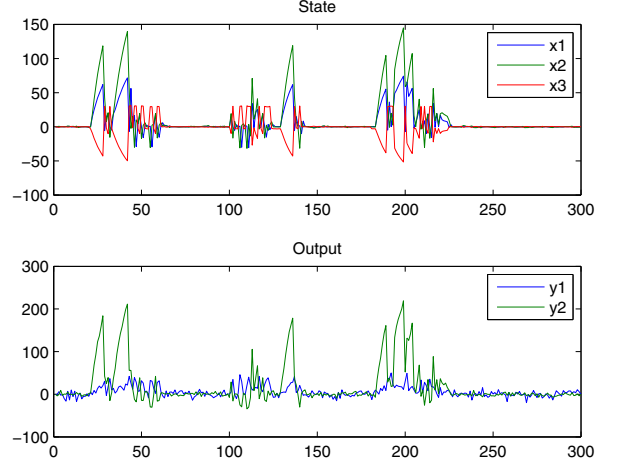Fig. 2. System states and outputs.

$$\Pi = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}, \ u(k) = \begin{cases} 1, & \begin{cases} \text{if, } 21 \leq k \leq 60; \\ \text{if, } 101 \leq k \leq 140; \\ \text{if, } 181 \leq k \leq 220; \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

We will denote $x_i, i = 1,2,3, y_1, y_2$ as the components of vector x and y, respectively in the above figures.

The simulation runs from $k = 1$ till $k = 300$. The IMM estimator for hybrid descriptor systems proposed in this paper is applied to the simulated system for state estimation. The Markov mode sequence can be seen in Fig. 1. System states and outputs are presented in Fig. 2, and the state estimates in Fig. 3.

In Fig. 2, the results attained by the time varying descriptor system for Kalman filter [Ali et al. (2014); Nikoukhah et al. (1992)] are also presented in the aim of comparison. This descriptor system Kalman filter assumes that the actual mode sequence is known, which is not the case of the IMM estimator.

The results shown in Fig. 3 are based on one random realization of the simulated stochastic system. In order to illustrate the statistic properties of the proposed IMM estimator, 1000 random realizations have been made, and the histograms of the state estimation errors at instants $k = 50, 140, 200$, are shown in Figs. 4, 5, 6.

6. CONCLUSIONS

In this paper the existing interacting multiple model estimator has been extended to the case of stochastic hybrid systems with modes described by descriptor equations. We have presented a numerical example to show the performance of our method. Among future research directions,
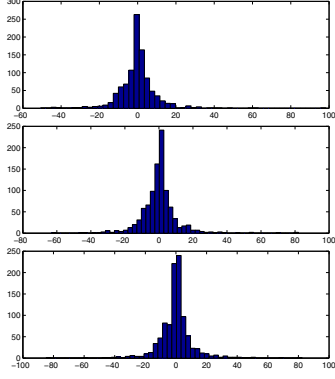
Fig. 4. Histogram for $k = 50, 140, 200$ of the state estimation error of x_1, respectively.
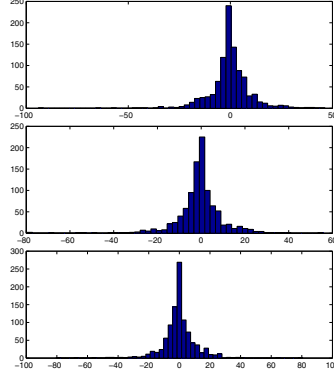

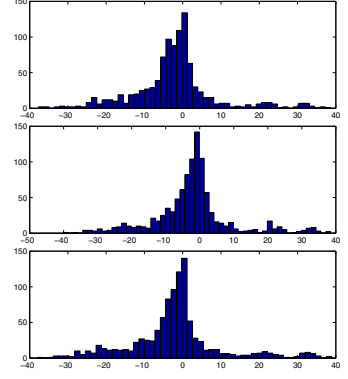Fig. 5. Histogram for $k = 50, 140, 200$ of the state estimation error of x_2, respectively.

Fig. 6. Histogram for $k = 50, 140, 200$ of the state estimation error of x_3, respectively.
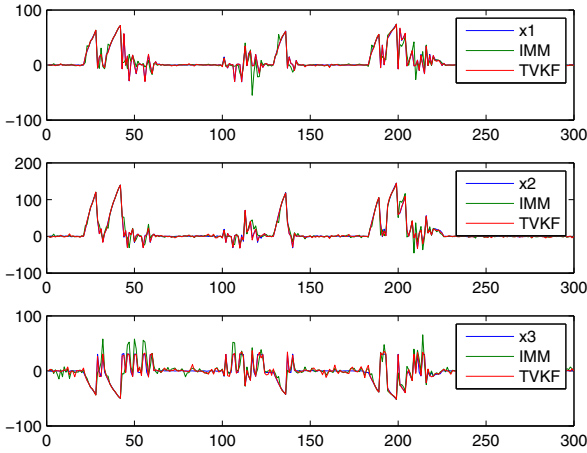

Fig. 3. States estimates by descriptor IMM estimator and by descriptor Kalman filter (TVKF for short).

joint estimation of state and parameters in descriptor hybrid systems will be studied, notably for the purpose of fault diagnosis.

Appendix A. GAIN MATRICES AND ESTIMATED STATE

In this appendix, we will check that the state estimation $\hat{x}_d(k + 1|k + 1)$ computed in (7) is indeed the filtered state, not the predicted state, by showing that, in the particular case of $E_{k+1} = I$, it coincides with the filtered state of the classical Kalman filter. The explicit forms of the gain matrices $G_d(k + 1)$ and $L_d(k + 1)$ in (7) are also derived in this appendix. For simplicity of notation, we omit $M_i(k + 1)$, and set

$$[L_{k+1}, K_{k+1}] = \left[L_{M_i(k+1)}, K_{M_i(k+1)}\right], \text{ etc,}$$

and $B_k = D_k = 0$. We are going to compute explicitly L_k, and K_k which were defined in [Ali et al. (2014)] as

$$[L_{k+1}, K_{k+1}] = (H_{k+1}^T R_k^{-1} H_{k+1})^{-1} H_{k+1}^T R_k^{-1}, \quad \text{(A.1)}$$

where

$$R_k = \begin{bmatrix} \Sigma_k & O \\ O & S_k \end{bmatrix}$$

and

$$\Sigma_k = A_k P_k A_k^T + Q_k.$$

Then, we have

$$H_{k+1}^T R_k^{-1} H_{k+1}$$
$$= \left[E_{k+1}^T, C_{k+1}^T\right] \begin{bmatrix} \Sigma_k^{-1} & O \\ O & S_k^{-1} \end{bmatrix} \begin{bmatrix} E_{k+1} \\ C_{k+1} \end{bmatrix}$$
$$= E_{k+1}^T \Sigma_k^{-1} E_{k+1} + C_{k+1}^T S_k^{-1} C_{k+1}.$$

Recall the matrix inverse formula,

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B$$
$$\cdot (DA^{-1}B + C^{-1})^{-1} DA^{-1}$$

where A^{-1}, C^{-1}, and $(DA^{-1}B + C^{-1})$ are assumed to exist.

Using the matrix inverse formula, we obtain

$$(H_{k+1}^T R_k^{-1} H_{k+1})^{-1}$$
$$= (E_{k+1}^T \Sigma_k^{-1} E_{k+1} + C_{k+1}^T S_k^{-1} C_{k+1})^{-1}$$
$$= (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1}$$
$$- (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T$$
$$\cdot \left(C_{k+1} \left(E_{k+1}^T \Sigma_k^{-1} E_{k+1}\right)^{-1} C_{k+1}^T + S_k\right)^{-1}$$
$$\cdot C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1}.$$

Let us define

$$G_{k+1} = (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T$$
$$\cdot (C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T$$
$$+ S_k)^{-1}. \quad \text{(A.2)}$$

Further,

$$(H_{k+1}^T R_k^{-1} H_{k+1})^{-1} H_{k+1}^T R_k^{-1}$$
$$= \left[(E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1}\right.$$
$$\left. - G_{k+1} C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1}\right]$$
$$\cdot \left[E_{k+1}^T, C_{k+1}^T\right] \begin{bmatrix} \Sigma_k^{-1} & O \\ O & S_k^{-1} \end{bmatrix}$$
$$= \left[(E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} E_{k+1}^T \Sigma_k^{-1}\right.$$
$$\left. - G_{k+1} C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} E_{k+1}^T \Sigma_k^{-1},\right.$$
$$(E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T S_k^{-1}$$
$$\left. - G_{k+1} C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T S_k^{-1}\right].$$

Immediately, from (A.1), we get

$$L_{k+1} = (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} E_{k+1}^T \Sigma_k^{-1}$$
$$- G_{k+1} C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} E_{k+1}^T \Sigma_k^{-1},$$

and

$$K_{k+1} = (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T S_k^{-1}$$
$$- G_{k+1} C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1}$$
$$\cdot C_{k+1}^T S_k^{-1}. \tag{A.3}$$

Substituting (A.2) into (A.3), we have

$$K_{k+1} = (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T$$
$$\cdot (C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T$$
$$+ S_k)^{-1},$$

which is exactly equals to G_{k+1} expressed by (A.2). To obtain this result, the following equalities are used

$$\left(C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T + S_k \right)^{-1}$$
$$= S_k^{-1} - (C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T + S_k)^{-1}$$
$$\cdot C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T S_k^{-1}. \tag{A.4}$$

Indeed,

$$(C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T + S_k) \cdot$$
$$\left((S_k^{-1} - (C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T + S_k)^{-1} \right.$$
$$\left. \cdot C_{k+1} (E_{k+1}^T \Sigma_k^{-1} E_{k+1})^{-1} C_{k+1}^T S_k^{-1} \right)$$
$$= C_{k+1} \left(E_{k+1}^T \Sigma_k^{-1} E_{k+1} \right)^{-1} C_{k+1}^T S_k^{-1} + I$$
$$- C_{k+1} \left(E_{k+1}^T \Sigma_k^{-1} E_{k+1} \right)^{-1} C_{k+1}^T S_k^{-1}$$
$$= I.$$

Hence, in particular, if we set $E_{k+1} = I$, one can get

$$L_{k+1} = I - G_{k+1} C_{k+1},$$
$$G_{k+1} = \Sigma_k C_{k+1}^T (C_{k+1} \Sigma_k C_{k+1}^T + S_k)^{-1},$$

which is just the Kalman gain. At last, we have

$$\hat{x}(k+1) = L_{k+1} A_k \hat{x}(k) + G_{k+1} y(k+1)$$
$$= A_k \hat{x}(k) + G_{k+1} [y(k+1)$$
$$- C_{k+1} A_{k+1} \hat{x}(k)].$$

The associated variance can be verified similarly.

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