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# ON SYMMETRIC NORM INEQUALITIES AND HERMITIAN BLOCK-MATRICES

ANTOINE MHANNA<sup>1\*</sup>

ABSTRACT. The main purpose of this paper is to englobe some new and known types of Hermitian block-matrices  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  satisfying or not the inequality  $\|M\| \leq \|A + B\|$  for all symmetric norms.

## 1. INTRODUCTION AND PRELIMINARIES

The first section will deal with some known results on the inequality and some preliminaries we used in the second section to derive some new generalization results. For positive semi-definite block-matrix  $M$ , we say that  $M$  is P.S.D. and we write  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$ , with  $A \in \mathbb{M}_n^+$ ,  $B \in \mathbb{M}_m^+$ . Let  $A$  be an  $n \times n$  matrix and  $F$  an  $m \times m$  matrix, ( $m > n$ ) written by blocks such that  $A$  is a diagonal block and all entries other than those of  $A$  are zeros, then the two matrices have the same singular values and for all unitarily invariant norms  $\|A\| = \|F\| = \|A \oplus 0\|$ , we say then that the symmetric norm on  $\mathbb{M}_m$  induces a symmetric norm on  $\mathbb{M}_n$ , so for square matrices we may assume that our norms are defined on all spaces  $\mathbb{M}_n$ ,  $n \geq 1$ . The spectral norm is denoted by  $\|\cdot\|_s$ , the Frobenius norm by  $\|\cdot\|_{(2)}$ , and the Ky Fan  $k$ -norms by  $\|\cdot\|_k$ . Let  $Im(X) := \frac{X - X^*}{2i}$  respectively  $Re(X) = \frac{X + X^*}{2}$  be the imaginary part respectively the real part of a matrix  $X$  and let  $\mathbb{M}_n^+$  denote the set of positive and semi-definite part of the space of  $n \times n$  complex matrices and  $M$  be any positive semi-definite block-matrices; that is,  $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{n+m}^+$ , with  $A \in \mathbb{M}_n^+$ ,  $B \in \mathbb{M}_m^+$ .

**Lemma 1.1.** [1] *For every matrix in  $\mathbb{M}_{2n}^+$  written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + Im(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - Im(X) \end{pmatrix} V^*$$

for some unitaries  $U, V \in \mathbb{M}_{2n}$ .

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\* Corresponding author.

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**Lemma 1.2.** [1] *For every matrix in  $\mathbb{M}_{2n}^+$  written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + \operatorname{Re}(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - \operatorname{Re}(X) \end{pmatrix} V^*$$

for some unitaries  $U, V \in \mathbb{M}_{2n}$ .

*Remark 1.3.* The proofs of Lemma 1.1 respectively Lemma 1.2 suggests that we have  $A + B \geq -\frac{(X - X^*)}{i}$  and  $A + B \geq \frac{(X - X^*)}{i}$ , respectively  $A + B \geq -(X + X^*)$  and  $A + B \geq (X + X^*)$ .

**Lemma 1.4.** [3] *Let  $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be any square matrix written by blocks of same size, if  $AC = CA$  then  $\det(M) = \det(AD - CB)$ .*

## 2. MAIN RESULTS

**2.1. Symmetric norm inequality.** Hereafter our block matrices are such their diagonal blocks are of equal size.

**Lemma 2.1.** [1] *Let  $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^+$ , if  $X$  is Hermitian or Skew-Hermitian then*

$$\|M\| \leq \|A + B\| \tag{2.1}$$

for all symmetric norms.

The fact is that there exist P.S.D. matrices with non Hermitian or Skew-Hermitian off-diagonal blocks satisfying (2.1).

**Definition 2.2.** A block matrix  $N = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  is said to be a Hermitio matrix if it is unitarily congruent to a matrix  $M = \begin{bmatrix} A & Y \\ Y & B \end{bmatrix}$  with Hermitian off diagonal blocks or to  $M = \begin{bmatrix} A & Y \\ -Y & B \end{bmatrix}$  with Skew-hermitian off diagonal blocks.

Clearly if  $N$  is P.S.D. it satisfies  $\|N\| \leq \|A + B\|$  for all symmetric norms (by Lemma 2.1).

**Proposition 2.3.** *Let  $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^+$  be a given positive semi-definite matrix. If  $X^*$  commute with  $A$ , or  $X$  commute with  $B$ , then  $\|M\| \leq \|A + B\|$  for all symmetric norms.*

*Proof.* We will assume without loss of generality that  $X^*$  commute with  $A$ , as the other case is similar. We show that such  $M$  is a Hermitio matrix. Take the right polar decomposition of  $X^*$  so  $X = U|X|$  and  $X^* = |X|U^*$ . Since  $U^*$  is unitary and  $X^*$  commute with  $A$ ,  $X$  and  $|X|$

commute with  $A$  thus  $AU^* = U^*A$ . If  $I_n$  is the identity matrix of order  $n$ , a direct computation shows that

$$\begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A & |X| \\ |X| & B \end{bmatrix},$$

consequently by Lemma 2.1,  $\|M\| \leq \|A + B\|$  for all symmetric norms and that completes the proof.  $\square$

*Remark 2.4.* It is easily seen that if  $X$  commute with the Hermitian matrix  $A$  so is  $X^*$  and conversely.

Let  $M_1 = \frac{A+B}{2} + Im(X)$ ,  $M_2 = \frac{A+B}{2} - Im(X)$ ,  $N_1 = \frac{A+B}{2} + Re(X)$  and  $N_2 = \frac{A+B}{2} - Re(X)$ . The following is a slight generalization of Lemma 2.1 unless one proves that this is a case of a Hermitio matrix.

**Theorem 2.5.** *Let  $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  be a positive semi definite matrix, if  $Im(X) = rI_n$  or  $Re(X) = rI_n$  for some  $r$ , then  $\|M\| \leq \|A + B\|$  for all symmetric norms.*

*Proof.* Let  $\sigma_i(H)$  denote the singular values of a matrix  $H$  ordered in decreasing order, by Remark 1.3 the matrices  $M_1 = \frac{A+B}{2} + Im(X)$  and  $M_2 = \frac{A+B}{2} - Im(X)$  are positive semi definite since  $Im(X) = rI_n$  we have:

$$\sum_{i=1}^k \sigma_i \left( \frac{A+B}{2} + Im(X) \right) + \sum_{i=1}^k \sigma_i \left( \frac{A+B}{2} - Im(X) \right) = \sum_{i=1}^k \sigma_i(A+B).$$

In other words by Lemma 1.1  $\|M\|_k \leq \|M_1\|_k + \|M_2\|_k = \|A + B\|_k$  for all Ky-Fan  $k$ -norms which completes the proof, using Lemma 1.2 the other case is similarly proven.  $\square$

**Theorem 2.6.** *Let  $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$  be a positive semi definite matrix, then  $\|M\| \leq 2\|A + B\|$  for all symmetric norms. Furthermore if  $M_1$  or  $M_2$  are positive definite then the large inequality is a strict one.*

*Proof.* The proof is very close to that of the previous theorem since  $M_1, M_2$  are two positive semi definite matrices we have  $\|M_1\|_k \leq \|A + B\|_k$  and  $\|M_2\|_k \leq \|A + B\|_k$  for all  $k \leq n$ , thus we derive the following inequality:

$$\|M\|_k \leq \|M_1\|_k + \|M_2\|_k \leq 2\|A + B\|_k$$

for all Ky-Fan  $k$ -norms. It is easily seen that if  $M_1$  or  $M_2$  are P.D. then  $\|M_1\|_k + \|M_2\|_k < 2\|A + B\|_k$   $\square$

*Remark 2.7.* Note by the decompositions in Lemma 1.1 and Lemma 1.2 if  $M > 0$  then all of  $M_1, N_1, M_2$  and  $N_2$  are positive definite.

Here's a counter example:

**Example 2.8.** Let

$$T = \begin{bmatrix} \frac{3}{10} & 0 & 0 & \frac{i}{11} \\ 0 & 5 & i & 0 \\ 0 & -i & 5 & 0 \\ \frac{-i}{11} & 0 & 0 & \frac{3}{10} \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

where  $X = \begin{bmatrix} 0 & \frac{i}{11} \\ i & 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} \frac{3}{10} & 0 \\ 0 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 5 & 0 \\ 0 & \frac{3}{10} \end{bmatrix}$ . The eigenvalues of  $T$  are the positive numbers:  $\lambda_1 = 6$ ,  $\lambda_2 = 4$ ,  $\lambda_3 \approx 0.39$ ,  $\lambda_4 \approx 0.21$ ,  $T \geq 0$ , but  $6 = \|T\|_s > \|A + B\|_s = \frac{53}{10}$ .

**Lemma 2.9.** Let

$$N = \begin{bmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} & D \\ D^* & \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \end{bmatrix}$$

where  $a_1, \dots, a_n$  respectively  $b_1, \dots, b_n$  are nonnegative respectively negative real numbers,  $A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$ ,  $B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$  and  $D$  is any diagonal matrix, then nor  $N$  neither  $-N$  is positive semi-definite. Set  $(d_1, \dots, d_n)$  as the diagonal entries of  $D^*D$ , if  $a_i + b_i \geq 0$  and  $a_i b_i - d_i < 0$  for all  $i \leq n$ , then  $\|N\| > \|A + B\|$ . for all symmetric norms

*Proof.* The diagonal of  $N$  has negative and positive numbers, thus nor  $N$  neither  $-N$  is positive semi-definite, now any two diagonal matrices will commute, in particular  $D^*$  and  $A$ , by applying Theorem 1.4 we get that the eigenvalues of  $N$  are the roots of

$$\det((A - \mu I_n)(B - \mu I_n) - D^*D) = 0$$

Equivalently the eigenvalues are all the solutions of the  $n$  equations:

$$\begin{aligned} 1) & \quad (a_1 - \mu)(b_1 - \mu) - d_1 = 0 \\ 2) & \quad (a_2 - \mu)(b_2 - \mu) - d_2 = 0 \\ 3) & \quad (a_3 - \mu)(b_3 - \mu) - d_3 = 0 \\ & \quad \vdots \\ i) & \quad (a_i - \mu)(b_i - \mu) - d_i = 0 \\ & \quad \vdots \\ n) & \quad (a_n - \mu)(b_n - \mu) - d_n = 0 \end{aligned} \quad (S)$$

Let us denote by  $x_i$  and  $y_i$  the two solutions of the  $i^{th}$  equation then:

$$\begin{array}{rclcl} x_1 + y_1 & = & a_1 + b_1 & \geq & 0 & \quad & x_1 y_1 & = & a_1 b_1 - d_1 & < & 0 \\ x_2 + y_2 & = & a_2 + b_2 & \geq & 0 & \quad & x_2 y_2 & = & a_2 b_2 - d_2 & < & 0 \\ & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ x_n + y_n & = & a_n + b_n & \geq & 0 & \quad & x_n y_n & = & a_n b_n - d_n & < & 0 \end{array}$$

This implies that each equation of  $(S)$  has one negative and one positive solution, their sum is positive, thus the positive root is bigger or equal than the negative one. Since  $A + B = \begin{pmatrix} a_1+b_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n+b_n \end{pmatrix}$ , summing over indexes we see that  $\|N\|_k > \|A + B\|_k$  for  $k = 1, \dots, n$  which yields to  $\|N\| > \|A + B\|$  for all symmetric norms.  $\square$

It seems easy to construct examples of non P.S.D matrices  $N$ , such that  $\|N\|_s > \|A + B\|_s$ , let us have a look of such inequality for P.S.D. matrices.

**Example 2.10.** Let

$$N_y = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

where  $A = \begin{bmatrix} 2 & 0 \\ 0 & y \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ . The eigenvalues of  $N_y$  are the numbers:  $\lambda_1 = 4$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = y$ ,  $\lambda_4 = 0$ , thus if  $y \geq 0$ ,  $N_y$  is positive semi-definite and for all  $y$  such that  $0 \leq y < 1$  we have

- (1)  $4 = \|N_y\|_s > \|A + B\|_s = 3$ .
- (2)  $16 + y^2 + 1 = \|N\|_{(2)}^2 > \|A + B\|_{(2)}^2 = 4(3 + y) + y^2 + 1$ .

REFERENCES

1. J. C. Bourin, E. Y. Lee, and M. Lin, *On a decomposition lemma for positive semi-definite block-matrices*, Linear Algebra and its Applications 437, pp.1906–1912, (2012).
2. F.Zhang, *Matrix Theory Basic Results and Techniques*, 2<sup>nd</sup> Edition (Universitext) Springer, 2013.
3. J. R. Silvester, *Determinant of Block Martrices*, Math. Gazette, 84 (2000), pp. 460-467.

<sup>1</sup> DEPARTMENT OF MATHEMATICS, LEBANESE UNIVERSITY HADATH, BEIRUT, LEBANON  
*E-mail address:* [tmhanat@yahoo.com](mailto:tmhanat@yahoo.com)