



# Computing Periodic Triangulations

Flat case vs. Hyperbolic case

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informatiques mathématiques

Shape Up - Berlin - September 2015

# Outline

- 1 Motivation - State of the art
- 2 The Flat Torus
- 3 The Hyperbolic Plane
- 4 The Bolza Surface
- 5 Covering spaces

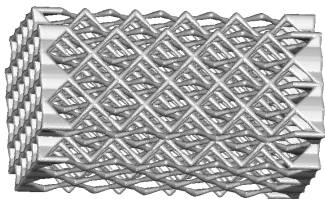
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# Motivation: applications

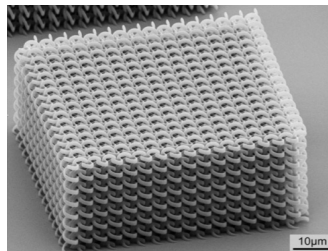
## Flat case

material engineering



bone scaffolding  
*M. Moesen, K.U. Leuven*

nano-structures



photonic crystal  
*M. Blome, Zuse Institut Berlin*

also cosmology, biology, etc

# Motivation: applications

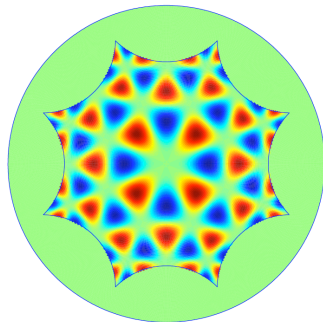
## Hyperbolic case

physics



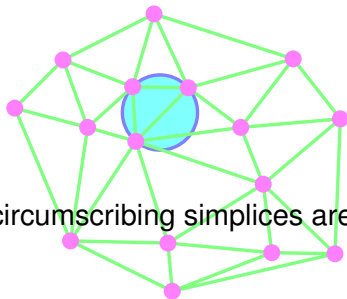
[Sausset, Tarjus, Viot]

neuromathematics



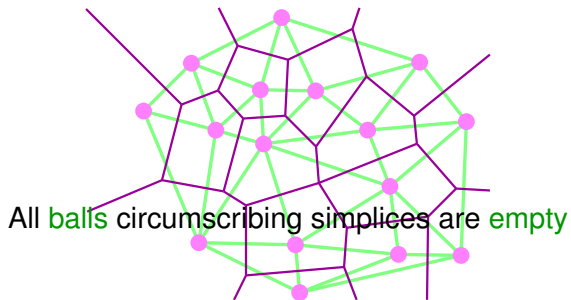
[Chossat, Faye, Faugeras]

# Delaunay triangulation

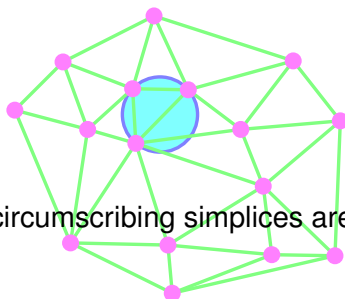


All **balls** circumscribing simplices are **empty**

# Delaunay triangulation



# Delaunay triangulation



All **balls** circumscribing simplices are **empty**

in  $\mathbb{R}^d$

- many algorithms *[...]*
- software  
*QHull, CGAL*

in  $\mathbb{H}^d$

- algorithms *[...]*  
*[Bogdanov, Devillers, T.]*
- software  
 *$d = 2$ : CGAL, in progress*

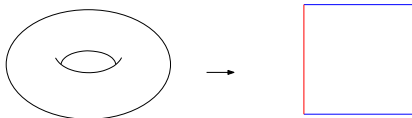


# Delaunay triangulation

From 1 to 2 handles

## (2D) Flat torus

locally Euclidean metric



- algorithms

  - 2D [Mazón, Recio]*

  - 3D torus [Dolbilin, Huson]*

  - dD closed [Caroli, T.]*

- software

  - CGAL 2D torus [Kruithof]*

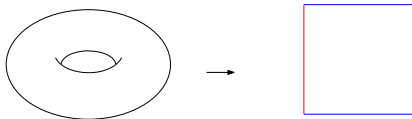
  - 3D torus [Caroli, T.]*

# Delaunay triangulation

From 1 to 2 handles

(2D) Flat torus

locally Euclidean metric



- algorithms

*2D [Mazón, Recio]*

*3D torus [Dolbilin, Huson]*

*dD closed [Caroli, T.]*

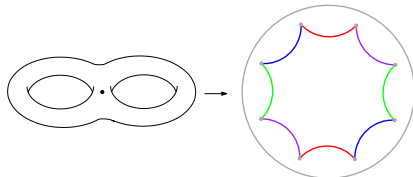
- software

*CGAL 2D torus [Kruithof]*

*3D torus [Caroli, T.]*

(2D) Bolza surface

locally hyperbolic metric



- algorithm ?

*2D Bolza surface*

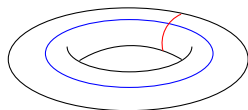
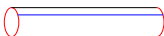
*[Bogdanov, T.]*

- software ?

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# The Flat Torus



locally Euclidean metric

$$G = \langle t_x, t_y \rangle$$

$$T = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \rightarrow T$$

# The Flat Torus

Delaunay triangulation

 $\mathcal{P}$  finite point set

$$G = \langle t_x, t_y \rangle$$

$$\mathbb{T} = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \rightarrow \mathbb{T}$$



# The Flat Torus

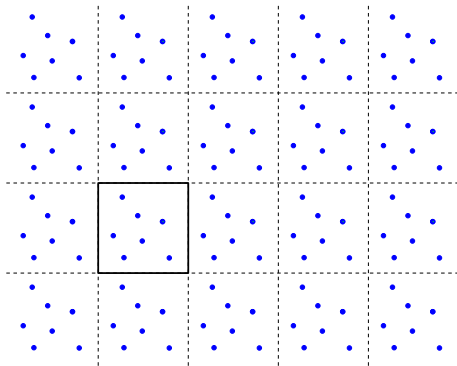
Delaunay triangulation

 $GP$  infinite point set

$$G = \langle t_x, t_y \rangle$$

$$\mathbb{T} = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \rightarrow \mathbb{T}$$



# The Flat Torus

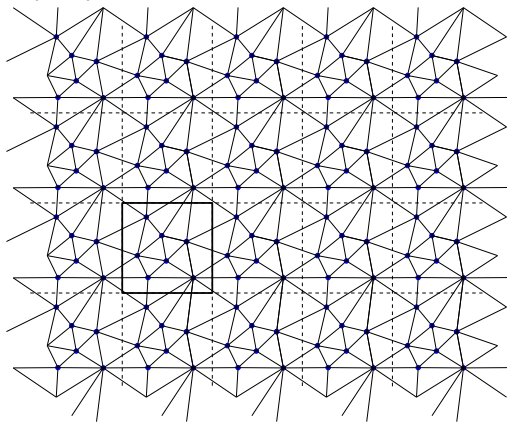
Delaunay triangulation

$$G = \langle t_x, t_y \rangle$$

$$\mathbb{T} = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \rightarrow \mathbb{T}$$

$DT_{\mathbb{R}}(GP)$



# The Flat Torus

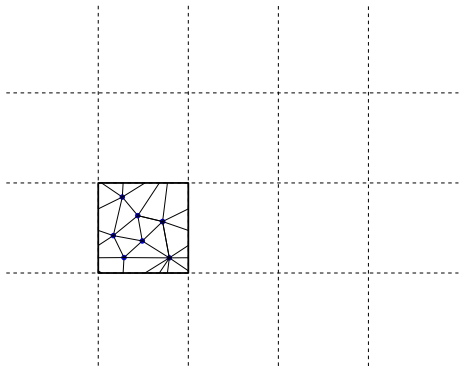
Delaunay triangulation

$$DT_{\mathbb{T}}(\pi(\mathcal{P})) = \pi(DT_{\mathbb{R}}(\mathcal{GP}))$$

$$G = \langle t_x, t_y \rangle$$

$$\mathbb{T} = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \rightarrow \mathbb{T}$$





# The Flat Torus

Delaunay triangulation

$$DT_{\mathbb{T}}(\pi(\mathcal{P})) = \pi(DT_{\mathbb{R}}(G\mathcal{P}))$$

*(under some conditions)*

$$G = \langle t_x, t_y \rangle$$

$$\mathbb{T} = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \rightarrow \mathbb{T}$$

Algorithm uses **covering spaces**

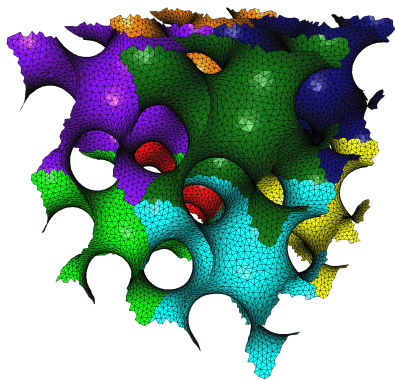
(3D) [Caroli, T.] ESA'09, SoCG'11, CGAL

video with astrophysicists (see abstract)

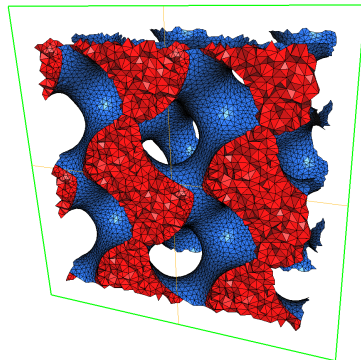
# The Flat Torus

CGAL 3D periodic volume meshes

*[Pellé, T.] (in progress)*



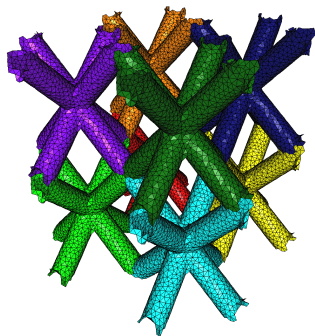
gyroid (TPMS)



# The Flat Torus

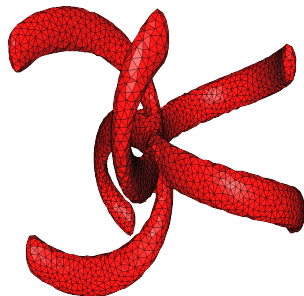
CGAL 3D periodic volume meshes

[Pellé, T.] (in progress)



bone scaffolding

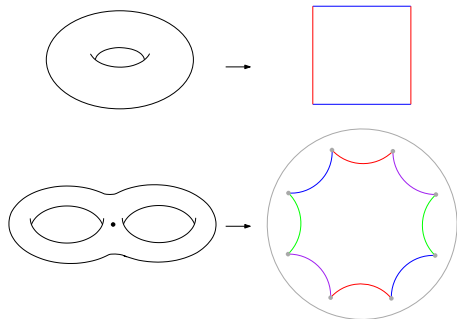
*data M. Moesen  
K.U. Leuven*



photonic crystal

*data M. Blome  
Zuse Institut Berlin*

## (2D) From 1 to 2 handles



Flat torus

$$\mathbb{T} = \mathbb{R}^2 / \mathcal{G}$$

$$\mathcal{G} = \langle t_x, t_y \rangle$$

Bolza surface

$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

$$\mathcal{G} = \langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

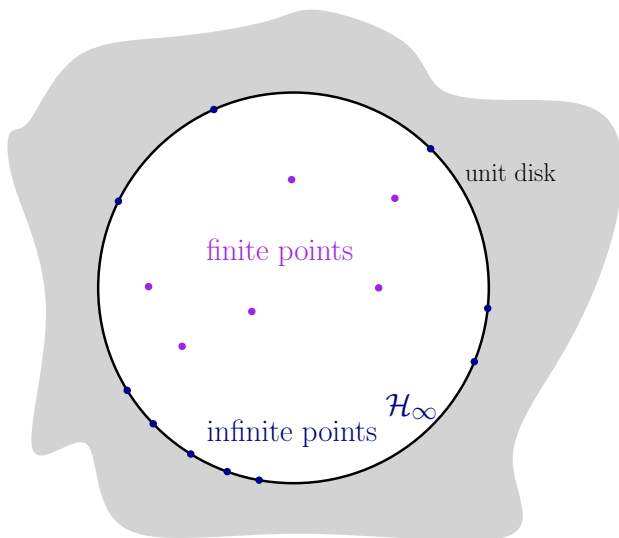
finitely presented group

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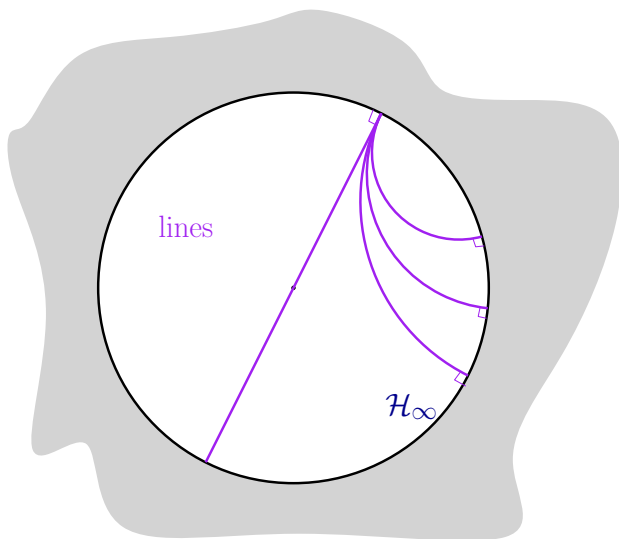
$\mathbb{H}^2$ 

## Poincaré disk model



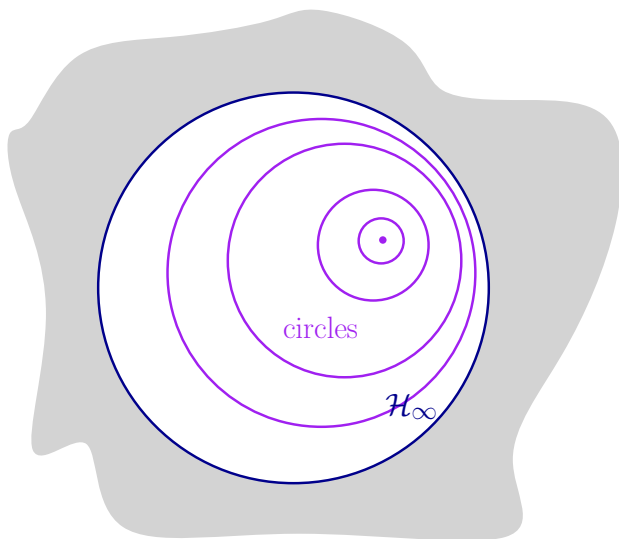
$\mathbb{H}^2$ 

## Poincaré disk model



$\mathbb{H}^2$ 

## Poincaré disk model

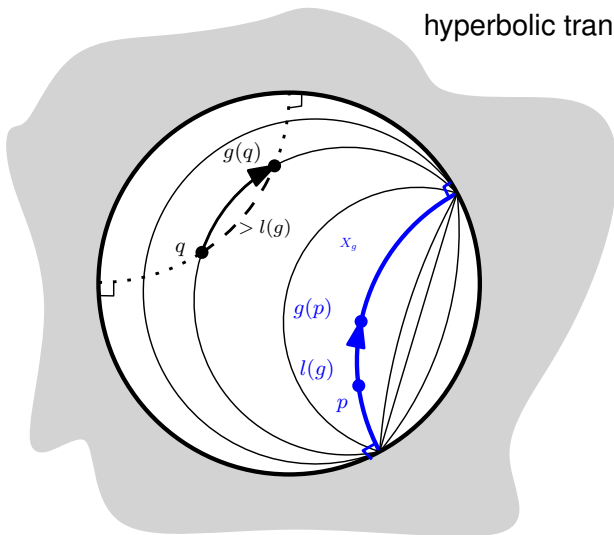






# Poincaré disk model

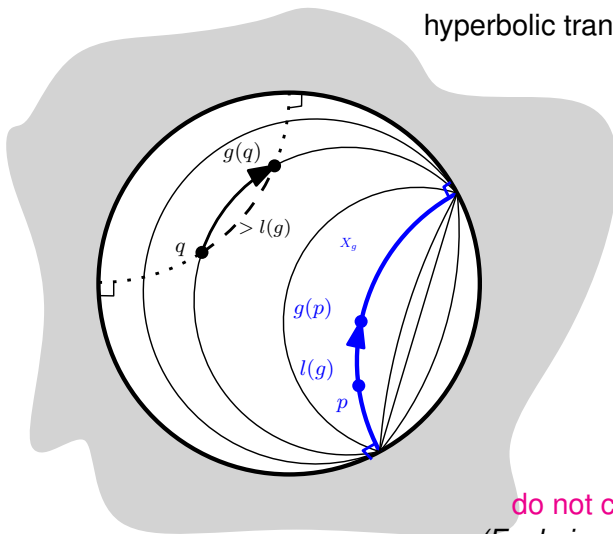
hyperbolic translations



$\mathbb{H}^2$ 

## Poincaré disk model

hyperbolic translations

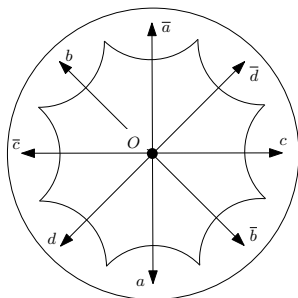


do not commute  
(Fuchsian groups)

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# The Bolza Surface

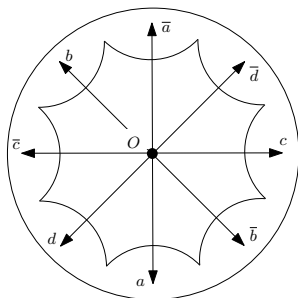


notation:  $gO = g, g \in \mathcal{G}$

$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

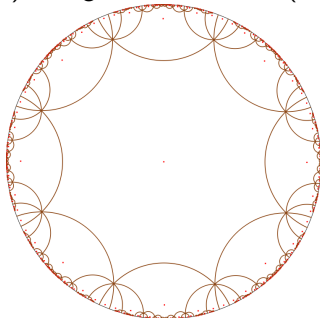
$$\mathcal{G} = \langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

# The Bolza Surface



notation:  $gO = g, g \in \mathcal{G}$

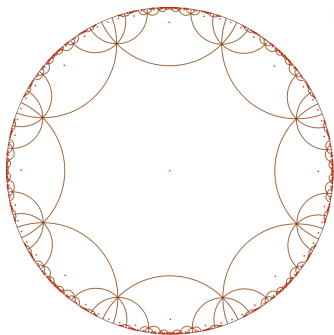
$\mathcal{D}_O(\mathcal{G}) = \text{region of } O \text{ in } \text{Vor}(\mathcal{G}O)$



$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

$$\mathcal{G} = \langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

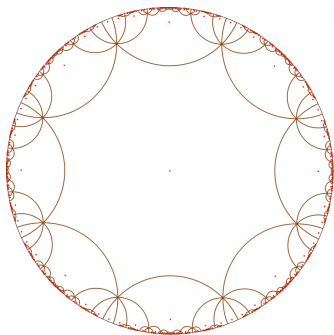
region of  $p$  in  $\text{Vor}(\mathcal{G}p)$ ,  $p \in \mathbb{H}^2$



8 sides

[Naatanen] [Bogdanov, T.]

region of  $p$  in  $Vor(\mathcal{G}p)$ ,  $p \in \mathbb{H}^2$

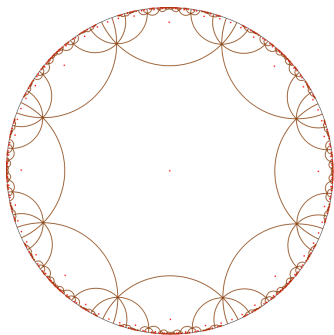


8 sides

sometimes 14 sides

[Naatanen] [Bogdanov, T.]

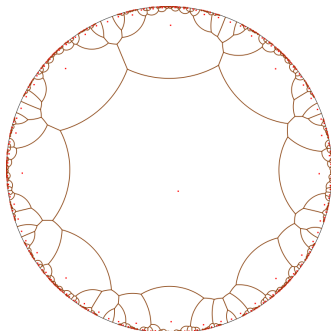
region of  $p$  in  $Vor(\mathcal{G}p)$ ,  $p \in \mathbb{H}^2$



8 sides

sometimes 14 sides

[Naatanen] [Bogdanov, T.]

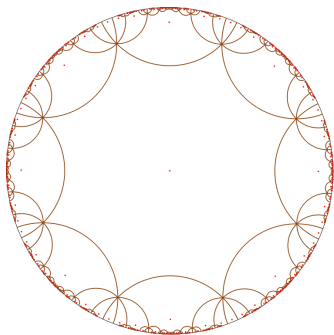


18 sides



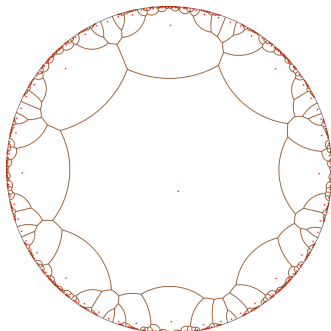
region of  $p$  in  $Vor(\mathcal{G}p)$ ,  $p \in \mathbb{H}^2$

[Naatanen] [Bogdanov, T.]



8 sides

sometimes 14 sides



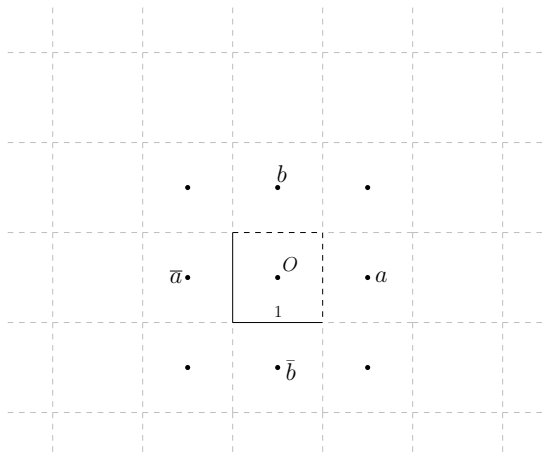
18 sides

relation with **orbits** under  $T(2, 3, 8)$

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## Flat torus



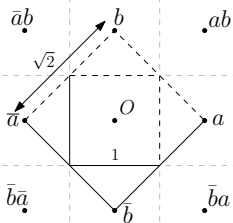
$$G = \langle a, b \rangle$$

$$\mathbb{T} = \mathbb{R}^2 / G$$

$$\pi : \mathbb{R}^2 \rightarrow \mathbb{T}$$

## 2-sheeted covering space

## Flat torus



$$G_2 = \langle \bar{a}b, ab \rangle$$

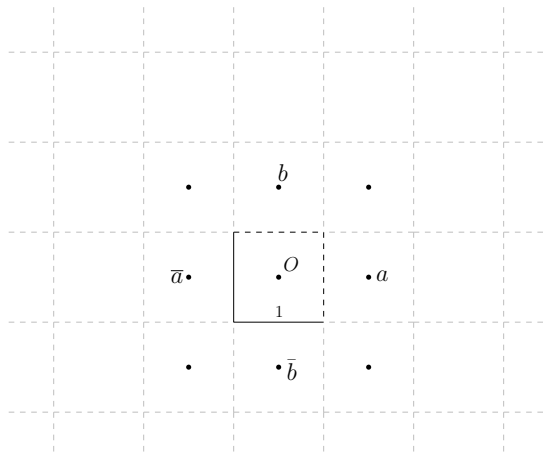
$$\mathbb{T}_2 = \mathbb{R}^2 / G_2$$

$$\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{T}_2$$

$2^k$ -sheeted covering spaces

## Flat torus

*sysole* of a surface = length of shortest non-contractible loop



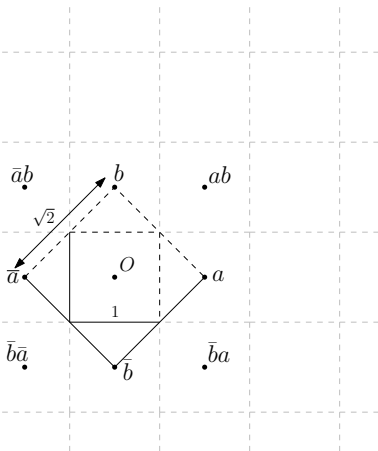
$$G = \langle a, b \rangle$$

$$\text{sys}(\mathbb{T}) = 1$$

$2^k$ -sheeted covering spaces

## Flat torus

*sys*tole of a surface = length of shortest non-contractible loop



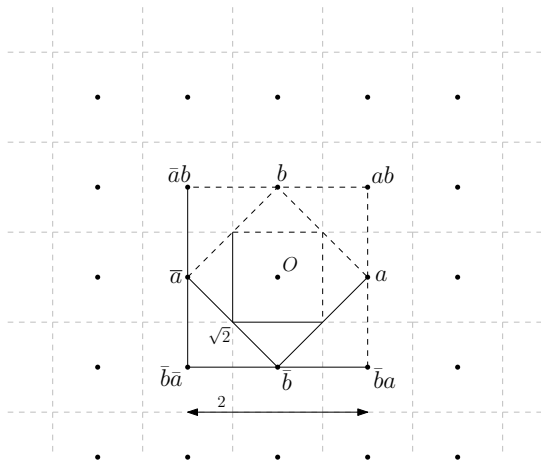
$$G_2 = \langle \bar{a}b, ab \rangle$$

$$\text{sys}(\mathbb{T}_2) = \sqrt{2}$$

$2^k$ -sheeted covering spaces

## Flat torus

*sysole* of a surface = length of shortest non-contractible loop



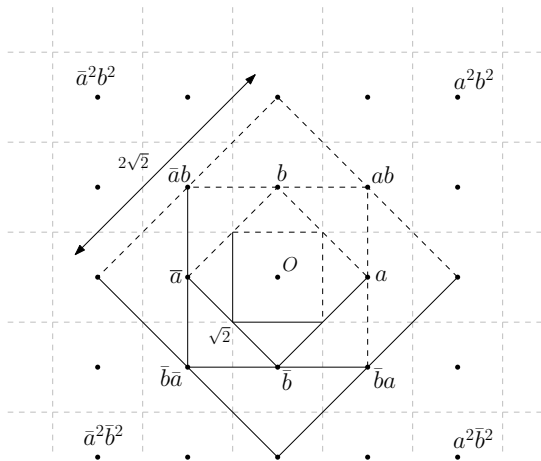
$$G_4 = \langle a^2, b^2 \rangle$$

$$\text{sys}(\mathbb{T}_4) = 2$$

# $2^k$ -sheeted covering spaces

## Flat torus

*sysole* of a surface = length of shortest non-contractible loop



$$G_8 = \langle \bar{a}^2 b^2, a^2 \bar{b}^2 \rangle$$

$$\text{sys}(\mathbb{T}_8) = 2\sqrt{2}$$



# $2^k$ -sheeted covering spaces

## Flat torus

$$G_{2^{k-1}} = \langle g, h \rangle$$

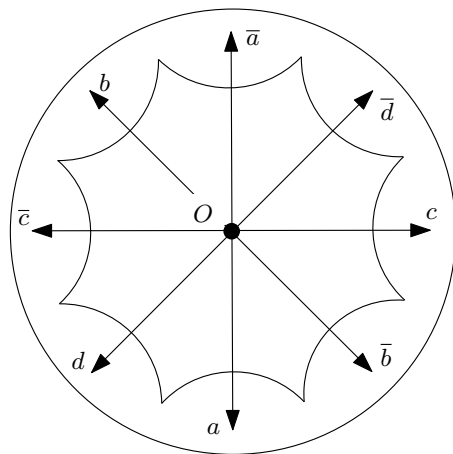
$$G_{2^k} = \langle \bar{g}h, gh \rangle$$

= elements of even size in elements of  $G_{2^{k-1}}$

$$\text{sys}(\mathbb{T}_{2^k}) = \sqrt{2}^k$$

[Bogdanov, T., Vegter] EuroCG'12

## Bolza surface



$$\mathcal{G} = \langle a, b, c, d \mid abc\bar{d}\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

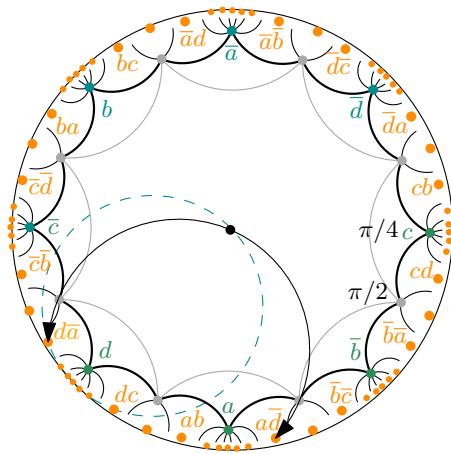
$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

$$\text{sys}(\mathcal{M}) = \text{translation\_length}(a) \simeq 3.06$$

$2^k$ -sheeted covering spaces

## Bolza surface

$\mathcal{G}_{k+1}$  generated by words of even size in the generators of  $\mathcal{G}_k$



$$\mathcal{G}_2 = \langle ab, cd, \bar{a}\bar{b}, \bar{c}\bar{d}, bc, d\bar{a}, \bar{b}\bar{c}, \bar{d}\bar{a} \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

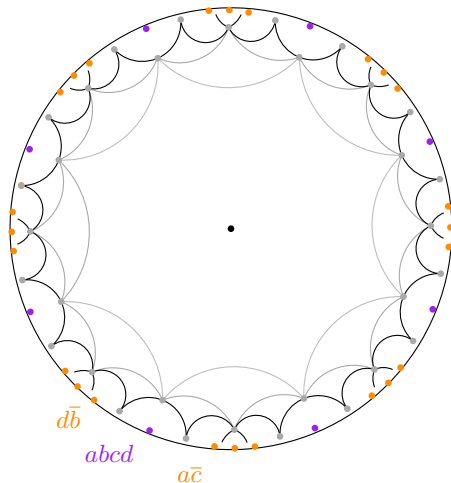
= only subgroup of  
index 2 of  $\mathcal{G}$  s.t.  
 $\mathcal{D}_O$  invariant by  $\rho_{\pi/4}$

$\text{sys}(\mathcal{M}_2) = \text{sys}(\mathcal{M})$   
(but fewer shortest loops)

$2^k$ -sheeted covering spaces

## Bolza surface

$\mathcal{G}_{k+1}$  generated by words of even size in the generators of  $\mathcal{G}_k$



$\mathcal{G}_4$  generated by  
 $d\bar{b}$ ,  $abcd$ ,  $a\bar{c}$   
 and their conjugates by  $\rho_{\pi/4}$

= only subgroup of  
 index 2 of  $\mathcal{G}_2$  s.t.  
 $\mathcal{D}_O$  invariant by  $\rho_{\pi/4}$

$\text{sys}(\mathcal{M}_4) > \text{sys}(\mathcal{M})$

# $2^k$ -sheeted covering spaces

## Bolza surface

[Bogdanov, T.]

$\delta_p(\mathcal{G})$  = diameter of largest empty disk in  $\mathcal{G}_p$

$$\delta_{\mathcal{M}} = \sup_p \delta_p(\mathcal{G})$$

$$4.89 < \delta_{\mathcal{M}} < 6.62$$

$2^k$ -sheeted covering spaces

## Bolza surface

[Bogdanov, T.]

 $\delta_p(\mathcal{G}) = \text{diameter of largest empty disk in } \mathcal{G}_p$ 

$$\delta_{\mathcal{M}} = \sup_p \delta_p(\mathcal{G})$$

$$4.89 < \delta_{\mathcal{M}} < 6.62$$

$$\text{sys}(\mathcal{M}_N) > 2 \delta_{\mathcal{M}} ?$$

$$\implies N > 32$$

 $N = 8$  for the flat torus

$$\impliedby N \geq 128$$

(using GAP)

## From 1 to 2 handles

Flat torus

Bolza surface

*quotient under the action of an*

Abelian group

non-Abelian group

*region of  $p$  in  $\text{Vor}(\mathcal{G}p)$*

does not depend on  $p$

depends on  $p$

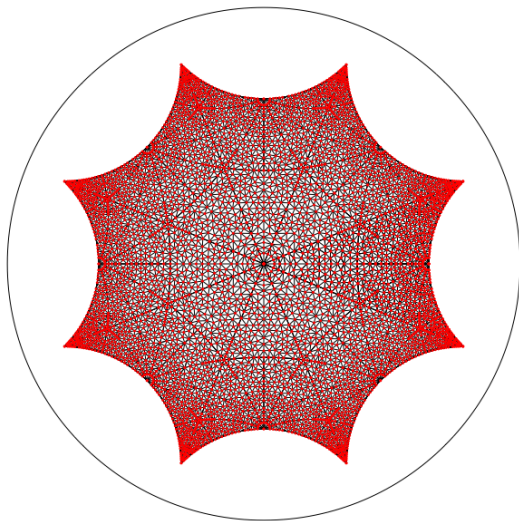
*systole of the  $2^k$  covering surfaces*

grows at each  $k$

does not grow at each  $k$

*etc*

Thank you



[Schmitt, T.]