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► **To cite this version:**

Alexandre Chotard, Anne Auger. Verifiable Conditions for Irreducibility, Aperiodicity and T-chain Property of a General Markov Chain. 2015. hal-01222222v1

HAL Id: hal-01222222

<https://inria.hal.science/hal-01222222v1>

Preprint submitted on 29 Oct 2015 (v1), last revised 23 Nov 2017 (v2)

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Verifiable Conditions for Irreducibility, Aperiodicity and T-chain Property of a General Markov Chain

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We consider in this paper Markov chains on a state space being an open subset of \mathbb{R}^n that obey the following general non linear state space model: $\Phi_{t+1} = F(\Phi_t, \alpha(\Phi_t, \mathbf{U}_{t+1}))$, $t \in \mathbb{N}$, where $(\mathbf{U}_t)_{t \in \mathbb{N}^*}$ (each $\mathbf{U}_t \in \mathbb{R}^p$) are i.i.d. random vectors, the function α , taking values in \mathbb{R}^m , is a measurable typically discontinuous function and $(\mathbf{x}, \mathbf{w}) \mapsto F(\mathbf{x}, \mathbf{w})$ is a C^1 function. In the spirit of the results presented in the chapter 7 of the Meyn and Tweedie book on “*Markov Chains and Stochastic Stability*”, we use the underlying deterministic control model to provide sufficient conditions that imply that the chain is a φ -irreducible, aperiodic T-chain with the support of the maximal irreducibility measure that has a non empty interior. To be able to show the same properties using previous results would require that the overall update function $(\mathbf{x}, \mathbf{u}) \mapsto F(\mathbf{x}, \alpha(\mathbf{x}, \mathbf{u}))$ is C^∞ and that \mathbf{U}_1 admits a lower semi-continuous density. In contrast, we assume that the function $(\mathbf{x}, \mathbf{w}) \mapsto F(\mathbf{x}, \mathbf{w})$ is C^1 , and that for all \mathbf{x} , $\alpha(\mathbf{x}, \mathbf{U}_1)$ admits a density $p_{\mathbf{x}}$ such that the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous. Hence the function $(\mathbf{x}, \mathbf{u}) \mapsto F(\mathbf{x}, \alpha(\mathbf{x}, \mathbf{u}))$ may have discontinuities captured by the function α .

We introduce the notion of a *strongly globally attracting state* and we prove that if there exists a strongly globally attracting state and a time step k , such that we find a k -path such that the k^{th} transition function starting from \mathbf{x}^* , $F^k(\mathbf{x}^*, \cdot)$, is a submersion at this k -path, the chain is a φ -irreducible, aperiodic, T-chain.

We present two applications of our results to Markov chains arising in the context of adaptive stochastic search algorithms to optimize continuous functions in a black-box scenario.

Keywords: Markov Chains, Irreducibility, Aperiodicity, T-chain, Control model, Optimization.

1. Introduction

Let X be an open subset of \mathbb{R}^n and O an open subset of \mathbb{R}^m equipped with their Borel sigma-algebra $\mathcal{B}(X)$ and $\mathcal{B}(O)$ for n, m two integers. This paper considers Markov chains $\Phi = (\Phi_t)_{t \in \mathbb{N}}$ defined on X via a multidimensional non-linear state space model

$$\Phi_{t+1} = G(\Phi_t, \mathbf{U}_{t+1}), \quad t \in \mathbb{N} \tag{1}$$

where $G : X \times \mathbb{R}^p \rightarrow X$ (for $p \in \mathbb{N}^*$) is a measurable function (\mathbb{R}^p being equipped of the Borel sigma-algebra) and $(\mathbf{U}_t)_{t \in \mathbb{N}^*}$ is an i.i.d. sequence of random vectors valued in \mathbb{R}^p

and defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ independent of Φ_0 also defined on the same probability space, and valued in X . In addition, we assume that Φ admits an alternative representation under the form

$$\Phi_{t+1} = F(\Phi_t, \alpha(\Phi_t, \mathbf{U}_{t+1})) \quad , \quad (2)$$

where $F : X \times O \rightarrow X$ is in a first time assumed measurable, but will typically be C^1 unless explicitly stated and $\alpha : X \times \mathbb{R}^p \rightarrow O$ is measurable and can be discontinuous. The functions F , G and α are connected via $G(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}, \alpha(\mathbf{x}, \mathbf{u}))$ for any \mathbf{x} in X and $\mathbf{u} \in \mathbb{R}^p$ such that G can also be *discontinuous*.

Deriving φ -irreducibility and aperiodicity of a general chain defined via (1) can sometimes be relatively challenging. An attractive way to do so is to investigate the underlying deterministic *control model* and use the results presented in [8, Chapter 7] that connect properties of the control model to the φ -irreducibility, aperiodicity and T -chain property of the chain. Indeed, it is typically easy to manipulate deterministic trajectories and prove properties related to this deterministic path. Unfortunately, the conditions developed in [8, Chapter 7] assume in particular that G is C^∞ and \mathbf{U}_t admits a lower semi-continuous density such that they cannot be applied to settings where G is discontinuous.

In this paper, following the approach to investigate the underlying control model for chains defined with (2), we develop general conditions that allow to easily verify φ -irreducibility, aperiodicity, the fact that the chain is a T -chain and identify that compact sets are small sets for the chain. Our approach relies on the fundamental assumptions that while α can be discontinuous, given $\mathbf{x} \in X$, $\alpha(\mathbf{x}, \mathbf{U})$ for \mathbf{U} distributed as \mathbf{U}_t admits a density $p_{\mathbf{x}}(\mathbf{w})$ where $\mathbf{w} \in O$ such that $p(\mathbf{x}, \mathbf{w}) = p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous. Hence we “pass” the discontinuity of G coming from the discontinuity of α into this density.

The model (2) is motivated by Markov chains arising in the stochastic black-box optimization context. Generally, Φ_t represents the state of a stochastic algorithm, for instance mean and covariance matrix of a multivariate normal distribution used to sample candidate solutions, \mathbf{U}_{t+1} contains the random inputs to sample the candidate solutions and $\alpha(\Phi_t, \mathbf{U}_{t+1})$ models the selection of candidate solutions according to a black-box function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ to be optimized. This selection step is usually discontinuous as points having similar function values can stem from different sampled vectors \mathbf{U}_{t+1} pointing to different solutions $\alpha(\Phi_t, \mathbf{U}_{t+1})$ belonging however to the same level set. The function F corresponds then to the update of the state of the algorithm given the selected solutions, and this update can be chosen to be at least C^1 . Some more detailed examples will be presented in Section 4. For some specific functions to be optimized, proving the *linear convergence* of the optimization algorithm can be done by investigating stability properties of a Markov chain underlying the optimization algorithm and following (2) [1, 2, 3]. Aperiodicity and φ -irreducibility are then two basic properties that generally need to be verified. This verification can turn out to be very challenging without the results developed in this paper. In addition, Foster-Lyapunov drift conditions are usually used to prove properties like Harris-recurrence, positivity or geometric ergodicity. Those drift conditions hold outside *small sets*. It is thus necessary to identify some small sets for the Markov chains.

Overview of the main results and structure of the paper The results we present stating the φ -irreducibility of a Markov chain defined via (2) use the concept of global attractiveness of a state—also used in [8]—that is a state that can be approached infinitely close from any initial state. We prove in Theorem 2 that if F is C^1 and the density $p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous, then the existence of a globally attractive state \mathbf{x}^* for which at some point in time, say k , we have a deterministic path such that the k^{th} transition function starting from \mathbf{x}^* , $F^k(\mathbf{x}^*, \cdot)$ is a submersion at this path, implies the φ -irreducibility of the chain. If we moreover assume that F is C^∞ , we can transfer the Theorem 7.2.6 of [8] to our setting and show that if the model is *forward accessible*, then φ -irreducibility is equivalent to the existence of a globally attracting state.

To establish the aperiodicity, we introduce the notion of a *strongly globally attracting state*, that is, informally speaking, a globally attracting state, \mathbf{x}^* , for which for any initial state \mathbf{y} and any distance $\epsilon > 0$, there exists a time step, say $t_{\mathbf{y}, \epsilon}$, such that we find for all time step larger than $t_{\mathbf{y}, \epsilon}$ a deterministic path that puts the chain within distance ϵ of \mathbf{x}^* . We then prove in Theorem 3 that under the same conditions than for the φ -irreducibility but holding at a strongly globally attracting state (instead of only a globally attracting state), the chain is φ -irreducible and aperiodic.

Those two theorems contain the main ingredients to prove the main theorem of the paper, Theorem 1, that under the same conditions than for the aperiodicity states that the chain is a φ -irreducible aperiodic T -chain for which compact sets are small sets.

This paper is structured as follows. In Section 2, we introduce and remind several definitions related to the Markov chain model of the paper needed all along the paper. We also present a series of technical results that are necessary in the next sections. In Section 3 we present the main result, i.e. Theorem 1, that states sufficient conditions for a Markov chain to be a φ -irreducible aperiodic T -chain for which compact sets are small sets. This result is a consequence of the propositions established in the subsequent subsections, namely Theorem 2 for the φ -irreducibility, Theorem 3 for the aperiodicity and Proposition 5 for the weak-Feller property. We also derive intermediate propositions and corollaries that clarify the connection between our results and the ones of [8, Chapter 7] (Proposition 3, Corollary 1) and that characterize the support of the maximal irreducibility measure (Proposition 4). We present in Section 4 two applications of our results. We detail two homogeneous Markov chains associated to two adaptive stochastic search algorithms aiming at optimizing continuous functions, sketch why establishing their φ -irreducibility, aperiodicity and identifying some small sets is important while explaining why existing tools cannot be applied. We then illustrate how the assumptions of Theorem 1 can be easily verified and establish thus that the chains are φ -irreducible, aperiodic, T -chains for which compact sets are small sets.

Notations

For A and B subsets of X , $A \subset B$ denotes that A is included in B (\subsetneq denotes the strict inclusion). We denote \mathbb{R}^n the set of n -dimensional real vectors, \mathbb{R}_+ the set of non-negative real numbers, \mathbb{N} the set of natural numbers $\{0, 1, \dots\}$, and for $(a, b) \in \mathbb{N}^2$,

$[a..b] = \bigcup_{i=a}^b \{i\}$. For $A \subset \mathbb{R}^n$, A^* denotes $A \setminus \{\mathbf{0}\}$. For X a metric space, $\mathbf{x} \in X$ and $\epsilon > 0$, $B(\mathbf{x}, \epsilon)$ denotes the open ball of center \mathbf{x} and radius ϵ . For $X \subset \mathbb{R}^n$ a topological space, $\mathcal{B}(X)$ denotes the Borel σ -algebra on X . We denote Λ_n the Lebesgue measure on \mathbb{R}^n , and for $B \in \mathcal{B}(\mathbb{R}^n)$, μ_B denotes the trace-measure $A \in \mathcal{B}(\mathbb{R}^n) \mapsto \Lambda_n(A \cap B)$. For $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y}$ denotes the scalar product of \mathbf{x} and \mathbf{y} , and $[\mathbf{x}]_i$ denotes the i^{th} coordinate of the vector \mathbf{x} and \mathbf{x}^T denotes the transpose of the vector \mathbf{x} . For a function $f : X \rightarrow \mathbb{R}^n$, we say that f is C^p if f is continuous, and its p -first derivatives exist and are continuous. For $f : X \rightarrow \mathbb{R}^n$ a differentiable function and $\mathbf{x} \in X$, $D_{\mathbf{x}}f$ denotes the differential of f at \mathbf{x} . A multivariate normal distribution with mean vector zero and covariance matrix identity is called a *standard multivariate normal distribution*, a standard normal distribution correspond to the case of the dimension 1. We use the notation $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ for the standard multivariate normal distribution where \mathbf{I}_n is the identity matrix in dimension n . We use the acronym i.i.d. for independent identically distributed.

2. Definitions and Preliminary Results

The random vectors defined in the previous section are assumed measurable with respect to the Borel σ -algebras of their codomain. We denote for all $t \in \mathbb{N}$ the random vector $\alpha(\Phi_t, \mathbf{U}_{t+1})$ of O as \mathbf{W}_{t+1} , i.e.

$$\mathbf{W}_{t+1} := \alpha(\Phi_t, \mathbf{U}_{t+1}) \quad (3)$$

such that Φ satisfies

$$\Phi_{t+1} = F(\Phi_t, \mathbf{W}_{t+1}) \quad (4)$$

Given $\Phi_t = \mathbf{x}$, the vector \mathbf{W}_t is assumed to admit a probability density $p_{\mathbf{x}}(\mathbf{w})$. The function $p(\mathbf{x}, \mathbf{w}) = p_{\mathbf{x}}(\mathbf{w})$ will be assumed lower semi-continuous in the whole paper.

We remind the definition of a substochastic transition kernel as well as of a transition kernel. Let $K : X \times \mathcal{B}(X) \rightarrow \mathbb{R}_+$ such that for all $A \in \mathcal{B}(X)$, the function $\mathbf{x} \in X \mapsto K(\mathbf{x}, A)$ is a non-negative measurable function, and for all $\mathbf{x} \in X$, $K(\mathbf{x}, \cdot)$ is a measure on $\mathcal{B}(X)$. If for all $\mathbf{x} \in X$, $K(\mathbf{x}, X) \leq 1$ then K is called a *substochastic transition kernel*, and if for all $\mathbf{x} \in X$, $K(\mathbf{x}, X) = 1$ then K is called a *transition kernel*.

Given F and $p_{\mathbf{x}}$ we define for all $\mathbf{x} \in X$ and all $A \in \mathcal{B}(X)$

$$P(\mathbf{x}, A) = \int 1_A(F(\mathbf{x}, \mathbf{w})) p_{\mathbf{x}}(\mathbf{w}) d\mathbf{w} \quad (5)$$

Then the function $\mathbf{x} \in X \mapsto P(\mathbf{x}, A)$ is measurable for all $A \in \mathcal{B}(X)$ (as a consequence of Fubini's theorem) and for all \mathbf{x} , $P(\mathbf{x}, \cdot)$ defines a measure on $(X, \mathcal{B}(X))$. Hence $P(\mathbf{x}, A)$ defines a transition kernel. It is immediate to see that this transition kernel corresponds to the transition kernel of the Markov chain defined in (2) or (4).

For $\mathbf{x} \in X$, we denote $O_{\mathbf{x}}$ the set of \mathbf{w} such that $p_{\mathbf{x}}$ is strictly positive, i.e.

$$O_{\mathbf{x}} := \{\mathbf{w} \in O \mid p_{\mathbf{x}}(\mathbf{w}) > 0\} = p_{\mathbf{x}}^{-1}((0, +\infty)) \quad (6)$$

that we call support of $p_{\mathbf{x}}$ ¹. Similarly to [8, Chapter 7] we consider the recursive functions F^t for $t \in \mathbb{N}^*$ such that $F^1 := F$ and for $\mathbf{x} \in X$ and $(\mathbf{w}_i)_{i \in [1..t+1]} \in O^{t+1}$

$$F^{t+1}(\mathbf{x}, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{t+1}) := F(F^t(\mathbf{x}, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_t), \mathbf{w}_{t+1}) . \quad (7)$$

The function F^t is connected to the Markov chain $\Phi = (\Phi_t)_{t \in \mathbb{N}}$ defined via (4) in the following manner

$$\Phi_t = F^t(\Phi_0, \mathbf{W}_1, \dots, \mathbf{W}_t) . \quad (8)$$

In addition, we define $p_{\mathbf{x},t}$ as $p_{\mathbf{x}}$ for $t = 1$ and for $t > 1$

$$p_{\mathbf{x},t}((\mathbf{w}_i)_{i \in [1..t]}) := p_{\mathbf{x},t-1}((\mathbf{w}_i)_{i \in [1..t-1]}) p_{F^{t-1}(\mathbf{x}, (\mathbf{w}_i)_{i \in [1..t-1]})}(\mathbf{w}_t) , \quad (9)$$

that is

$$p_{\mathbf{x},t}((\mathbf{w}_i)_{i \in [1..t]}) = p_{\mathbf{x}}(\mathbf{w}_1) p_{F(\mathbf{x}, \mathbf{w}_1)}(\mathbf{w}_2) \dots p_{F^{t-1}(\mathbf{x}, \mathbf{w}_1, \dots, \mathbf{w}_{t-1})}(\mathbf{w}_t) . \quad (10)$$

Then $p_{\mathbf{x},t}$ is measurable as the composition and product of measurable functions. Let $O_{\mathbf{x},t}$ be the support of $p_{\mathbf{x},t}$

$$O_{\mathbf{x},t} := \{\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_t) \in O^t \mid p_{\mathbf{x},t}(\mathbf{w}) > 0\} = p_{\mathbf{x},t}^{-1}((0, +\infty)) . \quad (11)$$

Then by the measurability of $p_{\mathbf{x},t}$, $O_{\mathbf{x},t}$ is a Borel set of O^t (endowed with the Borel σ -algebra). Note that $O_{\mathbf{x},1} = O_{\mathbf{x}}$. Given $\Phi_0 = \mathbf{x}$, the function $p_{\mathbf{x},t}$ is the joint probability distribution function of $(\mathbf{W}_1, \dots, \mathbf{W}_t)$.

Since $p_{\mathbf{x},t}$ is the joint probability distribution of $(\mathbf{W}_1, \dots, \mathbf{W}_t)$ given $\Phi_0 = \mathbf{x}$ and because Φ_t is linked to F^t via (8), the t -steps transition kernel P^t of Φ writes

$$P^t(\mathbf{x}, A) = \int_{O_{\mathbf{x},t}} \mathbf{1}_A(F^t(\mathbf{x}, \mathbf{w}_1, \dots, \mathbf{w}_t)) p_{\mathbf{x},t}(\mathbf{w}) d\mathbf{w} , \quad (12)$$

for all $\mathbf{x} \in X$ and all $A \in \mathcal{B}(X)$.

The deterministic system with trajectories

$$\mathbf{x}_t = F^t(\mathbf{x}_0, \mathbf{w}_1, \dots, \mathbf{w}_t) = F^t(\mathbf{x}_0, \mathbf{w})$$

for $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_t) \in O_{\mathbf{x},t}$ and for any $t \in \mathbb{N}^*$ is called the *associated control model* and is denoted $\text{CM}(F)$. Using a similar terminology to Meyn and Tweedie's [8], we say that $O_{\mathbf{x}}$ is a control set for $\text{CM}(F)$. We introduce the notion of t -steps path from a point $\mathbf{x} \in X$ to a set $A \in \mathcal{B}(X)$ as follows:

Definition 1 (*t*-steps path). *For $\mathbf{x} \in X$, $A \in \mathcal{B}(X)$ and $t \in \mathbb{N}^*$, we say that $\mathbf{w} \in O^t$ is a *t*-steps path from \mathbf{x} to A if $\mathbf{w} \in O_{\mathbf{x},t}$ and $F^t(\mathbf{x}, \mathbf{w}) \in A$.*

¹Note that the support is often defined as the closure of what we call support here.

Similarly to chapter 7 of Meyn-Tweedie, we define $A_+^0(\mathbf{x}) := \{\mathbf{x}\}$ and for $k \in \mathbb{N}^*$

$$A_+^k(\mathbf{x}) := \{F^k(\mathbf{x}, \mathbf{w}) \mid \mathbf{w} \in O_{\mathbf{x},k}\}$$

that is the set of all states that can be reached from \mathbf{x} after k steps.

Note that this definition depends on the probability density function $p_{\mathbf{x}}$ that determines the set of control sequences $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_k)$ via the definition of $O_{\mathbf{x},k}$. More precisely, several density functions equal almost everywhere can be associated to a random vector $\alpha(\mathbf{x}, \mathbf{U}_1)$. However, they can generate different sets $A_+^k(\mathbf{x})$.

Following [8], the set of states that can be reached starting from \mathbf{x} at some time in the future from \mathbf{x} is defined as

$$A_+(\mathbf{x}) = \bigcup_{k=0}^{+\infty} A_+^k(\mathbf{x}) .$$

The associated control model $\text{CM}(F)$ is *forward accessible* if for all \mathbf{x} , $A_+(\mathbf{x})$ has non empty interior [6].

Finally, a point \mathbf{x}^* is called a *globally attracting state* if for all $\mathbf{y} \in X$,

$$\mathbf{x}^* \in \Omega_+(\mathbf{y}) := \bigcap_{N=1}^{+\infty} \overline{\bigcup_{k=N}^{+\infty} A_+^k(\mathbf{y})} . \quad (13)$$

Although in general $\Omega_+(\mathbf{y}) \neq \overline{A_+(\mathbf{y})}$, these two sets can be used to define globally attracting states, as shown in the following proposition.

Proposition 1. *A point $\mathbf{x}^* \in X$ is a globally attracting state if and only if one of the three following equivalent conditions hold:*

- (1) for all $\mathbf{y} \in X$, $\mathbf{x}^* \in \overline{A_+(\mathbf{y})}$,
- (2) for all $\mathbf{y} \in X$ and any $U \in \mathcal{B}(X)$ neighbourhood of \mathbf{x}^* , there exists $t \in \mathbb{N}^*$ such that there exists a t -steps path from \mathbf{y} to U ,
- (3) for all $\mathbf{y} \in X$ there exists a sequence $(\mathbf{y}_t)_{t \in \mathbb{N}^*} \in \prod_{t \in \mathbb{N}^*} A_+^t(\mathbf{y})$ from which a subsequence converging to \mathbf{x}^* can be extracted.

Proof. Let us prove that (13) implies (1). Let \mathbf{x}^* be a globally attracting state. According to (13) for all $\mathbf{y} \in X$, $\mathbf{x}^* \in \bigcup_{k=1}^{+\infty} A_+^k(\mathbf{y}) \subset \overline{A_+(\mathbf{y})}$, so $\mathbf{x}^* \in \overline{A_+(\mathbf{y})}$.

Now we show that (1) implies (2). Suppose (1), take $\mathbf{y} \in X$ and take and U a neighbourhood of \mathbf{x}^* . Since $\mathbf{x}^* \in \overline{A_+(\mathbf{y})}$, there exists a sequence $(\mathbf{y}_t)_{t \in \mathbb{N}}$ of points of $A_+(\mathbf{y})$ converging to \mathbf{x}^* , and so there exists $t \in \mathbb{N}$ such that $\mathbf{y}_t \in U$. Since $\mathbf{y}_t \in A_+(\mathbf{y})$, either $\mathbf{y}_t = \mathbf{y}$ or there exists $k \in \mathbb{N}^*$ and $\mathbf{w} \in O_{\mathbf{y},k}$ such that $F^k(\mathbf{y}, \mathbf{w}) = \mathbf{y}_t \in U$. In the second case, \mathbf{w} is a k -steps path from \mathbf{y} to U . In the first case where $\mathbf{y}_t = \mathbf{y}$, take any $\mathbf{u} \in O_{\mathbf{y}}$. If $F(\mathbf{y}, \mathbf{u}) \in U$ then \mathbf{u} is a 1-step path from \mathbf{y} to U . Else, since $\mathbf{x}^* \in \overline{A_+(F(\mathbf{y}, \mathbf{u}))}$ and that $F(\mathbf{y}, \mathbf{u}) \notin U$, from the same reasoning as before there exists $k \in \mathbb{N}^*$ and \mathbf{v} a k -steps path from $F(\mathbf{y}, \mathbf{u})$ to U , and so (\mathbf{u}, \mathbf{v}) is a $k+1$ -steps path from \mathbf{y} to U .

Now we show that (2) implies (3). Suppose that (2) holds. Hence there exists $\tau_1 \in \mathbb{N}^*$ and \mathbf{w}_{τ_1} a τ_1 -steps path from \mathbf{y} to $B(\mathbf{x}^*, 1)$. Let \mathbf{y}_{τ_1} denote $F^{\tau_1}(\mathbf{y}, \mathbf{w}_{\tau_1})$. Similarly, there exists $\tau_2 \in \mathbb{N}^*$ and \mathbf{w}_2 a τ_2 -steps path from \mathbf{y}_{τ_1} to $B(\mathbf{x}^*, 1/2)$ and so on. We then have $\mathbf{y}_{\tau_t} \in A_+^{\tau_1 + \dots + \tau_t}(\mathbf{y})$ and $\mathbf{y}_{\tau_t} \in B(\mathbf{x}^*, 1/t)$. Hence we have a sequence $(\mathbf{y}_{\tau_t})_{t \in \mathbb{N}^*}$ converging to \mathbf{x}^* which is a subsequence of a sequence of $\prod_{t \in \mathbb{N}^*} A_+^t(\mathbf{y})$.

Finally we show that (3) implies that (13) holds for all $\mathbf{y} \in X$. Suppose that (3) holds. So there exists $(\mathbf{y}_t)_{t \in \mathbb{N}^*}$ a sequence of $\prod_{t \in \mathbb{N}^*} A_+^t(\mathbf{y})$ from which we can extract a subsequence converging to \mathbf{x}^* . Since $\mathbf{y}_t \in A_+^t(\mathbf{y}) \subset \bigcup_{k=t}^{\infty} A_+^k(\mathbf{y})$, and that for any $N \in \mathbb{N}^*$ the state \mathbf{x}^* is the limit of a subsequence of $(\mathbf{y}_t)_{t \geq N}$, we have $\mathbf{x}^* \in \overline{\bigcup_{t \geq N} A_+^t(\mathbf{y})}$ for any $N \in \mathbb{N}^*$, and so (13) holds. \square

The existence of a globally attracting state is linked in [8, Proposition 7.2.5] with φ -irreducibility. We will show that this link extends to our context.

We now define the notion of *strongly globally attractive state* that is needed for our result on the aperiodicity. More precisely we define:

Definition 2 (Strongly globally attracting state). *A point $\mathbf{x}^* \in X$ is called a strongly globally attracting state if for all $\mathbf{y} \in X$, for all $\epsilon \in \mathbb{R}_+^*$, there exists $t_{\mathbf{y}, \epsilon} \in \mathbb{N}^*$ such that for all $t \geq t_{\mathbf{y}, \epsilon}$, there exists a t -steps path from \mathbf{y} to $B(\mathbf{x}^*, \epsilon)$. That is, for all $(\mathbf{y}, \epsilon) \in X \times \mathbb{R}_+^*$*

$$\exists t_{\mathbf{y}, \epsilon} \in \mathbb{N}^* \text{ such that } \forall t \geq t_{\mathbf{y}, \epsilon}, A_+^t(\mathbf{y}) \cap B(\mathbf{x}^*, \epsilon) \neq \emptyset. \quad (14)$$

Equivalently, $\mathbf{x}^ \in X$ is a strongly globally attracting state if and only if for all $\mathbf{y} \in X$ there exists a sequence $(\mathbf{y}_t)_{t \in \mathbb{N}^*} \in \prod_{t \in \mathbb{N}^*} A_+^t(\mathbf{y})$ converging to \mathbf{x}^* .*

Proof. Let $\mathbf{x}^* \in X$ such that for all $\mathbf{y} \in X$ and $\epsilon \in \mathbb{R}_+^*$, there exists $t_{\mathbf{y}, \epsilon} \in \mathbb{N}^*$ such that for all $t \geq t_{\mathbf{y}, \epsilon}$, there exists a t -steps path from \mathbf{y} to $B(\mathbf{x}^*, \epsilon)$. So there exists $(t_{\mathbf{y}, 1/k})_{k \in \mathbb{N}^*}$ a sequence of \mathbb{N}^* such that for any $t \geq t_{\mathbf{y}, 1/k}$, there exists \mathbf{w}_t a t -steps path from \mathbf{y} to $B(\mathbf{x}^*, 1/k)$. For $t \geq t_{\mathbf{y}, 1}$ let \mathbf{w}_t be a t -step path from \mathbf{y} to $B(\mathbf{x}^*, \epsilon)$, where $\epsilon := 1/\sup\{k \in \mathbb{N}^* | t_{\mathbf{y}, 1/k} \leq t\}$, and for $t < t_{\mathbf{y}, 1}$ let $\mathbf{w}_t \in O_{\mathbf{y}, t}$. This way we take ϵ as small as we can to ensure the convergence of $(F^t(\mathbf{y}, \mathbf{w}_t))_{t \in \mathbb{N}^*}$ to \mathbf{x}^* . Let us denote $\mathbf{y}_t := F^t(\mathbf{y}, \mathbf{w}_t)$. For any $k \in \mathbb{N}^*$ and for t large enough, $t_{\mathbf{y}, 1/k} \leq t$ and so $1/\epsilon = \sup\{a \in \mathbb{N}^* | t_{\mathbf{y}, 1/a} \leq t\} \geq k$, hence by definition of \mathbf{w}_t and \mathbf{y}_t , $\mathbf{y}_t \in B(\mathbf{x}^*, \epsilon) \subset B(\mathbf{x}^*, 1/k)$. Hence the sequence $(\mathbf{y}_t)_{t \in \mathbb{N}^*}$ converges to \mathbf{x}^* .

Now suppose that there exists $\mathbf{x}^* \in X$ such that for all $\mathbf{y} \in X$ there exists $(\mathbf{y}_t)_{t \in \mathbb{N}^*}$ a sequence of $\prod_{t \in \mathbb{N}^*} A_+^t(\mathbf{y})$ converging to \mathbf{x}^* . Hence for all $\epsilon \in \mathbb{R}_+^*$, there exists $t_{\mathbf{y}, \epsilon} \in \mathbb{N}^*$ such that for all $t \geq t_{\mathbf{y}, \epsilon}$, $\mathbf{y}_t \in B(\mathbf{x}^*, \epsilon)$. And since $\mathbf{y}_t \in A_+^t(\mathbf{y}, \cdot)$, there exists \mathbf{w}_t such that $F^t(\mathbf{y}, \mathbf{w}_t) = \mathbf{y}_t \in B(\mathbf{x}^*, \epsilon)$, and so \mathbf{w}_t is a t -steps path from \mathbf{y} to $B(\mathbf{x}^*, \epsilon)$. \square

The following proposition connects globally and strongly globally attracting states.

Proposition 2. *Let $\mathbf{x}^* \in X$ be a strongly globally attracting state, then \mathbf{x}^* is a globally attracting state.*

Proof. This is immediate from comparing (3) of Proposition 1 and the second definition of strongly globally attracting states. \square

Our aim is to derive conditions for proving φ -irreducibility, aperiodicity and prove that compacts of X are small sets. We remind below the formal definitions associated to those notions as well as the definition of a weak Feller chain and a T-chain. A Markov chain Φ is φ -irreducible if there exists a measure φ on $\mathcal{B}(X)$ such that for all $A \in \mathcal{B}(X)$

$$\varphi(A) > 0 \Rightarrow \sum_{t=1}^{\infty} P^t(\mathbf{x}, A) > 0 \text{ for all } \mathbf{x} . \quad (15)$$

A set C is *small* if there exists $t \in \mathbb{N}^*$ and a non-trivial measure ν_t on $\mathcal{B}(X)$ such that for all $\mathbf{z} \in C$

$$P^t(\mathbf{z}, A) \geq \nu_t(A), \quad A \in \mathcal{B}(X) . \quad (16)$$

The small set is then called a ν_t -small set. Consider a small set C satisfying the previous equation with $\nu_t(C) > 0$ and denote $\nu_t = \nu$. The chain is called aperiodic if the g.c.d. of the set

$$E_C = \{k \geq 1 : C \text{ is a } \nu_k\text{-small set with } \nu_k = \alpha_k \nu \text{ for some } \alpha_k > 0\}$$

is one for some (and then for every) small set C .

The transition kernel of Φ is acting on bounded functions $f : X \rightarrow \mathbb{R}$ via the following operator

$$Pf(\mathbf{x}) \mapsto \int f(\mathbf{y})P(\mathbf{x}, d\mathbf{y}), \quad \mathbf{x} \in X . \quad (17)$$

Let $\mathcal{C}(X)$ be the class of bounded continuous functions from X to \mathbb{R} , then Φ is *weak Feller* if P maps $\mathcal{C}(X)$ to $\mathcal{C}(X)$. This definition is equivalent to $P1_O$ is lower semicontinuous for every open set $O \in \mathcal{B}(X)$.

Let a be a probability distribution on \mathbb{N} , we denote

$$K_a : (\mathbf{x}, A) \in X \times \mathcal{B}(X) \mapsto \sum_{i \in \mathbb{N}} a(i)P^i(\mathbf{x}, A) \quad (18)$$

the transition kernel, the associated Markov chain being called the K_a chain with *sampling distribution* a . When a satisfy the geometric distribution

$$a_\epsilon(i) = (1 - \epsilon)\epsilon^i \quad (19)$$

for $i \in \mathbb{N}$, then the transition kernel K_{a_ϵ} is called the resolvent. If there exists a sub-stochastic transition kernel T satisfying

$$K_a(\mathbf{x}, A) \geq T(\mathbf{x}, A)$$

for all $\mathbf{x} \in X$ and $A \in \mathcal{B}(X)$ with $T(\cdot, A)$ a lower semi-continuous function, then T is called a *continuous component* of K_a ([8, p.124]). If there exists a sampling distribution

a and T a continuous component of K_a such that $T(\mathbf{x}, X) > 0$ for all $\mathbf{x} \in X$, then the Markov chain Φ is called a T -chain ([8, p.124]). We say that $B \in \mathcal{B}(X)$ is *uniformly accessible using a* from $A \in \mathcal{B}(X)$ if there exists $\delta \in \mathbb{R}_+^*$ such that

$$\inf_{\mathbf{x} \in A} K_a(\mathbf{x}, B) > \delta ,$$

which is written as $A \overset{a}{\rightsquigarrow} B$ ([8, p.116]).

2.1. Technical results

We present in this section a series of technical results that will be needed to establish the main results of the paper.

Lemma 1. *Let $A \in \mathcal{B}(X)$ with X an open set of \mathbb{R}^n . If for all $\mathbf{x} \in A$ there exists $V_{\mathbf{x}}$ an open neighbourhood of \mathbf{x} such that $A \cap V_{\mathbf{x}}$ is Lebesgue negligible, then A is Lebesgue negligible.*

Proof. For $\mathbf{x} \in A$, let $r_{\mathbf{x}} > 0$ be such that $B(\mathbf{x}, r_{\mathbf{x}}) \subset V_{\mathbf{x}}$, and take $\epsilon > 0$. The set $\bigcup_{\mathbf{x} \in A} B(\mathbf{x}, r_{\mathbf{x}}/2) \cap B(\mathbf{0}, \epsilon)$ is closed and bounded, so it is a compact, and $\bigcup_{\mathbf{x} \in A} V_{\mathbf{x}}$ is an open cover of this compact. Hence we can extract a finite subcover $(V_{\mathbf{x}_i})_{i \in I}$, and so $\bigcup_{i \in I} V_{\mathbf{x}_i} \supset A \cap B(\mathbf{0}, \epsilon)$. Hence, it also holds that $A \cap B(\mathbf{0}, \epsilon) = \bigcup_{i \in I} A \cap B(\mathbf{0}, \epsilon) \cap V_{\mathbf{x}_i}$. Since by assumption $\Lambda_n(A \cap V_{\mathbf{x}_i}) = 0$, from the sigma-additivity property of measures we deduce that $\Lambda_n(A \cap B(\mathbf{0}, \epsilon)) = 0$. So with Fatou's lemma $\int_X \mathbf{1}_A(\mathbf{x}) d\mathbf{x} \leq \liminf_{k \rightarrow +\infty} \int_X \mathbf{1}_{A \cap B(\mathbf{0}, k)}(\mathbf{x}) d\mathbf{x} = 0$, which shows that $\Lambda_n(A) = 0$. \square

Lemma 2. *Suppose that $F : X \times O \rightarrow X$ is C^p for $p \in \mathbb{N}$, then for all $t \in \mathbb{N}^*$, $F^t : X \times O^t \rightarrow X$ defined as in (7) is C^p .*

Proof. By hypothesis, $F^1 = F$ is C^p . Suppose that F^t is C^p . Then the function $h : (\mathbf{x}, (\mathbf{w}_i)_{i \in [1..t+1]}) \in X \times O^{t+1} \mapsto (F^t(\mathbf{x}, \mathbf{w}_1, \dots, \mathbf{w}_t), \mathbf{w}_{t+1})$ is C^p , and so is $F^{t+1} = F \circ h$. \square

Lemma 3. *Suppose that the function $p : (\mathbf{x}, \mathbf{w}) \in X \times O \mapsto p_{\mathbf{x}}(\mathbf{w}) \in \mathbb{R}_+$ is lower semi-continuous and the function $F : (\mathbf{x}, \mathbf{w}) \in X \times O \mapsto F(\mathbf{x}, \mathbf{w}) \in X$ is continuous, then for all $t \in \mathbb{N}^*$ the function $(\mathbf{x}, \mathbf{w}) \in X \times O^t \mapsto p_{\mathbf{x}, t}(\mathbf{w})$ defined in (9) is lower semi-continuous.*

Proof. According to Lemma 2, F^t is continuous. By hypothesis, the function p is lower semi-continuous, which is equivalent to the fact that $p^{-1}((a, +\infty))$ is an open set for all $a \in \mathbb{R}$. Let $t \in \mathbb{N}^*$. Suppose that $(\mathbf{x}, \mathbf{w}) \in X \times O^t \mapsto p_{\mathbf{x}, t}(\mathbf{w})$ is lower semi-continuous. Let $a \in \mathbb{R}$, then the set $B_{a, t} := \{(\mathbf{x}, \mathbf{w}) \in X \times O^t \mid p_{\mathbf{x}, t}(\mathbf{w}) > a\}$ is an open set. We will show that then $B_{a, t+1}$ is also an open set.

First, suppose that $a > 0$. With (9),

$$\begin{aligned} B_{a,t+1} &= \{(\mathbf{x}, \mathbf{w}, \mathbf{u}) \in X \times O^t \times O \mid p_{\mathbf{x},t}(\mathbf{w})p_{F^t(\mathbf{x},\mathbf{w})}(\mathbf{u}) > a\} \\ &= \bigcup_{b \in \mathbb{R}_+^*} \{(\mathbf{x}, \mathbf{w}, \mathbf{u}) \in B_{b,t} \times O \mid p_{F^t(\mathbf{x},\mathbf{w})}(\mathbf{u}) > a/b\} \\ &= \bigcup_{b \in \mathbb{R}_+^*} \{(\mathbf{x}, \mathbf{w}, \mathbf{u}) \in B_{b,t} \times O \mid (F^t(\mathbf{x}, \mathbf{w}), \mathbf{u}) \in B_{a/b,1}\} \end{aligned}$$

The function F^t being continuous and $B_{a/b,1}$ being an open set, the set $B_{a/b,t+1}^F := \{(\mathbf{x}, \mathbf{w}, \mathbf{u}) \in X \times O^t \times O \mid (F^t(\mathbf{x}, \mathbf{w}), \mathbf{u}) \in B_{a/b,1}\}$ is also an open set. Therefore and as $B_{b,t}$ is an open set so is the set $(B_{b,t} \times O) \cap B_{a/b,t+1}^F$ for any $b \in \mathbb{R}_+^*$, and hence so is $B_{a,t+1} = \bigcup_{b \in \mathbb{R}_+^*} (B_{b,t} \times O) \cap B_{a/b,t+1}^F$.

If $a = 0$, note that $p_{\mathbf{x},t}(\mathbf{w})p_{F^t(\mathbf{x},\mathbf{w})}(\mathbf{u}) > 0$ is equivalent to $p_{\mathbf{x},t}(\mathbf{w}) > 0$ and $p_{F^t(\mathbf{x},\mathbf{w})}(\mathbf{u}) > 0$; hence $B_{0,t+1} = \{(\mathbf{x}, \mathbf{w}, \mathbf{u}) \in B_{0,t} \times O \mid (F^t(\mathbf{x}, \mathbf{w}), \mathbf{u}) \in B_{0,1}\}$, so the same reasoning holds.

If $a < 0$, then $B_{a,t+1} = X \times O^{t+1}$ which is an open set.

So we have proven that for all a , $B_{a,t+1}$ is an open set and hence the function $(\mathbf{x}, \mathbf{w}) \in X \times O^{t+1} \mapsto p_{\mathbf{x},t+1}(\mathbf{w})$ is lower semi-continuous. \square

Lemma 4. *Suppose that the function $F : X \times O \rightarrow X$ is C^0 , and that the function $p : (\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous. Then for any $\mathbf{x}^* \in X$, $t \in \mathbb{N}^*$, $\mathbf{w}^* \in O_{\mathbf{x}^*,t}$ and V an open neighbourhood of $F^t(\mathbf{x}^*, \mathbf{w}^*)$, $P^t(\mathbf{x}^*, V) > 0$.*

Proof. Since F is C^0 , from Lemma 2 F^t is also C^0 . Similarly, since p is lower semi-continuous, according to Lemma 3 so is the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x},t}(\mathbf{w})$, and so the set $O_{\mathbf{x},t} = p_{\mathbf{x},t}^{-1}((0, +\infty))$ is open for all \mathbf{x} and thus also for $\mathbf{x} = \mathbf{x}^*$. Let $B_V := \{\mathbf{w} \in O_{\mathbf{x}^*,t} \mid F^t(\mathbf{x}^*, \mathbf{w}) \in V\}$. Since F^t is continuous and $O_{\mathbf{x}^*,t}$ is open, the set B_V is open, and as $\mathbf{w}^* \in B_V$, it is non-empty. Furthermore

$$\begin{aligned} P^t(\mathbf{x}^*, V) &= \int_{O_{\mathbf{x}^*,t}} \mathbf{1}_V(F^t(\mathbf{x}^*, \mathbf{w}))p_{\mathbf{x}^*,t}(\mathbf{w})d\mathbf{w} \\ &= \int_{B_V} p_{\mathbf{x}^*,t}(\mathbf{w})d\mathbf{w} . \end{aligned}$$

As $p_{\mathbf{x}^*,t}$ is a strictly positive function over $B_V \subset O_{\mathbf{x}^*,t}$, and that B_V has positive Lebesgue measure, $P^t(\mathbf{x}^*, V) > 0$. \square

The following lemma establishes useful properties on a C^1 function $f : X \times O \rightarrow X$ for which there exists $\mathbf{x}^* \in X$ and $\mathbf{w}^* \in O$ such that $f(\mathbf{x}^*, \cdot)$ is a submersion at \mathbf{w}^* , and show in particular that a limited inverse function theorem and implicit function theorem can be expressed for submersions. These properties rely on the fact that a submersion can be seen locally as the composition of a diffeomorphism by a projection, as shown in [10].

Lemma 5. *Let $f : X \times O \rightarrow X$ be a C^1 function where $X \subset \mathbb{R}^n$ and $O \subset \mathbb{R}^m$ are open sets with $m \geq n$. If there exists $\mathbf{x}^* \in X$ and $\mathbf{w}^* \in O$ such that $f(\mathbf{x}^*, \cdot)$ is a submersion at \mathbf{w}^* , then*

1. *there exists N an open neighbourhood of $(\mathbf{x}^*, \mathbf{w}^*)$ such that for all $(\mathbf{y}, \mathbf{u}) \in N$, $f(\mathbf{y}, \cdot)$ is a submersion at \mathbf{u} ,*
2. *there exists $U_{\mathbf{w}^*} \subset O$ an open neighbourhood of \mathbf{w}^* , and $V_{f(\mathbf{x}^*, \mathbf{w}^*)}$ an open neighbourhood of $f(\mathbf{x}^*, \mathbf{w}^*)$, such that $V_{f(\mathbf{x}^*, \mathbf{w}^*)} = f(\mathbf{x}^*, U_{\mathbf{w}^*})$,*
3. *there exists g a C^1 function from $\tilde{V}_{\mathbf{x}^*}$ an open neighbourhood of \mathbf{x}^* to $\tilde{U}_{\mathbf{w}^*}$ an open neighbourhood of \mathbf{w}^* such that for all $\mathbf{y} \in \tilde{V}_{\mathbf{x}^*}$*

$$f(\mathbf{y}, g(\mathbf{y})) = f(\mathbf{x}^*, \mathbf{w}^*) .$$

Proof. Let $(\mathbf{e}_i)_{i \in [1..m]}$ be the canonical basis of \mathbb{R}^m and let us denote $f = (f_1, \dots, f_n)^T$ the representation of f (in the canonical basis of \mathbb{R}^n). Similarly, $\mathbf{u} \in O$ writes in the canonical basis $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m)^T$.

We start by proving the second point of the lemma. Since $f(\mathbf{x}^*, \cdot)$ is a submersion at \mathbf{w}^* , the matrix composed by the vectors $(D_{\mathbf{w}^*} f(\mathbf{x}^*, \cdot)(\mathbf{e}_i))_{i \in [1..m]}$ is of full rank n , hence there exists σ a permutation of $[1..m]$ such that the vectors $(D_{\mathbf{w}^*} f(\mathbf{x}^*, \cdot)(\mathbf{e}_{\sigma(i)}))_{i \in [1..n]}$ are linearly independent. We suppose that σ is the identity (otherwise we consider a reordering of the basis $(\mathbf{e}_i)_{i \in [1..m]}$ via σ). Let

$$h_{\mathbf{x}^*} : \mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m)^T \in O \mapsto (f_1(\mathbf{x}^*, \mathbf{u}), \dots, f_n(\mathbf{x}^*, \mathbf{u}), \mathbf{u}_{n+1}, \dots, \mathbf{u}_m)^T \in \mathbb{R}^m .$$

The Jacobian matrix of $h_{\mathbf{x}^*}$ taken at the vector \mathbf{w}^* writes

$$Jh_{\mathbf{x}^*}(\mathbf{w}^*) = \begin{pmatrix} \nabla_{\mathbf{w}^*} f_1(\mathbf{x}^*, \cdot)^T \\ \vdots \\ \nabla_{\mathbf{w}^*} f_n(\mathbf{x}^*, \cdot)^T \\ E_{n+1} \\ \vdots \\ E_m \end{pmatrix}$$

where $E_i \in \mathbb{R}^m$ is the (line) vector with a 1 at position i and zeros everywhere else. The matrix of the differential of $D_{\mathbf{w}^*} f(\mathbf{x}^*, \cdot)$ expressed in the canonical basis correspond to the n first lines of the above Jacobian matrix, such that the matrix $(D_{\mathbf{w}^*} f(\mathbf{x}^*, \cdot)(\mathbf{e}_i))_{i \in [1..n]}$ corresponds to the n times n first block. Hence the Jacobian matrix $Jh_{\mathbf{x}^*}(\mathbf{w}^*)$ is invertible. In addition, $h_{\mathbf{x}^*}$ is C^1 . Therefore we can apply the inverse function theorem to $h_{\mathbf{x}^*}$: there exists $U_{\mathbf{w}^*} \subset O$ a neighbourhood of \mathbf{w}^* and $V_{h_{\mathbf{x}^*}(\mathbf{w}^*)}$ a neighbourhood of $h_{\mathbf{x}^*}(\mathbf{w}^*)$ such that $h_{\mathbf{x}^*}$ is a bijection from $U_{\mathbf{w}^*}$ to $V_{h_{\mathbf{x}^*}(\mathbf{w}^*)}$. Let π_n denote the projection

$$\pi_n : \mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_m)^T \in \mathbb{R}^m \mapsto (\mathbf{y}_1, \dots, \mathbf{y}_n)^T \in \mathbb{R}^n .$$

Then $f(\mathbf{x}^*, \mathbf{u}) = \pi_n \circ h_{\mathbf{x}^*}(\mathbf{u})$ for all $\mathbf{u} \in O$, and so $f(\mathbf{x}^*, U_{\mathbf{w}^*}) = \pi_n(V_{h_{\mathbf{x}^*}(\mathbf{w}^*)})$. The set $V_{h_{\mathbf{x}^*}(\mathbf{w}^*)}$ being an open set, so is $V_{f(\mathbf{x}^*, \mathbf{w}^*)} := \pi_n(V_{h_{\mathbf{x}^*}(\mathbf{w}^*)})$ which is therefore an open

neighbourhood of $f(\mathbf{x}^*, \mathbf{w}^*) = \pi_n \circ h_{\mathbf{x}^*}(\mathbf{w}^*)$, that satisfies $V_{f(\mathbf{x}^*, \mathbf{w}^*)} = f(\mathbf{x}^*, U_{\mathbf{w}^*})$, which shows 2.

We are now going to prove the first point of the lemma. Since f is C^1 , the coefficients of the Jacobian matrix of $h_{\mathbf{x}^*}$ at \mathbf{w}^* are continuous functions of \mathbf{x}^* and \mathbf{w}^* , and as the Jacobian determinant is a polynomial in those coefficients, it is also a continuous function of \mathbf{x}^* and \mathbf{w}^* . The Jacobian determinant of $h_{\mathbf{x}^*}$ at \mathbf{w}^* being non-zero (since we have seen when proving the second point above that the Jacobian matrix at \mathbf{w}^* is invertible), the continuity of the Jacobian determinant implies the existence of N an open neighbourhood of $(\mathbf{x}^*, \mathbf{w}^*)$ such that for all $(\mathbf{y}, \mathbf{u}) \in N$, the Jacobian determinant of $h_{\mathbf{y}}$ at \mathbf{u} is non-zero. Since the matrix $(D_{\mathbf{u}}f(\mathbf{y}, \cdot)(\mathbf{e}_i))_{i \in [1..n]}$ corresponds to the n times n first block of the Jacobian matrix $Jh_{\mathbf{y}}(\mathbf{u})$, it is invertible which shows that $D_{\mathbf{u}}f(\mathbf{y}, \cdot)$ is of rank n which proves that $f(\mathbf{y}, \cdot)$ is a submersion at \mathbf{u} for all $(\mathbf{y}, \mathbf{u}) \in N$, which proves 1.

We may also apply the implicit function theorem to the function $(\mathbf{y}, \mathbf{u}) \in (X \times O) \mapsto h_{\mathbf{y}}(\mathbf{u}) \in \mathbb{R}^m$: there exists g a C^1 function from $\tilde{V}_{\mathbf{x}^*}$ an open neighbourhood of \mathbf{x}^* to $\tilde{U}_{\mathbf{w}^*}$ a open neighbourhood of \mathbf{w}^* such that $h_{\mathbf{y}}(\mathbf{u}) = h_{\mathbf{x}^*}(\mathbf{w}^*) \Leftrightarrow \mathbf{u} = g(\mathbf{y})$ for all $(\mathbf{y}, \mathbf{u}) \in \tilde{V}_{\mathbf{x}^*} \times \tilde{U}_{\mathbf{w}^*}$. Then $f(\mathbf{y}, g(\mathbf{y})) = \pi_n \circ h_{\mathbf{y}}(g(\mathbf{y})) = \pi_n \circ h_{\mathbf{x}^*}(\mathbf{w}^*) = f(\mathbf{x}^*, \mathbf{w}^*)$, proving 3. \square

The following lemma is a generalization of [8, Proposition 7.1.4] to our setting.

Lemma 6. *Suppose that F is C^∞ and that the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous. Then the control model is forward accessible if and only if for all $\mathbf{x} \in X$ there exists $t \in \mathbb{N}^*$ and $\mathbf{w} \in O_{\mathbf{x}, t}$ such that $F^t(\mathbf{x}, \cdot)$ is a submersion at \mathbf{w} .*

Proof. Suppose that the control model is forward accessible. Then, for all $\mathbf{x} \in X$, $A_+(\mathbf{x})$ is not Lebesgue negligible. Since $\sum_{i \in \mathbb{N}} \Lambda_n(A_+^i(\mathbf{x})) \geq \Lambda_n(A_+(\mathbf{x})) > 0$, there exists $i \in \mathbb{N}^*$ such that $\Lambda_n(A_+^i(\mathbf{x})) > 0$ ($i \neq 0$ because $A_+^0(\mathbf{x}) = \{\mathbf{x}\}$ is Lebesgue negligible). Suppose that for all $\mathbf{w} \in O_{\mathbf{x}, i}$, \mathbf{w} is a critical point for $F^i(\mathbf{x}, \cdot)$, that is the differential of $F^i(\mathbf{x}, \cdot)$ at \mathbf{w} is not surjective. According to Lemma 2 the function F^t is C^∞ , so we can apply Sard's theorem [13, Theorem II.3.1] to $F^i(\mathbf{x}, \cdot)$ which implies that the image of the critical points is Lebesgue negligible, hence $F^i(\mathbf{x}, O_{\mathbf{x}, i}) = A_+^i(\mathbf{x})$ is Lebesgue negligible. We have a contradiction, so there exists $\mathbf{w} \in O_{\mathbf{x}, i}$ for which $F^i(\mathbf{x}, \cdot)$ is a submersion at \mathbf{w} .

Suppose now that for all $\mathbf{x} \in X$, there exists $t \in \mathbb{N}^*$ and $\mathbf{w} \in O_{\mathbf{x}, t}$ such that $F^t(\mathbf{x}, \cdot)$ is a submersion at \mathbf{w} and let us prove that the control model is forward accessible. Since the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower continuous and that F is continuous, according to Lemma 3, then $p_{\mathbf{x}, t}$ is lower semi-continuous and hence $O_{\mathbf{x}, t}$ is an open set. Then according to Lemma 5, point 2) applied to the function F^t restricted to the open set $X \times O_{\mathbf{x}, t}$, there exists $U_{\mathbf{w}} \subset O_{\mathbf{x}, t}$ and $V_{F^t(\mathbf{x}, \mathbf{w})}$ non-empty open sets such that $F^t(\mathbf{x}, U_{\mathbf{w}}) \supset V_{F^t(\mathbf{x}, \mathbf{w})}$. Since $A_+(\mathbf{x}) \supset F^t(\mathbf{x}, O_{\mathbf{x}, t}) \supset F^t(\mathbf{x}, U_{\mathbf{w}})$, $A_+(\mathbf{x})$ has non-empty interior for all $\mathbf{x} \in X$, meaning the control model is forward accessible. \square

The following lemma treats of the preservation of Lebesgue null sets by a locally Lipschitz continuous function on spaces of equal dimension.

Lemma 7. (From [7, Corollary 5.9]) Take U an open set of \mathbb{R}^n and $f : U \rightarrow \mathbb{R}^n$ a locally Lipschitz-continuous function. Take $A \subset U$ a set of zero Lebesgue measure. Then its image $f(A)$ is also of zero Lebesgue measure.

Lemma 7 requires the dimensions of the domain and codomain to be equal. When the dimension of the domain is lower or equal than the dimension of the codomain, a generalization of Lemma 7 is presented in [11] for the preimage of sets via submersions. The authors of [11] investigate the so-called 0-property: a continuous function $f : Z \subset \mathbb{R}^m \rightarrow X \subset \mathbb{R}^n$ has the 0-property if the preimage of any set of Lebesgue measure 0 has Lebesgue measure 0. They show in [11, Theorem 2 and Theorem 3] that if f is a continuous function and that for almost all $\mathbf{z} \in Z$ it is a submersion at \mathbf{z} , then it has the 0-property. They also show in [11, Theorem 1] that for f a C^r function with $r \geq m - n + 1$ (this inequality coming from Sard's theorem [13, Theorem II.3.1]), then the 0-property is equivalent to f being a submersion at \mathbf{z} for almost all $\mathbf{z} \in Z$. In the following lemma, we establish conditions for a function f to have a stronger form of 0-property, for which the preimage of a set has Lebesgue measure 0 if and only if the set has measure 0.

Lemma 8. Let $g : Z \subset \mathbb{R}^m \rightarrow X \subset \mathbb{R}^n$ be a C^1 function where Z and X are open sets with $m \geq n$. Let $A \in \mathcal{B}(X)$ and let us assume that for almost all $\mathbf{z} \in g^{-1}(A)$, g is a submersion at \mathbf{z} , i.e. the differential of g at \mathbf{z} is surjective.

Then (i) $\Lambda_n(A) = 0$ implies that $\Lambda_m(g^{-1}(A)) = 0$, and (ii) if $A \subset g(Z)$ and if g is a submersion at \mathbf{z} for all $\mathbf{z} \in g^{-1}(A)$, then $\Lambda_n(A) = 0$ if and only if $\Lambda_m(g^{-1}(A)) = 0$.

Proof. This first part of the proof is similar to the proof of Lemma 5. Let $N \in \mathcal{B}(Z)$ be a Λ_m -negligible set such that g is a submersion at all points of $g^{-1}(A) \setminus N$, and take $\mathbf{z} \in g^{-1}(A) \setminus N$ and $(\mathbf{e}_i)_{i \in [1..m]}$ the canonical basis of \mathbb{R}^m . For $\mathbf{y} \in \mathbb{R}^m$, we denote $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_m)^T$ its expression in the canonical basis. In the canonical basis of \mathbb{R}^n we denote $g(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_n(\mathbf{x}))^T$. Since g is a submersion at \mathbf{z} , $D_{\mathbf{z}}g$ the differential of g at \mathbf{z} has rank n so there exists a permutation $\sigma_{\mathbf{z}} : [1..m] \rightarrow [1..n]$ such that the matrix formed by the vectors $(D_{\mathbf{z}}g(\mathbf{e}_{\sigma(i)}))_{i \in [1..n]}$ has rank n . We assume that this permutation is the identity (otherwise we consider a reordering of the canonical basis via σ). Let

$$h_{\mathbf{z}} : \mathbf{y} \in \mathbb{R}^m \mapsto (g_1(\mathbf{y}), \dots, g_n(\mathbf{y}), \mathbf{y}_{n+1}, \dots, \mathbf{y}_m)^T$$

Similarly as in the proof of Lemma 5, by expressing the differential of $h_{\mathbf{z}}$ in the basis $(\mathbf{e}_i)_{i \in [1..m]}$ we can see that the Jacobian determinant of $h_{\mathbf{z}}$ equals to the determinant of the matrix composed of the vectors $(D_{\mathbf{z}}g(\mathbf{e}_i))_{i \in [1..n]}$, which is non-zero, multiplied by the determinant of the identity matrix, which is one. Hence the Jacobian determinant of $h_{\mathbf{z}}$ is non-zero, and so we can apply the inverse function theorem to $h_{\mathbf{z}}$ (which inherits the C^1 property from g). We hence obtain that there exists $U_{\mathbf{z}}$ an open neighbourhood of \mathbf{z} , $V_{h_{\mathbf{z}}(\mathbf{z})}$ an open neighbourhood of $h_{\mathbf{z}}(\mathbf{z})$ such that the function $h_{\mathbf{z}}$ is a diffeomorphism from $U_{\mathbf{z}}$ to $V_{h_{\mathbf{z}}(\mathbf{z})}$. Then, denoting π_n the projection

$$\pi_n : \mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_m)^T \in \mathbb{R}^m \mapsto (\mathbf{z}_1, \dots, \mathbf{z}_n)^T ,$$

we have $g(\mathbf{u}) = \pi_n \circ h_{\mathbf{z}}(\mathbf{u})$ for all $\mathbf{u} \in Z$.

Then $g^{-1}(A) \cap U_{\mathbf{z}} = h_{\mathbf{z}}^{-1} \circ \pi_n^{-1}(A) \cap h_{\mathbf{z}}^{-1}(V_{h_{\mathbf{z}}(\mathbf{z})}) = h_{\mathbf{z}}^{-1}(A \times \mathbb{R}^{m-n} \cap V_{h_{\mathbf{z}}(\mathbf{z})})$. Since $h_{\mathbf{z}}$ is a diffeomorphism from $U_{\mathbf{z}}$ to $V_{h_{\mathbf{z}}(\mathbf{z})}$, $h_{\mathbf{z}}$ and $h_{\mathbf{z}}^{-1}$ are locally Lipschitz continuous. So we can use Lemma 7 with $h_{\mathbf{z}}^{-1}$ and its contrapositive with $h_{\mathbf{z}}$ and obtain that $\Lambda_m(A \times \mathbb{R}^{m-n} \cap V_{h_{\mathbf{z}}(\mathbf{z})}) = 0$ if and only if $\Lambda_m(h_{\mathbf{z}}^{-1}(A \times \mathbb{R}^{m-n} \cap V_{h_{\mathbf{z}}(\mathbf{z})})) = 0$, which implies that

$$\Lambda_m(A \times \mathbb{R}^{m-n} \cap V_{h_{\mathbf{z}}(\mathbf{z})}) = 0 \text{ if and only if } \Lambda_m(g^{-1}(A) \cap U_{\mathbf{z}}) = 0 . \quad (20)$$

If $\Lambda_n(A) = 0$ then $\Lambda_m(A \times \mathbb{R}^{m-n}) = 0$ and thus $\Lambda_m(A \times \mathbb{R}^{m-n} \cap V_{h_{\mathbf{z}}(\mathbf{z})}) = 0$ which in turns implies with (20) that $\Lambda_m(g^{-1}(A) \cap U_{\mathbf{z}}) = 0$. This latter statement holds for all $\mathbf{z} \in g^{-1}(A) \setminus N$, which with Lemma 1 implies that $\Lambda_m(g^{-1}(A) \setminus N) = 0$, and since N is a Lebesgue negligible set $\Lambda_m(g^{-1}(A)) = 0$. We have then proven the statement (i) of the lemma.

We will now prove the second statement. Suppose that $\Lambda_n(A) > 0$, so there exists $\mathbf{x} \in A$ such that for all $\epsilon > 0$, $\Lambda_n(B(\mathbf{x}, \epsilon) \cap A) > 0$ (this is implied by the contrapositive of Lemma 1). Assume that $A \subset g(Z)$, i.e. g is surjective on A , then there exists $\mathbf{z} \in Z$ such that $g(\mathbf{z}) = \mathbf{x}$. Since in the second statement we suppose that g is a submersion at \mathbf{u} for all $\mathbf{u} \in g^{-1}(A)$, we have that g is a submersion at \mathbf{z} , and so $h_{\mathbf{z}}$ is a diffeomorphism from $U_{\mathbf{z}}$ to $V_{h_{\mathbf{z}}(\mathbf{z})}$ and (20) holds. Since $V_{h_{\mathbf{z}}(\mathbf{z})}$ is an open neighbourhood of $h_{\mathbf{z}}(\mathbf{z}) = (g(\mathbf{z}), \mathbf{z}_{n+1}, \dots, \mathbf{z}_m)$, there exists (r_1, r_2) such that $B(g(\mathbf{z}), r_1) \times B((\mathbf{z}_i)_{i \in [n+1..m]}, r_2) \subset V_{h_{\mathbf{z}}(\mathbf{z})}$. Since $\Lambda_m(A \times \mathbb{R}^{m-n} \cap B(\mathbf{x}, r_1) \times B((\mathbf{z}_i)_{i \in [n+1..m]}, r_2)) = \Lambda_m((A \cap B(\mathbf{x}, r_1)) \times B((\mathbf{z}_i)_{i \in [n+1..m]}, r_2)) > 0$, we have $\Lambda_m(A \times \mathbb{R}^{m-n} \cap V_{h_{\mathbf{z}}(\mathbf{z})}) > 0$. This in turn implies through (20) that $\Lambda_m(g^{-1}(A) \cap U_{\mathbf{z}}) > 0$ and thus $\Lambda_m(g^{-1}(A)) > 0$. We have thus proven that if $\Lambda_n(A) > 0$ then $\Lambda_m(g^{-1}(A)) > 0$, which proves the lemma. \square

3. Main Results

We present here our main result. Its proof will be established in the following subsections.

Theorem 1. *Let $\Phi = (\Phi_t)_{t \in \mathbb{N}}$ be a time-homogeneous Markov chain on an open state space $X \subset \mathbb{R}^n$, defined via*

$$\Phi_{t+1} = F(\Phi_t, \alpha(\Phi_t, \mathbf{U}_{t+1})) \quad (21)$$

where $(\mathbf{U}_t)_{t \in \mathbb{N}^*}$ is a sequence of i.i.d. random vectors in \mathbb{R}^p , $\alpha : X \times \mathbb{R}^p \rightarrow O$ and $F : X \times O \rightarrow X$ are two measurable functions with O an open subset of \mathbb{R}^m . For all $\mathbf{x} \in X$, we assume that $\alpha(\mathbf{x}, \mathbf{U}_1)$ admits a probability density function that we denote $\mathbf{w} \in O \mapsto p_{\mathbf{x}}(\mathbf{w})$. We define the function $F^t : X \times O^t \rightarrow X$ via (7), the probability density function $p_{\mathbf{x}, t}$ via (9), and the sets $O_{\mathbf{x}}$ and $O_{\mathbf{x}, t}$ via (6) and (11). For $B \in \mathcal{B}(X)$, we denote μ_B the trace measure $A \in \mathcal{B}(X) \mapsto \Lambda_n(A \cap B)$, where Λ_n denotes the Lebesgue measure on \mathbb{R}^n . Suppose that

1. the function $(\mathbf{x}, \mathbf{w}) \in (X \times O) \mapsto F(\mathbf{x}, \mathbf{w})$ is C^1 ,
2. the function $(\mathbf{x}, \mathbf{w}) \in (X \times O) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous,

3. there exists $\mathbf{x}^* \in X$ a strongly globally attracting state, $k \in \mathbb{N}^*$ and $\mathbf{w}^* \in O_{\mathbf{x}^*, k}$ such that the function $\mathbf{w} \in O^k \mapsto F^k(\mathbf{x}^*, \mathbf{w})$ is a submersion at \mathbf{w}^* .

Then there exists B_0 a non-empty open subset of $A_+^k(\mathbf{x}^*)$ containing $F^k(\mathbf{x}^*, \mathbf{w}^*)$ such that Φ is a μ_{B_0} -irreducible aperiodic T-chain, and compact sets of X are small sets.

Before providing the proof of this theorem, we discuss its assumptions with respect to the chapter 7 of the Meyn and Tweedie book. Results similar to Theorem 1 are presented in [8, Chapter 7]. The underlying assumptions in [8, Chapter 7] translate to our setting as (i) the overall update function $(\mathbf{x}, \mathbf{u}) \mapsto F(\mathbf{x}, \alpha(\mathbf{x}, \mathbf{u}))$ is C^∞ , (ii) $\mathbf{w} \mapsto p(\mathbf{w})$ is lower semi-continuous. In contrast, in our context we do not need the overall update function $(\mathbf{x}, \mathbf{u}) \mapsto F(\mathbf{x}, \alpha(\mathbf{x}, \mathbf{u}))$ to even be continuous, and instead we need F to be C^1 . We also need the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ to be lower semi-continuous. In [8], assuming (i) and (ii) and the forward accessibility of the control model, the Markov chain is proved to be a T-chain [8, Proposition 7.1.5]; this property is then used to prove that under these assumptions the existence of a globally attracting state is equivalent to the φ -irreducibility of the Markov chain [8, Proposition 7.2.5 and Theorem 7.2.6]. The T-chain property is a strong property and in our context, we prove in Proposition 3 that if Φ is a T-chain, then we also get the equivalence between φ -irreducibility and the existence of a globally attracting state. We develop another approach in Lemma 9, relying on the submersion property of point 3) of Theorem 1 rather than on the T-chain property. This approach is used in Theorem 2 to prove that the existence of a globally attracting state $\mathbf{x}^* \in X$ for which there exists $k \in \mathbb{N}^*$ and $\mathbf{w}^* \in O_{\mathbf{x}^*, k}$ such that $F^k(\mathbf{x}^*, \cdot)$ is a submersion at \mathbf{w}^* implies the φ -irreducibility of the Markov chain. The approach developed in Lemma 9 allows for a finer control of the transition kernel than with the T-chain property, which is then used to get aperiodicity in Theorem 3 by assuming the existence of a strongly attracting state on which the submersion property of 3) of Theorem 1 holds. In the applications of Section 4, the existence of a strongly attracting state is immediately derived from the proof of the existence of a globally attracting state. In contrast in [8, Theorem 7.3.5], assuming (i), (ii), the forward accessibility of the control model, the existence of a globally attracting state \mathbf{x}^* and the connexity of $O_{\mathbf{x}}$, aperiodicity is proven to be a consequence of the connexity of $\overline{A_+(\mathbf{x}^*)}$.

Proof. (of Theorem 1) From Theorem 3, there exists B_0 a non-empty open subset of $A_+^k(\mathbf{x}^*)$ containing $F^k(\mathbf{x}^*, \mathbf{w}^*)$ such that Φ is a μ_{B_0} -irreducible aperiodic chain. With Proposition 5 the chain is also weak Feller. Since B_0 is a non-empty open set $\text{supp } \mu_{B_0}$ has non empty interior, so from [8, Theorem 6.0.1] with (iii) Φ is a μ_{B_0} -irreducible T-chain and with (ii) compact sets are petite sets. Finally, since the chain is μ_{B_0} -irreducible and aperiodic, with [8, Theorem 5.5.7] petite sets are small sets. \square

Assuming that F is C^∞ we showed in Lemma 6 that the forward accessibility of the control model is equivalent to assuming that for all $\mathbf{x} \in X$ there exists $t \in \mathbb{N}^*$ and $\mathbf{w} \in O_{\mathbf{x}, t}$ such that $F^t(\mathbf{x}, \cdot)$ is a submersion at \mathbf{w} , which satisfies a part of condition 3. of Theorem 1. Hence, we can use Lemma 6 and Theorem 1 to derive Corollary 1.

Corollary 1. *Suppose that*

1. *the function $(\mathbf{x}, \mathbf{w}) \mapsto F(\mathbf{x}, \mathbf{w})$ is C^∞ ,*
2. *the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous,*
3. *the control model $CM(F)$ is forward accessible,*
4. *there exists \mathbf{x}^* a strongly globally attracting state.*

Then there exists B_0 a non-empty open subset of $A_+^k(\mathbf{x}^)$ containing $F^k(\mathbf{x}^*, \mathbf{w}^*)$ such that Φ is a μ_{B_0} -irreducible aperiodic T -chain, and compacts sets of X are small sets.*

Proof. From Lemma 6, the second part of the assumption 3. of Theorem 1 is satisfied such that the conclusions of Theorem 1 hold. \square

3.1. φ -Irreducibility

When (i) the function $(\mathbf{x}, \mathbf{w}) \mapsto p(\mathbf{x}, \mathbf{w})$ is independent of \mathbf{x} , that is $p(\mathbf{x}, \mathbf{w}) = p(\mathbf{w})$, (ii) the function $\mathbf{w} \mapsto p(\mathbf{w})$ for all $\mathbf{x} \in X$ is lower semi-continuous, (iii) F is C^∞ and (iv) the control model is forward accessible, it is shown in [8, Proposition 7.1.5] that Φ is a T -chain. This is a strong property that is then used to show the equivalence of the existence of a globally attracting state and the φ -irreducibility of the Markov chain Φ in [8, Theorem 7.2.6]. In our context where the function $(\mathbf{x}, \mathbf{w}) \mapsto p(\mathbf{x}, \mathbf{w})$ varies with \mathbf{x} , the following proposition shows that the equivalence still holds assuming that the Markov chain Φ is a T -chain.

Proposition 3. *Suppose that*

1. *the Markov chain Φ is a T -chain,*
2. *the function F is continuous,*
3. *the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous*

Then the Markov chain Φ is φ -irreducible if and only if there exists \mathbf{x}^ a globally attracting state.*

Proof. Suppose that there exists \mathbf{x}^* a globally attracting state. Since Φ is a T -chain, there exists a a sampling distribution such that K_a possesses a continuous component T such that $T(\mathbf{x}, X) > 0$ for all $\mathbf{x} \in X$.

Take $A \in \mathcal{B}(X)$ such that $T(\mathbf{x}^*, A) > 0$ (such a A always exists because we can for instance take $A = X$). The function $T(\cdot, A)$ being lower semi-continuous, there exists $\delta > 0$ and $r > 0$ such that for all $\mathbf{y} \in B(\mathbf{x}^*, r)$, $T(\mathbf{y}, A) > \delta$, hence $B(\mathbf{x}^*, r) \overset{a}{\rightsquigarrow} A$. Since \mathbf{x}^* is a globally attracting state, for all $\mathbf{y} \in X$, $\mathbf{x}^* \in \bigcup_{k \in \mathbb{N}^*} A_+^k(\mathbf{y})$ so there exists points of $\bigcup_{k \in \mathbb{N}^*} A_+^k$ arbitrarily close to \mathbf{x}^* . Hence there exists $t_{\mathbf{y}}$ and $\mathbf{w} \in O_{\mathbf{y}, t_{\mathbf{y}}}$ such that $F^{t_{\mathbf{y}}}(\mathbf{y}, \mathbf{w}) \in B(\mathbf{x}^*, r)$. Furthermore, since $O_{\mathbf{y}, t_{\mathbf{y}}}$ is an open set (by the lower semi-continuity of $p_{\mathbf{x}, t_{\mathbf{y}}}(\cdot)$ which in turn is implied by the lower semi-continuity of the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ and the continuity of F with Lemma 3) and $F^{t_{\mathbf{y}}}(\mathbf{y}, \cdot)$ is continuous (as implied by the continuity of F with Lemma 2) the set $E := \{\mathbf{u} \in O_{\mathbf{y}, t_{\mathbf{y}}} | F^{t_{\mathbf{y}}}(\mathbf{y}, \mathbf{u}) \in$

$B(\mathbf{x}^*, r)$ is an open set, and as $\mathbf{w} \in E$ it is non empty. Since $P^{t\mathbf{y}}(\mathbf{y}, B(\mathbf{x}^*, r)) = \int_E p_{\mathbf{y}, t\mathbf{y}}(\mathbf{u}) d\mathbf{u}$ and that $p_{\mathbf{y}, t\mathbf{y}}(\mathbf{u}) > 0$ for all $\mathbf{u} \in E \subset O_{\mathbf{y}, t\mathbf{y}}$, $P^{t\mathbf{y}}(\mathbf{y}, B(\mathbf{x}^*, r)) > 0$ as the integral of a positive function over a set of positive Lebesgue measure is positive. Hence $K_{a_\epsilon}(\mathbf{y}, B(\mathbf{x}^*, r)) > 0$ (where K_{a_ϵ} is the transition kernel defined in (18) with the geometric distribution (19)), and so $\{\mathbf{y}\} \overset{a_\epsilon}{\rightsquigarrow} B(\mathbf{x}^*, r)$. Hence with [8, Lemma 5.5.2] $\{\mathbf{y}\} \overset{a^*a_\epsilon}{\rightsquigarrow} A$ which implies that for some $t \in \mathbb{N}^*$, $P^t(\mathbf{y}, A) > 0$. Therefore, $T(\mathbf{x}^*, A) > 0$ implies that $\sum_{t \in \mathbb{N}^*} P^t(\mathbf{y}, A) > 0$ for all $\mathbf{y} \in X$. And since $T(\mathbf{x}^*, X) > 0$, $T(\mathbf{x}^*, \cdot)$ is not a trivial measure, so the Markov chain Φ is $T(\mathbf{x}^*, \cdot)$ -irreducible.

Suppose that Φ is φ -irreducible, then φ is non-trivial and according to Proposition 4 any point of $\text{supp } \varphi$ is a globally attracting state, so there exists a globally attracting state. \square

Although the T -chain property allows for a simple proof of the equivalence between the existence of a globally attracting state and the φ -irreducibility of the Markov chain. The T -chain property is not needed for Theorem 2, which instead relies on the following lemma. Interestingly, not using the T -chain in the lemma allows some control on the transition kernel, which is then used for Theorem 3 for aperiodicity.

Lemma 9. *Let $A \in \mathcal{B}(X)$ and suppose that*

1. *the function F is C^1 ,*
2. *the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous,*
3. *there exists $\mathbf{x}^* \in X$, $k \in \mathbb{N}^*$ and $\mathbf{w}^* \in O_{\mathbf{x}^*, k}$ such that $F^k(\mathbf{x}^*, \cdot)$ is a submersion at \mathbf{w}^* .*

Then there exists $B_0 \subset A_+^k(\mathbf{x}^)$ a non-empty open set containing $F^k(\mathbf{x}^*, \mathbf{w}^*)$ and such that for all $\mathbf{z} \in B_0$, there exists $U_{\mathbf{x}^*}$ an open neighbourhood of \mathbf{x}^* that depends on \mathbf{z} and having the following property: for $\mathbf{y} \in X$ if there exists a t -steps path from \mathbf{y} to $U_{\mathbf{x}^*}$, then for any $A \in \mathcal{B}(X)$*

$$P^{t+k}(\mathbf{y}, A) = 0 \Rightarrow \exists V_{\mathbf{z}} \text{ an open neighbourhood of } \mathbf{z} \text{ such that } \Lambda_n(V_{\mathbf{z}} \cap A) = 0 \quad (22)$$

or equivalently,

$$\text{for all } V_{\mathbf{z}} \text{ open neighbourhood of } \mathbf{z}, \Lambda_n(V_{\mathbf{z}} \cap A) > 0 \Rightarrow P^{t+k}(\mathbf{y}, A) > 0 \quad (23)$$

Proof. (i) We will need through this proof a set $N = N_1 \times N_2$ which is an open neighbourhood of $(\mathbf{x}^*, \mathbf{w}^*)$, such that for all $(\mathbf{x}, \mathbf{w}) \in N$ we have $p_{\mathbf{x}, k}(\mathbf{w}) > 0$ and that $F^k(\mathbf{x}, \cdot)$ is a submersion at \mathbf{w} . To obtain N , first let us note that since F is C^1 , according to Lemma 7 so is F^t for all $t \in \mathbb{N}^*$; and since the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous, according to Lemma 3 so is the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}, t}(\mathbf{w})$ for all $t \in \mathbb{N}^*$. Hence the set $\{(\mathbf{x}, \mathbf{w}) \in X \times O^t | p_{\mathbf{x}, k}(\mathbf{w}) > 0\}$ is an open set, and since $\mathbf{w}^* \in O_{\mathbf{x}^*, k}$, there exists $M_1 \times M_2$ a neighbourhood of $(\mathbf{x}^*, \mathbf{w}^*)$ such that for all $(\mathbf{x}, \mathbf{w}) \in M_1 \times M_2$, $p_{\mathbf{x}, k}(\mathbf{w}) > 0$. Furthermore, according to point 1. of Lemma 5, there exists $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ an open

neighbourhood of $(\mathbf{x}^*, \mathbf{w}^*)$ such that for all $(\mathbf{x}, \mathbf{w}) \in \tilde{M}_1 \times \tilde{M}_2$, $F^k(\mathbf{x}, \cdot)$ is a submersion at \mathbf{w} . Then the set $N := M \cap \tilde{M}$ has the desired property.

(ii) We now prove that for all $\mathbf{y} \in X$, U any open neighbourhood of \mathbf{x}^* and $A \in \mathcal{B}(X)$, if there exists \mathbf{v} a t -steps path from \mathbf{y} to U and if $P^{t+k}(\mathbf{y}, A) = 0$ then there exists $\mathbf{x}_0 \in U$ such that $P^k(\mathbf{x}_0, A) = 0$. Indeed, U being open containing $F^t(\mathbf{y}, \mathbf{v})$ there exists $\epsilon > 0$ such that $B(F^t(\mathbf{y}, \mathbf{v}), \epsilon) \subset U$, and by continuity of $F^t(\mathbf{y}, \cdot)$, there exists $\eta > 0$ such that $F^t(\mathbf{y}, B(\mathbf{v}, \eta)) \subset B(F^t(\mathbf{y}, \mathbf{v}), \epsilon) \subset U$; furthermore, $P^{t+k}(\mathbf{y}, A) = 0$ implies that

$$P^{t+k}(\mathbf{y}, A) = \int_{O_{\mathbf{y},t}} p_{\mathbf{y},t}(\mathbf{u}) P^k(F^t(\mathbf{y}, \mathbf{u}), A) d\mathbf{u} = 0 .$$

Since for all $\mathbf{u} \in O_{\mathbf{y},t}$, $p_{\mathbf{y},t}(\mathbf{u}) > 0$, this implies that for almost all $\mathbf{u} \in O_{\mathbf{y},t}$, $P^k(F^t(\mathbf{y}, \mathbf{u}), A) = 0$. Since $\mathbf{v} \in O_{\mathbf{y},t}$, the set $O_{\mathbf{y},t} \cap B(\mathbf{v}, \eta)$ is a non-empty open set and therefore has positive Lebesgue measure; so there exists $\mathbf{u}_0 \in O_{\mathbf{y},t} \cap B(\mathbf{v}, \eta)$ such that $P^k(F^t(\mathbf{y}, \mathbf{u}_0), A) = 0$. Let \mathbf{x}_0 denote $F^t(\mathbf{y}, \mathbf{u}_0)$. By choice of η , we also have $\mathbf{x}_0 \in F^t(\mathbf{y}, B(\mathbf{v}, \eta)) \subset U$.

(iii) Now let us construct the set B_0 mentioned in the lemma. We consider the function F^k restricted to $X \times N_2$. According to assumption 3. and (i), we have $\mathbf{x}^* \in X$ and \mathbf{w}^* in N_2 such that $F^k(\mathbf{x}^*, \cdot)$ is a submersion at \mathbf{w}^* . Hence using point 2. of Lemma 5 on the function F^k restricted to $X \times N_2$, we obtain that there exists $V_{\mathbf{w}^*} \subset N_2$ an open neighbourhood of \mathbf{w}^* and $U_{F^k(\mathbf{x}^*, \mathbf{w}^*)}$ an open neighbourhood of $F^k(\mathbf{x}^*, \mathbf{w}^*)$ such that $U_{F^k(\mathbf{x}^*, \mathbf{w}^*)} \subset F^k(\mathbf{x}^*, V_{\mathbf{w}^*})$. We take $B_0 = U_{F^k(\mathbf{x}^*, \mathbf{w}^*)}$ and will prove in what follows that it satisfies the properties announced. Note that since $B_0 \subset F^k(\mathbf{x}^*, V_{\mathbf{w}^*})$, that $V_{\mathbf{w}^*} \subset N_2$ and that $\mathbf{x}^* \in N_1$, $V_{\mathbf{x}^*} \subset O_{\mathbf{x}^*,k}$ and so $B_0 \subset A_+^k(\mathbf{x}^*)$.

(iv) Now, for $\mathbf{z} \in B_0$, let us construct the set $U_{\mathbf{x}^*}$ mentioned in the lemma. We will make it so that there exists a C^1 function g valued in O and defined on a set containing $U_{\mathbf{x}^*}$, such that $F^k(\mathbf{x}, g(\mathbf{x})) = \mathbf{z}$ for all $\mathbf{x} \in U_{\mathbf{x}^*}$. First, since $\mathbf{z} \in B_0$ and $B_0 = U_{F^k(\mathbf{x}^*, \mathbf{w}^*)} \subset F^k(\mathbf{x}^*, V_{\mathbf{w}^*})$, there exists $\mathbf{w}_{\mathbf{z}} \in V_{\mathbf{w}^*}$ such that $F^k(\mathbf{x}^*, \mathbf{w}_{\mathbf{z}}) = \mathbf{z}$. Since $V_{\mathbf{w}^*} \subset N_2$, the function $F^k(\mathbf{x}^*, \cdot)$ is a submersion at $\mathbf{w}_{\mathbf{z}}$, so we can apply point 3. of Lemma 5 to the function F^k restricted to $X \times N_2$: there exists g a C^1 function from $\tilde{U}_{\mathbf{x}^*}^g$ an open neighbourhood of \mathbf{x}^* to $\tilde{V}_{\mathbf{w}_{\mathbf{z}}}^g \subset N_2$ an open neighbourhood of $\mathbf{w}_{\mathbf{z}}$ such that for all $\mathbf{x} \in \tilde{U}_{\mathbf{x}^*}^g$, $F^k(\mathbf{x}, g(\mathbf{x})) = F^k(\mathbf{x}^*, \mathbf{w}_{\mathbf{z}}) = \mathbf{z}$. We now take $U_{\mathbf{x}^*} := \tilde{U}_{\mathbf{x}^*}^g \cap N_1$; it is an open neighbourhood of \mathbf{x}^* and for all $\mathbf{x} \in U_{\mathbf{x}^*}$, $F^k(\mathbf{x}, g(\mathbf{x})) = \mathbf{z}$.

(v) We now construct the set $V_{\mathbf{z}}$. For $\mathbf{y} \in X$, if there exists a t -steps path from \mathbf{y} to $U_{\mathbf{x}^*}$ and that $P^{t+k}(\mathbf{y}, A) = 0$, then we showed in (ii) that there exists $\mathbf{x}_0 \in U_{\mathbf{x}^*}$ such that $P^k(\mathbf{x}_0, A) = 0$. Since $\mathbf{x}_0 \in U_{\mathbf{x}^*} \subset \tilde{U}_{\mathbf{x}^*}^g \cap N_1$ and that $g(\mathbf{x}_0) \in \tilde{V}_{\mathbf{w}_{\mathbf{z}}}^g \subset N_2$, the function $F^k(\mathbf{x}_0, \cdot)$ is a submersion at $g(\mathbf{x}_0)$. Therefore, we can apply point 2) of Lemma 5 to F restricted to $X \times N_2$, and so there exists $U_{g(\mathbf{x}_0)} \subset N_2$ an open neighbourhood of $g(\mathbf{x}_0)$ and $V_{\mathbf{z}}$ an open neighbourhood of $F^k(\mathbf{x}_0, g(\mathbf{x}_0)) = \mathbf{z}$ such that $V_{\mathbf{z}} \subset F^k(\mathbf{x}_0, U_{g(\mathbf{x}_0)})$.

(vi) Finally we will show that $\Lambda_n(V_{\mathbf{z}} \cap A) = 0$. Let $\tilde{B} := \{\mathbf{w} \in U_{g(\mathbf{x}_0)} | F^k(\mathbf{x}_0, \mathbf{w}) \in V_{\mathbf{z}} \cap A\}$. Then

$$P^k(\mathbf{x}_0, A) = \int_{O_{\mathbf{x}_0,k}} \mathbf{1}_A(F^k(\mathbf{x}_0, \mathbf{w})) p_{\mathbf{x}_0,k}(\mathbf{w}) d\mathbf{w} \geq \int_{\tilde{B}} p_{\mathbf{x}_0,k}(\mathbf{w}) d\mathbf{w} ,$$

so $\int_{\tilde{B}} p_{\mathbf{x}_0, k}(\mathbf{w}) d\mathbf{w} = 0$. As $\mathbf{x}_0 \in U_{\mathbf{x}^*} \subset N_1$ and $\tilde{B} \subset U_{g(\mathbf{x}_0)} \subset N_2$, $p_{\mathbf{x}_0, k}(\mathbf{w}) > 0$ for all $\mathbf{w} \in \tilde{B}$, which implies with the fact that $\int_{\tilde{B}} p_{\mathbf{x}_0, k}(\mathbf{w}) d\mathbf{w} = 0$ that \tilde{B} is Lebesgue negligible. Now let h denote the function $F^k(\mathbf{x}_0, \cdot)$ restricted to $U_{g(\mathbf{x}_0)}$. The function h is a C^1 function and $V_{\mathbf{z}}$ is included into the image of h . Both $U_{g(\mathbf{x}_0)}$ to $V_{\mathbf{z}}$ are open sets. Furthermore $\mathbf{x}_0 \in N_1$ and for all $\mathbf{u} \in h^{-1}(V_{\mathbf{z}})$ since $h^{-1}(V_{\mathbf{z}}) \subset U_{g(\mathbf{x}_0)} \subset N_2$ we have $\mathbf{u} \in N_2$ so the function h is a submersion at \mathbf{u} . Therefore we can apply Lemma 8 to h , and so if $\Lambda_m(h^{-1}(V_{\mathbf{z}} \cap A)) = 0$ then $\Lambda_n(V_{\mathbf{z}} \cap A) = 0$. Since $h^{-1}(V_{\mathbf{z}}) = U_{g(\mathbf{x}_0)}$, we have $h^{-1}(V_{\mathbf{z}} \cap A) = \tilde{B}$, so we do have $\Lambda_m(h^{-1}(V_{\mathbf{z}} \cap A)) = 0$ which implies $\Lambda_n(V_{\mathbf{z}} \cap A) = 0$.

(vii) The equivalent formulation between (22) and (23) is simply obtained by taking the contrapositive. \square

If the function F is C^∞ , then the condition of Lemma 9 on the differential of $F(\mathbf{x}^*, \cdot)$ can be relaxed by asking the control model to be forward accessible using Lemma 6. If the point \mathbf{x}^* used in Lemma 9 is a globally attracting state it follows from Lemma 9 that the chain Φ is irreducible, as stated in the following theorem.

Theorem 2. *Suppose that F is C^1 , the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous and there exists a globally attracting state $\mathbf{x}^* \in X$, $k \in \mathbb{N}^*$ and $\mathbf{w}^* \in O_{\mathbf{x}^*, k}$ such that the function $\mathbf{w} \in \mathbb{R}^{m_k} \mapsto F^k(\mathbf{x}^*, \mathbf{w}) \in \mathbb{R}^n$ is a submersion at \mathbf{w}^* . Then Φ is a μ_{B_0} -irreducible Markov chain, where B_0 is a non empty open subset of $A_+^k(\mathbf{x}^*)$ containing $F^k(\mathbf{x}^*, \mathbf{w}^*)$.*

Furthermore if F is C^∞ , the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ lower semi-continuous, and the control model is forward accessible, then the existence of a globally attracting state is equivalent to the φ -irreducibility of the Markov chain Φ .

Proof. We want to show that for φ a non-trivial measure, Φ is φ -irreducible; i.e. for any $A \in \mathcal{B}(X)$, we need to prove that $\varphi(A) > 0$ implies that $\sum_{t \in \mathbb{N}^*} P^t(\mathbf{x}, A) > 0$ for all $\mathbf{x} \in X$.

According to Lemma 9 there exists a non-empty open set $B_0 \subset A_+^k(\mathbf{x}^*)$ containing $F^k(\mathbf{x}^*, \mathbf{w}^*)$, such that for all $\mathbf{z} \in B_0$ there exists $U_{\mathbf{x}^*}$ a neighbourhood of \mathbf{x}^* that depends on \mathbf{z} having the following property: if for $\mathbf{y} \in X$ there exists a t -steps path from \mathbf{y} to $U_{\mathbf{x}^*}(\mathbf{z})$, and if for all $V_{\mathbf{z}}$ neighbourhood of \mathbf{z} , $V_{\mathbf{z}} \cap A$ has positive Lebesgue measure, then $P^{t+k}(\mathbf{y}, A) > 0$. Since B_0 is a non-empty open set, the trace-measure μ_{B_0} is non-trivial. Suppose that $\mu_{B_0}(A) > 0$, then there exists $\mathbf{z}_0 \in B_0 \cap A$ such that for all $V_{\mathbf{z}_0}$ neighbourhood of \mathbf{z}_0 , $V_{\mathbf{z}_0} \cap A$ has positive Lebesgue measure². And since \mathbf{x}^* is globally attractive, according to Proposition 1 for all $\mathbf{y} \in X$ there exists $t_{\mathbf{y}} \in \mathbb{N}^*$ such that there exists a $t_{\mathbf{y}}$ -steps path from \mathbf{y} to the set $U_{\mathbf{x}^*}$ corresponding to \mathbf{z}_0 . Hence, with Lemma 9, $P^{t_{\mathbf{y}}+k}(\mathbf{y}, A) > 0$ for all $\mathbf{y} \in X$ and so Φ is μ_{B_0} -irreducible.

If F is C^∞ , according to Lemma 6, forward accessibility implies that for all $\mathbf{x} \in X$ there exists $k \in \mathbb{N}^*$ and $\mathbf{w} \in O_{\mathbf{x}, k}$ such that the function $F^k(\mathbf{x}, \cdot)$ is a submersion at

²If not, it would mean that for all $\mathbf{z} \in B_0 \cap A$, there exists $V_{\mathbf{z}}$ a neighbourhood of \mathbf{z} such that $B_0 \cap A \cap V_{\mathbf{z}}$ is Lebesgue-negligible, which with Lemma 1 would imply that $B_0 \cap A$ is Lebesgue negligible and bring a contradiction.

\mathbf{w} , which, using the first part of the proof of Theorem 2, shows that the existence of a globally attracting state implies the irreducibility of the Markov chain.

Finally, if Φ is φ -irreducible, take $\mathbf{x}^* \in \text{supp } \varphi$. By definition of the support of a measure, for all U neighbourhood of \mathbf{x}^* , $\varphi(U) > 0$. This imply through (15) that for all $\mathbf{y} \in X$ there exists $t \in \mathbb{N}^*$ such that $P^t(\mathbf{y}, U) > 0$. Since

$$P^t(\mathbf{y}, U) = \int_{O_{\mathbf{y},t}} 1_U(F^t(\mathbf{y}, \mathbf{w})) p_{\mathbf{y},t}(\mathbf{w}) d\mathbf{w} > 0$$

this implies the existence of a t -steps path from \mathbf{y} to U . Then, according to Proposition 1, \mathbf{x}^* is a globally attracting state. \square

Let $\mathbf{x}^* \in X$ be the globally attracting state used in Theorem 2. The support of the irreducibility measure used in Theorem 2 is a subset of $\overline{A_+(\mathbf{x}^*)}$. In the following proposition, we expend on this and show that when F is continuous and $p_{\mathbf{x}}$ lower semi-continuous, the support of the maximal irreducibility measure is exactly $\overline{A_+(\mathbf{x}^*)}$ for any globally attractive state \mathbf{x}^* .

Proposition 4. *Suppose that the function F is continuous, that the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous, and that the Markov chain Φ is φ -irreducible. Take ψ the maximal irreducibility measure of Φ . Then*

$$\text{supp } \psi = \{\mathbf{x}^* \in X \mid \mathbf{x}^* \text{ is a globally attracting state}\} ,$$

and so, for $\mathbf{x}^* \in X$ a globally attracting state,

$$\text{supp } \psi = \overline{A_+(\mathbf{x}^*)} .$$

Proof. Take $\mathbf{x}^* \in \text{supp } \psi$, we will show that it is a globally attracting state. By definition of the support of a measure, for all U neighbourhood of \mathbf{x}^* , $\psi(U) > 0$. The measure ψ being a irreducibility measure, this imply through (15) that for all $\mathbf{y} \in X$ there exists $t \in \mathbb{N}^*$ such that $P^t(\mathbf{y}, U) > 0$, which in turns imply the existence of a t -steps path from \mathbf{y} to U . Then, according to Proposition 1, \mathbf{x}^* is a globally attracting state, and so $\text{supp } \psi \subset \{\mathbf{x}^* \in X \mid \mathbf{x}^* \text{ is a globally attracting state}\}$.

Take $\mathbf{x}^* \in X$ a globally attracting state, then according to Proposition 1, for all $\mathbf{y} \in X$ there exists $t_{\mathbf{y}} \in \mathbb{N}^*$ and $\mathbf{w} \in O_{\mathbf{y},t_{\mathbf{y}}}$ such that $F^{t_{\mathbf{y}}}(\mathbf{y}, \mathbf{w}) \in B(\mathbf{x}^*, \epsilon)$. And since according to Lemma 2, $F^{t_{\mathbf{y}}}$ is continuous and that $B(\mathbf{x}^*, \epsilon)$ is an open, there exists $\eta > 0$ such that for all $\mathbf{u} \in B(\mathbf{w}, \eta)$, $F^{t_{\mathbf{y}}}(\mathbf{y}, \mathbf{u}) \in B(\mathbf{x}^*, \epsilon)$. Since p is lower semi-continuous and F continuous, according to Lemma 3 so is the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x},t_{\mathbf{y}}}(\mathbf{w})$ and so the set $O_{\mathbf{y},t_{\mathbf{y}}}$ is an open set. We can then chose the value of η small enough such that $B(\mathbf{w}, \eta) \subset O_{\mathbf{y},t_{\mathbf{y}}}$. Hence

$$P^{t_{\mathbf{y}}}(\mathbf{y}, B(\mathbf{x}^*, \epsilon)) \geq \int_{B(\mathbf{w}, \eta)} \mathbf{1}_{B(\mathbf{x}^*, \epsilon)}(F^{t_{\mathbf{y}}}(\mathbf{y}, \mathbf{u})) p_{\mathbf{y},t_{\mathbf{y}}}(\mathbf{u}) d\mathbf{u} = \int_{B(\mathbf{w}, \eta)} p_{\mathbf{y},t_{\mathbf{y}}}(\mathbf{u}) d\mathbf{u} > 0 .$$

The measure ψ being the maximal irreducibility measure, then

$$\psi(A) > 0 \Leftrightarrow \sum_{t \in \mathbb{N}^*} P^t(\mathbf{y}, A) > 0, \text{ for all } \mathbf{y} \in X. \text{ }^3$$

Since we proved that for all $\mathbf{y} \in X$, $P^{t_{\mathbf{y}}}(\mathbf{y}, B(\mathbf{x}^*, \epsilon)) > 0$, we have $\psi(B(\mathbf{x}^*, \epsilon)) > 0$. Finally, since we can chose ϵ arbitrarily small, this implies that $\mathbf{x}^* \in \text{supp } \psi$.

Let $(\mathbf{x}^*, \mathbf{y}^*) \in X^2$ be globally attracting states, then $\mathbf{y}^* \in \Omega_+(\mathbf{x}^*) \subset \overline{A_+(\mathbf{x}^*)}$, so $\{\mathbf{y}^* \in X | \mathbf{y}^* \text{ is a globally attracting state}\} \subset \overline{A_+(\mathbf{x}^*)}$.

Conversely, take $\mathbf{y}^* \in \overline{A_+(\mathbf{x}^*)}$, we will show that \mathbf{y}^* is a globally attracting state. Since $\mathbf{y}^* \in \overline{A_+(\mathbf{x}^*)}$, for all $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}^*$ and \mathbf{w}_ϵ a k_ϵ -steps path from \mathbf{x}^* to $B(\mathbf{y}^*, \epsilon)$. Take $\mathbf{x} \in X$. Since \mathbf{x}^* is a globally attracting state, according to Proposition 1 for all $\eta > 0$ there exists $t \in \mathbb{N}^*$ and \mathbf{u}_η a t -steps path from \mathbf{x} to $B(\mathbf{x}^*, \eta)$. And since F^{k_ϵ} is continuous, there exists $\eta_0 > 0$ such that for all $\mathbf{z} \in B(\mathbf{x}^*, \eta_0)$, $F^{k_\epsilon}(\mathbf{z}, \mathbf{w}_\epsilon) \in B(\mathbf{y}^*, \epsilon)$. Furthermore, since the set $\{(\mathbf{x}, \mathbf{w}) \in X \times O^{k_\epsilon} | p_{\mathbf{x}, k_\epsilon}(\mathbf{w}) > 0\}$ is an open set we can take η_0 small enough to ensure that $\mathbf{w}_\epsilon \in O_{F^t(\mathbf{x}, \mathbf{u}_{\eta_0}), k_\epsilon}$. Hence for any $\mathbf{x} \in X$, $\epsilon > 0$, $(\mathbf{u}_{\eta_0}, \mathbf{w}_\epsilon)$ is a $t + k_\epsilon$ -steps path from \mathbf{x} to $B(\mathbf{y}^*, \epsilon)$, which with Proposition 1 proves that \mathbf{y}^* is a globally attracting state. Hence $\overline{A_+(\mathbf{x}^*)} \subset \{\mathbf{y}^* | \mathbf{y}^* \text{ is a globally attracting state}\}$. \square

3.2. Aperiodicity

The results of Lemma 9 give the existence of a non-empty open set B_0 such that for all $\mathbf{z} \in B_0$ there exists $U_{\mathbf{x}^*}$ a neighbourhood of \mathbf{x}^* which depends of \mathbf{z} . And if $V_{\mathbf{z}} \cap A$ has positive Lebesgue measure for all $V_{\mathbf{z}}$ neighbourhood of \mathbf{z} , then for all $\mathbf{y} \in X$ the existence of a t -steps path from \mathbf{y} to $U_{\mathbf{x}^*}$ implies that $P^{t+k}(\mathbf{y}, A) > 0$. Note that $P^t(\mathbf{y}, A) > 0$ holds true for any $t \in \mathbb{N}^*$ such that there exists a t -steps path from \mathbf{y} to $U_{\mathbf{x}^*}$.

The global attractivity of \mathbf{x}^* gives for any $\mathbf{y} \in X$ the existence of one such t for which there exists a t -step path from \mathbf{y} to $U_{\mathbf{x}^*}$; and as seen in Theorem 2 this can be exploited to prove the irreducibility of the Markov chain. However, the strong global attractivity of \mathbf{x}^* gives for all $\mathbf{y} \in X$ the existence of a $t_{\mathbf{y}}$ such that for all $t \geq t_{\mathbf{y}}$ there exists a t -step path from \mathbf{y} to $U_{\mathbf{x}^*}$, which implies that $P^t(\mathbf{y}, A) > 0$ for all $t \geq t_{\mathbf{y}}$ and for all $\mathbf{y} \in X$. We will see in the following theorem that this implies the aperiodicity of the Markov chain.

Theorem 3. *Suppose that*

1. *the function $(\mathbf{x}, \mathbf{w}) \mapsto F(\mathbf{x}, \mathbf{w})$ is C^1 ,*
2. *the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous,*
3. *there exists $\mathbf{x}^* \in X$ a strongly globally attractive state, $k \in \mathbb{N}^*$ and $\mathbf{w}^* \in O_{\mathbf{x}^*, k}$ such that $F^k(\mathbf{x}^*, \cdot)$ is a submersion at \mathbf{w}^* .*

³The implication \Rightarrow is by definition of a irreducibility measure. For the converse suppose that A is a set such that $\sum_{t \in \mathbb{N}^*} P^t(\mathbf{y}, A) > 0$ for all $\mathbf{y} \in X$, so the set $\{\mathbf{y} \in X | \sum_{t \in \mathbb{N}^*} P^t(\mathbf{y}, A) > 0\}$ equals X . If $\psi(A) = 0$, from [8, Theorem 4.0.1] this would imply that the set $\{\mathbf{y} \in X | \sum_{t \in \mathbb{N}^*} P^t(\mathbf{y}, A) > 0\}$, which equals X , is also ψ -null, which is impossible since by definition ψ is a non-trivial measure. Therefore $\sum_{t \in \mathbb{N}^*} P^t(\mathbf{y}, A) > 0$ for all $\mathbf{y} \in X$ implies that $\psi(A) > 0$.

Then there exists B_0 a non-empty open subset of $A_+^k(\mathbf{x}^*)$ containing $F^k(\mathbf{x}^*, \mathbf{w}^*)$ such that Φ is a μ_{B_0} -irreducible aperiodic Markov chain.

Proof. According to Theorem 2 there exists B_0 an open neighbourhood of $F^k(\mathbf{x}^*, \mathbf{w}^*)$ such that the chain Φ is μ_{B_0} -irreducible. Let ψ be its maximal irreducibility measure (which exists according to [8, Theorem 4.0.1]). According to [8, Theorem 5.4.4.] there exists $d \in \mathbb{N}^*$ and a sequence $(D_i)_{i \in [0..d-1]} \in \mathcal{B}(X)^d$ of sets such that

1. for $i \neq j$, $D_i \cap D_j = \emptyset$
2. $\mu_{B_0}((\bigcup_{i=0}^{d-1} D_i)^c) = 0$
3. for $i = 0, \dots, d-1 \pmod{d}$, for $\mathbf{x} \in D_i$, $P(\mathbf{x}, D_{i+1}) = 1$

Note that 2. is usually stated with the maximal measure ψ but then of course also holds for μ_{B_0} . We will prove that $d = 1$.

From 3. we deduce that for $\mathbf{x} \in D_i$ and $j \in \mathbb{N}^*$, $P^j(\mathbf{x}, D_{i+j \pmod{d}}) = 1$. And with the first point for $l \neq j \pmod{d}$, $P^j(\mathbf{x}, D_{i+l \pmod{d}}) = 0$.

From Lemma 9, there exists \tilde{B}_0 , an open neighbourhood of $F^k(\mathbf{x}^*, \mathbf{w}^*)$ such that for all $\mathbf{z} \in \tilde{B}_0$ there exists $U_{\mathbf{x}^*}$ an open neighbourhood of \mathbf{x}^* having the following property: for $\mathbf{y} \in X$ if there exists a t -steps path from \mathbf{y} to $U_{\mathbf{x}^*}$ and if given A in $\mathcal{B}(X)$, for all $V_{\mathbf{z}}$ open neighbourhood of \mathbf{z} , $V_{\mathbf{z}} \cap A$ has positive Lebesgue measure then $P^{t+k}(\mathbf{y}, A) > 0$. We did not show that $B_0 = \tilde{B}_0$, but we can consider the set $B_1 = B_0 \cap \tilde{B}_0$ which is also an open neighbourhood of $F^k(\mathbf{x}^*, \mathbf{w}^*)$.

Then with 2., $\mu_{B_0}((\bigcup_{i=0}^{d-1} D_i)^c) \geq \mu_{B_1}((\bigcup_{i=0}^{d-1} D_i)^c) = 0$, and since B_1 is a non-empty open set, μ_{B_1} is not trivial hence $\mu_{B_1}(\bigcup_{i=0}^{d-1} D_i) > 0$. So there exists $i \in [0..d-1]$ and $\mathbf{z} \in B_1$ such that for all $V_{\mathbf{z}}$ open neighbourhood of \mathbf{z} , $V_{\mathbf{z}} \cap D_i$ has positive Lebesgue measure (as implied by the contrapositive of Lemma 1).

Since \mathbf{x}^* is a strongly globally attracting state, for all $\mathbf{y} \in X$ there exists $t_{\mathbf{y}} \in \mathbb{N}^*$ such that for all $t \geq t_{\mathbf{y}}$ there exists a t -steps path from \mathbf{y} to $U_{\mathbf{x}^*}$. Using the property of $U_{\mathbf{x}^*}$, this implies that $P^{t+k}(\mathbf{y}, D_i) > 0$. Since this holds for any $t \geq t_{\mathbf{y}}$, it also holds for $t = d(t_{\mathbf{y}} + k) + 1 - k$, and so $P^{d(t_{\mathbf{y}}+k)+1-k+k}(\mathbf{y}, D_i) > 0$. As we had deduced that for $l \neq j \pmod{d}$, $P^j(\mathbf{y}, D_{i+l \pmod{d}}) = 0$, we can conclude that $d(t_{\mathbf{y}} + k) + 1 = 0 \pmod{d}$, hence $1 = 0 \pmod{d}$ meaning $d = 1$ and so Φ is aperiodic. \square

In [8, Proposition 7.3.4 and Theorem 7.3.5] in the context of the function $(\mathbf{x}, \mathbf{w}) \mapsto p_{\mathbf{x}}(\mathbf{w})$ being independent of \mathbf{x} , F being C^∞ and $p_{\mathbf{x}}$ lower semi-continuous, under the assumption that the control model is forward accessible, that there exists a globally attracting state $\mathbf{x}^* \in X$ and that the set $O_{\mathbf{x}}$ is connected, the connexity of $\overline{A_+(\mathbf{x}^*)}$ implies the aperiodicity of the Markov chain. Although in most practical cases the set $O_{\mathbf{x}}$ is connected, it is good to keep in mind that when $O_{\mathbf{x}}$ is not connected, $\overline{A_+(\mathbf{x}^*)}$ can also not be connected and yet the Markov chain can be aperiodic (e.g. any sequence of i.i.d. random variables with non-connected support is a φ -irreducible aperiodic Markov chain). In such problems our approach still offer conditions to show the aperiodicity of the Markov chain.

3.3. Weak-Feller

Our main result summarized in Theorem 1 uses the fact that the chain is weak Feller. Our experience is that this property can be often easily verified by proving that if f is continuous and bounded then

$$\mathbf{x} \in X \mapsto \int f(F(\mathbf{x}, \mathbf{w}))p_{\mathbf{x}}(\mathbf{w})d\mathbf{w}$$

is continuous and bounded. This latter property often deriving from the dominated convergence theorem. We however provide below another result to automatically prove the weak Feller property.

Proposition 5. *Suppose that*

- for all $\mathbf{w} \in O$ the function $\mathbf{x} \in X \mapsto F(\mathbf{x}, \mathbf{w})$ is continuous,
- for all $\mathbf{x} \in X$ the function $\mathbf{w} \in O \mapsto F(\mathbf{x}, \mathbf{w})$ is measurable,
- for all $\mathbf{w} \in O$, the function $\mathbf{x} \in X \mapsto p_{\mathbf{x}}(\mathbf{w})$ is lower semi-continuous.
- for all $\mathbf{x} \in X$ the function $\mathbf{w} \in O \mapsto p_{\mathbf{x}}(\mathbf{w})$ is measurable,

Then the Markov chain Φ is weak-Feller.

Proof. To be weak-Feller means that for any open set $U \in \mathcal{B}(X)$ the function $\mathbf{x} \in X \mapsto P(\mathbf{x}, U)$ is lower semi-continuous.

Take $\mathbf{x} \in X$ and $\mathbf{w} \in O$. If $F(\mathbf{x}, \mathbf{w}) \notin U$ then $\forall \mathbf{y} \in X$, $\mathbf{1}_U(F(\mathbf{y}, \mathbf{w})) \geq \mathbf{1}_U(F(\mathbf{x}, \mathbf{w})) = 0$. If $F(\mathbf{x}, \mathbf{w}) \in U$ as U is an open set there exists $\epsilon > 0$ such that $B(F(\mathbf{x}, \mathbf{w}), \epsilon) \subset U$, and as the function $\mathbf{y} \mapsto F(\mathbf{y}, \mathbf{w})$ is continuous for all $\epsilon > 0$ there exists $\eta > 0$ such that if $\mathbf{y} \in B(\mathbf{x}, \eta)$ then $F(\mathbf{y}, \mathbf{w}) \in B(F(\mathbf{x}, \mathbf{w}), \epsilon) \subset U$. Therefore for all \mathbf{y} in the neighbourhood $B(\mathbf{x}, \eta)$ we have $\mathbf{1}_U(F(\mathbf{y}, \mathbf{w})) = \mathbf{1}_U(F(\mathbf{x}, \mathbf{w})) \geq \mathbf{1}_U(F(\mathbf{x}, \mathbf{w})) - \epsilon$, meaning the function $\mathbf{x} \in X \mapsto \mathbf{1}_U(F(\mathbf{x}, \mathbf{w}))$ is lower semi-continuous. For $\mathbf{w} \in O$ the function $\mathbf{x} \mapsto p_{\mathbf{x}}(\mathbf{w})$ is assumed lower semi-continuous, hence so is $\mathbf{x} \mapsto \mathbf{1}_U(F(\mathbf{x}, \mathbf{w}))p_{\mathbf{x}}(\mathbf{w})$.

Finally we can apply Fatou's Lemma for all sequence $(\mathbf{x}_t)_{t \in \mathbb{N}} \in X^{\mathbb{N}}$ converging to \mathbf{x} :

$$\begin{aligned} \liminf P(\mathbf{x}_t, U) &= \liminf \int_O \mathbf{1}_U(F(\mathbf{x}_t, \mathbf{w}))p_{\mathbf{x}_t}(\mathbf{w})d\mathbf{w} \\ &\geq \int_O \liminf \mathbf{1}_U(F(\mathbf{x}_t, \mathbf{w}))p_{\mathbf{x}_t}(\mathbf{w})d\mathbf{w} \\ &\geq \int_O \mathbf{1}_U(F(\mathbf{x}, \mathbf{w}))p_{\mathbf{x}}(\mathbf{w})d\mathbf{w} = P(\mathbf{x}, U) . \end{aligned}$$

□

4. Applications

We illustrate now the usefulness of Theorem 1. For this, we present two examples of Markov chains that can be modeled via (2) and detail how to apply Theorem 1 to

prove their φ -irreducibility, aperiodicity and the fact that compact sets are small sets. Those Markov chains stem from adaptive stochastic algorithms aiming at optimizing continuous optimization problems. Their stability study implies the linear convergence (or divergence) of the underlying algorithm. Those examples are not artificial: in both cases, showing the φ -irreducibility, aperiodicity and the fact that compact sets are small sets by hand without the results of the current paper seem to be very difficult. They actually motivated the development of the theory of this paper.

4.1. A step-size adaptive randomized search on scaling-invariant functions

We consider first a step-size adaptive stochastic search algorithm optimizing an objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ without constraints. The algorithm pertains to the class of so-called *Evolution Strategies* (ES) algorithms [12] that date back to the 70's. The algorithm is however related to information geometry. It was recently derived from taking the natural gradient of a joint objective function defined on the Riemannian manifold formed by the family of Gaussian distributions [4, 9]. More precisely, let $\mathbf{X}_0 \in \mathbb{R}^n$ and let $(\mathbf{U}_t)_{t \in \mathbb{N}^*}$ be an i.i.d. sequence of random vectors where each \mathbf{U}_t is composed of $\lambda \in \mathbb{N}^*$ components $\mathbf{U}_t = (\mathbf{U}_t^1, \dots, \mathbf{U}_t^\lambda) \in (\mathbb{R}^n)^\lambda$ with $(\mathbf{U}_t^i)_{i \in [1..\lambda]}$ i.i.d. and following each a standard multivariate normal distribution $\mathcal{N}(0, \mathbf{I}_n)$. Given $(\mathbf{X}_t, \sigma_t) \in \mathbb{R}^n \times \mathbb{R}_+^*$, the current state of the algorithm, λ candidate solutions centered on \mathbf{X}_t are sampled using the vector \mathbf{U}_{t+1} , i.e. for i in $[1..\lambda]$

$$\mathbf{X}_t + \sigma_t \mathbf{U}_{t+1}^i, \quad (24)$$

where σ_t called the step-size of the algorithm corresponds to the overall standard deviation of $\sigma_t \mathbf{U}_{t+1}^i$. Those solutions are ranked according to their f -values. More precisely, let \mathcal{S} be the permutation of λ elements such that

$$f(\mathbf{X}_t + \sigma_t \mathbf{U}_{t+1}^{\mathcal{S}(1)}) \leq f(\mathbf{X}_t + \sigma_t \mathbf{U}_{t+1}^{\mathcal{S}(2)}) \leq \dots \leq f(\mathbf{X}_t + \sigma_t \mathbf{U}_{t+1}^{\mathcal{S}(\lambda)}) . \quad (25)$$

To break the possible ties and have an uniquely defined permutation \mathcal{S} , we can simply consider the natural order, i.e. if for instance $\lambda = 2$ and $f(\mathbf{X}_t + \sigma_t \mathbf{U}_{t+1}^1) = f(\mathbf{X}_t + \sigma_t \mathbf{U}_{t+1}^2)$, then $\mathcal{S}(1) = 1$ and $\mathcal{S}(2) = 2$. The new estimate of the optimum \mathbf{X}_{t+1} is formed by taking a weighted average of the μ best directions (typically $\mu = \lambda/2$), that is

$$\mathbf{X}_{t+1} = \mathbf{X}_t + \kappa_m \sum_{i=1}^{\mu} w_i \mathbf{U}_{t+1}^{\mathcal{S}(i)} \quad (26)$$

where the sequence of weights $(w_i)_{i \in [1..\mu]}$ sums to 1, and $\kappa_m > 0$ is called a learning rate. The step-size is adapted according to

$$\sigma_{t+1} = \sigma_t \exp \left(\frac{\kappa_\sigma}{2n} \left(\sum_{i=1}^{\mu} w_i \left(\|\mathbf{U}_{t+1}^{\mathcal{S}(i)}\|^2 - n \right) \right) \right) , \quad (27)$$

where $\kappa_\sigma > 0$ is a learning rate for the step-size. The equations (26) and (27) correspond to the so-called xNES algorithm with covariance matrix restricted to $\sigma_t^2 \mathbf{I}_n$ [4]. One crucial question in optimization is related to the convergence of an algorithm.

On the class of so-called scaling-invariant functions (see below for the definition) with optimum in $\mathbf{x}^* \in \mathbb{R}^n$, a proof of the *linear convergence* of the aforementioned algorithm can be obtained if the normalized chain $\mathbf{Z}_t = (\mathbf{X}_t - \mathbf{x}^*)/\sigma_t$ —which turns out to be an homogeneous Markov chain—is stable enough to satisfy a Law of Large Numbers. This result is explained in details in [1] but in what follows we remind for the sake of completeness the definition of a scaling invariant function and detail the expression of the chain \mathbf{Z}_t .

A function is scaling-invariant with respect to \mathbf{x}^* if for all $\rho > 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$f(\mathbf{x}) \leq f(\mathbf{y}) \Leftrightarrow f(\mathbf{x}^* + \rho(\mathbf{x} - \mathbf{x}^*)) \leq f(\mathbf{x}^* + \rho(\mathbf{y} - \mathbf{x}^*)) . \quad (28)$$

Examples of scaling-invariant functions include $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}^*\|$ for any arbitrary norm on \mathbb{R}^n . It also includes non continuous functions or functions with non-convex sublevel sets, i.e. non-quasi-convex functions.

As mentioned above, on this class of functions, $\mathbf{Z}_t = (\mathbf{X}_t - \mathbf{x}^*)/\sigma_t$ is an homogeneous Markov chain that can be defined independently of the Markov chain (\mathbf{X}_t, σ_t) in the following manner. Given $\mathbf{Z}_t \in \mathbb{R}^n$, sample λ candidate solutions centered on \mathbf{Z}_t using a vector \mathbf{U}_{t+1} , i.e. for i in $[1..\lambda]$

$$\mathbf{Z}_t + \mathbf{U}_{t+1}^i , \quad (29)$$

where similarly as for the chain (\mathbf{X}_t, σ_t) , $(\mathbf{U}_t)_{t \in \mathbb{N}}$ are i.i.d. and each \mathbf{U}_t is a vector of λ i.i.d. components following a standard multivariate normal distribution. Those λ solutions are evaluated and ranked according to their f -values. Similarly to (25), the permutation \mathcal{S} containing the order of the solutions is extracted. This permutation can be uniquely defined if we break the ties as explained below (25). The update of \mathbf{Z}_t then reads

$$\mathbf{Z}_{t+1} = \frac{\mathbf{Z}_t + \kappa_m \sum_{i=1}^{\mu} w_i \mathbf{U}_{t+1}^{\mathcal{S}(i)}}{\exp\left(\frac{\kappa_\sigma}{2n} \left(\sum_{i=1}^{\mu} w_i (\|\mathbf{U}_{t+1}^{\mathcal{S}(i)}\|^2 - n)\right)\right)} . \quad (30)$$

We refer to [1, Proposition 1] for the details. Let us now define $\mathbf{W}_{t+1} = (\mathbf{U}_{t+1}^{\mathcal{S}(1)}, \dots, \mathbf{U}_{t+1}^{\mathcal{S}(\mu)}) \in \mathbb{R}^{n \times \mu}$ and for $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{y} \in (\mathbb{R}^n)^\mu$ (with $\mathbf{y} = (\mathbf{y}^1, \dots, \mathbf{y}^\mu)$)

$$F_{\text{xNES}}(\mathbf{z}, \mathbf{y}) = \frac{\mathbf{z} + \kappa_m \sum_{i=1}^{\mu} w_i \mathbf{y}^i}{\exp\left(\frac{\kappa_\sigma}{2n} \left(\sum_{i=1}^{\mu} w_i (\|\mathbf{y}^i\|^2 - n)\right)\right)} , \quad (31)$$

such that

$$\mathbf{Z}_{t+1} = F_{\text{xNES}}(\mathbf{Z}_t, \mathbf{W}_{t+1}) .$$

Also there exists a function $\alpha : (\mathbb{R}^n, \mathbb{R}^{n \times \lambda}) \rightarrow \mathbb{R}^{n \times \mu}$ such that $\mathbf{W}_{t+1} = \alpha(\mathbf{Z}_t, \mathbf{U}_{t+1})$. Indeed, given \mathbf{z} and \mathbf{u} in $\mathbb{R}^{n \times \lambda}$ we have explained how the permutation giving the ranking of the candidate solutions $\mathbf{z} + \mathbf{u}^i$ on f can be uniquely defined. Then $\alpha(\mathbf{z}, \mathbf{u}) = (\mathbf{u}^{\mathcal{S}(1)}, \dots, \mathbf{u}^{\mathcal{S}(\lambda)})$. Hence we have just explained why the Markov chain defined via (30)

fits the Markov chain model underlying this paper, that is (2). If we assume that the level sets of the function f are Lebesgue negligible, then a density $p : (\mathbf{z}, \mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^{n \times \mu} \mapsto \mathbb{R}_+$ associated to \mathbf{W}_{t+1} writes

$$p(\mathbf{z}, \mathbf{w}) = \frac{\lambda!}{(\lambda - \mu)!} \mathbf{1}_{\{f(\mathbf{z}+\mathbf{w}^1) < \dots < f(\mathbf{z}+\mathbf{w}^\mu)\}} (1 - Q_{\mathbf{z}}^f(\mathbf{w}^\mu))^{\lambda - \mu} p_{\mathcal{N}}(\mathbf{w}^1) \dots p_{\mathcal{N}}(\mathbf{w}^\mu) \quad (32)$$

with each $\mathbf{w}^i \in \mathbb{R}^n$ and $\mathbf{w} = (\mathbf{w}^1, \dots, \mathbf{w}^\mu)$, where $Q_{\mathbf{z}}^f(\mathbf{w}^\mu) = \Pr(f(\mathbf{z} + \mathcal{N}) \leq f(\mathbf{z} + \mathbf{w}^\mu))$ with \mathcal{N} following a standard multivariate normal distribution and

$$p_{\mathcal{N}}(\mathbf{y}) = \frac{1}{(\sqrt{2\pi})^n} \exp(-\mathbf{y}^T \mathbf{y} / 2)$$

the density of a standard multivariate normal distribution in dimension n . If the objective function f is continuous, then the density $p(\mathbf{z}, \mathbf{w})$ is lower semi-continuous.

We now prove by applying Theorem 1 that the Markov chain \mathbf{Z}_t is a φ -irreducible aperiodic T -chain and compact sets are small sets for the chain. Those properties together with a drift for positivity will imply the linear convergence of the xNES algorithm [1].

Proposition 6. *Suppose that the scaling invariant function f is continuous, and that its level sets are Lebesgue negligible. Then the Markov chain $(\mathbf{Z}_t)_{t \in \mathbb{N}}$ defined in (30) is a φ -irreducible aperiodic T -chain and compact sets are small sets for the chain.*

Proof. It is not difficult to see that $p(\mathbf{z}, \mathbf{w})$ is lower-semi continuous since f is continuous and that F_{xNES} is a C^1 function. We remind that $O_{\mathbf{z}} = \{\mathbf{w} \in \mathbb{R}^{n \times \mu} | p(\mathbf{z}, \mathbf{w}) > 0\}$ hence with (32) $O_{\mathbf{z}} = \{\mathbf{w} \in \mathbb{R}^{n \times \mu} | \mathbf{1}_{\{f(\mathbf{z}+\mathbf{w}^1) < \dots < f(\mathbf{z}+\mathbf{w}^\mu)\}} > 0\}$.

We will now prove that the point $\mathbf{z}^* := \mathbf{0}$ is a strongly globally attracting state. For $\mathbf{y} \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}_+^*$, this means there exists a $t_{\mathbf{y}, \epsilon} \in \mathbb{N}^*$ such that for all $t \geq t_{\mathbf{y}, \epsilon}$, there exists a t -steps path from \mathbf{y} to $B(\mathbf{0}, \epsilon)$. Note that $\lim_{\|\mathbf{w}\| \rightarrow +\infty} F_{\text{xNES}}(\mathbf{y}, \mathbf{w}) = \mathbf{0}$, meaning that there exists a $r \in \mathbb{R}_+^*$ such that if $\|\mathbf{w}\| \geq r$ then $F_{\text{xNES}}(\mathbf{z}, \mathbf{w}) \in B(\mathbf{0}, \epsilon)$. Therefore, and since $O_{\mathbf{y}} \cap \{\mathbf{w} \in \mathbb{R}^{n \times \mu} | \|\mathbf{w}\| \geq r\}$ is non empty, there exists a $\mathbf{w}_{\mathbf{y}, \epsilon} \in O_{\mathbf{y}}$ which is a 1-step path from \mathbf{y} to $B(\mathbf{0}, \epsilon)$. Now, showing that there is such a path from $\tilde{\mathbf{y}} \in \mathbb{R}^n$ for all $t \geq 1$ is trivial: take $\mathbf{w} \in O_{\mathbf{y}, t-1}$, and denote $\mathbf{y} = F^{t-1}(\tilde{\mathbf{y}}, \mathbf{w})$; $(\mathbf{w}, \mathbf{w}_{\mathbf{y}, \epsilon})$ is a t -steps path from $\tilde{\mathbf{y}}$ to $B(\mathbf{0}, \epsilon)$.

We now prove that there exists $\mathbf{w}^* \in O_{\mathbf{0}}$ such that $F(\mathbf{0}, \cdot)$ is a submersion at \mathbf{w}^* , by proving that the differential of $F(\mathbf{0}, \cdot)$ at \mathbf{w}^* is surjective. Take $\mathbf{w}_0 = (\mathbf{0}, \dots, \mathbf{0}) \in \mathbb{R}^{n \times \mu}$ and $\mathbf{h} = (\mathbf{h}_i)_{i \in [1.. \mu]} \in \mathbb{R}^{n \times \mu}$, then

$$\begin{aligned} F_{\text{xNES}}(\mathbf{0}, \mathbf{0} + \mathbf{h}) &= \frac{\kappa_m \sum_{i=1}^{\mu} w_i \mathbf{h}_i}{\exp\left(\frac{\kappa_\sigma}{2n} \left(\sum_{i=1}^{\mu} w_i (\|\mathbf{h}_i\|^2 - n)\right)\right)} \\ &= \mathbf{0} + \kappa_m \left(\sum_{i=1}^{\mu} w_i \mathbf{h}_i \right) \exp\left(\frac{\kappa_\sigma}{2}\right) \exp\left(-\frac{\kappa_m}{2n} \sum_{i=1}^{\mu} w_i \|\mathbf{h}_i\|^2\right) \\ &= F_{\text{xNES}}(\mathbf{0}, \mathbf{0}) + \kappa_m \left(\sum_{i=1}^{\mu} w_i \mathbf{h}_i \right) \exp\left(\frac{\kappa_\sigma}{2}\right) (1 + o(\|\mathbf{h}\|)) \quad . \end{aligned}$$

Hence $D_{\mathbf{w}_0} F_{\text{xNES}}(\mathbf{0}, \cdot)(\mathbf{h}) = \kappa_m \exp(\kappa_\sigma/2) \sum_{i=1}^\mu w_i \mathbf{h}_i$, and is therefore a surjective linear map. The point \mathbf{w}_0 is not in $O_{\mathbf{0}}$, but according to Lemma 5 since $F_{\text{xNES}}(\mathbf{0}, \cdot)$ is a submersion at \mathbf{w}_0 there exists $V_{\mathbf{w}_0}$ an open neighbourhood of \mathbf{w}_0 such that for all $\mathbf{v} \in V_{\mathbf{w}_0}$, $F_{\text{xNES}}(\mathbf{0}, \cdot)$ is a submersion at \mathbf{v} . Finally since $V_{\mathbf{w}_0} \cap O_{\mathbf{0}}$ is not empty, there exists $\mathbf{w}^* \in O_{\mathbf{0}}$ such that $F_{\text{xNES}}(\mathbf{0}, \cdot)$ is a submersion at \mathbf{w}^* .

We can then apply Theorem 1 which shows that $(\mathbf{Z}_t)_{t \in \mathbb{N}}$ is a ψ -irreducible aperiodic T-chain, and that compact sets are small sets for the chain. \square

4.2. A step-size adaptive randomized search on a simple constraint optimization problem

We now consider a similar algorithm belonging to the class of evolution strategies optimizing a linear function under a linear constraint. The goal for the algorithm is to diverge as fast as possible as the optimum of the problem is at infinity. More precisely let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be two linear functions (w.l.o.g. we take $f(\mathbf{x}) = [\mathbf{x}]_1$, and $g(\mathbf{x}) = -\cos \theta [\mathbf{x}]_1 - \sin \theta [\mathbf{x}]_2$) with $\theta \in (0, \pi/2)$. The goal is to maximize f while respecting the constraint $g(\mathbf{x}) > 0$.

As for the previous algorithm, the state of the algorithm is reduced to $(\mathbf{X}_t, \sigma_t) \in \mathbb{R}^n \times \mathbb{R}_+^*$ where \mathbf{X}_t represents the favorite solution and σ_t is the step-size controlling the standard deviation of the sampling distribution used to generate new solutions. From \mathbf{X}_t , λ new solutions are sampled

$$\mathbf{Y}_{t+1}^i = \mathbf{X}_t + \sigma_t \mathbf{V}_{t+1}^i, \quad (33)$$

where each $\mathbf{V}_t = (\mathbf{V}_t^1, \dots, \mathbf{V}_t^\lambda)$ with $(\mathbf{V}_t^i)_i$ i.i.d. following a standard multivariate normal distribution in dimension n . Those solutions may lie in the infeasible domain, that is they might violate the constraint, i.e. $g(\mathbf{Y}_{t+1}^i) \leq 0$. Hence a specific mechanism is added to ensure that we have λ solutions within the feasible domain. Here this mechanism is very simple, it consists in resampling a solution till it lies in the feasible domain. We denote $\tilde{\mathbf{Y}}_{t+1}^i$ the candidate solution i that satisfies the constraint. While the resampling of a candidate solution can possibly call for an infinite numbers of multivariate normal distribution, it can be shown in our specific case that this candidate solution can be generated using a single random vector \mathbf{U}_{t+1}^i and is a function of the normalized distance to the constraint $\delta_t = g(\mathbf{X}_t)/\sigma_t$. This is due to the fact that the distribution of the feasible candidate solution orthogonal to the constraint direction follows a truncated Gaussian distribution and orthogonal to the constraint a Gaussian distribution (we refer to [2, Lemma 2] for the details). Hence overall,

$$\tilde{\mathbf{Y}}_{t+1}^i = \mathbf{X}_t + \sigma_t \tilde{\mathcal{G}}(\delta_t, \mathbf{U}_{t+1}^i)$$

where $[\mathbf{U}_t = (\mathbf{U}_t^1, \dots, \mathbf{U}_t^\lambda)]_t$ are i.i.d. (see [2, Lemma 2]) and the function $\tilde{\mathcal{G}}$ is defined in [2, equation (15)]. Those λ feasible candidate solutions are ranked on the objective function f and as before, the permutation \mathcal{S} containing the ranking of the solutions is

extracted. The update of \mathbf{X}_{t+1} then reads

$$\mathbf{X}_{t+1} = \mathbf{X}_t + \sigma_t \tilde{\mathcal{G}}(\delta_t, \mathbf{U}_{t+1}^{S(1)}) , \quad (34)$$

that is the best solution is kept. The update of the step-size satisfies

$$\sigma_{t+1} = \sigma_t \exp \left(\frac{1}{2d_\sigma} \left(\frac{\|\tilde{\mathcal{G}}(\delta_t, \mathbf{U}_{t+1}^{S(1)})\|^2}{n} - 1 \right) \right) , \quad d_\sigma \in \mathbb{R}_+^* . \quad (35)$$

This algorithm corresponds to a so-called $(1, \lambda)$ -ES with resampling using the cumulative step-size adaptation mechanism of the covariance matrix adaptation ES (CMA-ES) algorithm [5].

It is not difficult to show that $(\delta_t)_{t \in \mathbb{N}}$ is an homogeneous Markov chain (see [2, Proposition 5]) whose update reads

$$\delta_{t+1} = \left(\delta_t + g(\tilde{\mathcal{G}}(\delta_t, \mathbf{U}_{t+1}^{S(1)})) \right) \exp \left(-\frac{1}{2d_\sigma} \left(\frac{\|\tilde{\mathcal{G}}(\delta_t, \mathbf{U}_{t+1}^{S(1)})\|^2}{n} - 1 \right) \right) . \quad (36)$$

and that the divergence of the algorithm can be proven if $(\delta_t)_{t \in \mathbb{N}}$ satisfies a Law of Large Numbers. Given that typical conditions to prove that an homogeneous Markov chain satisfies a LLN is φ -irreducibility, aperiodicity, Harris-recurrence and positivity and that those latter two conditions are practical to verify with drift conditions that hold outside a small set, we see the interest to be able to prove the irreducibility aperiodicity and identify that compact sets are small sets for $(\delta_t)_{t \in \mathbb{N}}$.

With respect to the modeling of the paper, let $\mathbf{W}_t = \tilde{\mathcal{G}}(\delta_t, \mathbf{U}_{t+1}^{S(1)})$, then there is a well-defined function α such that $\mathbf{W}_t = \alpha(\delta_t, \mathbf{U}_{t+1})$ and according to [2, Lemma 3] the density $p(\delta, \mathbf{w})$ of \mathbf{W}_t knowing that $\delta_t = \delta$ equals

$$p(\delta, \mathbf{w}) = \lambda \frac{p_{\mathcal{N}}(\mathbf{w}) \mathbf{1}_{\mathbb{R}_+^*}(\delta + g(\mathbf{w}))}{\mathcal{F}_{p_{\mathcal{N}}}(\delta)} \left(\int_{-\infty}^{[\mathbf{w}]_1} p_1(u) \frac{\mathcal{F}_{p_{\mathcal{N}}}(\frac{\delta - u \cos \theta}{\sin \theta})}{\mathcal{F}_{p_{\mathcal{N}}}(\delta)} du \right)^{\lambda-1} , \quad (37)$$

where p_1 is the density of a one dimensional normal distribution, $p_{\mathcal{N}}$ the density of a n-dimensional multivariate normal distribution and $\mathcal{F}_{p_{\mathcal{N}}}$ its associated cumulative distribution function.

The state space X for the Markov chain $(\delta_t)_{t \in \mathbb{N}}$ is \mathbb{R}_+^* , the set O equals to \mathbb{R}^n and the function F implicitly given in (36):

$$F(\delta, \mathbf{w}) = (\delta + g(\mathbf{w})) \exp \left(-\frac{1}{2d_\sigma} \left(\frac{\|\mathbf{w}\|^2}{n} - 1 \right) \right) . \quad (38)$$

The control set $O_{x,t}$ equals

$$O_{x,t} := \{(\mathbf{w}_1, \dots, \mathbf{w}_t) \in \mathbb{R}^{nt} | x > -g(\mathbf{w}_1), \dots, F^{t-1}(x, \mathbf{w}_1, \dots, \mathbf{w}_{t-1}) > -g(\mathbf{w}_t)\} .$$

We are now ready to apply the results develop within the paper to prove that the chain $(\delta_t)_{t \in \mathbb{N}}$ is a φ -irreducible aperiodic T -chain and that compact sets are small sets.

Proposition 7. *The Markov chain $(\delta_t)_{t \in \mathbb{N}}$ is a φ -irreducible aperiodic T-chain and compact sets of \mathbb{R}_+^* are small sets.*

Proof. The function $p(\delta, \mathbf{w})$ defined in (37) is lower semi-continuous, and the function F defined in (38) is C^1 .

We now prove that any point $\delta^* \in \mathbb{R}_+^*$ is a strongly globally attracting state, i.e. for all $\delta_0 \in \mathbb{R}_+^*$ and $\epsilon \in \mathbb{R}_+^*$ small enough there exists $t_0 \in \mathbb{N}^*$ such that for all $t \geq t_0$ there exists $\mathbf{w} \in O_{\delta_0, t}$ such that $F^t(\delta_0, \mathbf{w}) \in B(\delta^*, \epsilon)$. Let $\delta_0 \in \mathbb{R}_+^*$. Let $k \in \mathbb{N}^*$ be such that $\delta_0 \exp(k/(2d_\sigma)) > \delta^*$. We take $\mathbf{w}_i = \mathbf{0}$ for all $i \in [1..k]$ and define $\delta_k := F^k(\delta_0, \mathbf{w}_1, \dots, \mathbf{w}_k)$. By construction of k , we have $\delta_k = \delta_0 \exp(-k/(2d_\sigma)(-1)) > \delta^*$. Now, take $\mathbf{u} = (-1, \dots, -1)$ and note that the limit $\lim_{\alpha \rightarrow +\infty} F(\delta_k, \alpha \mathbf{u}) = 0$. Since the function F is continuous and that $F(\delta_k, \mathbf{0}\mathbf{u}) > \delta_k$, this means that the set $(0, \delta_k)$ is included into the image of the function $\alpha \mapsto F(\delta_k, \alpha \mathbf{u})$. And since $\delta^* \in (0, \delta_k)$, there exists $\alpha_0 \in \mathbb{R}_+$ such that $F(\delta_k, \alpha_0 \mathbf{u}) = \delta^*$. Now let $\bar{\mathbf{w}} = (\mathbf{w}_1, \dots, \mathbf{w}_k, \alpha_0 \mathbf{u})$, and note that since $g(\mathbf{u}) > 0$ and g is linear, $\alpha \mathbf{u} \in O_\delta = \{\mathbf{v} \in \mathbb{R}^n \mid \delta + g(\mathbf{v}) > 0\}$ for all $\alpha \in \mathbb{R}_+$ and all $\delta \in \mathbb{R}_+^*$; hence $\alpha_0 \mathbf{u} \in O_{\delta_k}$ and $\mathbf{w}_i = \mathbf{0}\mathbf{u} \in O_\delta$ for all $\delta \in \mathbb{R}_+^*$. Therefore $\bar{\mathbf{w}} \in O_{\delta_0, k+1}$ and $F^{k+1}(\delta_0, \bar{\mathbf{w}}) = \delta^*$, so $\bar{\mathbf{w}}$ is a $k+1$ -steps path from δ_0 to $B(\delta^*, \epsilon)$. As the proof stand for all k large enough, δ^* is a strongly globally attractive state.

We will now show that $F(0, \cdot)$ is a submersion at some point $\mathbf{w} \in \mathbb{R}^n$. To do so we compute the differential $D_{\mathbf{w}}F(0, \cdot)$ of $F(0, \cdot)$ at \mathbf{w} :

$$\begin{aligned} F(0, \mathbf{w} + \mathbf{h}) &= g(\mathbf{w} + \mathbf{h}) \exp\left(-\frac{1}{2d_\sigma} \left(\frac{\|\mathbf{w} + \mathbf{h}\|^2}{n} - 1\right)\right) \\ &= g(\mathbf{w} + \mathbf{h}) \exp\left(-\frac{1}{2d_\sigma} \left(\frac{\|\mathbf{w}\|^2 + 2\mathbf{w} \cdot \mathbf{h} + \|\mathbf{h}\|^2}{n} - 1\right)\right) \\ &= g(\mathbf{w} + \mathbf{h}) \exp\left(-\frac{1}{2d_\sigma} \left(\frac{\|\mathbf{w}\|^2}{n} - 1\right)\right) \exp\left(-\frac{1}{2d_\sigma} \left(\frac{2\mathbf{w} \cdot \mathbf{h} + \|\mathbf{h}\|^2}{n}\right)\right) \\ &= g(\mathbf{w} + \mathbf{h}) \exp\left(-\frac{1}{2d_\sigma} \left(\frac{\|\mathbf{w}\|^2}{n} - 1\right)\right) \left(1 - \frac{2\mathbf{w} \cdot \mathbf{h}}{2d_\sigma n} + o(\|\mathbf{h}\|)\right) \\ &= F(0, \mathbf{w}) - F(0, \mathbf{w}) \frac{\mathbf{w} \cdot \mathbf{h}}{d_\sigma n} + g(\mathbf{h}) \exp\left(-\frac{1}{2d_\sigma} \left(\frac{\|\mathbf{w}\|^2}{n} - 1\right)\right) + o(\|\mathbf{h}\|). \end{aligned}$$

Hence for $\mathbf{w} = (-\sqrt{n}, 0, \dots, 0)$ and $\mathbf{h} = (0, \alpha, 0, \dots, 0)$, $D_{\mathbf{w}}F(0, \cdot)(\mathbf{h}) = -\alpha \sin \theta \exp(0)$. Hence for α spanning \mathbb{R} , $D_{\mathbf{w}}F(0, \cdot)(\mathbf{h})$ spans \mathbb{R} such that the image of $D_{\mathbf{w}}F(0, \cdot)$ equals \mathbb{R} , i.e. $D_{\mathbf{w}}F(0, \cdot)$ is surjective meaning $F(0, \cdot)$ is a submersion at \mathbf{w} . According to Lemma 5 this means there exists N an open neighbourhood of $(0, \mathbf{w})$ such that for all $(\delta, \mathbf{u}) \in N$, $F(\delta, \cdot)$ is a submersion at \mathbf{u} . So for $\delta^* \in \mathbb{R}_+^*$ small enough, $F(\delta^*, \cdot)$ is a submersion at $\mathbf{w} = (-\sqrt{n}, 0, \dots, 0) \in O_{\delta^*}$.

Adding this with the fact that δ^* is a strongly globally attracting state, we can then apply Theorem 1 which concludes the proof. \square

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