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► **To cite this version:**

Reynald Lercier, Christophe Ritzenthaler, Florent Rovetta, Jeroen Sijlsing, Benjamin Smith. Distributions of traces of Frobenius for smooth plane curves over finite fields. *Experimental Mathematics*, 2019, 28 (1), pp.39-48. 10.1080/10586458.2017.1328321 . hal-01217995

HAL Id: hal-01217995

<https://inria.hal.science/hal-01217995v1>

Submitted on 6 Jan 2025

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DISTRIBUTIONS OF TRACES OF FROBENIUS FOR SMOOTH PLANE CURVES OVER FINITE FIELDS

REYNALD LERCIER, CHRISTOPHE RITZENTHALER, FLORENT ROVETTA, JEROEN SIJSLING,
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ABSTRACT. In a previous article, we obtained data on the distribution of traces of Frobenius of non-hyperelliptic genus 3 curves over small finite fields. In the present one, we give a heuristic explanation of these data, by extrapolating from results on the distribution of traces of Frobenius for plane curves whose degree is small with respect to the cardinality of their finite base field. In particular, our methods shed some new light on the asymmetry of the distribution around its mean value, which is related to the Serre obstruction.

1. INTRODUCTION

More than 30 years ago, Serre found closed formulæ for the maximal possible number of rational points on a curve of genus $g \leq 2$ over a finite field \mathbb{F}_q . The same article [28] also considered the problem of obtaining a similar formula for curves of genus 3. This problem is more involved, as Serre indeed noted at the time. The main obstruction to obtaining a closed formula is now known as the *Serre obstruction*: a principally polarized abelian threefold over \mathbb{F}_q that becomes the Jacobian of a genus 3 curve after base extension to $\overline{\mathbb{F}}_q$ need not be the Jacobian of such a curve over \mathbb{F}_q . There is no such obstruction in dimension 1, because elliptic curves are their own Jacobians; nor in dimension 2, because all genus 2 curves are hyperelliptic (and thus their automorphism groups are isomorphic to those of their Jacobians, which implies in general that such a descent is possible). The Serre obstruction appears in dimension 3 precisely because of the existence of non-hyperelliptic curves of genus 3.

There have been many computational and conceptual approaches to the Serre obstruction. Partial results can be found in [13, 19, 22, 23, 24, 27], while data for small fields can be found at manypoints.org. More general approaches were considered in [2, 17, 18]. However, so far none of these results can be used to give a closed formula in the sense of [28].

More precisely, the issue is as follows. The number of rational points on a given curve C of genus g over a finite field \mathbb{F}_q is given by

$$\#C(\mathbb{F}_q) = q + 1 - t .$$

Here t is the trace of Frobenius acting on the first étale cohomology group $H^1(J_{\text{ét}}, \mathbb{Z}_\ell)$ of the Jacobian J of C , with ℓ some prime not equal to p . The classical Hasse–Weil–Serre bound shows that the absolute value of t is bounded by $2g\sqrt{q}$. The usual strategy for constructing maximal curves is to construct a principally polarized abelian variety A over \mathbb{F}_q with a trace t close to $-2g\sqrt{q}$, so as to obtain a number of rational points that is close to the upper bound $q + 1 + 2g\sqrt{q}$.

The Serre obstruction amounts to the fact that A is not necessarily the Jacobian of a curve C over \mathbb{F}_q . If it is not, then the quadratic twist of A will be a Jacobian instead. However, taking such a quadratic twist changes the trace of Frobenius t to $-t$, so that the resulting C has few points instead of many points.

This article was motivated by the hope of finding a new inroad to the problem, not by studying all descent problems for individual abelian varieties, but rather by an indirect approach: namely, by proving that for large negative values of t there are more curves with trace t than with trace $-t$.

Date: June 9, 2022.

2010 Mathematics Subject Classification. 14Q05; 14H10; 14H25; 14H37; 14H45; 14H50.

Key words and phrases. Genus 3 curves; plane quartics; moduli; families; enumeration; finite fields.

The authors acknowledge support by grant ANR-09-BLAN-0020-01.

More precisely, let $\mathcal{N}_{q,3}(t)$ be the number of non-hyperelliptic genus 3 curves over \mathbb{F}_q of trace t up to \mathbb{F}_q -isomorphism. We are interested in studying the difference

$$\mathcal{V}_{q,3}(t) := \mathcal{N}_{q,3}(t) - \mathcal{N}_{q,3}(-t)$$

for $0 \leq t \leq 6\sqrt{q}$, and more specifically in proving that $\mathcal{V}_{q,3}(t) \leq 0$ for large enough t . This would be enough to show (using [19]) that there always exists a curve C such that $\#C(\mathbb{F}_q) \geq q + 1 + 3\lfloor 2\sqrt{q} \rfloor - 3$, and would moreover allow us to give the precise maximal value (see [26, Prop.4.1.7]).

A numerical study of $\mathcal{V}_{q,3}(t)$ for prime fields \mathbb{F}_q with $11 \leq q \leq 53$ appears in [20]. Over these small fields, it was possible to construct all smooth non-hyperelliptic curves of genus 3 (that is, all smooth plane quartics) up to \mathbb{F}_q -isomorphism, and to compute the trace of a representative of each \mathbb{F}_q -isomorphism class. The resulting functions $\mathcal{V}_{q,3}(t)$ had some remarkable properties: for example, it always appeared that $\mathcal{V}_{q,3}(t)$ was negative for $t > 1.7\sqrt{q}$, so that the corresponding number of curves with many points was larger than the number of curves with few points. Our original hope that $\mathcal{V}_{q,3}(t)$ would always be negative for large enough t turned out to be false in general, as was noted in [20]. Nevertheless, for each q the data obtained fitted simple and pleasing graphs, and moreover these graphs almost coincided after normalizing by an explicit power of q .

This phenomenon seemed interesting enough to merit further investigation. Moreover, it fits into a larger framework of results on the distribution of the number of rational points on curves over finite fields, as studied in [6, 7, 8, 9, 10, 16, 21, 33, 34] and especially [5]. The results of Bucur–David–Feigon–Lalín in [5] show that the number of rational points on plane curves of degree d over \mathbb{F}_q is distributed according to an explicit and intuitive binomial law as long as d is large enough compared to q . We study this binomial law, or rather the normalized variation of this law around its mean, in Section 2. By the central limit theorem, this variation converges to 0; however, we show that after multiplying by a factor of \sqrt{q} , its behavior as q tends to infinity can be approximated by the function

$$\psi : x \mapsto \frac{1}{3\sqrt{2\pi}} x(3-x^2)e^{-x^2/2}.$$

In Section 3, using [5], we study the variation around the mean of the distribution of the number of rational points on plane curves of degree d over \mathbb{F}_q as $q \rightarrow \infty$ for both general curves (Corollary 3.4) and smooth curves (Corollary 3.6). Along the way, we show by elementary methods that the bound $d \geq q^2 + q$ of [5, Prop. 1.6] can be improved to $d \geq 2q - 1$ (in Proposition 3.3). We note that in these cases, where d is large compared to q , the fact that there are more curves with large negative trace t than with trace $-t$ is easy to see, since a curve with a large positive trace would have a negative number of rational points.

In Section 4, in contrast to the approach for sufficiently large d above, we instead fix the particular small value $d = 4$, while still letting q tend to ∞ . Here it remains a challenge to prove any exact results, but the comparison with our experiments in [20] is impressive. It would be especially interesting to see whether the observed phenomena persist for larger q than those considered in these experiments. Our results are also consistent with those of [1], where it is conjectured that the distribution of the fraction of curves of genus g over a fixed finite field \mathbb{F}_q whose number of points equals n tends to a Poisson distribution as g tends to infinity. More precisely, it is suggested as one runs over a set of curves over \mathbb{F}_q that represent the \mathbb{F}_q -rational points of the moduli stack of curves \mathbf{M}_g , one should have the following limit behavior:

$$\lim_{g \rightarrow \infty} \# \{C \in \mathbf{M}_g(\mathbf{F}_q) : \#C(\mathbb{F}_q) = n\} = \frac{\lambda^n e^{-\lambda}}{n!} \quad \text{where } \lambda = q + 1 + \frac{1}{q-1}.$$

The same argument as in the proof of Proposition 2.2 shows that the variation around the mean gives rise to the same distribution that we obtain in our paper.

Acknowledgments. We are very grateful to Mohamed Barakat for his helpful comments, Masaaki Homma for his proof of Lemma 3.2, Everett Howe for email exchanges which led us to the heuristic interpretation described in this article, and Atilla Yilmaz for pointing out the link between our statistical result and Edgeworth series.

2. SOME REMARKS ON THE BINOMIAL DISTRIBUTION

We begin by analyzing the asymmetry of the binomial distribution around its mean in a general context. We will then adapt the parameters according to our arithmetical problems.

Let us consider the the binomial random variable of standard deviation σ

$$S_\sigma := \sum_{i=1}^{N_\sigma} B_{\sigma,i}$$

given by N_σ Bernoulli random variables $B_{\sigma,i}$ taking the value 1 with probability μ_σ . Let

$$b_\sigma(m) := \text{Prob}(S_\sigma = m) = \binom{N_\sigma}{m} \mu_\sigma^m (1 - \mu_\sigma)^{(N_\sigma - m)}$$

be the corresponding binomial mass function of mean $E_\sigma = N_\sigma \mu_\sigma$. We have the usual relation

$$\sigma = \sqrt{N_\sigma \mu_\sigma (1 - \mu_\sigma)} .$$

Let us now consider an increasing sequence of σ . For any fixed real $\alpha > 0$, we define a sequence of intervals $I_\sigma = [-\sigma^{1-\alpha}, \sigma^{1-\alpha}]$. In order to deal with approximations in both σ and $x \in I_\sigma$, we need one more piece of notation.

Definition 2.1. Suppose that for all (sufficiently large) σ we are given an interval I_σ along with functions f_σ and g_σ on I_σ . We write $f_\sigma = o(g_\sigma)$ if for all $\epsilon > 0$, there exists a σ_0 such that

$$|f_\sigma(x)| \leq \epsilon |g_\sigma(x)| \quad \forall x \in I_\sigma, \quad \forall \sigma > \sigma_0 .$$

The notation $f_\sigma = O(g_\sigma)$ is defined similarly.

The triangular central limit theorem [4, Th.27.3] shows¹ that the normalized average sequence $Y_\sigma := (S_\sigma - E_\sigma)/\sigma$ converges in law to the standard normal distribution: that is, the cumulative distribution law of the random variable Y_σ converges to the integral of the Gaussian density. We therefore see that $Y_\sigma(x) - Y_\sigma(-x)$ tends to 0 as $\sigma \rightarrow \infty$.

In order to deal with asymptotic approximations (in σ) of S_σ , we introduce a new continuous function that we will call b . We define

$$\begin{aligned} m(\sigma, x) &:= E_\sigma - \sigma x , \\ n(\sigma, x) &:= N_\sigma(1 - \mu_\sigma) + \sigma x = N_\sigma - m(\sigma, x) . \end{aligned}$$

Then if σ is large enough, we define the aforementioned continuous function b by

$$b(\sigma, x) := \frac{\Gamma(N_\sigma + 1)}{\Gamma(n(\sigma, x) + 1)\Gamma(m(\sigma, x) + 1)} \mu_\sigma^{m(\sigma, x)} (1 - \mu_\sigma)^{n(\sigma, x)} . \quad (2.1)$$

Observe that if $m(\sigma, x)$ happens to be integral, then

$$b(\sigma, x) = b_\sigma(m(\sigma, x)) ;$$

as such, $b(\sigma, x)$ should be thought of as the collection of continuous interpolations of these values, with the difference that we have switched to using the normalized parameter x instead of m .

Proposition 2.2. *With the previous notation, assume that $E_\sigma = \sigma^2 + 1 + O(1/\sigma^2)$ when σ tends to infinity, and let $\alpha > 0$ be any fixed real. Then for $x \in [-\sigma^{1-\alpha}, \sigma^{1-\alpha}]$, we have*

$$b(\sigma, x) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \right) \frac{1}{\sigma} - \left(\frac{1}{6\sqrt{2\pi}} x(x^2 - 3) e^{-\frac{1}{2}x^2} \right) \frac{1}{\sigma^2} + O\left(\frac{1}{\sigma^3} \right) .$$

We defer the proof, which is long and technical, to Section 5.

Corollary 2.3. *With the notation of Proposition 2.2,*

$$b(\sigma, x) - b(\sigma, -x) = \left(\frac{1}{3\sqrt{2\pi}} x(3 - x^2) e^{-\frac{1}{2}x^2} \right) \frac{1}{\sigma^2} + O\left(\frac{1}{\sigma^3} \right) .$$

¹ This can also be deduced from the De Moivre–Laplace theorem [4, Ex. 25.11], which claims that if $(m_\sigma)_\sigma$ is any sequence of integers such that $(m_\sigma - E_\sigma)/\sigma$ tends to some real x as $\sigma \rightarrow \infty$ then $\lim_{\sigma \rightarrow \infty} \sigma \text{Prob}(S_\sigma = m_\sigma) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

Remark 2.4. The error terms in the central limit theorem are well-studied, and there are powerful tools to deal with them in the literature. However, the usual Edgeworth series techniques do not seem to apply here. Indeed, while the expressions involved are identical (for example, the polynomial $x(3 - x^2)$ appearing in Proposition 2.2 is the negative of the third Hermite polynomial [15, Sec. 3.4]), the interval I_σ of convergence we obtain, which is optimal, is not the interval predicted by a formal application of Edgeworth series we found in the literature.

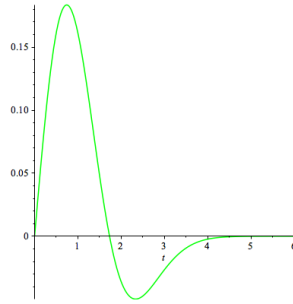


FIGURE 1. The graph of the approximation $\psi : x \mapsto \frac{1}{3\sqrt{2\pi}}x(x^2 - 3)e^{-x^2/2}$.

We will use these results in Sections 2 and 3 with the following parameters. We assume that the probabilities for a plane curve over \mathbb{F}_q to pass through any one of the $q^2 + q + 1$ points of the projective plane are described by independent and identically distributed random variables with probability $1/q + O(1/q^2)$. More precisely, we will apply the results for the following values.

- (1) In Corollary 3.4 we will have $\sigma = \sqrt{q - 1/q^2} = \sqrt{q} + O(q^{-5/2})$, $\mu_\sigma = 1/q$, and

$$E_\sigma = (q^2 + q + 1)/q = \sigma^2 + 1 + O(1/q) = \sigma^2 + 1 + O(1/\sigma^2).$$

- (2) In Corollary 3.6 we will have $\sigma = \sqrt{q(1 - \frac{1}{q^2+q+1})}$, $\mu_\sigma = (q + 1)/(q^2 + q + 1)$, and

$$E_\sigma = q + 1 = \sigma^2 + 1 + O(1/\sigma^2).$$

In both situations, we meet the hypotheses of Proposition 2.2. Note that we can also apply our results to affine plane curves. Indeed, we pass through q^2 points with probability $\mu_\sigma = 1/q$ so we have $\sigma = \sqrt{q - 1}$ and $E_\sigma = \sigma^2 + 1$.

In the arithmetic setting below, the normalization will not be $m(\sigma, x) = E_\sigma - \sigma x$ but $q + 1 - \sqrt{q}x$. Hence we need to check that we can transfer the perturbation on x induced by this transformation into the $O(1/\sigma^3)$. Since the main term of the expansion in Corollary 2.3 is $x(x^2 - 3)e^{-\frac{1}{2}x^2/\sigma^2}$, if we replace x by $(m(\sigma, x) - (q + 1))/\sqrt{q} = x + O(1/\sigma)$, then the result still holds. This is the case in our applications.

3. RELATION WITH THE NUMBER OF POINTS ON PLANE CURVES OVER FINITE FIELDS.

The relation between the considerations in the previous section and the distribution of traces of Frobenius of plane curves over finite fields was first described by Bucur–David–Feigon–Lalín [5]. Let \mathbb{F}_q be the finite field with q elements, and let $R = \mathbb{F}_q[x, y, z]$ be the homogeneous coordinate ring of the projective plane \mathbb{P}^2 over \mathbb{F}_q . For f in R , we let $C_f \subset \mathbb{P}^2$ be the plane curve defined by $f = 0$. Intuitively, the probability that a given point P in $\mathbb{P}^2(\mathbb{F}_q)$ lies on C_f should equal $1/q$, since *a priori* the set of possible values for f at P has q elements (and P is on C if and only if $f(P) = 0$). Supposing furthermore that the probabilities are independent as P varies over $\mathbb{P}^2(\mathbb{F}_q)$, we essentially find ourselves in the situation of Section 2.

More precisely, let $R_d \subset R$ be the subset of homogeneous polynomials of degree d , and consider curves defined by polynomials in R_d . Then the following seminal result from [5] shows that the intuition above is accurate as long as d is large enough with respect to q .

Proposition 3.1 ([5, Prop. 1.6]). *Let B_1, \dots, B_{q^2+q+1} be i.i.d. Bernoulli random variables that assume the value 1 with probability $1/q$. If $d \geq q^2 + q$, then*

$$\frac{\#\{f \in R_d : \#C_f(\mathbb{F}_q) = n\}}{\#R_d} = \text{Prob}(B_1 + \dots + B_{q^2+q+1} = n) .$$

The original proof of Proposition 3.1 is based on a general result of Poonen [25]. However, we will give another proof by elementary means, improving the bound on d in the process.

We choose an enumeration P_1, \dots, P_{q^2+q+1} of the set $\mathbb{P}^2(\mathbb{F}_q)$, which we in turn lift to \mathbb{F}_q -rational affine representatives $\tilde{P}_1, \dots, \tilde{P}_{q^2+q+1}$ in $\mathbb{A}^3(\mathbb{F}_q)$. This allows us to define the linear map

$$\begin{aligned} L : R_d(\mathbb{F}_q) &\longrightarrow \mathbb{F}_q^{q^2+q+1} , \\ f &\longmapsto (f(\tilde{P}_1), \dots, f(\tilde{P}_{q^2+q+1})) . \end{aligned}$$

The forms in the kernel of L define the plane curves that pass through all rational points of the projective plane; these are called *plane filling curves*. It is known that the kernel of L is $R_d(\mathbb{F}_q) \cap J$, where J is the ideal generated by $x^q y - y^q x$, $y^q z - z^q y$ and $z^q x - x^q z$ (see [30, 31] and [11, Prop.2.1]).

Lemma 3.2. *For $d \geq 2q - 1$ the map L is surjective.*

Proof. This proof is due to Masaaki Homma (see also his forthcoming article [12]). For any $d \geq 2q - 1$, the degree d polynomial $x^{d-(2q-2)}(y^{q-1} - x^{q-1})(z^{q-1} - x^{q-1})$ in R_d takes a non-zero value at $(1 : 0 : 0)$, and 0 at every other point in $\mathbb{P}^2(\mathbb{F}_q)$. Using the transitive action of $\text{PGL}_3(\mathbb{F}_q)$, we see that we can construct degree d polynomials having the same properties for any rational point of the plane. Therefore the evaluation map L is surjective. \square

Proposition 3.3. *Proposition 3.1 holds for $d \geq 2q - 1$.*

Proof. Lemma 3.2 shows that under the given hypothesis on d we can always find a homogeneous polynomial with a prescribed zero set of cardinality n and given non-zero values on the complement. Hence, the cardinality of the set of such polynomials is the order of the kernel of L , and does not depend on the prescribed zero set. This implies that the proportion of polynomials with a zero set of cardinality n follows a binomial distribution $\text{Prob}(B_1 + \dots + B_{q^2+q+1} = n)$. \square

For a (possibly singular) plane curve C_f , we let $t = q + 1 - \#C_f(\mathbb{F}_q)$, and we define the *normalized trace* $x := t/\sqrt{q}$ in analogy with Section 2. Note that x is not bounded as $q \rightarrow \infty$. Let

$$N_{q,d}(x) := \sqrt{q} \cdot \frac{\#\{f \in R_d : \#C_f(\mathbb{F}_q) = q + 1 - \sqrt{q}x\}}{\#R_d} ;$$

we want to approximate the difference

$$V_{q,d}(x) := \sqrt{q} \cdot (N_{q,d}(x) - N_{q,d}(-x))$$

(note the normalization factor \sqrt{q} which appears once in $N_{q,d}(x)$ and once in $V_{q,d}$).

Corollary 3.4. *For $d \geq 2q - 1$, any $\alpha > 0$, and $x \in [-(\sqrt{q})^{1-\alpha}, (\sqrt{q})^{1-\alpha}]$, we get the following approximation of $V_{q,d}(x)$ (in the sense of Definition 2.1):*

$$V_{q,d}(x) = \frac{1}{3\sqrt{2\pi}} x(3 - x^2) \cdot e^{-x^2/2} + O\left(\frac{1}{\sqrt{q}}\right) .$$

Proof. This follows from Proposition 3.3 and Section 2. \square

We now restrict our considerations to nonsingular curves. Let $R_d^{\text{ns}} \subset R_d$ be the subset of homogeneous polynomials corresponding to nonsingular plane curves. Since we have restricted to a further subset defined by the non-vanishing of the rather complicated discriminant form, the sieving process to get the distribution is correspondingly more involved, and no elementary method seems to apply. Instead we use another result in [5].

Theorem 3.5 ([5, Th.1.1]). *Let B_1, \dots, B_{q^2+q+1} be i.i.d. Bernoulli random variables taking the value 1 with probability $(q+1)/(q^2+q+1)$. If $0 \leq n \leq q^2+q+1$, then*

$$\frac{\#\{f \in R_d^{ns} : \#C_f(\mathbb{F}_q) = n\}}{\#R_d^{ns}} = \text{Prob}(B_1 + \dots + B_{q^2+q+1} = n) \\ \times \left(1 + O\left(q^n \left(d^{-1/3} + (d-1)^2 q^{-\min(\lfloor \frac{d}{p} \rfloor + 1, \frac{d}{3})} + dq^{-\lfloor \frac{d-1}{p} \rfloor - 1} \right) \right) \right).$$

As before, we let

$$N_{q,d}^{ns}(x) := \sqrt{q} \cdot \frac{\#\{f \in R_d^{ns} : \#C_f(\mathbb{F}_q) = q+1 - \sqrt{q}x\}}{\#R_d^{ns}};$$

we want to analyze the difference

$$V_{q,d}^{ns}(x) := \sqrt{q} \cdot (N_{q,d}^{ns}(x) - N_{q,d}^{ns}(-x)).$$

Using our Lemma, we see that Theorem 3.5 implies in particular that the same asymptotic formula from Corollary 3.4 holds for smooth curves, except that the lower bound on d is now doubly exponential instead of linear in q . More precisely, we have the following.

Corollary 3.6. *For $d \geq q^3(q^2+q+2)$, any $\alpha > 0$ and $x \in [-(\sqrt{q})^{1-\alpha}, (\sqrt{q})^{1-\alpha}]$, we get the following approximation of $V_{q,d}^{ns}(x)$ (in the sense of Definition 2.1):*

$$V_{q,d}^{ns}(x) = \frac{1}{3\sqrt{2\pi}} x(3-x^2) \cdot e^{-x^2/2} + O\left(\frac{1}{\sqrt{q}}\right).$$

Proof. When $d \geq q^3(q^2+q+2)$ the $O()$ term in Theorem 3.5 is $O(q^{-1/2})$; so applying Theorem 3.5 and Section 2 yields the result. \square

4. EXPERIMENTAL RESULTS AND THEIR LIMITATIONS

We consider the special case $d = 4$. The smooth plane quartic curves C_f defined by $f \in R_4^{ns}(\mathbb{F}_q)$ are precisely the non-hyperelliptic curves of genus 3 over \mathbb{F}_q . Since d is now fixed, our previous results cannot be applied directly. However, as mentioned in the introduction, we wish to compare the statistical distributions obtained in this way with the experimental results obtained in [20].

In order to do so, let us recall the notation of [20]. Let $\mathcal{N}_{q,3}(t)$ denote the number of \mathbb{F}_q -isomorphism classes of non-hyperelliptic curves of genus 3 over \mathbb{F}_q of trace t , weighted by the order of their \mathbb{F}_q -automorphism group:

$$\mathcal{N}_{q,3}(t) := \sum_{\substack{\{C/\mathbb{F}_q \text{ n.h. genus 3} \\ \text{curve with trace } t\}} / \simeq} \frac{1}{\#\text{Aut}_{\mathbb{F}_q}(C)}. \quad (4.1)$$

This is the most natural way to define $\mathcal{N}_{q,3}(t)$ (c.f. [3, 32]). In order to compare our results for different values of q , we renormalize $\mathcal{N}_{q,3}(t)$ to

$$\mathcal{N}_{q,3}^{\text{KS}}(x) := \frac{\sqrt{q}}{q^6+1} \mathcal{N}_{q,3}(t) \quad \text{where } t := \lfloor \sqrt{q}x \rfloor \text{ for } x \in [-6, 6].$$

This coincides with the normalization of the trace distribution in Katz–Sarnak [14] (the factor q^6+1 being the number of \mathbb{F}_q -points of the moduli space of non hyperelliptic genus 3 curves). The numerical results in [20] on the quantity $\mathcal{N}_{q,3}^{\text{KS}}(x)$ and the difference

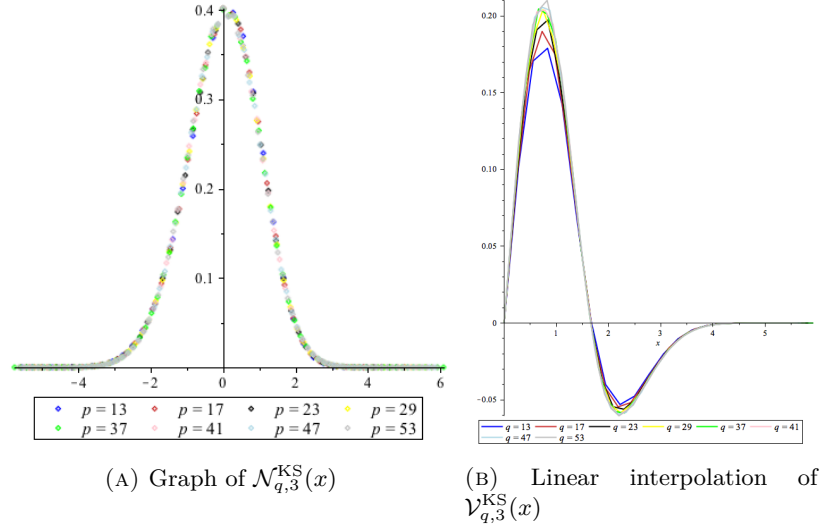
$$\mathcal{V}_{q,3}^{\text{KS}}(x) := \sqrt{q} (\mathcal{N}_{q,3}^{\text{KS}}(x) - \mathcal{N}_{q,3}^{\text{KS}}(-x))$$

are graphically summarized in Figure 2.

In order to apply our previous results, we need to understand the connection between $N_{q,4}^{ns}(x)$ and $\mathcal{N}_{q,3}^{\text{KS}}(x)$.

Lemma 4.1. *With $\mathcal{N}_{q,3}(t)$ defined as above, we have*

$$\#\text{GL}_3(\mathbb{F}_q) \cdot \mathcal{N}_{q,3}(t) = \#\{f \in R_4^{ns} : \#C_f(\mathbb{F}_q) = q+1-t\}.$$

FIGURE 2. The numerical trace distributions $\mathcal{N}_{q,3}^{\text{KS}}(x)$ and $\mathcal{V}_{q,3}^{\text{KS}}(x)$.

Proof. As smooth plane quartics are isomorphic to their canonical embeddings, an \mathbb{F}_q -rational isomorphism between two quartics is induced by an element of $\text{PGL}_3(\mathbb{F}_q)$. But on the other hand, two ternary forms define the same subvariety of \mathbb{P}^2 if and only if they differ by scalar multiplication by an element of \mathbb{F}_q^\times , so

$$\begin{aligned} \#\{f \in R_4^{\text{ns}} : \#C_f(\mathbb{F}_q) = q + 1 - t\} &= \#\mathbb{F}_q^\times \sum_{\{C/\mathbb{F}_q\}/\simeq} \frac{\#\text{PGL}_3(\mathbb{F}_q)}{\#\text{Aut}_{\mathbb{F}_q}(C)} \\ &= \#\text{GL}_3(\mathbb{F}_q) \sum_{\{C/\mathbb{F}_q\}/\simeq} \frac{1}{\#\text{Aut}_{\mathbb{F}_q}(C)} \\ &= \#\text{GL}_3(\mathbb{F}_q) \cdot \mathcal{N}_{q,3}(t), \end{aligned}$$

where the sums are taken over the same set of curves as in Equation (4.1). \square

By Lemma 4.1,

$$\mathcal{N}_{q,3}^{\text{KS}}(x) = \frac{\#R_4^{\text{ns}}}{(q^6 + 1)\#\text{GL}_3(\mathbb{F}_q)} N_{q,4}^{\text{ns}}(x).$$

The singular quartic curves in the 14-dimensional space $\mathbb{P}R_4$ are contained in the discriminant locus D , a hypersurface of degree 27. By [29, Th.2.1], we have $\#D(\mathbb{F}_q) \leq 27q^{13} + \frac{q^{13}-1}{q-1}$, so

$$\frac{\#R_4^{\text{ns}}}{(q^6 + 1)\#\text{GL}_3(\mathbb{F}_q)} = \frac{\#R_4 - (q-1)\#D(\mathbb{F}_q)}{(q^6 + 1)(q^3 - 1)(q^3 - q)(q^3 - q^2)} = 1 + O\left(\frac{1}{q}\right).$$

Since $\mathcal{N}_{q,3}^{\text{KS}}(x)$ (and therefore $N_{q,4}^{\text{ns}}(x)$) is uniformly bounded, this implies

$$\mathcal{N}_{q,3}^{\text{KS}}(x) = N_{q,4}^{\text{ns}}(x) \left(1 + O\left(\frac{1}{q}\right)\right) = N_{q,4}^{\text{ns}}(x) + O\left(\frac{1}{q}\right).$$

We can now make a graphical comparison of the experimental distribution of $\mathcal{V}_{q,3}^{\text{KS}}(x)$ for the largest value of q (to wit, $q = 53$) with the results from Section 3. Interpolating the binomial coefficient as in Equation (2.1), which defined the function b in Section 2, we define B_1 , B_2 , and B_3 by

$$B_i(x) := \sigma_i \binom{N}{N\mu_i - \sigma_i x} \mu_i^{N\mu_i - \sigma_i x} (1 - \mu_i)^{N - (N\mu_i - \sigma_i x)}, \quad (4.2)$$

where $N = q^2 + q + 1$ and

- (1) $\sigma_1 = \sqrt{q - 1/q^2}$ and $\mu_1 = 1/q$ (corresponding to Corollary 3.4, *i.e.*, possibly singular plane quartics);
- (2) $\sigma_2 = \sqrt{q(1 - \frac{1}{q^2+q+1})}$ and $\mu_2 = (q+1)/(q^2+q+1)$ (corresponding to Corollary 3.6, *i.e.*, smooth plane quartics);
- (3) $\sigma_3 = \sqrt{q}$ and $\mu_3 = 1/q$ (a simplified version of these models).

In both cases (1) and (2) we have blithely ignored the constraints on the degree that were required for the proofs of the corresponding results. Yet as Figure 3 shows, the plots of B_1, B_2, B_3 and the Gaussian density function are almost indistinguishable, and all of these functions interpolate the distribution $\mathcal{N}_{53,3}^{\text{KS}}(x)$ quite well.

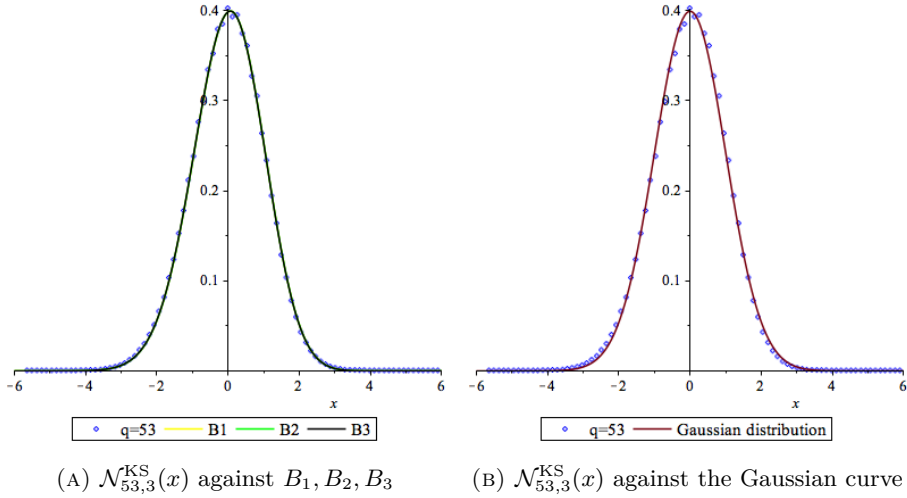


FIGURE 3. Comparisons between $\mathcal{N}_{53,3}^{\text{KS}}(x)$ and its approximations.

As above, we are led to define

$$V_i(x) := \sigma(B_i(x) - B_i(-x)) \quad \text{for } i = 1, 2, 3 \quad (4.3)$$

and

$$V^{\text{lim}}(x) := \frac{1}{3\sqrt{2\pi}}x(3-x^2)e^{-x^2/2}. \quad (4.4)$$

This gives rise to the plots in Figure 4. Once more we observe that the various curves closely

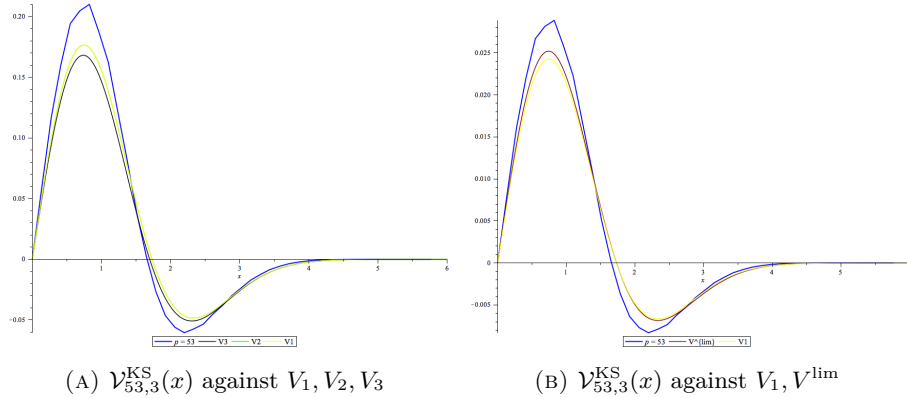


FIGURE 4. Comparisons between $\mathcal{V}_{53,3}^{\text{KS}}(x)$ and its approximations.

resemble the distribution of $\mathcal{V}_{53,3}^{\text{KS}}(x)$. This time around, the agreement is all the more remarkable

in that the limit distribution V^{lim} does not depend on any parameters, and hence cannot be adjusted.

5. PROOF OF PROPOSITION 2.2

For notational convenience, throughout this proof we write N , E , μ , m , and n for N_σ , E_σ , μ_σ , $m(\sigma, x)$ and $n(\sigma, x)$.

Since $E = \sigma^2 + 1 + O(1/\sigma^2)$ and $x = o(\sigma)$ at worst, we have

$$\begin{aligned} N &= \sigma^4 + O(\sigma^2) , \\ \mu &= \frac{1}{\sigma^2} + O\left(\frac{1}{\sigma^4}\right) , \\ m &:= E - \sigma x = \sigma^2 - s\sigma + 1 + O\left(\frac{1}{\sigma^2}\right) , \\ n &:= (N - E) + \sigma x = N - m = \sigma^4 + O(\sigma^2) . \end{aligned}$$

Stirling's approximation of the Gamma function

$$\Gamma(z+1) = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z e^{\theta z} \quad \text{with } |\theta z| \leq \frac{1}{12z}$$

yields an approximation

$$b(\sigma, x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{N}{mn}} \left(\frac{E}{m}\right)^m \left(\frac{N-E}{n}\right)^n e^\theta \quad \text{with } \theta := \theta_N - \theta_m - \theta_n . \quad (5.1)$$

Claims 1 through 6 below approximate the individual factors in Equation (5.1).

Claim 1: $\sqrt{\frac{N}{mn}} = \frac{1}{\sigma} \left(1 + \frac{x}{2\sigma} + O\left(\frac{x^2}{\sigma^2}\right)\right)$.

Proof. Note that

$$\frac{N}{mn} = \frac{1}{\sigma^2} \times \frac{1}{1 + (2\mu - 1)\frac{x}{\sigma} - \frac{x^2}{N}} = \frac{1}{\sigma^2} \times \frac{1}{1 - (1 + O(\frac{1}{\sigma^2}))\frac{x}{\sigma} + O(\frac{1}{\sigma^2})} ,$$

and the Taylor expansion of the monotone function $z \mapsto 1/\sqrt{1+z}$ yields Claim 1. \square

Claim 2: $e^\theta = 1 + O\left(\frac{1}{\sigma^2}\right)$.

Proof. Estimating θ , we have

$$|\theta| = |\theta_N - \theta_m - \theta_n| \leq \frac{1}{12} \left| \frac{1}{N} + \frac{N}{nm} \right| .$$

From Claim 1 and $N = \sigma^4 + O(\sigma^2)$ we obtain $\theta = O(1/\sigma^2)$, which is enough to deduce Claim 2. \square

We will approximate the product of the two middle terms of Equation (5.1) using

$$\left(\frac{E}{m}\right)^m \left(\frac{N-E}{n}\right)^m = e^{-(A+B)} \quad \text{where } A := -\ln\left(\frac{E}{m}\right)^m \quad \text{and } B := -\ln\left(\frac{N-E}{n}\right)^n .$$

Claim 3: $A = -\sigma x + \frac{x^2}{2} + \frac{x^3}{6} \frac{1}{\sigma} + O\left(\frac{x^4}{\sigma^2}\right)$.

Proof. Indeed,

$$\begin{aligned} A &= (E - \sigma x) \ln\left(\frac{E - \sigma x}{E}\right) = E \left(1 - \frac{\sigma}{E} x\right) \ln\left(1 - \frac{\sigma}{E} x\right) \\ &= E \left(-\frac{\sigma}{E} x + \frac{1}{2} \left(\frac{\sigma}{E} x\right)^2 + \frac{1}{6} \left(\frac{\sigma}{E} x\right)^3 + O\left(\left(\frac{\sigma}{E} x\right)^4\right)\right) . \end{aligned}$$

The claim follows because $E = \sigma^2 + 1 + O(1/\sigma^2)$ and $\sigma/E = 1/\sigma - 1/\sigma^3 + O(1/\sigma^5)$. \square

Claim 4: $B = \sigma x + \frac{x^2}{2} \frac{1}{\sigma^2} + O\left(\frac{x^2}{\sigma^4}\right)$.

Proof. The proof is analogous to that of Claim 3, using $B = (N - E)(1 + \frac{\sigma}{N-E} x) \ln(1 + \frac{\sigma}{N-E} x)$ and $\sigma/(N - E) = 1/\sigma^3 + O(1/\sigma^5)$. \square

Claim 5: $A + B = \frac{x^2}{2} + \frac{x^3}{6} \frac{1}{\sigma} + O\left(\frac{x^4}{\sigma^2}\right)$.

Proof. This is immediate from Claims 3 and 4. \square

Claim 6: $e^{-(A+B)} = e^{-\frac{x^2}{2}} \left(1 - \frac{x^3}{6\sigma}\right) + O\left(\frac{1}{\sigma^2}\right)$.

Proof. Taking a first-order Taylor approximation of the monotone function $z \mapsto e^z$ shows that

$$e^z = 1 + z + R(z),$$

where the remainder term R satisfies $|R(z)| \leq (z^2/2)e^{|z|}$. Thus

$$\begin{aligned} e^{-(A+B)} &= e^{-\frac{x^2}{2}} e^{-\frac{x^3}{6\sigma} + O\left(\frac{x^4}{\sigma^2}\right)} \\ &= e^{-\frac{x^2}{2}} \left(1 - \frac{x^3}{6\sigma} + O\left(\frac{x^4}{\sigma^2}\right)\right) + e^{-\frac{x^2}{2}} R\left(-\frac{x^3}{6\sigma} + O\left(\frac{x^4}{\sigma^2}\right)\right). \end{aligned}$$

The last term can be estimated by

$$\left| e^{-\frac{x^2}{2}} R\left(-\frac{x^3}{6\sigma} + O\left(\frac{x^4}{\sigma^2}\right)\right) \right| \leq \left| \left(\frac{1}{72\sigma^2} + O\left(\frac{1}{\sigma^4}\right)\right) x^6 e^{(|\frac{x^3}{6\sigma} + O(\frac{x^4}{\sigma^2})| - \frac{1}{2})x^2} \right|.$$

Here the assumption $x = o(\sigma)$ plays a role, bounding the product of the last factors as σ grows, so in fact a stronger estimate holds:

$$e^{-\frac{x^2}{2}} R\left(\frac{x^3}{6\sigma} + O\left(\frac{x^4}{\sigma^2}\right)\right) = O\left(\frac{1}{\sigma^2}\right).$$

Since $x^4 e^{-\frac{x^2}{2}}$ is also bounded, regardless of any growth assumptions on x , we also have

$$e^{-\frac{x^2}{2}} \left(O\left(\frac{x^4}{\sigma^2}\right)\right) = O\left(\frac{1}{\sigma^2}\right), \quad (5.2)$$

which proves the claim. \square

Putting all of our estimates together, we obtain a good approximation for b as σ tends to infinity. First,

$$\begin{aligned} \sqrt{\frac{N}{mn}} e^\theta &= \frac{1}{\sigma} \left(1 + \frac{x}{2\sigma} + O\left(\frac{x^2}{\sigma^2}\right)\right) \left(1 + O\left(\frac{1}{\sigma^2}\right)\right) \\ &= \frac{1}{\sigma} + \frac{x}{2\sigma^2} + O\left(\frac{x^2}{\sigma^3}\right). \end{aligned}$$

Now consider the product

$$\begin{aligned} \sqrt{2\pi} b(\sigma, x) &= \left(\frac{1}{\sigma} + \frac{x}{2\sigma^2} + O\left(\frac{x^2}{\sigma^3}\right)\right) e^{-(A+B)} \\ &= e^{-\frac{x^2}{2}} \left(\frac{1}{\sigma} + \frac{x}{2\sigma^2} + O\left(\frac{x^2}{\sigma^3}\right)\right) \left(1 - \frac{x^3}{6\sigma} + O\left(\frac{1}{\sigma^2}\right)\right); \end{aligned} \quad (5.3)$$

its main contribution is given by

$$e^{-\frac{x^2}{2}} \left(\frac{1}{\sigma} + \frac{x}{2\sigma^2}\right) \left(1 - \frac{x^3}{6\sigma}\right) = e^{-\frac{x^2}{2}} \left(\frac{1}{\sigma} - \frac{x(x^2 - 3)}{6\sigma^2} - \frac{x^4}{12\sigma^3}\right).$$

As in the derivation of Equation (5.2), the final term in this sum is $O(1/\sigma^3)$. The same technique shows that the other cross-terms in the product of Equation (5.3) are $O(1/\sigma^3)$, which concludes the proof of Proposition 2.2.

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