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# The Collatz Conjecture.

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## Abstract

As the title indicates this is an attempt to prove the Collatz Conjecture also known as the  $3x + 1$  problem.

**Keywords:** Collatz conjecture, congruences.

## 1. INTRODUCTION

For any positive integer  $x$ , we will proceed in the following algorithm, if  $x$  is pair divide it by two, if not multiply it by 3 and add 1 then divide it by 2 to obtain a certain integer that will follow the same procedure, the Collatz conjecture states that no matter what we have chosen as  $x$ , the algorithm will end up giving the integer 1. In other words for any positive integer  $x$  iterations of the following function:

$$C(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \text{ (so called the } \frac{3}{2} \text{ step),} \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} \text{ (so called the } \frac{1}{2} \text{ step)} \end{cases}$$

will eventually reach the number 1. It is well known (see [1]) that the conjecture is verified to be true for any number less than  $5.10^{18}$ .

## 2. THE INITIAL APPROACH

Basically my approach was: given any number  $x$ , if we proved when applying the algorithm that at sometime the process gives a integer less strictly than the starting  $x$  then we have literally proven the conjecture, we refer to this condition as the stuck condition, but what if this is obviously true for some  $x$ 's; as we will see in this section, what about the rest of the integers?

**Definition 2.1.** *An integer  $x$  can be written uniquely as  $2i+h \equiv h \pmod{2}$ ,  $h = 0, 1$ , this is called the first position of  $x$ , the same is also applied to  $i$ , the second position of  $x$  and so on, a class of integers denoted  $C_j$  is a set of integers such that we have defined  $j$  positions of  $x$ .*

In the iteration process of  $C(x)$ , starting by  $x$ , the obtained integers after say  $q$  steps are of the form  $\frac{3^n}{2^q}x + f = rx + f$ , where  $f$  is a positive integer. If the stuck condition is satisfied for  $x$  then necessarily  $q = n + d$  and  $r < 1$  or equivalently if  $d$  and  $n$  are the numbers of the  $\frac{1}{2}$  and the  $\frac{3}{2}$  steps respectively:

$$n \ln(3) < (n + d) \ln(2) \iff \left\lceil n \frac{(\ln(3) - \ln(2))}{\ln(2)} \right\rceil \leq d$$

The following lemma states the converse:

**Lemma 2.1.** *For a positive integer  $x$  in  $C_j$ , if  $r < 1$  then there exist a positive number  $y$  large enough, belonging to the same class  $C_j$  such that for all  $t > y$  and  $t \in C_j$ ,  $t$  satisfies the stuck condition.*

*Proof.* Suppose that at the  $i^{\text{th}}$  step  $r < 1$ , write the integer  $g$  at the  $i^{\text{th}}$  iteration as  $g = rx + f$ , to prove the lemma choose  $x$  large enough so that  $(1 - r)x > f$ .  $\square$

**Corollary 2.1.** *Let  $x > 1$  be a positive integer then*

- (1) *If  $x$  is pair,  $x$  satisfies the stuck condition,*
- (2) *If  $x = 4k + 1$  for some  $k$ ,  $x$  satisfies the stuck condition,*
- (3) *If  $x = 16k + 3$ , for some  $k$ ,  $x$  satisfies the stuck condition.*

*Proof.* The first case is pretty obvious, divide  $x$  by two to obtain a smaller number, for the second case it is a consequence of what we will have after just two steps,  $3x + 1 = 12k + 4$  divided by 4 gives  $3k + 1 < 4k + 1 = x$ . The last one have a similar proof, writing  $x$  as it is given, iterating the steps, verifying that  $r < 1$  is obtained and that the stuck condition is satisfied.  $\square$

As one can see there are other integers that will have issues and supplementary conditions regarding the satisfaction of the stuck condition, the last section will get this solved.

### 3. THE IDEA

**Theorem 3.1.** *The Collatz Conjecture is true.*

*Proof.* The straight forward proof is this one; as seen before we have a finite number  $x$ , we started the algorithm but we need at some point more  $\frac{1}{2}$  steps than the  $\frac{3}{2}$  steps to get to the stuck condition, now here is why the stuck condition will be satisfied no matter what  $x$  we have chosen the proof relies on induction: we have  $x$ , take the biggest  $z = 2^i$  less than  $x$  and write  $x \equiv a \pmod{z} \equiv a \pmod{\alpha z}$ , note that  $a < x$ , the Collatz conjecture is proven for  $a$  and suppose it took  $a, m$  steps in the algorithm to reach 1, what's  $\alpha$ ? it is an  $2^j$  integer for some  $j$  sufficiently large to be fixed later on of course the number  $a + \alpha z$  will follow the same iteration steps as if it was  $x$  so the steps we had to apply on  $x$  we can apply them on  $a$ . After  $m$  steps we will be facing the number  $y \equiv 1 \pmod{2^h}$  for a certain  $h < i + j$  here we will make sure of two essential things, first in these  $m$  steps any outcome number will not exceed the modulus part that is  $2^s$  for some  $s$  second when reaching the number  $y$  we must still have a large  $2^h$  in the modulus section, from this point it will be easy to see that the number  $y$  will get alternation of steps each time between the  $\frac{1}{2}$  step and  $\frac{3}{2}$  step because  $y_1 = 3y + 1$  is divisible by 4 and  $z_1 = \frac{y_1}{4} \equiv 1 \pmod{2^{h-2}}$ , consequently  $j$  must also be taken in a way that the modulus sector will not vanish until the stuck condition is verified for  $x$ , this is sufficient to prove that we are strictly decreasing the arbitrary chosen number  $x$  each time which prove the induction hypothesis and the conjecture as well.  $\square$

## REFERENCES

- [1] Toms Oliveira e Silva, *Empirical Verification of the  $3x+1$  and Related Conjectures*. In *The Ultimate Challenge: The  $3x+1$  Problem* (book edited by Jeffrey C. Lagarias), pp. 189-207, American Mathematical Society, 2010