The Collatz Conjecture

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Abstract

As the title indicates this is an attempt to prove the Collatz Conjecture also known as the $3x+1$ problem.
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1. Introduction

For any positive integer x, we will proceed in the following algorithm, if x is pair divide it by two, if not multiply it by 3 and add 1 then divide it by 2 to obtain a certain integer that will follow the same procedure, the Collatz conjecture states that no mater what we have chosen as x, the algorithm will end up giving the integer 1. In other words for any positive integer x iterations of the following function:

$$C(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2 \quad \text{(so called the } \tfrac{3}{2} \text{ step)}, \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2 \quad \text{(so called the } \tfrac{1}{2} \text{ step)} \end{cases}$$

will eventually reach the number 1. It is well known (see [1]) that the conjecture is verified to be true for any number less than 5.10^{18}.

2. The initial approach

Basically my approach was: given any number x, if we proved when applying the algorithm that at sometime the process gives a integer less strictly than the starting x then we have literally proven the conjecture, we refer to this condition as the stuck condition, but what if this is obviously true for some x's; as we will see in this section, what about the rest of the integers?

Definition 2.1. *An integer x can be written uniquely as $2i+h \equiv h \pmod 2$, $h = 0, 1$, this is called the first position of x, the same is also applied to i, the second position of x and so on, a class of integers denoted C_j is a set of integers such that we have defined j positions of x.*

In the iteration process of $C(x)$, starting by x, the obtained integers after say q steps are of the form $\dfrac{3^n}{2^q}x + f = rx + f$, where f is a positive integer. If the stuck condition is satisfied for x then necessarily $q = n + d$ and $r < 1$ or equivalently if d and n are respectively the $\frac{1}{2}$ and the $\frac{3}{2}$ steps:

$$nln(3) < (n+d)ln(2) \iff \left\lceil n\frac{(ln(3) - ln(2))}{ln(2)} \right\rceil \leq d$$

The following lemma states the converse:

Lemma 2.1. *For a positive integer x in C_j , if $r < 1$ then there exist a positive number y large enough, belonging to the same class C_j such that for all $t > y$ and $t \in C_j$, t satisfies the stuck condition.*

Proof. Suppose that at the i^{th} step $r < 1$, write the integer g at the i^{th} iteration as $g = rx + f$, to prove the lemma choose x large enough such that $(1 - r)x > f$. □

Corollary 2.1. *Let $x > 1$ be a positive integer then*

(1) If x is pair, x satisfies the stuck condition,
(2) If $x = 4k + 1$ for some k, x satisfies the stuck condition,
(3) If $x = 16k + 3$, for some k, x satisfies the stuck condition.

Proof. The first case is pretty obvious, divide x by two to obtain a smaller number, for the second case it is a consequence of what we will have after just two steps, $3x + 1 = 12k + 4$ divided by 4 gives $3k + 1 < 4k + 1 = x$. The last one have a similar proof, writing x as it is given, iterating the steps, verifying that $r < 1$ is obtained and that the stuck condition is satisfied. □

As one can see there are other integers that will have issues and supplementary conditions regarding the satisfaction of the stuck condition, the last section will get this solved.

3. The idea

Theorem 3.1. *The Collatz Conjecture is true.*

Proof. The straight forward proof is this one; as seen before we have a finite number x, we started the algorithm but we need at some point more $\frac{1}{2}$ steps than the $\frac{3}{2}$ steps to get to the stuck condition, now here is why the stuck condition will be satisfied, no matter what x we have chosen: the proof will rely on induction: we have x, take the biggest $z = 2^i$ less than x and write $x \equiv a \pmod{z} \equiv a \pmod{\alpha z}$, note that $a < x$, the Collatz conjecture is proven for a and suppose it took a, m steps in the algorithm to reach 1, what's α? it is an 2^j integer for some j sufficiently large to be fixed later on of course the number $a + \alpha z$ will follow the same iteration steps as if it was x so the steps we had to apply on x we can apply them on a. After m steps we will be facing the number $y \equiv 1 \pmod{2^h}$ for a certain $h < i + j$ here we will make sure of two essential things, first in these m steps any outcome number will not exceed the modulus part that is 2^s for some s second when reaching the number y we must still have a large 2^h in the modulus section, from this point it will be easy to see that the number y will get alternation steps each time between the $\frac{1}{2}$ step and $\frac{3}{2}$ step because $y_1 = 3y + 1$ is divisible by 4 and $z_1 = \dfrac{y_1}{4} \equiv 1 \pmod{2^{h-2}}$, consequently j must also be taken in a way that the modulus sector will not vanish until the stuck condition is verified for x, this is sufficient to prove that we are strictly decreasing the arbitrary chosen number x each time which prove the induction hypothesis and the conjecture as well. □

References

[1] Toms Oliveira e Silva, *Empirical Verification of the 3x+1 and Related Conjectures.* In The Ultimate Challenge: The 3x+1 Problem (book edited by Jeffrey C. Lagarias), pp. 189-207, American Mathematical Society, 2010