



HAL
open science

Aperiodic Subshifts on Polycyclic Groups

Emmanuel Jeandel

► **To cite this version:**

| Emmanuel Jeandel. Aperiodic Subshifts on Polycyclic Groups. 2015. hal-01213364v1

HAL Id: hal-01213364

<https://inria.hal.science/hal-01213364v1>

Preprint submitted on 8 Oct 2015 (v1), last revised 18 Aug 2016 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Aperiodic Subshifts on Polycyclic Groups

Emmanuel Jeandel
LORIA, UMR 7503 - Campus Scientifique, BP 239
54 506 VANDOEUVRE-LÈS-NANCY, FRANCE
emmanuel.jeandel@loria.fr

8th October 2015

Abstract

We prove that every polycyclic group of nonlinear growth admits a strongly aperiodic SFT and has an undecidable domino problem. This answers a question of [4] and generalizes the result of [2].

Subshifts of finite type (SFT for short) in a group G are coloring of the elements of G subject to local constraints. They have been studied extensively for the free abelian group \mathbb{Z} [13] and on the free abelian group \mathbb{Z}^2 , where they correspond to tilings of the discrete plane [12].

One of the main question about SFT is the existence of strongly aperiodic SFT, that is finding a finite set of local constraints so that the only way to color the group G is to do so aperiodically. Strongly aperiodic SFT are known to exists for the group \mathbb{Z}^2 [3] and the author [9] has provided examples on many groups of the form $\mathbb{Z} \times G$.

A related question is the decidability of the domino problem: Given a SFT X on a group G , decide if X is empty. This question is somewhat related to the existence of *weakly* aperiodic SFT, i.e. where no coloring is periodic along a subgroup of finite index in G . The undecidability of the domino problem has been established on \mathbb{Z}^2 [3], on Baumslag-Solitar groups [1] and on any nilpotent group of nonlinear growth [2].

In this article we introduce a few elementary techniques that show how to exhibit strongly aperiodic SFT on any polycyclic group which is not of linear growth (i.e. not virtually \mathbb{Z}), and we also prove that any such group has an undecidable domino problem. The first result answers a question of Carroll and Penland [4], while the second generalizes the result of Ballier and Stein from nilpotent groups to polycyclic groups.

A full characterization of groups (or even finitely presented groups) with strongly aperiodic SFT is still for now out of reach.

1 Definitions

We assume some familiarity with group theory and actions of groups. See [5] for a good reference on symbolic dynamics on groups.

All groups below are implicitly supposed to be finitely generated (f.g. for short).

Let A be a finite set and G a group. We denote by A^G the set of all functions from G to A . For $x \in A^G$, we write x_g instead of $x(g)$ for the value of x in g .

G acts on A^G by

$$(g \cdot x)_h = x_{g^{-1}h}$$

A *pattern* is a partial function P of G to A with finite support. The support of P will be denoted by $\text{Supp}(P)$.

A *subshift* of A^G is a subset X of A^G which is topologically closed (for the product topology on A^G) and invariant under the action of G .

A subshift can also be defined in terms of forbidden patterns. If \mathcal{P} is a collection of patterns, the subshift defined by \mathcal{P} is

$$X_{\mathcal{P}} = \{x \in A^G \mid \forall g \in G, \forall P \in \mathcal{P} \exists h \in \text{Supp}(P), (g \cdot x)_h \neq P_h\}$$

Every such set is a subshift, and every subshift can be defined this way. If X can be defined by a finite set \mathcal{P} , X is said to be a subshift of finite type, or for short a SFT.

For a point $x \in X$, the stabilizer of x is $\text{Stab}(x) = \{g \mid g \cdot x = x\}$

A subshift X is *strongly aperiodic* if it is nonempty and every point of X has a finite stabilizer. Some authors require that the stabilizer of each point is trivial, this will not make any difference in this article.

A f.g. group G is said to have decidable domino problem if there is an algorithm that, given a description of a finite set of patterns \mathcal{P} , decides if $X_{\mathcal{P}}$ is empty.

In the remaining, we are interested in groups G which admit strongly aperiodic SFTs or have an undecidable domino problem.

We now summarize previous theorems:

Theorem 1. • \mathbb{Z} does not admit strongly aperiodic SFT and has decidable word problem

- Free groups of rank ≥ 2 have decidable domino problem [11]
- The free abelian group \mathbb{Z}^2 [3] has a strongly aperiodic SFT and an undecidable domino problem
- The free abelian group \mathbb{Z}^3 [8] has a strongly aperiodic SFT.
- f.g. nilpotent groups have an undecidable word problem unless they are virtually cyclic [2]

We also give a few structural results:

- Theorem 2.**
- *Let G, H be f. g. commensurable groups. Then G admits a strongly aperiodic SFT (resp. has an undecidable domino problem) if and only if H does. [4]*
 - *Let G, H be finitely presented groups that are quasi-isometric. Then G admits a weakly aperiodic SFT (resp. has an undecidable domino problem) if and only if H does. [6]*
 - *Finitely presented groups with a strongly aperiodic SFT have decidable word problem [9].*

The first of these results will be used almost everywhere in what follow.

2 Strongly aperiodic SFTs

We start by a few easy propositions that explain how to build SFT in G from a SFT in H or a SFT in G/H , provided in the second case that H is finitely generated.

Proposition 2.1. *Let G be a f.g. group and H a f.g. normal subgroup of G . Let $\phi : G \rightarrow G/H$ the corresponding morphism. Let X be a SFT on G/H over the alphabet A . Let*

$$Y = \{y \in A^G \mid \exists x \in X \forall g \in G, y_g = x_{\phi(g)}\}$$

Then Y is a SFT.

Furthermore, for every $y \in Y$, there exists $x \in X$ s.t. $\text{Stab}(y) = H\text{Stab}(x)$.

See e.g. [4] for a proof. The idea is to lift the forbidden patterns defined on G/H to forbidden patterns defined on G , and to use additional forbidden patterns to force $y_g = y_{g'}$ whenever $gg'^{-1} \in H$. This is possible as H is supposed to be finitely generated.

Proposition 2.2. *Let G be a f.g. group and H a f.g. subgroup of G .*

Let X be a SFT on H over the alphabet A defined by the set of forbidden patterns \mathcal{P} . Using the same forbidden patterns, we may define a SFT Y on G .

Then Y is nonempty iff X is nonempty. More precisely, let K be a left transversal of H in G . Then $y \in Y$ iff there exists points $(x^k)_{k \in K}$ in X s.t. $y_{kh} = x_h^k$.

In particular, for all $y \in Y$, there exists $x \in X$ s.t. $\text{Stab}(y) \cap H \subseteq \text{Stab}(x)$

Proof. Write $G = KH$ for some transversal K . Each element of $g \in G$ can therefore be written in a unique way in the form $g = kh$.

Let $y \in Y$ and let $k \in K$. Define $(x^k)_h = y_{kh}$. Let $h' \in H$ and P a forbidden pattern of \mathcal{P} . We will prove there exists p s.t. $(h' \cdot x^k)_p \neq P_p$ which proves that $x^k \in X$.

By definition of Y , there exists p in the support of \mathcal{P} s.t. $(h'k^{-1} \cdot y)_p \neq P_p$. Therefore $y_{kh'^{-1}p} \neq P_p$. But $(h' \cdot x^k)_p = (x^k)_{h'^{-1}p} = y_{kh'^{-1}p}$, the result is proven.

Conversely, take some points $(x^k)_{k \in K}$ in X and define $y_{kh} = x_h^k$. Let $g \in G$. Write $g^{-1} = kh^{-1}$ for some k and h . Let $P \in \mathcal{P}$. As $x^k \in X$, there exists $p \in H$ s.t. $(h \cdot x^k)_p \neq P_p$. But then $(g \cdot y)_p = y_{g^{-1}p} = y_{kh^{-1}p} = x_{h^{-1}p}^k = (h \cdot x^k)_p \neq P_p$. Therefore $y \in Y$. \square

We now start with the first proposition that gives a natural way to prove that a group G has a strongly aperiodic SFT.

Proposition 2.3. *Let G be a f.g. group. Suppose that G contains two f.g. groups H_1, H_2 s.t.*

- $H_1 \subseteq H_2$.
- H_1 is normal and G/H_1 admits a strongly aperiodic SFT X_1
- H_2 admits a strongly aperiodic SFT X_2

Then G admits a strongly aperiodic SFT Y . Furthermore, if each point in X_1 and X_2 have trivial stabilizers, then every point of Y has a trivial stabilizer.

It is a good but nontrivial exercise to show, under the conditions of the proposition, that if H_1 and H_2 have a decidable word problem, then G does.

Proof. Use the two previous propositions to build Y_1 and Y_2 and consider $Y = Y_1 \times Y_2$.

Let $y = (y_1, y_2) \in Y$ and consider its stabilizer $K = \text{Stab}(y)$. By definition of Y_1 , $K \subseteq H_1 F$ for some finite set F . By definition of Y_2 , $K \cap H_2$ is finite. In particular $K \cap H_1$ is finite.

This implies that K is finite. Indeed, for each $f \in F$, $K \cap fH_1 = K \cap H_1 f$ is finite: If $x, y \in K \cap fH_1$, then $xy^{-1} \in K \cap H_1$.

Furthermore if F is trivial and $K \cap H_2$ is trivial, then K is trivial. \square

Corollary 2.4. *Let G be a f.g. group. If G contains a f.g. normal subgroup H_1 s.t. G/H_1 admits a strongly aperiodic SFT and H_1 admits a strongly aperiodic SFT, then G admits a strongly aperiodic SFT.*

In particular, if G_1 and G_2 are f.g. groups that admit strongly aperiodic SFTs, then $G_1 \times G_2$ does.

Therefore groups with strongly aperiodic SFT are closed under direct product. This seems quite natural but does not seem to have been known.

Using the previous proposition, we will be able to prove that nontrivial nilpotent and polycyclic groups have strongly aperiodic SFT. We start with nilpotent groups.

Theorem 3. *Let G be a finitely generated group of nonlinear polynomial growth. Then G admits a strongly aperiodic SFT.*

Proof. Such groups are exactly the f.g. virtually nilpotent groups of Hirsch number different from 1. The proof will use repeatedly that if G is a f.g. group, and H a subgroup of finite index, then G has a strongly aperiodic SFT iff H does, see [4] for details. In particular it is sufficient to prove the theorem for f.g. nilpotent groups.

First, a few definitions, see [14, Chapter 1] for details.

The *Hirsch number* $h(G)$ of G is the number of infinite factors in a series with cyclic or finite factors. It is defined for nilpotent groups, and more generally for polycyclic groups. A nilpotent (or a polycyclic) group of Hirsch number 0 is a finite group. A nilpotent (or a polycyclic) group of Hirsch number 1 is a finite-by-cyclic-by-finite group, hence virtually \mathbb{Z} .

The only thing we will need about nilpotent groups is that (a) they have a non trivial center (b) quotients and subgroups of nilpotent groups are nilpotent and finitely generated (c) $h(G) = h(G/H) + h(H)$ (which make sense due to the previous point) (d) every nilpotent group contains a torsion-free nilpotent group of finite index.

Note that the rank and the Hirsch number coincide for free abelian groups.

The result is clearly true for Hirsch number 0. We now prove the theorem by induction on the Hirsch number $h(G) \geq 2$.

We start with Hirsch number 2. A nilpotent group of Hirsch number 2 is virtually \mathbb{Z}^2 , hence has a strongly aperiodic SFT.

Now let G be a nilpotent group of Hirsch number $n > 2$. We may suppose wlog that G is torsion free, as any nilpotent group contains a torsion-free nilpotent group of finite index.

G has a non trivial center, hence contains a copy of $H_1 = \mathbb{Z}$ in its center. By induction, G/H_1 , which is of Hirsch number at least 2, has a strongly aperiodic SFT.

Furthermore, G/H_1 is a nilpotent group, hence contains a torsion-free nilpotent group of finite index, hence contains a torsion-free element x . Let y be a representative of x in G . Then the group H_2 generated by H_1 and y is by construction of Hirsch length 2, hence has a strongly aperiodic SFT.

We have produced our required groups H_1 and H_2 . □

We now proceed to the proof for a polycyclic group. The proof is roughly similar, except there is an additional case to work out, which is the case of group G that admit an exact sequence $1 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$. If we denote $H = \mathbb{Z}^n$, we cannot use the previous proposition, as $G/H = \mathbb{Z}$ does not have a strongly aperiodic SFT. We will however prove that a well chosen aperiodic SFT on H remains aperiodic when seen as a SFT over G (assuming H does not contain any proper nontrivial normal subgroup of G).

Lemma 2.5. *Let $n > 1$. There exists a strongly aperiodic SFT on \mathbb{Z}^n over the alphabet A and a map ϕ from A to $\{0, 1\}$ with the following property: If $x \in X$ and $v \in \mathbb{Z}^n$ then $\phi(x_{iv})_{i \in \mathbb{Z}}$ is monochromatic if and only if $v_1 = 0$.*

Proof. Kari [10] has built a SFT X in \mathbb{Z}^2 with the following properties: Once applying ϕ on each symbol of a configuration of X , the resulting configuration in $\{0, 1\}^{\mathbb{Z}^2}$ consists in monochromatic rows. Furthermore, each column represents a Sturmian word of slope α for some irrational $\alpha = \frac{\log 3/2}{2 \log 2}$

As a consequence, if $v = (x, y)$ is a vector which is not horizontal ($y \neq 0$), then it is not possible for all symbols along the axis v to be monochromatic.

If we exchange row and columns in the above discussion, we obtain a SFT over \mathbb{Z}^2 with the property.

We now obtain a SFT with the same property in \mathbb{Z}^n by first converting this one by repeating it periodically along all other axes, to obtain a SFT Y (using Prop 2.1), and then we do a cartesian product between this SFT and any strongly aperiodic SFT on \mathbb{Z}^n , which exists by Corollary 2.4. \square

Theorem 4. *Let G be a virtually polycyclic group of Hirsch number different from 1.*

Then G admits a strongly aperiodic SFT.

Proof. Polycyclic groups are group which admit a series with cyclic factors. All properties above are still true when nilpotent is replaced by polycyclic, except the first one. However we know that polycyclic groups admit a nontrivial normal free abelian subgroup.

We start with Hirsch number 2. A polycyclic group is virtually \mathbb{Z}^2 , hence has a strongly aperiodic SFT.

For the induction, consider G a polycyclic group of Hirsch number $n+1 \geq 2$. G contains a nontrivial normal free abelian subgroup H_1 . We take H_1 as small as possible. Furthermore, by possibly changing G to a subgroup of G of finite index, we may suppose that no subgroup of G of finite index contains a free normal abelian subgroup of rank less than the rank of H_1 .

There are four cases:

- $h(H_1) = 1$. We use the induction hypothesis on G/H_1 and we proceed as in the previous proof to build H_2 .
- $1 < h(H_1) < n - 1$. We use the induction hypothesis on G/H_1 and we take $H_2 = H_1$.
- $h(H_1) = n + 1$. Then G is virtually \mathbb{Z}^n , this is easy.
- $h(H_1) = n$. G/H_1 is of Hirsch length 1, hence is virtually cyclic. By replacing G by a finite index subgroup of G , we may suppose that G/H_1 is \mathbb{Z} . Let b be a representative of the generator of G/H_1 . Let $a_1 \dots a_n$ be the generators of $H_1 = \mathbb{Z}^n$. Every element of G can therefore be written $a_1^{p_1} \dots a_n^{p_n} b^p$ or in the form $b^q a_1^{q_1} \dots a_n^{q_n}$

Now we take the SFT X from the previous lemma 2.5 over \mathbb{Z}^n , that we extend as a SFT Y over G by Proposition 2.2.

Suppose that Y has a point y with nontrivial stabilizer g , that is $g \cdot y = y$. Write $g = b^p a$ for $a \in H_1$. As X was chosen strongly aperiodic, $p \neq 0$. Then

$$(x^0)_{a_2^k} = y_{a_2^k} = (g \cdot y)_{a_2^k} = (a \cdot y)_{b^{-p} a_2^k} = (a \cdot y)_{a_1^{kq_1} \dots a_n^{kq_n} b^{-p}} = (a \cdot x^{-p})_{a_1^{kq_1} \dots a_n^{kq_n}}$$

for some $q_1 \dots q_n$. (recall the notation x^{-p} from Proposition 2.2. In particular x^{-p} and x^0 are in X).

Now the set $\phi(y_{a_2^k}) = \phi((x^0)_{a_2^k})$ is monochromatic, hence by the previous lemma applied to $a \cdot x^{-p}$, we deduce that $q_1 = 0$.

We can do the same with $a_3, a_4 \dots a_n$ and we deduce that, if we call A the group generated by $a_2 \dots a_n$, we have that $b^p A \subseteq Ab^p$, from which it follows that $b^p A = Ab^p$.

However this implies that the group generated by H_1 and b^p , which is a subgroup of finite index of G , has a normal free abelian subgroup of rank smaller than H_1 , a contradiction.

□

3 The domino problem

We now prove that the domino problem is undecidable for all virtually polycyclic groups which are not virtually cyclic (i.e. which are of Hirsch number greater than 2).

Proposition 3.1. *Let G be a virtually polycyclic group of Hirsch number greater than 2. Then G admits a f.g. subgroup H that factors onto \mathbb{Z}^2 .*

Proof. By recurrence on the Hirsch number, it is sufficient to prove the result for polycyclic groups.

Every polycyclic group of Hirsch number 2 is virtually \mathbb{Z}^2 , there is nothing to prove.

Let G be a polycyclic group of Hirsch number at least 3 and let N be a nontrivial normal free abelian subgroup of G . If N is of rank greater than 2, G contains a subgroup isomorphic to \mathbb{Z}^2 , there is nothing to prove.

Otherwise, G/N is a polycyclic group of Hirsch number at least 2, and admits a subgroup H that factors onto \mathbb{Z}^2 . Therefore G admits a subgroup (the preimage of H that factors onto \mathbb{Z}^2).

□

Corollary 3.2. *Every virtually polycyclic group which is not virtually cyclic has an undecidable domino problem.*

Proof. As G is polycyclic, H is polycyclic. Then $H/N = \mathbb{Z}^2$ for a normal subgroup N of H , which is finitely generated as G (therefore H) is polycyclic.

Therefore H has an undecidable domino problem: Given a SFT X on H/N , we may obtain (constructively) a SFT Y on H s.t. X is empty iff Y is empty by Proposition 2.1.

Therefore G has an undecidable domino problem: Given a SFT X on H , we may obtain (constructively) a SFT Y on G s.t. X is empty iff Y is empty by Proposition 2.2. \square

4 Conclusion

Proposition 2.2 gives a way to transform a SFT X on H to a SFT Y on $G \supseteq H$. In many cases, Y will not be strongly aperiodic. However, surprisingly, it is the case in some situations. We already have used this fact in the proof of our main theorem. Another example is that $\mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$ has a strongly aperiodic SFT Y , which is basically obtained from any strongly aperiodic SFT X on \mathbb{Z}^2 . The idea is that if $g \in \mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$ is nontrivial, then some power of g is in $\mathbb{Z} \times \mathbb{Z}$, therefore no nontrivial element of g can be stabilizer of some point of Y .

Of course $\mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$ is an infinitely generated group, and the study of SFT is primarily interesting in finitely generated groups. However this observation might be useful for example to produce a strongly aperiodic SFT in some solvable groups, e.g. $BS(1, 2) \times BS(1, 2)$ ($BS(1, 2)$ is basically a semidirect product of $\mathbb{Z}[1/2]$ and \mathbb{Z}).

An obvious natural generalization of the main theorem would be to deal with solvable groups rather than polycyclic groups. The difficulty is that solvable groups that are not polycyclic always have abelian subgroups which are not finitely generated. This adds additional cases that we do not know how to treat. Of particular interest is the lamplighter group, or solvable Baumslag-Solitar groups (for which only weakly aperiodic SFTs are known). Furthermore, it is not true that every solvable group has a strongly aperiodic SFT as there exist solvable finitely presented group with an undecidable word problem [7].

Another possible generalization which is promising is Noetherian groups, which are groups where every subgroup is finitely generated. The wilder examples of Noetherian groups, Tarski monsters, do admit strongly aperiodic SFT X [9] (where points in X might have finite, non trivial stabilizer), so it is quite possible that all Noetherian groups (which are not virtually cyclic) do admit strongly aperiodic SFT.

References

- [1] Nathalie Aubrun and Jarkko Kari. Tiling Problems on Baumslag-Solitar groups. In *Machines, Computations and Universality (MCU)*, number 128

- in Electronic Proceedings in Theoretical Computer Science, pages 35–46, 2013.
- [2] Alexis Ballier and Maya Stein. The domino problem on groups of polynomial growth. arXiv:1311.4222.
 - [3] Robert Berger. *The Undecidability of the Domino Problem*. PhD thesis, Harvard University, 1964.
 - [4] David Carroll and Andrew Penland. Periodic Points on Shifts of Finite Type and Commensurability Invariants of Groups. arXiv:1502.03195.
 - [5] Tullio Ceccherini-Silberstein and Michel Coornaert. *Cellular Automata on Groups*. Springer Monographs in Mathematics. Springer, 2010.
 - [6] David Bruce Cohen. The large scale geometry of strongly aperiodic subshifts of finite type. arXiv:1412.4572, 2014.
 - [7] Dion Gildenhuys Gilbert Baumslag and Ralph Strebel. Algorithmically Insoluble Problems about Finitely Presented Solvable Groups, Lie and Associative Algebras. I. *Journal of Pure and Applied Algebra*, 39:53–94, 1986.
 - [8] Karel Culik II and Jarkko Kari. An aperiodic set of Wang cubes. *Journal of Universal Computer Science*, 1(10):675–686, 1995.
 - [9] Emmanuel Jeandel. Some Notes about Subshifts on Groups. preprint.
 - [10] Jarkko Kari. A small aperiodic set of Wang tiles. *Discrete Mathematics*, 160:259–264, 1996.
 - [11] Dietrich Kuske and Markus Lohrey. Logical aspects of Cayley-graphs: the group case. *Annals of Pure and Applied Logic*, 131(1-3):263–286, January 2005.
 - [12] Douglas A. Lind. Multi-Dimensional Symbolic Dynamics. In Susan G. Williams, editor, *Symbolic Dynamics and its Applications*, number 60 in Proceedings of Symposia in Applied Mathematics, pages 61–79. American Mathematical Society, 2004.
 - [13] Douglas A. Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, New York, NY, USA, 1995.
 - [14] Daniel Segal. *Polycyclic groups*. Number 82 in Cambridge Tracts in Mathematics. Cambridge University Pres, 1983.