Modeling of elastic Helmholtz equations by hybridizable discontinuous Galerkin method (HDG) for geophysical applications

M. Bonnasse-Gahot\textsuperscript{1,2}, H. Calandra\textsuperscript{3}, J. Diaz\textsuperscript{1} and S. Lanteri\textsuperscript{2}

\textsuperscript{1} INRIA Bordeaux-Sud-Ouest, team-project Magique 3D
\textsuperscript{2} INRIA Sophia-Antipolis-Méditerranée, team-project Nachos
\textsuperscript{3} TOTAL Exploration-Production
Motivation

Examples of seismic applications
Motivation

Imaging method : the full wave inversion

- Iterative procedure
- Inverse problem requiring to solve many direct problems
Motivation

Imaging method: the full wave inversion

- Iterative procedure
- Inverse problem requiring to solve many direct problems

Seismic imaging: time-domain or harmonic-domain?

- Time-domain: imaging condition complicated but low computational cost
- Harmonic-domain: imaging condition simple but huge computational cost
Motivation

Imaging method: the full wave inversion
  ▶ Iterative procedure
  ▶ Inverse problem requiring to solve many direct problems

Seismic imaging: time-domain or harmonic-domain?
  ▶ Time-domain: imaging condition complicated but low computational cost
  ▶ Harmonic-domain: imaging condition simple but huge computational cost

Forward problem of the inversion process
  ▶ Elastic wave propagation in harmonic domain: Helmholtz equation
  ▶ Reduction of the size of the linear system
Motivation

Seismic imaging in heterogeneous complex media

- Complex topography
- High heterogeneities
Motivation

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Use of unstructured meshes with FE methods
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DG method

- Flexible choice of interpolation orders ($p$ – adaptativity)
- Highly parallelizable method
- Increased computational cost as compared to classical FEM
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Seismic imaging in heterogeneous complex media

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DG method

- Flexible choice of interpolation orders ($p$ – adaptativity)
- Highly parallelizable method
- Increased computational cost as compared to classical FEM

DOF of classical FEM

DOF of DGM
Motivation

Objective of this work

- Development of an hybridizable DG (HDG) method
- Comparison with a reference method: a standard nodal DG method

**Figure:** Degrees of freedom of DGM

**Figure:** Degrees of freedom of HDGM
HDG methods

HDG methods


Contents

3D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

2D Numerical results

3D preliminary numerical results

Conclusions-Perspectives
3D Helmholtz elastic equations

First order formulation of Helmholtz wave equations

\[ \mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3, \]

\[
\begin{cases}
  i \omega \rho (\mathbf{x}) \mathbf{v}(\mathbf{x}) = \nabla \cdot \sigma(\mathbf{x}) + f_s(\mathbf{x}) \\
  i \omega \sigma(\mathbf{x}) = C(\mathbf{x}) \varepsilon(\mathbf{v}(\mathbf{x}))
\end{cases}
\]

- Free surface condition: \( \sigma \mathbf{n} = 0 \) on \( \Gamma_l \)
- Absorbing boundary condition: \( \sigma \mathbf{n} = PAP^T \mathbf{v} \) on \( \Gamma_a \)

- \( \mathbf{v} \): velocity vector
- \( \sigma \): stress tensor
- \( \varepsilon \): strain tensor
3D Helmholtz equations

First order formulation of Helmholtz wave equations

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\[
\begin{align*}
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    i\omega \mathbf{\sigma}(\mathbf{x}) &= \mathbf{C}(\mathbf{x}) \varepsilon(\mathbf{v}(\mathbf{x}))
\end{align*}
\]

- Free surface condition: \( \mathbf{\sigma} \mathbf{n} = 0 \) on \( \Gamma_l \)
- Absorbing boundary condition: \( \mathbf{\sigma} \mathbf{n} = PAP^T \mathbf{v} \) on \( \Gamma_a \)

- \( \rho \): mass density
- \( \mathbf{C} \): tensor of elasticity coefficients
- \( f_s \): source term, \( f_s \in L^2(\Omega) \)
Contents

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Notations and definitions

Hybridizable Discontinuous Galerkin method

2D Numerical results

3D preliminary numerical results

Conclusions-Perspectives
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Notations

- $\mathcal{T}_h$ mesh of $\Omega$ composed of tetrahedrons $K$
Notations and definitions

Notations

- $\mathcal{T}_h$ mesh of $\Omega$ composed of tetrahedrons $K$
- $\mathcal{F}_h$ set of all faces $F$ of $\mathcal{T}_h$
Notations

- $\mathcal{T}_h$ mesh of $\Omega$ composed of tetrahedrons $K$
- $\mathcal{F}_h$ set of all faces $F$ of $\mathcal{T}_h$
- $n$ the normal outward vector of an element $K$
Notations and definitions

Approximations spaces

- $P_p(K)$ set of polynomials of degree at most $p$ on $K$
Notations and definitions

Approximations spaces

- \( P_p(K) \) set of polynomials of degree at most \( p \) on \( K \)
- \( V_h^p = \{ v \in (L^2(\Omega))^3 : \ v|_K \in V^p(K) = (P_p(K))^3, \forall K \in T_h \} \)
Notations and definitions

Approximations spaces

- $P_p(K)$ set of polynomials of degree at most $p$ on $K$
- $\mathbf{V}_h^p = \{ \mathbf{v} \in (L^2(\Omega))^3 : \mathbf{v}|_K \in \mathbf{V}_h^p(K) = (P_p(K))^3, \forall K \in T_h \}$
- $\Sigma_h^p = \{ \underline{\sigma} \in (L^2(\Omega))^6 : \underline{\sigma}|_K \in \Sigma_h^p(K) = (P_p(K))^6, \forall K \in T_h \}$
Notations and definitions

Approximations spaces

- \( P_p(K) \) set of polynomials of degree at most \( p \) on \( K \)
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- \( \Sigma_h^p = \{ \underline{\sigma} \in (L^2(\Omega))^6 : \underline{\sigma}|_K \in \Sigma_h^p(K) = (P_p(K))^6, \forall K \in \mathcal{T}_h \} \)
- \( \mathbf{M}_h = \{ \eta \in (L^2(\mathcal{F}_h))^3 : \eta|_F \in (P_p(F))^3, \forall F \in \mathcal{F}_h \} \)
Notations and definitions

Definitions

► Jump $\llbracket \cdot \rrbracket$ of a vector $\mathbf{v}$ through $F$:

$$\llbracket \mathbf{v} \rrbracket = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^- = \mathbf{v}^+ \cdot \mathbf{n}^+ - \mathbf{v}^- \cdot \mathbf{n}^+$$

► Jump of a tensor $\sigma$ through $F$:

$$\llbracket \sigma \rrbracket = \sigma^+ \mathbf{n}^+ + \sigma^- \mathbf{n}^- = \sigma^+ \mathbf{n}^+ - \sigma^- \mathbf{n}^+$$
Contents

3D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method
  Formulation
  Algorithm

2D Numerical results

3D preliminary numerical results

Conclusions-Perspectives
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\imath \omega \rho \mathbf{v} - \nabla \cdot \sigma &= 0 \\
\imath \omega \sigma - \mathbf{C} \varepsilon (\mathbf{v}) &= 0
\end{align*}
\]
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\int_K i\omega \rho^K v^K \cdot w + \int_K \sigma^K : \nabla w - \int_{\partial K} \hat{\sigma}^{\partial K} \cdot n \cdot w &= 0 \\
\int_K i\omega \sigma^K : \xi + \int_K v^K \cdot \nabla \cdot \left( C^K \xi \right) - \int_{\partial K} \hat{v}^{\partial K} \cdot C^K \xi \cdot n &= 0
\end{align*}
\]

\(\hat{\sigma}^K\) and \(\hat{v}^K\) are numerical traces of \(\sigma^K\) and \(v^K\) respectively on \(\partial K\).
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \sigma^K : \nabla \mathbf{w} - \int_{\partial K} \hat{\sigma}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} &= 0 \\
\int_K i\omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \hat{\mathbf{v}}^{\partial K} \cdot C^K \xi \cdot \mathbf{n} &= 0
\end{align*}
\]

We define:

\[
\begin{align*}
\hat{\mathbf{v}}^{\partial K} &= \hat{\mathbf{v}}^F, \\
\hat{\mathbf{v}}^F &= \lambda^F, \\
\hat{\sigma}^{\partial K} \cdot \mathbf{n} &= \sigma^K \cdot \mathbf{n} - \tau I (\mathbf{v}^K - \lambda^F), \quad \forall F \in \mathcal{F}_h,
\end{align*}
\]

where \(\tau\) is the stabilization parameter \((\tau > 0)\)
HDG formulation of the equations

Local HDG formulation

We replace $\hat{v}^K$ and $(\hat{\sigma}^K \cdot n)$ by their definitions into the local equations

$$\begin{align*}
    \int_K i \omega \rho^K v^K \cdot w &+ \int_K \sigma^K : \nabla w - \int_{\partial K} \sigma^K \cdot n \cdot w \\
    &\quad + \int_{\partial K} \tau I (v^K - \lambda^F) \cdot w = 0 \\
    \int_K i \omega \sigma^K : \xi &+ \int_K v^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \lambda^F \cdot C^K \xi \cdot n = 0
\end{align*}$$
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\int_K i \omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \sigma^K) \cdot \mathbf{w} + \int_{\partial K} \tau I (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} &= 0 \\
\int_K i \omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \cdot (\mathbf{C}^K \xi) - \int_{\partial K} \lambda^F \cdot \mathbf{C}^K \xi \cdot \mathbf{n} &= 0
\end{align*}
\]
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\int_K i\omega \rho^K v^K \cdot w - \int_K (\nabla \cdot \sigma^K) \cdot w + \int_{\partial K} \tau I (v^K - \lambda^F) \cdot w &= 0 \\
\int_K i\omega \sigma^K : \xi + \int_K v^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \lambda^F \cdot C^K \xi \cdot n &= 0
\end{align*}
\]

We define:

\[W^K = (V^K_x, V^K_y, V^K_z, \sigma^K_{xx}, \sigma^K_{yy}, \sigma^K_{zz}, \sigma^K_{xy}, \sigma^K_{xz}, \sigma^K_{yz})^T\]

\[\Lambda = (\Lambda^F_1, \Lambda^F_2, ..., \Lambda^F_{n_f})^T, \text{ where } n_f = \text{card}(F_h)\]

Discretization of the local HDG formulation

\[\Lambda^K W^K + \sum_{F \in \partial K} C^{K,F} \Lambda = 0\]
HDG formulation of the equations

Local HDG formulation

\[
\begin{aligned}
\int_K i\omega \rho^K v^K \cdot w - \int_K (\nabla \cdot \sigma^K) \cdot w + \int_{\partial K} \tau I (v^K - \lambda^F) \cdot w &= 0 \\
\int_K i\omega \sigma^K : \xi + \int_K v^K \cdot \nabla \cdot \left( C^K \xi \right) - \int_{\partial K} \lambda^F \cdot \xi_n &= 0
\end{aligned}
\]

We define:

\[
W^K = \left( V^K_x, V^K_y, V^K_z, \sigma^K_{xx}, \sigma^K_{yy}, \sigma^K_{zz}, \sigma^K_{xy}, \sigma^K_{xz}, \sigma^K_{yz} \right)^T
\]

\[
\Lambda = \left( \Lambda^F_1, \Lambda^F_2, \ldots, \Lambda^F_{n_f} \right)^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)
\]

Discretization of the local HDG formulation

\[
A^K W^K + C^K \Lambda = 0
\]
HDG formulation of the equations

Transmission condition

In order to determine $\lambda^K$, the continuity of the normal component of $\hat{\sigma}^K$ is weakly enforced, rendering this numerical trace conservative:

$$\int_F [\hat{\sigma}^K \cdot n] \cdot \eta = 0$$
HDG formulation of the equations

Transmission condition

In order to determine $\lambda^K$, the continuity of the normal component of $\hat{\sigma}^K$ is weakly enforced, rendering this numerical trace conservative:

$$\int_F \left[ \hat{\sigma}^K \cdot n \right] \cdot \eta = 0$$

Replacing $(\hat{\sigma}^K \cdot n)$ and summing over all faces, the transmission condition becomes:

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \left( \sigma^K \cdot n \right) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau I \left( \mathbf{v}^K - \lambda^F \right) \cdot \eta = 0$$
HDG formulation of the equations

Transmission condition

In order to determine $\lambda^K$, the continuity of the normal component of $\overline{\sigma}^K$ is weakly enforced, rendering this numerical trace conservative:

$$\int_F [\overline{\sigma}^K \cdot n] \cdot \eta = 0$$

Replacing $(\overline{\sigma}^K \cdot n)$ and summing over all faces, the transmission condition becomes:

$$\sum_{K \in T_h} \int_{\partial K} (\sigma^K \cdot n) \cdot \eta - \sum_{K \in T_h} \int_{\partial K} \tau I (v^K - \lambda^F) \cdot \eta = 0$$

Discretization of the transmission condition

$$\sum_{K \in T_h} [B^K W^K + L^K \Lambda] = 0$$
HDG formulation of the equations

Global HDG formulation

\[
\begin{align*}
\int_K i\omega \rho^K v^K \cdot w - \int_K (\nabla \cdot \sigma^K) \cdot w + \int_{\partial K} \tau I (v^K - \lambda^F) \cdot w &= 0 \\
\int_K i\omega \sigma^K : \xi + \int_K v^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \lambda^F \cdot C_K \xi \cdot n &= 0 \\
\sum_{K \in T_h} \int_{\partial K} (\sigma^K \cdot n) \cdot \eta - \sum_{K \in T_h} \int_{\partial K} \tau I (v^K - \lambda^F) \cdot \eta &= 0
\end{align*}
\]
HDG formulation of the equations

Global HDG formulation

\[ \int_K i\omega \rho^K v^K \cdot w - \int_K (\nabla \cdot \sigma^K) \cdot w + \int_{\partial K} \tau I (v^K - \lambda^F) \cdot w = 0 \]

\[ \int_K i\omega \sigma^K : \xi + \int_K v^K \cdot \nabla \cdot \left( C^K \xi \right) - \int_{\partial K} \lambda^F \cdot C^K \xi \cdot n = 0 \]

\[ \sum_{K \in T_h} \int_{\partial K} (\sigma^K \cdot n) \cdot \eta - \sum_{K \in T_h} \int_{\partial K} \tau I (v^K - \lambda^F) \cdot \eta = 0 \]

Global HDG discretization

\[ A^K W^K + C^K \Lambda = 0 \]

\[ \sum_{K \in T_h} [B^K W^K + L^K \Lambda] = 0 \]
HDG formulation of the equations

Global HDG formulation

\[
\begin{align*}
\int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} &= \int_K (\nabla \cdot \sigma^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\
\int_K i\omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \cdot \left( C^K \xi \right) - \int_{\partial K} \lambda^F \cdot C_K \xi \cdot \mathbf{n} &= 0 \\
\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\sigma^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \eta &= 0
\end{align*}
\]

Global HDG discretization

\[
\begin{align*}
\mathbf{W}^K &= -(A^K)^{-1} C^K \Lambda \\
\sum_{K \in \mathcal{T}_h} [\mathbf{B}^K \mathbf{W}^K + \mathbf{L}^K \Lambda] &= 0
\end{align*}
\]
HDG method Formulation

HDG formulation of the equations

Global HDG formulation

\[
\begin{cases}
\int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \sigma^K) \cdot \mathbf{w} + \int_{\partial K} \mathbf{I} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\
\int_K i\omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \lambda^F \cdot C^K \xi \cdot \mathbf{n} = 0 \\
\sum_{K \in T_h} \int_{\partial K} (\sigma^K \cdot \mathbf{n}) \cdot \eta - \sum_{K \in T_h} \int_{\partial K} \mathbf{I} (\mathbf{v}^K - \lambda^F) \cdot \eta = 0
\end{cases}
\]

Global HDG discretization

\[
\sum_{K \in T_h} \left[ -B^K (A^K)^{-1} C^K + L^K \right] \Lambda = 0
\]
Main idea of the algorithm using the HDG formulation

1. Construction of the linear system \( M \Lambda \)
with \( M = \sum_{K \in \mathcal{T}_h} \left[ -B^K (A^K)^{-1} C^K + L^K \right] \)

\[ \text{for } K = 1 \text{ to } Nb_{tri} \text{ do} \]
\[ \quad \text{Compute matrices } B^K, (A^K)^{-1}, C^K \text{ and } L^K \]
\[ \quad \text{Construction of the corresponding section of } M \]
\[ \text{end for} \]
Main idea of the algorithm using the HDG formulation

1. Construction of the linear system $\mathbf{M}\Lambda$
2. Construction of the right hand side $\mathbf{S}$
Main idea of the algorithm using the HDG formulation

1. Construction of the linear system $M\Lambda$
2. Construction of the right hand side $S$
3. Resolution $M\Lambda = S$
Main idea of the algorithm using the HDG formulation

1. Construction of the linear system $M\Lambda$
2. Construction of the right hand side $S$
3. Resolution $M\Lambda = S$
4. Computation of the solutions of the initial problem

\[
\text{for } K = 1 \text{ to } Nb_{tri} \text{ do }
\quad \text{Compute } W^K = -(A^K)^{-1} C^K \Lambda
\text{end for}
\]
Contents

3D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

2D Numerical results
   Plane wave in an homogeneous medium
   Marmousi test-case

3D preliminary numerical results

Conclusions-Perspectives
Plane wave

Physical parameters:
- \(\rho = 2000 \text{kg.m}^{-3}\)
- \(\lambda = 16 \text{GPa}\)
- \(\mu = 8 \text{GPa}\)

Plane wave:
\[
 u = \nabla e^{i(k \cos \theta x + k \sin \theta y)}
\]
where
\[
 k = \frac{\omega}{v_p}
\]
\(\theta = 0\)

Three meshes:
- 3000 elements
- 10000 elements
- 45000 elements
Plane wave

Convergence order of the HDG scheme
## Plane wave

### Memory used

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## Plane wave

### Memory used

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Marmousi test-case

Computational domain $\Omega$ composed of 235000 triangles
Parallel results for the Marmousi test-case with the HDG-P3 scheme, $f = 2$Hz
Caracteristics of the computing processors used

- **Plafrim** platform
- Hardware specification: 16 nodes, 12 cores by nodes
- Caracteristics of computing nodes:
  - 2 Hexa-core Westmere Intel® Xeon® X5670
  - Frequency: 2.93 GHz
  - Cache L3: 12 Mo
  - RAM: 96 Go
  - Infiniband DDR: 20Gb/s
  - Ethernet: 1Gb/s
Efficiency of the parallelism of the global matrix construction

\[
\text{Efficiency} = \frac{\text{sequential computational time}}{\text{nb\_cores} \times \text{computational time with one core}}
\]
Efficiency of the parallelism of the global matrix construction
Efficiency of the parallelism of the resolution part

![Graph showing the efficiency of parallelism for different numbers of cores, comparing HDGm and IPDGM methods. The graph displays a downward trend as the number of cores increases, indicating a decrease in efficiency.](attachment:graph.png)
Speed up for the global matrix construction

![Graph showing speed up for the global matrix construction](image-url)
Speed up (Total simulation time)

![Graph showing speed up over nb cores](image-url)
Memory required (GB) for the simulation

![Graph showing memory requirements for different numbers of cores for HDGm and IPDGm methods.](image)

- **HDGm**
- **IPDGm**

Memory (GB) vs. Nb cores (1 to 96)
Contents

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Notations and definitions

Hybridizable Discontinuous Galerkin method

2D Numerical results

3D preliminary numerical results
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Conclusions-Perspectives
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  - $\rho = 2000\, \text{kg}\cdot\text{m}^{-3}$
  - $\lambda = 16\, \text{GPa}$
  - $\mu = 8\, \text{GPa}$

- Plane wave:
  $$u = \nabla e^{i(k \cos \theta x + k \sin \theta y)}$$

where $k = \frac{\omega}{v_p}$

- $\theta = 0$

- Two meshes:
  - 2600 elements
  - 6400 elements
3D preliminary numerical results

**Figure**: Exact solution
3D preliminary numerical results

**Figure**: Numerical solution computed with $P_3$-HDG scheme on mesh composed of 6400 tetrahedrons
### 3D preliminary numerical results

**Plane wave in an homogeneous medium**

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### 3D preliminary numerical results

**Plane wave in an homogeneous medium**

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Conclusion

Contents

3D Helmholtz elastic equations

Notations and definitions

Hybridizable Discontinuous Galerkin method

2D Numerical results

3D preliminary numerical results

Conclusions-Perspectives
Conclusions-Perspectives

Conclusions

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- Computational time gain

Perspectives

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Conclusions-Perspectives

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Thank you!