



HAL
open science

Yamanouchi toppling

Robert Cori, Domenico Senato, Pasquale Petrullo

► **To cite this version:**

Robert Cori, Domenico Senato, Pasquale Petrullo. Yamanouchi toppling. 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014), 2014, Chicago, United States. pp.441-452, 10.46298/dmtcs.2413 . hal-01207603

HAL Id: hal-01207603

<https://inria.hal.science/hal-01207603>

Submitted on 1 Oct 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Yamanouchi toppling - *Extended abstract*

Robert Cori^{1*}Pasquale Petrullo^{2†}Domenico Senato^{2‡}¹*LaBRI Université Bordeaux I, France*²*Università degli Studi della Basilicata, Italy*

Abstract We study an extension of the chip-firing game. A given set of admissible moves, called Yamanouchi moves, allows the player to pass from a starting configuration α to a further configuration β . This can be encoded via an action of a certain group, the toppling group, associated with each connected graph. This action gives rise to a generalization of Hall-Littlewood symmetric polynomials and a new combinatorial basis for them. Moreover, it provides a general method to construct all orthogonal systems associated with a given random variable.

Résumé. On s'intéresse ici à une variante du modèle combinatoire du tas de sable. Un ensemble particulier de suites d'éboulements, les éboulements de Yamanouchi est défini. Les éléments de cet ensemble permettent de passer d'une configuration à une autre, ceci peut être représenté par l'action d'un certain groupe, le groupe des éboulements que l'on peut associer à tout graphe connexe. Cette action donne lieu à une généralisation de polynômes symétriques de Hall-Littlewood et un nouveau champ combinatoire pour ceux-ci. Une extension à la construction d'autres familles de polynômes orthogonaux est proposée.

Keywords: chip-firing game, Yamanouchi words, Young tableaux, orthogonal polynomials, Hall-Littlewood symmetric polynomials.

1 Introduction

In [3] A. Björner, L. Lovász and P. Schor have studied a solitary game called the chip-firing game which is closely related to the sandpile model of Dhar [6], arising in physics. In more recent papers some developments around this game were proposed. Musiker [11] introduced an unexpected relationship with elliptic curves, Norine and Baker [1] by means of an analogous game, proposed a Riemann-Roch formula for graphs, for which Cori and Le Borgne [5] presented a purely combinatorial description. An algebraic presentation of the theory can be found in [2] and [12]. In this paper we investigate this game, which we refer to as the toppling game, by exhibiting a wide range of connections with classical orthogonal polynomials and symmetric functions. We focus our attention on a noteworthy class of admissible moves, that we have called Yamanouchi moves. These moves are built up starting from suitable elementary moves, denoted by T_1, T_2, \dots, T_n and called topplings. A first crucial fact is that Yamanouchi moves are not invertible, this provides a partial order on the set of configurations and then principal order ideals may

*Email: cori@labri.u-bordeaux.fr

†Email: p.petrullo@gmail.com

‡Email: domenico.senato@unibas.it

be considered. A second interesting point is that topplings can be seen as operators acting on the ring of formal series $\mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]]$, and in particular on the ring of polynomials $\mathbb{Z}[x_1, x_2, \dots, x_n]$. Hence, each principal order ideal $\mathcal{H}_\alpha = \{\beta \mid \alpha \rightarrow_{\mathcal{Y}} \beta\}$, which stores all configurations obtained from a starting configuration α by means of a Yamanouchi move, may be identified with the series

$$\mathcal{H}_\alpha(x) = \sum_{\alpha \rightarrow_{\mathcal{Y}} \beta} x^\beta,$$

where $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$. Topplings generate a group G , the toppling group, attached to any graph \mathcal{G} . By setting $T_{[i]} = T_1 T_2 \dots T_i$ we obtain algebraically independent elements spanning a subalgebra $\mathbb{Z}[G]_{\geq}$ of the group algebra $\mathbb{Z}[G]$. We show there exists an operator τ , with $\tau^{-1} \in \mathbb{Z}[G]_{\geq}$, which depends on \mathcal{G} but not on α , such that $\tau \cdot x^\alpha = \mathcal{H}_\alpha(x)$ and

$$\tau = \prod_{i=1}^{n-1} \frac{1}{1 - T_{[i]}}.$$

By setting $T_{[i,j]} = T_{[i]} T_{[i+1]} \dots T_{[j-1]}$ we obtain a further set of generators of $\mathbb{Z}[G]_{\geq}$ yielding a deformation of τ and a weighted version of $\mathcal{H}_\alpha(x)$. More precisely, we recover

$$\hat{\tau} = \prod_{1 \leq i < j \leq n} \frac{1}{1 - T_{[i,j]}}$$

and $\hat{\mathcal{H}}_\alpha(x) = \hat{\tau} \cdot x^\alpha$, and also we obtain

$$\hat{\mathcal{H}}_\alpha(x) = \sum_{\alpha \rightarrow_{\mathcal{Y}} \beta} C(\lambda) x^\beta,$$

where $C(\lambda)$ counts the number of pairwise distinct decompositions in terms of the generators $T_{[i,j]}$'s of the unique element $T^\lambda = T_1^{\lambda_1} T_2^{\lambda_2} \dots T_n^{\lambda_n}$ in the toppling group G such that $\beta = T^\lambda(\alpha)$. Parameters z_1, z_2, z_3, q may be introduced in order to keep track of certain statistics $\ell_1, \ell_2, \ell_3, d$ defined on the set of all decompositions of a given T^λ . Then, the following series can be considered,

$$\hat{\mathcal{H}}_\alpha(x; z_1 z_2 z_3, q) = \sum_{\alpha \rightarrow_{\mathcal{Y}} \beta} \left(\sum_{T^\lambda = T_{[i_1, j_1]} T_{[i_2, j_2]} \dots} z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} q^d \right) x^\beta.$$

Such a series satisfies $\hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q) = \hat{\tau}(z_1, z_2, z_3, q) \cdot x^\alpha$, where

$$\hat{\tau}(z_1, z_2, z_3, q) = \prod_{1 \leq i < j \leq n} \frac{1 - (1 - q) T_{[i,j]} z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}}}{1 - T_{[i,j]} z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}}}.$$

Since $\hat{\tau}(z_1, z_2, z_3, q)^{-1} = \hat{\tau}(z_1, z_2, (1 - q)z_3, q(q - 1)^{-1})$, then a combinatorial description of $\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q) = \hat{\tau}(z_1, z_2, z_3, q)^{-1} \cdot x^\alpha$ is obtained for free. Moreover, $\ell_1 \geq \ell_2 \geq \ell_3 \geq d \geq 0$

implies that the coefficient of x^β in $\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q)$, as well as that in $\hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q)$, is a polynomial with integer coefficients. Then, one can consider a projection Π such that $\Pi x^\beta = x^\beta$ if $\beta \in \mathbb{N}^n$, and $\Pi x^\beta = 0$ otherwise. Hence both $\{\Pi \hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q)\}_\alpha$ and $\{\Pi \hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q)\}_\alpha$, with α ranging over \mathbb{N}^n , are bases of $\mathbb{Z}[z_1, z_2, z_3, q][x]$. Our interest in the toppling game is explained by focusing the attention on a very special family of graphs $\{\mathcal{L}_n\}_{n \geq 1}$. In this case, we prove that the polynomial sequence $\{\Pi \hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q)\}_\alpha$, via a suitable linear functional of umbral type [14], yields to a parametrized extension of classical orthogonal polynomials. More concretely, set $\alpha = (n - 1, n - 1, \dots, n - 1)$ and let $X = X_1, X_2, \dots, X_{n-1}$ denote i.i.d. random variables. Then, denote by $p_{n-1}(z_1, z_2, z_3, q; t)$ the expected value of the polynomial obtained by replacing $x_1 = t$ and $x_{i+1} = X_i$ in $\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q)$:

$$p_{n-1}(t; z_1, z_2, z_3, q) = \mathbf{E} \hat{\mathcal{K}}_\alpha(t, X_1, \dots, X_{n-1}; z_1, z_2, z_3, q).$$

The polynomial sequence $\{p_n(t; z_1, z_2, z_3, q)\}_n$ generalizes the orthogonal polynomial system associated with X [4, 13]. Classical cases, like Hermite, Laguerre, Poisson-Charlier, Chebyshev arise for $z_1 = z_2 = z_3 = q = 1$ when suitable random variables are chosen. Moreover, if we replace x^β with complete homogeneous symmetric functions $h_\beta(x)$ then an action of the toppling group on symmetric functions is obtained. In this context, when $\mathcal{G} = \mathcal{L}_n$ the action of the $T_{[i,j]}$'s turns out to be closely related to that of raising operators [8] and this leads to a generalization of Hall-Littlewood symmetric functions [9] $\{R_\lambda(x; t)\}_\lambda$. The toppling game not only provides a new elementary combinatorial basis for symmetric functions and orthogonal polynomials [10], but it also suggests further generalizations to analogous environments defined starting from further families of graphs $\{\mathcal{G}_n\}_n$.

2 The toppling game

Here and in the following, by a graph $\mathcal{G} = (V, E)$ we will mean a connected graph, with set of vertices $V = \{v_1, v_2, \dots, v_n\}$, and whose set of undirected edges E contains at most one edge $\{v_i, v_j\}$ for each pair (v_i, v_j) . Also, we say that v_i and v_j are neighbours whenever $\{v_i, v_j\} \in E$. A *configuration* on \mathcal{G} is a map, $\alpha: v_i \in V \mapsto \alpha(v_i) \in \mathbb{Z}$, associating each vertex v_i with an integral weight $\alpha(v_i)$. If we set $\alpha_i = \alpha(v_i)$ then we may identify any configuration α with the array $(\alpha_1, \alpha_2, \dots, \alpha_n)$. We set $\epsilon_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{in})$ so that ϵ_i is the configuration associating v_i with 1 and v_j with 0 if $j \neq i$. A *toppling* of the vertex v_i is a map $T_i^\mathcal{G}: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ defined by

$$T_i^\mathcal{G}(\alpha) = \alpha - d_i \epsilon_i + \sum_{\{v_j, v_i\} \in E} \epsilon_j \tag{1}$$

and $d_i = |\{v_j \mid \{v_i, v_j\} \in E\}|$ is the *degree* of v_i . Roughly speaking, the map $T_i^\mathcal{G}$ increases by 1 the weight α_j of each neighbour v_j of v_i , and simultaneously decreases by d_i the weight α_i . As a trivial consequence we deduce that the *size* $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ of any $\alpha \in \mathbb{Z}^n$ is preserved by each toppling $T_i^\mathcal{G}$. One may visualize topplings as special moves of a suitable combinatorial game defined on the graph \mathcal{G} . More precisely, fix a starting configuration $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ on the graph and label each vertex v_i with its own weight α_i . By “firing” the vertex v_i we change the configuration α into a new configuration $\beta = T_i^\mathcal{G}(\alpha)$. A *move* of the toppling game simply is a finite sequence of fired vertices $v_{i_1} v_{i_2} \dots v_{i_l}$, possibly not all distinct. Now, let us fix two configurations α and β on a graph \mathcal{G} and let us look for all possible moves changing α into β . Also, assume that certain moves are forbidden in the game, so that each player is forced to perform only those moves that are in a given set \mathcal{M} of

admissible moves. Henceforth, by writing $\alpha \rightarrow_{\mathcal{M}} \beta$ we will mean that α can be changed into β by means of a suitable admissible move in \mathcal{M} . When useful, \mathcal{M} will be identified with the corresponding set of monomials belonging to the free monoid \mathcal{T}^* generated by the alphabet $\mathcal{T} = \{T_1^{\mathcal{G}}, T_2^{\mathcal{G}}, \dots, T_n^{\mathcal{G}}\}$. At this point a possible strategy of a player passes through the determination of those moves of minimal length (i.e. minimal number of fired vertices) that are admissible and that change α into β . Therefore, the toppling game starts once a pair $(\mathcal{G}, \mathcal{M})$ is chosen, and two configurations $\alpha, \beta \in \mathbb{Z}^n$ are given so that $\alpha \rightarrow_{\mathcal{M}} \beta$. Hence, the idea is to characterize, in an explicit way, the subset $\mathcal{M}_{\alpha, \beta}$ of all admissible moves changing α into β , then to determine all moves in $\mathcal{M}_{\alpha, \beta}$ of minimal length. In the following we will focus our attention on a special set of admissible moves, which we name *Yamanouchi moves*, defined by $\mathcal{Y} = \{T_{i_1}^{\mathcal{G}} T_{i_2}^{\mathcal{G}} \dots T_{i_l}^{\mathcal{G}} \mid i_1 i_2 \dots i_l \text{ is Yamanouchi}\}$. Recall that a sequence, or word, $w = i_1 i_2 \dots i_l$ of positive integers is said to be *Yamanouchi* if and only if, for all k and for all i , the number $occ(i, k)$, of occurrences of i in $i_1 i_2 \dots i_k$, satisfies $occ(i, k) \geq occ(i + 1, k)$. If $w = i_1 i_2 \dots i_l$ then we set $T_w^{\mathcal{G}} = T_{i_1}^{\mathcal{G}} T_{i_2}^{\mathcal{G}} \dots T_{i_l}^{\mathcal{G}}$ and write $\alpha \rightarrow_w \beta$ if and only if $\beta = T_w^{\mathcal{G}}(\alpha)$. Associated with each Yamanouchi word $w = i_1 i_2 \dots i_l$ there is an integer partition $\lambda(w) = (\lambda_1, \lambda_2, \dots)$ whose i th part λ_i equals the number $occ(i, l)$ of occurrences of i in w . A suitable filling of the Young diagram of $\lambda(w)$ yields a coding of w in terms of a standard Young tableau. More precisely, the Young tableau associated with w is the unique tableaux of shape $\lambda(w)$ whose i th row (of length λ_i) stores all j such that $w_j = i$. This fixes a bijection between the set of all Yamanouchi words of l letters and the set of all standard Young tableaux of l boxes.

Proposition 1 *If $\alpha \rightarrow_w \beta$ and $\lambda = \lambda(w)$ then for all w' such that $\lambda = \lambda(w')$ we have $\alpha \rightarrow_{w'} \beta$.*

If $\alpha \rightarrow_w \beta$ then the set of all Yamanouchi words extracted from all standard Young tableaux of shape $\lambda(w)$ corresponds to moves in $\mathcal{Y}_{\alpha, \beta}$. On the other hand the converse is not true. Hence, in order to get an explicit characterization of $\mathcal{Y}_{\alpha, \beta}$ and of those moves in $\mathcal{Y}_{\alpha, \beta}$ of minimal length we need a slightly deeper investigation.

3 The toppling group

Assume a graph $\mathcal{G} = (V, E)$ is given and write T_1, T_2, \dots, T_n instead of $T_1^{\mathcal{G}}, T_2^{\mathcal{G}}, \dots, T_n^{\mathcal{G}}$. The *toppling group* associated with \mathcal{G} is the group G generated by T_1, T_2, \dots, T_n . Let 1 denote its unity. By virtue of (1) we have $T_i T_j = T_j T_i$ for all i, j , and then G is commutative. In particular, this says that all $g \in G$ may be expressed in terms of topplings as a product of type $T^a = T_1^{a_1} T_2^{a_2} \dots T_n^{a_n}$, for a suitable array of integers $a = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$. On the other hand, it is also easy to verify that $T_1 T_2 \dots T_n(\alpha) = \alpha$ for all α , and then we have $T_1 T_2 \dots T_n = 1$. As a consequence, all $g \in G$ admit a presentation T^a with $a \in \mathbb{N}^n$. In fact, if $g = T^a$ and if $a \notin \mathbb{N}^n$ then let $k = \min\{a_1, a_2, \dots, a_n\}$ and set $b = a - k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$. Then $b \in \mathbb{N}^n$ and $T^b = T^a (T_1 T_2 \dots T_n)^k = T^a = g$. The following lemma characterizes all presentations of the unity 1 in G .

Lemma 2 *For all graphs \mathcal{G} we have $T^a = 1$ if and only if $a = k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$.*

A first consequence of Lemma 2 is that apart from $T_i T_j = T_j T_i$ and $T_1 T_2 \dots T_n = 1$ there are not further relations satisfied by T_1, T_2, \dots, T_n . This provides a characterization of all distinct presentations of any element in the toppling group G .

Theorem 3 *For all graphs \mathcal{G} and for all $a, b \in \mathbb{N}^n$ we have $T^a = T^b$ if and only if there exists $k \in \mathbb{Z}$ such that $b = a + k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$.*

Now, once $a \in \mathbb{N}^n$ is chosen, we may set $k = \min\{a_1, a_2, \dots, a_n\}$ and define $b = a - k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$. Clearly $T^a = T^b$ and b is the unique array in \mathbb{N}^n of minimal size with this property. In other terms, T^b is the unique reduced decomposition of $g = T^a$.

Theorem 4 *Let $(\mathcal{G}, \mathcal{Y})$ be a toppling game. If $\alpha \rightarrow_{\mathcal{Y}} \beta$ then there exists a unique partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with $l \leq n - 1$ parts such that $T^\lambda(\alpha) = \beta$. Hence, $\mathcal{Y}_{\alpha, \beta}$ consists of exactly all Yamanouchi moves associated with standard Young tableaux of shape $\mu = \lambda + k(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$, for a suitable $k \in \mathbb{N}$. In particular, all minimal moves in $\mathcal{Y}_{\alpha, \beta}$ are the Yamanouchi moves associated with standard Young tableaux of shape λ .*

4 A partial order on \mathbb{Z}^n

Our next goal is the following: we want to characterize in an explicit way the sets $\mathcal{H}_\alpha = \{\beta \in \mathbb{Z}^n \mid \alpha \rightarrow_{\mathcal{Y}} \beta\}$ defined for all $\alpha \in \mathbb{Z}^n$. To this aim, we denote by I_n the set of all $a = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$ such that $a_i = 0$ for some i . Note that, for all $g \in G$ there is exactly one $a \in I_n$ such that $g = T^a$. In particular, T^a is the reduced decomposition of g . This gives the following characterization of the group algebra $\mathbb{Z}[G]$.

Theorem 5 *We have*

$$\mathbb{Z}[G] = \left\{ \sum_{a \in I_n} c_a T^a \mid c_a \in \mathbb{Z} \text{ and the sum involves a finite number of non-zero terms} \right\},$$

and in particular

$$\mathbb{Z}[G] \cong \frac{\mathbb{Z}[x_1, x_2, \dots, x_n]}{\langle 1 - x_1 x_2 \cdots x_n \rangle},$$

with $\langle 1 - x_1 x_2 \cdots x_n \rangle$ denoting the principal ideal generated by $1 - x_1 x_2 \cdots x_n$.

We consider not only elements in $\mathbb{Z}[G]$ but we also admit formal series in T_1, T_2, \dots, T_n . In particular we set

$$\mathbb{Z}[[G]] = \left\{ \sum_{a \in I_n} c_a T^a \mid c_a \in \mathbb{Z} \text{ and the sum involves a possibly infinite number of non-zero terms} \right\}.$$

If we associate each $\alpha \in \mathbb{Z}^n$ with a monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ then we may translate the action of G on \mathbb{Z}^n into an action of $\mathbb{Z}[[G]]$, and in particular of $\mathbb{Z}[G]$, on the ring of formal series

$$\mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]] = \left\{ \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \mid c_\alpha \in \mathbb{Z} \text{ and the sum involves a possibly infinite number of non-zero terms} \right\}.$$

This is done by setting $T^a \cdot x^\alpha = x^{T^a(\alpha)}$ for all $a \in \mathbb{N}^n$ and for all $\alpha \in \mathbb{Z}^n$. Clearly, knowing the set $\mathcal{H}_\alpha = \{\beta \in \mathbb{Z}^n \mid \alpha \rightarrow_{\mathcal{Y}} \beta\}$ is equivalent to knowing the formal series

$$\mathcal{H}_\alpha(x) = \sum_{\alpha \rightarrow_{\mathcal{Y}} \beta} x^\beta.$$

In the following we will determine a special element $\tau \in \mathbb{Z}[[G]]$ satisfying $\tau \cdot x^\alpha = \mathcal{H}_\alpha(x)$. Before doing that we introduce a new relation \leq_G on \mathbb{Z}^n : set $\beta \leq_G \alpha$ if and only if $\beta = T^\lambda(\alpha)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Clearly, if $\lambda_n \neq 0$ then we may define $\mu = \lambda - \lambda_n(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$ so that $T^\lambda(\alpha) = T^\mu(\beta)$

and T^μ is a reduced decomposition for T^λ . Note that μ is an integer partition with at most $n - 1$ parts. Henceforth we will denote by P_n the set of all integer partitions with at most $n - 1$ parts or, equivalently the subset of all λ 's in I_n such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The following proposition makes explicit the relation between the relation \leq_G and the toppling game.

Proposition 6 *For all graphs \mathcal{G} the relation \leq_G is a partial order on \mathbb{Z}^n . Moreover, \leq_G and \rightarrow_y agree on \mathbb{Z}^n .*

Proposition 6 assures us that the set \mathcal{H}_α is nothing but the order ideal generated by α , that is $\mathcal{H}_\alpha = \{\beta \mid \beta \leq_G \alpha\}$. Let us consider the following element in $\mathbb{Z}[[G]]$:

$$\tau = \sum_{\lambda \in P_n} T^\lambda,$$

and let it act on the monomial x^α . Being $\beta = T^\lambda(\alpha)$ for at most one $\lambda \in P_n$, then we recover

$$\tau \cdot x^\alpha = \sum_{\lambda \in P_n} T^\lambda \cdot x^\alpha = \sum_{\beta \leq_G \alpha} x^\beta = \mathcal{H}_\alpha(x).$$

The crucial point is that a closed expression of τ is available by introducing suitable elements of G . More precisely, we define

$$T_{[i]} = T_1 T_2 \dots T_i \text{ for all } i = 1, 2, \dots, n - 1.$$

Note that $T_{[i]}$ may be thought of as a Yamanouchi move associated with a Young standard tableau of one column filled with $1, 2, \dots, i$.

Theorem 7 *Let $\lambda \in P_n$ and denote by $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_i)$ the conjugate of λ . Then we have*

$$T^\lambda = T_{[\lambda'_1]} T_{[\lambda'_2]} \dots T_{[\lambda'_i]}.$$

Corollary 8 *We have*

$$\tau = \prod_{i=1}^{n-1} \frac{1}{1 - T_{[i]}}.$$

Now, we may write

$$\mathcal{H}_\alpha(x) = \prod_{i=1}^{n-1} \frac{1}{1 - T_{[i]}} \cdot x^\alpha.$$

Moreover, straightforward computations will prove the following theorem.

Theorem 9 *For each graph \mathcal{G} there exists a formal series $\tau(x) \in \mathbb{Z}[[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]]$ such that $\mathcal{H}_\alpha(x) = \tau(x)x^\alpha$, for all $\alpha \in \mathbb{Z}^n$.*

5 A special set of Yamanouchi moves

If G is the toppling group associated with a graph \mathcal{G} then we set

$$\mathbb{Z}[G]_{\geq} = \left\{ \sum_{\lambda \in P_n} c_\lambda T^\lambda \mid c_\lambda \in \mathbb{Z} \text{ and the sum involves a possibly infinite number of non-zero terms} \right\}.$$

Clearly $\mathbb{Z}[G]_{\geq}$ is a subalgebra of $\mathbb{Z}[G]$ and Theorem 7 assures us it is exactly the subalgebra generated by $T_{[1]}, T_{[2]}, \dots, T_{[n-1]}$. Let us consider a wider set of generators of $\mathbb{Z}[G]_{\geq}$ defined by $T_{[i,j]} = T_{[i]}T_{[i+1]} \cdots T_{[j-1]}$ for all $1 \leq i < j \leq n$. Note that each $T_{[i,j]}$ is a Yamanouchi move associated with a tableau whose shape consists of consecutive integers. Obviously $T_{[i]} = T_{[i,i+1]}$ so that the $T_{[i,j]}$'s generate the whole $\mathbb{Z}[G]_{\geq}$. On the other hand, the presentation of any $g \in G$ as a product of these generators is in general not unique. In order to find a *reduced decomposition* of $g = T^\lambda$, that is a presentation that involves a minimal number of generators, we write $g = T_{[\lambda_1]}T_{[\lambda_2]} \cdots T_{[\lambda_m]}$, then rearrange and associate so that $T^\lambda = (T_{[i_{11}]}T_{[i_{12}]} \cdots T_{[i_{1j_1}]} \cdots (T_{[i_{k1}]}(T_{[i_{k2}]} \cdots T_{[i_{kj_k}]})$, and each product $(T_{[i_{h1}]}(T_{[i_{h2}]} \cdots T_{[i_{hj_h}]})$ consists of an increasing sequence of consecutive integers. Thus, a reduced decomposition of T^λ is given by $T^\lambda = T_{[i_{11}, i_{1j_1}+1]}T_{[i_{21}, i_{2j_2}+1]} \cdots T_{[i_{k1}, i_{kj_k}+1]}$.

Example 1 If $\lambda = (8, 7, 4, 3, 2, 2, 1)$ then $\lambda' = (7, 6, 4, 3, 2, 2, 2, 1)$. We recover

$$T^\lambda = T_{[7]}T_{[6]}T_{[4]}T_{[3]}T_{[2]}T_{[2]}T_{[2]}T_{[1]} = (T_{[1]}T_{[2]}T_{[3]}T_{[4]})(T_{[2]})(T_{[2]})(T_{[6]}T_{[7]}) = T_{[1,5]}T_{[2,3]}^2T_{[6,8]}.$$

A reduced decomposition of T^λ is given $T_{[1,5]}T_{[2,3]}^2T_{[6,8]}$.

A decomposition $T^\lambda = T_{[i_1, j_1]}T_{[i_2, j_2]} \cdots T_{[i_l, j_l]}$ is said to be *square free* if and only if each generator occurs at most once. For instance, $T_{[1,2]}T_{[1,3]}T_{[2,5]}$ is square free but $T_{[1,2]}T_{[1,2]}T_{[2,3]}T_{[2,5]}$ is not square free. Also, note that $T_{[1,2]}T_{[1,3]}T_{[2,5]} = T_{[1,2]}T_{[1,2]}T_{[2,3]}T_{[2,5]}$ so that a given element in $\mathbb{Z}[G]_{\geq}$ may have both square free and not square free decompositions. Now, consider the operator $\hat{\tau}$ in $\mathbb{Z}[[G]]$ defined by

$$\hat{\tau} = \prod_{1 \leq i < j \leq n} \frac{1}{1 - T_{[i,j]}}. \tag{2}$$

Note that $\hat{\tau}$ is the analogue of τ for the generators $T_{[i,j]}$'s. Moreover, we recover

$$\hat{\tau} = \sum_{\lambda \in P_n} C(\lambda)T^\lambda,$$

with $C(\lambda)$ being the number of pairwise distinct presentations of $g = T^\lambda$ in terms of the generators $T_{[i,j]}$'s. Finally, we may write

$$\hat{H}_\alpha(x) = \hat{\tau} \cdot x^\alpha = \sum_{\beta \leq G\alpha} C(\lambda)x^\beta,$$

with λ denoting the unique element in P_n such that $\beta = T^\lambda(\alpha)$.

6 The graph \mathcal{L}_n and classical orthogonal polynomials

In this section we restrict our attention to a special family of graphs, denoted by $\{\mathcal{L}_n\}_n$, and defined by $\mathcal{L}_n = (V_n, E_n)$, with $V_n = \{v_1, v_2, \dots, v_n\}$ and $E_n = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}\}$, for all $n \geq 1$. Now, assume a linear functional $L: \mathbb{C}[t] \rightarrow \mathbb{C}$ is given. An *orthogonal polynomial system* associated with L is a polynomial sequence $\{p_n(t)\}_n$, with $\deg p_n = n$ for all $n \in \mathbb{N}$, such that $L p_n(t)p_m(t) = 0$ if and only if $n \neq m$. We refer to [4] for a background on this subject. The toppling game performed on

the \mathcal{L}_n 's gives rise to a nice combinatorial framework for orthogonal polynomial systems. Consider the unique $p_\alpha(x)$ such that $x^\alpha = \hat{\tau} \cdot p_\alpha(x)$, or equivalently

$$p_\alpha(x) = \hat{\tau}^{-1} \cdot x^\alpha. \quad (3)$$

Observe that

$$\hat{\tau}^{-1} = \prod_{1 \leq i < j \leq n} (1 - T_{[i,j]}) = \sum (-1)^l T_{[i_1, j_1]} T_{[i_2, j_2]} \cdots T_{[i_l, j_l]},$$

where the sum is taken over all square free decompositions in G . Because the size of a configuration is preserved by the toppling game, $p_\alpha(x)$ is a homogeneous polynomial of degree $|\alpha|$. Moreover, we have

$$p_\alpha(x) = \sum (-1)^l (T_{[i_1, j_1]} T_{[i_2, j_2]} \cdots T_{[i_l, j_l]} \cdot x^\alpha) = \sum_{\beta} p_{\alpha, \beta} x^\beta,$$

so that β ranges over all configurations that can be obtained from α by means of a square free Yamanouchi move $T_{[i_1, j_1]} T_{[i_2, j_2]} \cdots T_{[i_l, j_l]}$. Also, the coefficient $p_{\alpha, \beta}$ of x^β in $p_\alpha(x)$ has the following combinatorial description,

$$p_{\alpha, \beta} = \sum (-1)^l, \quad (4)$$

where l ranges over all lengths of all square free Yamanouchi moves changing α into β . In order to manipulate polynomials with an arbitrary large number of variables at the same time we set

$$\mathbb{C}[x_1, x_2, \dots] = \bigcup_{n \geq 1} \mathbb{C}[x_1, x_2, \dots, x_n].$$

Now, let $E: \mathbb{C}[x_1, x_2, \dots] \rightarrow \mathbb{C}$ denote a map such that

1. for all $n \geq 1$ the restriction $E: \mathbb{C}[x_1, x_2, \dots, x_n] \rightarrow \mathbb{C}$ is a linear functional;
2. for all $n \geq 1$, for all $p \in \mathbb{C}[x_1, x_2, \dots, x_n]$ and for all $\sigma \in \mathfrak{S}_n$, we have $Ep(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = Ep(x_1, x_2, \dots, x_n)$.

Henceforth, we will call any functional of this type a *symmetric functional*. Once a symmetric functional E is given then we may define *conditional operators* $E_i: \mathbb{C}[x_1, x_2, \dots] \rightarrow \mathbb{C}[x_i]$ for all $i \geq 1$. Each E_i is defined by $E_i x^\alpha = x_i^{\alpha_i} E x^\alpha x_i^{-\alpha_i}$ for all $\alpha \in \mathbb{N}^n$. Roughly speaking, each E_i acts on $\mathbb{C}[x_1, \dots, x_{i-1}, x_{i+1}, \dots]$ as E acts, and fixes each polynomial in $\mathbb{C}[x_i]$.

Remark 1 If X_1, X_2, \dots is an infinite sequence of i.i.d. random variables, and if \mathbf{E} is the expectation functional, then $\mathbf{E}: \mathbb{C}[X_1, X_2, \dots] \rightarrow \mathbb{C}$ gives an example of symmetric functional.

Set $\mathcal{G} = \mathcal{L}_n$ and choose $\alpha \in \mathbb{Z}^n$. In this case it is not too difficult to see that $T_{[i]} \cdot x^\alpha = x^\alpha x_{i+1} x_i^{-1}$ for all $i = 1, 2, \dots, n-1$. Hence, by means of straightforward computations we recover

$$\hat{\tau}^{-1} \cdot x^\alpha = \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_j}{x_i}\right) x^\alpha.$$

If $p_n(x)$ is defined as in (3) with $\alpha = (n-1, n-1, \dots, n-1)$, then we recover

$$p_n(x) = x_2 x_3^2 \cdots x_n^{n-1} \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Thus, if $E_1: \mathbb{C}[x_1, x_2, \dots] \rightarrow \mathbb{C}[x_1]$ is a conditional operator associated with a symmetric functional $E: \mathbb{C}[x_1, x_2, \dots] \rightarrow \mathbb{C}$, then we have

$$p_{n-1}(x_1) = E_1 p_n(x) = \sum_{k=0}^{n-1} x_1^k E x_2 x_3^2 \cdots x_n^{n-1} e_{n-k-1}(x_2, x_3, \dots, x_n) \prod_{1 \leq i < j \leq n} (x_i - x_j),$$

with $e_i(x_2, x_3, \dots, x_n)$ denoting the i th elementary symmetric polynomial in x_2, x_3, \dots, x_n . This implies that $p_{n-1}(x_1)$ is a polynomial in x_1 of degree at most $n - 1$. In particular, we have $\deg p_n = n - 1$ if and only if

$$E x_2 x_3^2 \cdots x_n^{n-1} \prod_{2 \leq i < j \leq n} (x_i - x_j) \neq 0 \text{ for all } n \geq 1. \tag{5}$$

On the other hand, since E is symmetric then we may replace each x_i with x_{i-1} in (5) without changing its value. Then we obtain

$$E x_1 x_2^2 \cdots x_{n-1}^n \prod_{1 \leq i < j \leq n-1} (x_i - x_j) = E(\hat{\tau}^{-1} \cdot (x_1 x_2 \cdots x_{n-1})^n) \neq 0.$$

This last condition, if satisfied for all n , will assure $\deg p_n = n$.

Theorem 10 (Orthogonal polynomial systems) *Let $E: \mathbb{C}[x_1, x_2, \dots] \rightarrow \mathbb{C}$ be a symmetric functional, set $\mathcal{G} = \mathcal{L}_n$, and assume*

$$E \hat{\tau}^{-1} \cdot (x_1 x_2 \cdots x_{n-1})^n \neq 0 \text{ for all } n \geq 1.$$

Then, the polynomial sequence $\{p_n(t)\}_n$ defined by

$$p_{n-1}(x_1) = E_1 \hat{\tau} \cdot (x_1 x_2 \cdots x_n)^{n-1} \neq 0 \text{ for all } n \geq 1,$$

satisfies $E p_n(x_1) p_m(x_1) = 0$ if and only $n \neq m$.

Theorem 10 gives us a general method to construct all orthogonal systems associated with a given linear functional L , provided they exist. In fact, given $L: \mathbb{C}[t] \rightarrow \mathbb{C}$, we consider the unique symmetric functional $E: \mathbb{C}[x_1, x_2, \dots] \rightarrow \mathbb{C}$ such that $E x_i^k = L t^k$, for all $i \geq 1$ and for all $k \geq 0$. Also, we set $a_\alpha = E x^\alpha$ for all $\alpha \in \mathbb{Z}^n$. Hence, an orthogonal polynomial system $\{p_n(t)\}_n$ associated with L exists if and only if $E \hat{\tau}^{-1}(x_1 x_2 \cdots x_{n-1})^n \neq 0$ for all $n \geq 1$. Moreover, it is obtained by setting $p_{n-1}(x_1) = E_1 \hat{\tau}^{-1}(x_1 x_2 \cdots x_n)^{n-1}$. So, the following combinatorial formula for $p_{n-1}(t)$ is provided:

$$p_{n-1}(t) = \sum_{\beta} p_{\alpha, \beta} a_{\beta_2, \dots, \beta_n} t^{\beta_1},$$

where $\alpha = (n - 1, n - 1, \dots, n - 1)$, β ranges over all configurations that can be obtained from α by means of square free Yamanouchi move, and $p_{\alpha, \beta}$ is the coefficient (4).

7 Statistics on the toppling group

In this section we focus our attention on the distribution of certain statistics $\ell_1, \ell_2, \ell_3, d$ defined on the decompositions of elements in the toppling group in terms of the three families of generators $\{T_i \mid i = 1, 2, \dots, n\}$, $\{T_{[i]} \mid i = 1, 2, \dots, n - 1\}$ and $\{T_{[i, j]} \mid 1 \leq i < j \leq n\}$. More precisely, each $T^\lambda \in G$,

$\lambda \in P_n$, admits a unique expression, up to order of the T_i 's, of type $g = T^\lambda$. This expression involves ℓ_1 generators, and in particular ℓ_1 equals the size of the partition λ . Analogously, T^λ can be written in a unique way, up to order, in terms of the $T_{[i]}$'s. In particular, we have $T^\lambda = T_{[\lambda'_1]}T_{[\lambda'_2]} \cdots$ and this involves $\ell_2 = \lambda_1$ generators. On the other hand, there are $C(\lambda)$ ways of writing T^λ as a product of generators $T_{[i,j]}$'s. Each of such expressions involves a certain number, say ℓ_3 , of generators. Among these $d \leq \ell_3$ are pairwise distinct.

Theorem 11 *Set*

$$\hat{\tau}(z_1, z_2, z_3, q) = \prod_{1 \leq i < j \leq n} \frac{1 - (1 - q)T_{[i,j]}z_3z_2^{j-i}z_1^{\binom{j}{2} - \binom{i}{2}}}{1 - T_{[i,j]}z_3z_2^{j-i}z_1^{\binom{j}{2} - \binom{i}{2}}}. \tag{6}$$

Then we have

$$\hat{\tau}(z_1, z_2, z_3, t) = \sum z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} q^d T_{[i_1, j_1]} T_{[i_2, j_2]} \cdots,$$

where the sum ranges over all pairwise distinct decompositions $T_{[i_1, j_1]} T_{[i_2, j_2]} \cdots$ in $\mathbb{Z}[G]_{\geq}$.

We may define a parametrized version of the series $\hat{\mathcal{H}}_\alpha(x)$ by setting

$$\hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q) = \hat{\tau}(z_1, z_2, z_3, q) \cdot x^\alpha. \tag{7}$$

We recover

$$\hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q) = \sum_{\alpha \rightarrow \gamma \beta} \left(\sum z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} q^d \right) x^\beta,$$

where the coefficient of x^β stores the value of $\ell_1, \ell_2, \ell_3, d$ relative to all pairwise distinct decompositions of the unique T^λ such that $\beta = T^\lambda(\alpha)$. As a consequence we obtain a parameterized extension of orthogonal polynomials. In fact, for all $n \geq 1$ and for all $\alpha \in \mathbb{Z}^n$ set $\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q) = \hat{\tau}(z_1, z_2, z_3, q)^{-1} \cdot x^\alpha$. Since $\hat{\tau}(z_1, z_2, z_3, q)^{-1} = \hat{\tau}(z_1, z_2, (1 - q)z_3, q(q - 1)^{-1})$ then the following combinatorial description is obtained:

$$\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q) = \sum_{\alpha \rightarrow \gamma \beta} \left(\sum z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} (1 - q)^{\ell_3 - d} (-q)^d \right) x^\beta.$$

Since $\ell_1 \geq \ell_2 \geq \ell_3 \geq d \geq 0$ then the coefficient of x^β in $\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q)$ is a polynomial with integer coefficients. By setting $z_1 = z_2 = z_3 = q = 1$ we have a non-zero contribution only for decompositions such that $\ell_3 - d = 0$, that is square free decompositions. At this point, one can consider a projection Π such that

$$\Pi x^\beta = \begin{cases} x^\beta & \text{if } \beta \in \mathbb{N}^n \\ 0 & \text{otherwise} \end{cases},$$

and then both $\{\Pi \hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q)\}_\alpha$ and $\{\Pi \hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q)\}_\alpha$, with α ranging over \mathbb{N}^n , are bases of $\mathbb{Z}[z_1, z_2, z_3, q][x]$. By defining a suitable symmetric functional E , and by setting $p_{n-1}(x_1; z_1, z_2, z_3, q) = E_1 \hat{\tau}(z_1, z_2, z_3, q)^{-1} \cdot (x_1 x_2 \cdots x_n)^{n-1}$ we obtain a polynomial sequence $\{p_n(t; z_1, z_2, z_3, q)\}_n$ that generalizes classical orthogonal polynomials.

Example 2 Hermite polynomials $\{H_n(t)\}_n$ forms an orthogonal polynomial system with respect to the linear functional L defined by $L t^k = \mathbf{E} X^k$, where X is a Gaussian random variable. Assume $X = X_1, X_2, \dots$ are independent Gaussian random variables and set $x_1 = t$ and $x_i = X_{i-1}$. Then the polynomial sequence $\{p_n(t; z_1, z_2, z_3, q)\}_n$ defined by

$$p_{n-1}(t; z_1, z_2, z_3, q) = \mathbf{E} \Pi \hat{\mathcal{H}}_\alpha \left(t, X_1, X_2, \dots, X_{n-1}; z_1, z_2, (1-q)z_3, \frac{q}{q-1} \right),$$

with $\alpha = (n-1, n-1, \dots, n-1)$ generalizes Hermite polynomials.

8 New combinatorics for H-L symmetric polynomials

The toppling group G acts on the ring $\Lambda(x; z_1, z_2, z_3, q)$, of symmetric polynomials in x_1, x_2, \dots, x_n with coefficients in $\mathbb{Z}[z_1, z_2, z_3, q]$. In fact, let $h_i(x)$ denote the i th complete homogeneous symmetric polynomial, and for all $\alpha \in \mathbb{N}^n$ set $h_\alpha(x) = h_{\alpha_1}(x)h_{\alpha_2}(x) \cdots h_{\alpha_n}(x)$. By assuming $h_i(x) = 0$ for $i < 0$ and $h_0(x) = 1$, we set

$$T_i \cdot h_\alpha(x) = h_{T_i(\alpha)}(x). \tag{8}$$

Finally, if α is a partition and if $s_\alpha(x)$ is a Schur polynomial then the symmetric polynomials $\hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q) = \hat{\tau}(z_1, z_2, z_3, q) \cdot s_\alpha(x)$ and $\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q) = \hat{\tau}^{-1}(z_1, z_2, z_3, q) \cdot s_\alpha(x)$ may be considered.

Theorem 12 For any graph \mathcal{G} the sets $\{\hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, t)\}_\alpha$ and $\{\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, t)\}_\alpha$, with α ranging over all integer partitions, are bases of the ring $\Lambda(x; z_1, z_2, z_3, q)$.

Again, the special case $\mathcal{G} = \mathcal{L}_n$ leads to some interesting consequences. In fact, we recover $T_{[i,j]} \cdot h_\alpha(x) = h_{\alpha - \epsilon_i + \epsilon_j}(x)$, so that $T_{[i,j]}$ acts as a “lowering operator” R_{ji} (see [9], p.214). In particular this implies

$$\hat{\mathcal{H}}_\alpha(x; z_1, z_2, z_3, q) = \prod_{1 \leq i < j \leq n} \left(\frac{1 - (1-q)z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}} R_{ji}}{1 - z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}} R_{ji}} \right) \cdot s_\alpha(x).$$

and

$$\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q) = \prod_{1 \leq i < j \leq n} \left(\frac{1 - z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}} R_{ji}}{1 - (1-q)z_3 z_2^{j-i} z_1^{\binom{j}{2} - \binom{i}{2}} R_{ji}} \right) \cdot s_\alpha(x).$$

Now, recall that the Hall-Littlewood symmetric polynomial $R_\alpha(x; t)$, α being a partition, satisfies

$$R_\alpha(x; t) = \prod_{1 \leq i < j \leq n} (1 - t R_{ji}) \cdot s_\alpha.$$

This implies

$$R_\alpha(x; t) = \hat{\mathcal{K}}_\alpha(x; 1, 1, t, 1) = \lim_{q \rightarrow 1} \hat{\mathcal{H}}_\alpha \left(x; 1, 1, (1-q)t, \frac{q}{q-1} \right).$$

Remark 2 Hence, $\hat{\mathcal{K}}_\alpha(x; z_1, z_2, z_3, q)$ extends to four parameters Hall-Littlewood symmetric polynomials, and then several classical bases of symmetric functions such as Schur functions ($t = 0$). From this point of view they share some analogy with Macdonald symmetric polynomials. Moreover, the toppling game also yields several families of symmetric functions, each one attached to a graph \mathcal{G} and then a general theory extending the classical case ($\mathcal{G} = \mathcal{L}_n$) could be carried out.

Remark 3 Lowering operators R_{j_i} 's have a counterpart R_{i_j} 's called "raising operators" [8, 9]. In our setting they will arise as generators, say the $T_{[j,i]}$'s, of the subalgebra $\mathbb{Z}[G]_{\leq}$ spanned over \mathbb{Z} by all T^λ , with $0 \leq \lambda_1 \leq \lambda_2 \dots \leq \lambda_n$.

References

- [1] M. Baker, S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*. Advances in Mathematics **215**, 766–788 (2007).
- [2] N. Biggs *Chip-firing and the critical group of a graph* J. of Algebraic Comb. **9**, 25–45 (1999).
- [3] A. Björner, L. Lovász, P. Schor, *Chip-firing game on graphs*. Europ. J. of Comb. **12**, 283–291 (1991).
- [4] T.S. Chihara, *An introduction to Orthogonal polynomials*. Mathematics and its applications - A series of monographs and texts **13**, (1978).
- [5] R. Cori, Y. Le Borgne, *The Riemann-Roch theorem for graphs and the rank in complete graphs*. Preprint, pp. 35 (2013).
- [6] D. Dhar, *Self-organized critical state of the sandpile automaton models*. Phys. Rev. Lett., **64**, 1613–1616 (1990.)
- [7] D. Dhar, P. Ruelle, S. Sen, and D. Verma. *Algebraic aspects of abelian sandpile model*. J. Phys. A **A28**, 805–831 (1995).
- [8] A. Garsia, *Orthogonality of Milne's polynomials and raising operators*. Discrete Mathematics **99**, 247–264 (1992).
- [9] I.G. Macdonald, *Symmetric functions and Hall polynomials*. Oxford Mathematical Monograph. Oxford University Press (1995).
- [10] I.G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*. Cambridge Tracts in Mathematics **157**, Cambridge University Press (2003).
- [11] G. Musiker, *The critical groups of a family of graphs and elliptic curves over finite fields*. Journal of Algebraic Combinatorics **30**, 255–276 (2009).
- [12] D. Perkinson, J. Perlman, and J. Wilmes. *Primer for the algebraic geometry of sandpiles*. preprint, arXiv:1112.6163, (2011).
- [13] P. Petrullo, D. Senato, R. Simone, *Orthogonal polynomials through the invarinat theory of binary forms*. Preprint (2014).
- [14] G.-C. Rota and B. Taylor, *The classical umbral calculus*. SIAM Journal of Mathematical Analysis **25**(2), 694–711 (1994).