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# 0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

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**Abstract.** We define an action of the 0-Hecke algebra of type A on the Stanley-Reisner ring of the Boolean algebra. By studying this action we obtain a family of multivariate noncommutative symmetric functions, which specialize to the noncommutative Hall-Littlewood symmetric functions and their  $(q, t)$ -analogues introduced by Bergeron and Zabrocki. We also obtain multivariate quasisymmetric function identities, which specialize to a result of Garsia and Gessel on the generating function of the joint distribution of five permutation statistics.

**Résumé.** Nous définissons une action de l'algèbre de Hecke-0 de type A sur l'anneau Stanley-Reisner de l'algèbre de Boole. En étudiant cette action, on obtient une famille de fonctions symétriques non commutatives multivariées, qui se spécialisent pour les non commutatives fonctions de Hall-Littlewood symétriques et leur  $(q, t)$ -analogues introduits par Bergeron et Zabrocki. Nous obtenons également des identités de fonction quasisymétrique multivariées, qui se spécialisent à la suite de Garsia et Gessel sur la fonction génératrice de la distribution conjointe de cinq statistiques de permutation.

**Keywords:** 0-Hecke algebra, Stanley-Reisner ring, Boolean algebra, noncommutative Hall-Littlewood symmetric function, multivariate quasisymmetric function.

## 1 Introduction

Let  $\mathbb{F}$  be any field. The symmetric group  $\mathfrak{S}_n$  naturally acts on the polynomial ring  $\mathbb{F}[X] := \mathbb{F}[x_1, \dots, x_n]$  by permuting the variables  $x_1, \dots, x_n$ . The invariant algebra  $\mathbb{F}[X]^{\mathfrak{S}_n}$ , which consists of all the polynomials fixed by this  $\mathfrak{S}_n$ -action, is a polynomial algebra generated by the elementary symmetric functions  $e_1, \dots, e_n$ . The coinvariant algebra  $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ , with  $(\mathbb{F}[X]_+^{\mathfrak{S}_n}) = (e_1, \dots, e_n)$ , is a vector space of dimension  $n!$  over  $\mathbb{F}$ , and when  $\mathbb{F}$  has characteristic larger than  $n$  the coinvariant algebra carries the regular representation of  $\mathfrak{S}_n$ . A well known basis for  $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$  consists of the descent monomials. Garsia [7] obtained this basis by transferring a natural basis from the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$  of the Boolean algebra  $\mathcal{B}_n$  to the polynomial ring  $\mathbb{F}[X]$ . Here the *Boolean algebra*  $\mathcal{B}_n$  is the set of all subsets of  $[n] := \{1, 2, \dots, n\}$  partially ordered by inclusion, and the *Stanley-Reisner ring*  $\mathbb{F}[\mathcal{B}_n]$  is the quotient of the polynomial algebra  $\mathbb{F}[y_A : A \subseteq [n]]$  by the ideal  $(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n)$ .

The 0-Hecke algebra  $H_n(0)$  (of type A) is a deformation of the group algebra of  $\mathfrak{S}_n$ . It acts on  $\mathbb{F}[X]$  by the Demazure operators, also known as the isobaric divided difference operators, having the same

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invariant algebra as the  $\mathfrak{S}_n$ -action on  $\mathbb{F}[X]$ . In our earlier work [13], we showed that the coinvariant algebra  $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$  is also isomorphic to the regular representation of  $H_n(0)$ , for any field  $\mathbb{F}$ , by constructing another basis for  $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$  which consists of certain polynomials whose leading terms are the descent monomials. This and the previously mentioned connection between the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$  and the polynomial ring  $\mathbb{F}[X]$  motivate us to define an  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$ .

It turns out that our  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$  has similar properties to the  $H_n(0)$ -action on  $\mathbb{F}[X]$ . It preserves an  $\mathbb{N}^{n+1}$ -multigrading of  $\mathbb{F}[\mathcal{B}_n]$  and has invariant algebra equal to a polynomial algebra  $\mathbb{F}[\Theta]$ , where  $\Theta$  is the set of *rank polynomials*  $\theta_i$  (the usual analogue of  $e_i$  in  $\mathbb{F}[\mathcal{B}_n]$ ). We show that the  $H_n(0)$ -action is  $\Theta$ -linear and thus descends to the coinvariant algebra  $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ . Using the descent monomials in  $\mathbb{F}[\mathcal{B}_n]$  it is not hard to see that  $\mathbb{F}[\mathcal{B}_n]/(\Theta)$  carries the regular representation of  $H_n(0)$ .

It is well known that every finite dimensional (complex)  $\mathfrak{S}_n$ -representation is a direct sum of simple (i.e. irreducible)  $\mathfrak{S}_n$ -modules, and the simple  $\mathfrak{S}_n$ -modules are indexed by partitions  $\lambda$  of  $n$ , which correspond to the Schur functions  $s_\lambda$  via the *Frobenius characteristic map*. Hotta-Springer [11] and Garsia-Procesi [9] discovered that the cohomology ring of the *Springer fiber* indexed by a partition  $\mu$  of  $n$  is isomorphic to certain quotient ring  $R_\mu$  of  $\mathbb{F}[X]$ , which admits a graded  $\mathfrak{S}_n$ -module structure corresponding to the *modified Hall-Littlewood symmetric function*  $\tilde{H}_\mu(x; t)$  via the Frobenius characteristic map. The coinvariant algebra of  $\mathfrak{S}_n$  is nothing but  $R_{1^n}$ .

In our previous work [13] we established a partial analogue of the above result by showing that the  $H_n(0)$ -action on  $\mathbb{F}[X]$  descends to  $R_\mu$  if and only if  $\mu = (1^k, n - k)$  is a hook, and if so then  $R_\mu$  has graded quasisymmetric characteristic equal to  $\tilde{H}_\mu(x; t)$  and graded noncommutative characteristic  $\tilde{\mathbf{H}}_\mu(\mathbf{x}; t)$ . Here  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$  is the noncommutative modified Hall-Littlewood symmetric function introduced by Bergeron and Zabrocki [3] for any composition  $\alpha$  of  $n$ . Using an analogue of the nabla operator Bergeron and Zabrocki [3] also introduced a  $(q, t)$ -analogue  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$  for any composition  $\alpha$ . Now we provide in Theorem 1.1 below a complete representation theoretic interpretation for  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$  and  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$  by the  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$ .

To state our result, we first recall the two characteristic maps for representations of  $H_n(0)$  introduced by Krob and Thibon [14], which we call the *quasisymmetric characteristic* and the *noncommutative characteristic*. The simple  $H_n(0)$ -modules are indexed by compositions  $\alpha$  of  $n$  and correspond to the *fundamental quasisymmetric functions*  $F_\alpha$  via the quasisymmetric characteristic; the projective indecomposable  $H_n(0)$ -modules are also indexed by compositions  $\alpha$  of  $n$  and correspond to the *noncommutative ribbon Schur functions*  $\mathbf{s}_\alpha$  via the noncommutative characteristic. See §2 for details.

**Theorem 1.1** *Let  $\alpha$  be a composition of  $n$ . Then there exists a homogeneous  $H_n(0)$ -invariant ideal  $I_\alpha$  of the multigraded algebra  $\mathbb{F}[\mathcal{B}_n]$  such that the quotient algebra  $\mathbb{F}[\mathcal{B}_n]/I_\alpha$  becomes a projective  $H_n(0)$ -module with multigraded noncommutative characteristic equal to*

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} \underline{t}^{D(\beta)} \mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t_1, \dots, t_{n-1}].$$

*One has  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t, t^2, \dots, t^{n-1}) = \tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ , and obtains  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$  from  $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$  by taking  $t_i = t^i$  for all  $i \in D(\alpha)$ , and  $t_i = q^{n-i}$  for all  $i \in [n-1] \setminus D(\alpha)$ .*

Here  $D(\alpha)$  is the set of partial sums of  $\alpha$ , the notation  $\beta \preceq \alpha$  means  $\alpha$  and  $\beta$  are compositions of  $n$  with  $D(\beta) \subseteq D(\alpha)$ , and  $\underline{t}^S$  denotes the product  $\prod_{i \in S} t_i$  over all elements  $i$  in a multiset  $S$ , including the repeated ones. Taking  $\alpha = (1^n)$  shows that  $\mathbb{F}[\mathcal{B}_n]/(\Theta)$  carries the regular representation of  $H_n(0)$ .

Specializations of  $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$  include not only  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ , but also a more general family of noncommutative symmetric functions depending on parameters associated with paths in binary trees introduced recently by Lascoux, Novelli, and Thibon [15].

Next we study the quasisymmetric characteristic of  $\mathbb{F}[\mathcal{B}_n]$ . We combine the usual  $\mathbb{N}^{n+1}$ -multigrading of  $\mathbb{F}[\mathcal{B}_n]$  (recorded by  $\underline{t} := t_0, \dots, t_n$ ) with the length filtration of  $H_n(0)$  (recorded by  $q$ ) and obtain an  $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic for  $\mathbb{F}[\mathcal{B}_n]$ .

**Theorem 1.2** *The  $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic of  $\mathbb{F}[\mathcal{B}_n]$  is*

$$\begin{aligned} \text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \text{Com}(n, k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\text{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\text{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

Here we identify  $F_I$  with  $F_\alpha$  if  $D(\alpha) = I \subseteq [n-1]$ . The set  $\text{Com}(n, k)$  consists of all *weak compositions of  $n$  with length  $k$* , i.e. all the sequences  $\alpha = (\alpha_1, \dots, \alpha_k)$  of  $k$  nonnegative integers with  $|\alpha| := \sum_{i=1}^k \alpha_i = n$ . The *descent multiset* of the weak composition  $\alpha$  is the *multiset*

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

We also define  $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$ . The set  $[k+1]^n$  consists of all words of length  $n$  on the alphabet  $[k+1]$ . Given  $\mathbf{p} = (p_1, \dots, p_n) \in [k+1]^n$ , we write  $p'_i := \#\{j : p_j \leq i\}$ ,  $\text{inv}(\mathbf{p}) := \#\{(i, j) : 1 \leq i < j \leq n : p_i > p_j\}$ , and  $D(\mathbf{p}) := \{i : p_i > p_{i+1}\}$ .

Let  $\mathbf{ps}_{q; \ell}(F_\alpha) := F_\alpha(1, q, q^2, \dots, q^{\ell-1}, 0, 0, \dots)$ . Applying the linear transformation  $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1; \ell+1}$  and the specialization  $t_i = q_2^i u_2$  for all  $i = 0, 1, \dots, n$  to Theorem 1.2, we recover a result of Garsia and Gessel [8, Theorem 2.2] on the generating function of the joint distribution of five permutation statistics:

$$\frac{\sum_{w \in \mathfrak{S}_n} q_0^{\text{inv}(w)} q_1^{\text{maj}(w^{-1})} u_1^{\text{des}(w^{-1})} q_2^{\text{maj}(w)} u_2^{\text{des}(w)}}{(u_1; q_1)_n (u_2; q_2)_n} = \sum_{\ell, k \geq 0} u_1^\ell u_2^k \sum_{(\lambda, \mu) \in B(\ell, k)} q_0^{\text{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|} \quad (1)$$

Here  $(u; q)_n := \prod_{0 \leq i \leq n} (1 - q^i u)$ , the set  $B(\ell, k)$  consists of pairs of weak compositions  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$  satisfying the conditions  $\ell \geq \lambda_1 \geq \cdots \geq \lambda_n$ ,  $\max\{\mu_i : 1 \leq i \leq n\} \leq k$ , and  $\lambda_i = \lambda_{i+1} \Rightarrow \mu_i \geq \mu_{i+1}$  (such pairs  $(\lambda, \mu)$  are sometimes called *bipartite partitions*), and  $\text{inv}(\mu)$  is the number of inversion pairs in  $\mu$ . Some further specializations of Theorem 1.2 imply identities of Carlitz-MacMahon [6, 17] and Adin-Brenti-Roichman [1].

The structure of this paper is as follows. Section 2 reviews the representation theory of the 0-Hecke algebra. Section 3 studies the Stanley-Reisner ring of the Boolean algebra. Section 4 defines a 0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra. The noncommutative and quasisymmetric characteristics are discussed in Section 5 and Section 6. Finally we give some remarks and questions for future research in Section 7, including a generalization to an action of the Hecke algebra of any finite Coxeter group on the Stanley-Reisner ring of the Coxeter complex.

## 2 Representation theory of the 0-Hecke algebra

We review the representation theory of the 0-Hecke algebra in this section. The (type A) *Hecke algebra*  $H_n(q)$  is the associative  $\mathbb{F}(q)$ -algebra generated by  $T_1, \dots, T_{n-1}$  with relations

$$\begin{cases} (T_i + 1)(T_i - q) = 0, & 1 \leq i \leq n-1, \\ T_i T_j = T_j T_i, & 1 \leq i, j \leq n-1, |i-j| > 1, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \leq i \leq n-2. \end{cases} \quad (2)$$

It has an  $\mathbb{F}(q)$ -basis  $\{T_w : w \in \mathfrak{S}_n\}$  where  $T_w := T_{i_1} \cdots T_{i_k}$  if  $w = s_{i_1} \cdots s_{i_k}$  is a reduced expression.

Specializing  $q = 1$  gives the group algebra of  $\mathfrak{S}_n$ , with  $s_i = T_i|_{q=1}$  and  $w = T_w|_{q=1}$ . Let  $w \in \mathfrak{S}_n$ . The *length* of  $w$  equals  $\text{inv}(w) := \#\{(i, j) : 1 \leq i < j \leq n, w(i) > w(j)\}$ , and the *descent set* of  $w$  is  $D(w) = \{i : 1 \leq i \leq n-1, w(i) > w(i+1)\}$ . We write  $\text{des}(w) := |D(w)|$  and  $\text{maj}(w) := \sum_{i \in D(w)} i$ .

Let  $\alpha$  be a (weak) composition of  $n$ , and let  $\alpha^c$  be the composition of  $n$  with  $D(\alpha^c) = [n-1] \setminus D(\alpha)$ . The *parabolic subgroup*  $\mathfrak{S}_\alpha$  is the subgroup of  $\mathfrak{S}_n$  generated by  $\{s_i : i \in D(\alpha^c)\}$ . The set of all minimal  $\mathfrak{S}_\alpha$ -coset representatives is  $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$ . The *descent class* of  $\alpha$  consists of the permutations in  $\mathfrak{S}_n$  with descent set equal to  $D(\alpha)$ , and turns out to be an interval under the (left) weak order of  $\mathfrak{S}_n$ , denoted by  $[w_0(\alpha), w_1(\alpha)]$ . One sees that  $w_0(\alpha)$  is the longest element of the parabolic subgroup  $\mathfrak{S}_{\alpha^c}$ , and  $w_1(\alpha)$  is the longest element in  $\mathfrak{S}^\alpha$  (c.f. Björner and Wachs [5, Theorem 6.2]).

Another interesting specialization of  $H_n(q)$  is the *0-Hecke algebra*  $H_n(0)$ , with generators  $\bar{\pi}_i = T_i|_{q=0}$  for  $i = 1, \dots, n-1$ , and an  $\mathbb{F}$ -basis  $\{\bar{\pi}_w = T_w|_{q=0} : w \in \mathfrak{S}_n\}$ . Let  $\pi_i := \bar{\pi}_i + 1$ . Then  $\pi_1, \dots, \pi_{n-1}$  form another generating set for  $H_n(0)$ , with the same relations as (2) except  $\pi_i^2 = \pi_i$ ,  $1 \leq i \leq n-1$ . The element  $\pi_w := \pi_{i_1} \cdots \pi_{i_k}$  is well defined for any  $w \in \mathfrak{S}_n$  with a reduced expression  $w = s_{i_1} \cdots s_{i_k}$ , and  $\{\pi_w : w \in \mathfrak{S}_n\}$  is another  $\mathbb{F}$ -basis for  $H_n(0)$ . One can check that  $\pi_w$  equals the sum of  $\bar{\pi}_u$  over all  $u$  less than or equal to  $w$  in the Bruhat order of  $\mathfrak{S}_n$ . In particular,  $\pi_{w_0(\alpha)}$  is the sum of  $\bar{\pi}_u$  for all  $u \in \mathfrak{S}_{\alpha^c}$ .

Norton [18] decomposed the 0-Hecke algebra  $H_n(0)$  into a direct sum of projective indecomposable submodules  $\mathbf{P}_\alpha := H_n(0) \cdot \bar{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)}$  for all  $\alpha \models n$  (i.e. compositions of  $n$ ). Each  $\mathbf{P}_\alpha$  has an  $\mathbb{F}$ -basis  $\{\bar{\pi}_w \pi_{w_0(\alpha^c)} : w \in [w_0(\alpha), w_1(\alpha)]\}$ . Its *radical*  $\text{rad } \mathbf{P}_\alpha$  is the unique maximal  $H_n(0)$ -submodule spanned by  $\{\bar{\pi}_w \pi_{w_0(\alpha^c)} : w \in (w_0(\alpha), w_1(\alpha)]\}$ . Although  $\mathbf{P}_\alpha$  itself is not necessarily simple, its *top*  $\mathbf{C}_\alpha := \mathbf{P}_\alpha / \text{rad } \mathbf{P}_\alpha$  is a one-dimensional simple  $H_n(0)$ -module with the action of  $H_n(0)$  given by

$$\bar{\pi}_i = \begin{cases} -1, & \text{if } i \in D(\alpha), \\ 0, & \text{if } i \notin D(\alpha). \end{cases}$$

It follows from general representation theory of algebras (see e.g. [2, §I.5]) that  $\{\mathbf{P}_\alpha : \alpha \models n\}$  and  $\{\mathbf{C}_\alpha : \alpha \models n\}$  are the complete lists of pairwise non-isomorphic projective indecomposable and simple  $H_n(0)$ -modules, respectively.

Krob and Thibon [14] introduced a correspondence between  $H_n(0)$ -representations and the dual Hopf algebras  $\text{QSym}$  and  $\text{NSym}$ , which we review next. The Hopf algebra  $\text{QSym}$  has a free  $\mathbb{Z}$ -basis of *fundamental quasisymmetric functions*  $F_\alpha$ , and the dual Hopf algebra  $\text{NSym}$  has a dual basis of *non-commutative ribbon Schur functions*  $\mathfrak{s}_\alpha$ , for all compositions  $\alpha$ .

Let  $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq M_{k+1} = 0$  be a composition series of  $H_n(0)$ -modules with simple factors  $M_i/M_{i+1} \cong \mathbf{C}_{\alpha^{(i)}}$  for  $i = 0, 1, \dots, k$ . Then the *quasisymmetric characteristic* of  $M$  is

$$\text{Ch}(M) := F_{\alpha^{(0)}} + \cdots + F_{\alpha^{(k)}}.$$

The *noncommutative characteristic* of a projective  $H_n(0)$ -module  $M \cong \mathbf{P}_{\alpha(1)} \oplus \cdots \oplus \mathbf{P}_{\alpha(k)}$  is

$$\mathbf{ch}(M) := \mathbf{s}_{\alpha(1)} + \cdots + \mathbf{s}_{\alpha(k)}.$$

It is not hard to extend these characteristic maps to  $H_n(0)$ -modules with gradings and filtrations.

### 3 Stanley-Reisner ring of the Boolean algebra

In this section we study the Stanley-Reisner ring of the Boolean algebra. The *Boolean algebra*  $\mathcal{B}_n$  is the ranked poset of all subsets of  $[n] := \{1, 2, \dots, n\}$  ordered by inclusion, with minimum element  $\emptyset$  and maximum element  $[n]$ . The rank of a subset of  $[n]$  is defined as its cardinality. The *Stanley-Reisner ring*  $\mathbb{F}[\mathcal{B}_n]$  of the Boolean algebra  $\mathcal{B}_n$  is the quotient of the polynomial algebra  $\mathbb{F}[y_A : A \subseteq [n]]$  by the ideal  $(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n)$ . It has an  $\mathbb{F}$ -basis  $\{y_M\}$  indexed by the multichains  $M$  in  $\mathcal{B}_n$ , and is multigraded by the rank multisets  $r(M)$  of the multichains  $M$ .

The symmetric group  $\mathfrak{S}_n$  acts on the Boolean algebra  $\mathcal{B}_n$  by permuting the integers  $1, \dots, n$ . This induces an  $\mathfrak{S}_n$ -action on the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$ , preserving its multigrading. The *invariant algebra*  $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$  consists of all elements in  $\mathbb{F}[\mathcal{B}_n]$  invariant under this  $\mathfrak{S}_n$ -action. One can show that  $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n} = \mathbb{F}[\Theta]$ , where  $\Theta := \{\theta_0, \dots, \theta_n\}$ . Garsia [7] showed that  $\mathbb{F}[\mathcal{B}_n]$  is a free  $\mathbb{F}[\Theta]$ -module on the basis of descent monomials

$$Y_w := \prod_{i \in D(w)} y_{\{w(1), \dots, w(i)\}}, \quad \forall w \in \mathfrak{S}_n. \quad (3)$$

There is an analogy between the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$  and the polynomial ring  $\mathbb{F}[X]$  via the *transfer map*  $\tau : \mathbb{F}[\mathcal{B}_n] \rightarrow \mathbb{F}[X]$  defined by

$$\tau(y_M) := \prod_{1 \leq i \leq k} \prod_{j \in A_i} x_j$$

for all multichains  $M = (A_1 \subseteq \cdots \subseteq A_k)$  in  $\mathcal{B}_n$ . It is *not* a ring homomorphism (e.g.  $y_{\{1\}} y_{\{2\}} = 0$  but  $x_1 x_2 \neq 0$ ). Nevertheless, it induces an isomorphism  $\tau : \mathbb{F}[\mathcal{B}_n]/(\theta_0) \cong \mathbb{F}[X]$  of  $\mathfrak{S}_n$ -modules. Moreover, it sends the rank polynomials  $\theta_1, \dots, \theta_n$  to the elementary symmetric polynomials  $e_1, \dots, e_n$ , and sends the descent monomials  $Y_w$  in  $\mathbb{F}[\mathcal{B}_n^*]$  defined by (3) to the corresponding descent monomials in  $\mathbb{F}[X]$  for all  $w \in \mathfrak{S}_n$ .

**Example 3.1** *The Boolean algebra  $\mathcal{B}_3$  consists of all subsets of  $\{1, 2, 3\}$ . Its Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_3]$  is a free  $\mathbb{F}[\Theta]$ -module with a basis of descent monomials  $Y_1 = 1$ ,  $Y_{s_1} = y_2$ ,  $Y_{s_2 s_1} = y_3$ ,  $Y_{s_2} = y_{13}$ ,  $Y_{s_1 s_2} = y_{23}$ ,  $Y_{s_1 s_2 s_1} = y_{23} y_3$ , where  $\Theta$  consists of the rank polynomials  $\theta_0 = y_\emptyset$ ,  $\theta_1 := y_1 + y_2 + y_3$ ,  $\theta_2 := y_{12} + y_{13} + y_{23}$ ,  $\theta_3 := y_{123}$ . The transfer map  $\tau$  sends  $\theta_1, \theta_2, \theta_3$  to  $e_1, e_2, e_3$ , and sends the six descent monomials in  $\mathbb{F}[\mathcal{B}_3]$  to the six descent monomials  $1, x_2, x_3, x_1 x_3, x_2 x_3, x_2 x_3^2$  in  $\mathbb{F}[x_1, x_2, x_3]$ .*

The homogeneous components of  $\mathbb{F}[\mathcal{B}_n]$  are indexed by multisets with elements in  $\{0, \dots, n\}$ , or equivalently by weak compositions  $\alpha$  of  $n$ . The  $\alpha$ -homogeneous component  $\mathbb{F}[\mathcal{B}_n]_\alpha$  has an  $\mathbb{F}$ -basis  $\{y_M : r(M) = D(\alpha)\}$ . Denote by  $\text{Com}(n, k)$  the set of all weak compositions of  $n$  with length  $k$ . If  $M = (A_1 \subseteq \cdots \subseteq A_k)$  is a multichain of length  $k$  in  $\mathcal{B}_n$  then we set  $A_0 := \emptyset$  and  $A_{k+1} := [n]$  by convention. Define  $\alpha(M) := (\alpha_1, \dots, \alpha_{k+1})$ , where  $\alpha_i = |A_i| - |A_{i-1}|$  for all  $i \in [k+1]$ . Then

$\alpha(M) \in \text{Com}(n, k+1)$  and  $D(\alpha(M)) = r(M)$ , i.e.  $\alpha(M)$  indexes the homogeneous component containing  $y_M$ . Define  $\sigma(M)$  to be the minimal element in  $\mathfrak{S}_n$  which sends the standard multichain  $[\alpha_1] \subseteq [\alpha_1 + \alpha_2] \subseteq \cdots \subseteq [\alpha_1 + \cdots + \alpha_k]$  with rank multiset  $D(\alpha(M))$  to  $M$ . Then  $\sigma(M) \in \mathfrak{S}^{\alpha(M)}$ .

The map  $M \mapsto (\alpha(M), \sigma(M))$  is a bijection between multichains of length  $k$  in  $\mathcal{B}_n$  and the pairs  $(\alpha, \sigma)$  of  $\alpha \in \text{Com}(n, k+1)$  and  $\sigma \in \mathfrak{S}^\alpha$ . A short way to write down this encoding of  $M$  is to insert bars at the descent positions of  $\sigma(M)$ . For example, the length-4 multichain  $\{2\} \subseteq \{2\} \subseteq \{1, 2, 4\} \subseteq [4]$  in  $\mathcal{B}_4$  is encoded by  $2||14|3|$ .

There is another way to encode the multichain  $M$ . Let  $p_i(M) := \min\{j \in [k+1] : i \in A_j\}$ ,  $1 \leq i \leq n$ . So  $p_i(M)$  is the first position where  $i$  appears in  $M$ . One checks that

$$\begin{cases} p_i(M) > p_{i+1}(M) \Leftrightarrow i \in D(\sigma(M)^{-1}), \\ p_i(M) = p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}), D(s_i\sigma(M)) \not\subseteq D(\alpha(M)), \\ p_i(M) < p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}), D(s_i\sigma(M)) \subseteq D(\alpha(M)). \end{cases} \quad (4)$$

The map  $M \mapsto p(M) := (p_1(M), \dots, p_n(M))$  is an bijection between the set of multichains with length  $k$  in  $\mathcal{B}_n$  and the set  $[k+1]^n$  of all words of length  $n$  on the alphabet  $[k+1]$ , for any fixed integer  $k \geq 0$ .

Let  $p(M) = \mathbf{p} = (p_1, \dots, p_n) \in [k+1]^n$ . Then  $\text{inv}(p(M)) = \text{inv}(\sigma(M))$ . Let  $\mathbf{p}' := (p'_1, \dots, p'_k)$  where  $p'_i := |\{j : p_j(M) \leq i\}| = |A_i|$ . Then the rank multiset of  $M$  consists of  $p'_1, \dots, p'_k$ . Define  $D(\mathbf{p}) := \{i \in [n-1] : p_i > p_{i+1}\}$ . For example, the multichain  $3|14||2|5$  corresponds to  $\mathbf{p} = (2, 4, 1, 2, 5) \in [5]^5$ , and one has  $\mathbf{p}' = (1, 3, 3, 4)$ ,  $D(2, 5, 1, 2, 4) = \{2\}$ .

These two encodings (with slightly different notation) were already used by Garsia and Gessel [8] in their work on generating functions of multivariate distributions of permutation statistics.

## 4 0-Hecke algebra action

We saw an analogy between  $\mathbb{F}[\mathcal{B}_n]$  and  $\mathbb{F}[X]$  in the last section. The usual  $H_n(0)$ -action on the polynomial ring  $\mathbb{F}[X]$  is via the *Demazure operators*

$$\bar{\pi}_i(f) := \frac{x_{i+1}f - x_{i+1}s_i f}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[X], 1 \leq i \leq n-1. \quad (5)$$

The above definition is equivalent to

$$\bar{\pi}_i(x_i^a x_{i+1}^b m) = \begin{cases} (x_i^{a-1} x_{i+1}^{b+1} + x_i^{a-2} x_{i+1}^{b+2} \cdots + \underline{x_i^b x_{i+1}^a})m, & \text{if } a > b, \\ 0, & \text{if } a = b, \\ -(x_i^a x_{i+1}^b + x_i^{a+1} x_{i+1}^{b-1} + \cdots + x_i^{b-1} x_{i+1}^{a+1})m, & \text{if } a < b. \end{cases} \quad (6)$$

Here  $m$  is any monomial in  $\mathbb{F}[X]$  containing neither  $x_i$  nor  $x_{i+1}$ . Denote by  $\bar{\pi}'_i$  the operator obtained from (6) by taking only the leading term (underlined) in the lexicographic order of the result. Then  $\bar{\pi}'_1, \dots, \bar{\pi}'_{n-1}$  realize another  $H_n(0)$ -action on  $\mathbb{F}[X]$ . We call it the *transferred  $H_n(0)$ -action* because it can be obtained by applying the transfer map  $\tau$  to our  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$ , which we now define.

Let  $M = (A_1 \subseteq \cdots \subseteq A_k)$  be a multichain in  $\mathcal{B}_n$ . Recall that  $p_i(M) := \min\{j \in [k+1] : i \in A_j\}$ ,  $1 \leq i \leq n$ . We define

$$\bar{\pi}_i(y_M) := \begin{cases} -y_M, & p_i(M) > p_{i+1}(M), \\ 0, & p_i(M) = p_{i+1}(M), \\ s_i(y_M), & p_i(M) < p_{i+1}(M) \end{cases} \quad (7)$$

for  $i = 1, \dots, n - 1$ . Applying the transfer map  $\tau$  one recovers  $\bar{\pi}'_i$ . For instance, when  $n = 4$  one has

$$\begin{aligned} \bar{\pi}_1(y_{1|34||2|}) &= y_{2|34||1|}, & \bar{\pi}'_1(x_1^4 x_2 x_3^3 x_4^3) &= x_1 x_2^4 x_3^3 x_4^3, \\ \bar{\pi}_2(y_{1|34||2|}) &= -y_{1|34||2|}, & \bar{\pi}'_2(x_1^4 x_2 x_3^3 x_4^3) &= -x_1^4 x_2 x_3^3 x_4^3, \\ \bar{\pi}_3(y_{1|34||2|}) &= 0, & \bar{\pi}'_3(x_1^4 x_2 x_3^3 x_4^3) &= 0. \end{aligned}$$

One can check that  $\bar{\pi}_1, \dots, \bar{\pi}_{n-1}$  realize an  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$  preserving the multigrading of  $\mathbb{F}[\mathcal{B}_n]$ . If one set  $t_i = t^i$  for  $i = 1, \dots, n$ , then there is an isomorphism  $\mathbb{F}[\mathcal{B}_n]/(\emptyset) \cong \mathbb{F}[X]$  of graded  $H_n(0)$ -modules (which can be given explicitly, but *not* via the transfer map  $\tau$ ).

It is not hard to show that  $\mathbb{F}[\mathcal{B}_n]^{H_n(0)} = \mathbb{F}[\Theta]$ , where  $\mathbb{F}[\mathcal{B}_n]^{H_n(0)}$  is the *invariant algebra* of the  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$ , defined as

$$\mathbb{F}[\mathcal{B}_n]^{H_n(0)} := \{f \in \mathbb{F}[\mathcal{B}_n] : \pi_i f = f, i = 1, \dots, n - 1\}.$$

**Proposition 4.1** *The  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$  is  $\Theta$ -linear.*

Therefore the *coinvariant algebra*  $\mathbb{F}[\mathcal{B}_n]/(\Theta)$  is a multigraded  $H_n(0)$ -module, which is isomorphic to the regular representation of  $H_n(0)$  by Theorem 1.1. This cannot be obtained simply by applying the transfer map  $\tau$ , since  $\tau$  is *not* a map of  $H_n(0)$ -modules.

## 5 Noncommutative characteristic

In this section we use the  $H_n(0)$ -action on the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$  of the Boolean algebra  $\mathcal{B}_n$  to provide a noncommutative analogue of the following remarkable result.

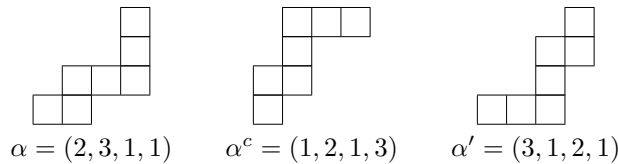
**Theorem 5.1 (Hotta-Springer [11], Garsia-Procesi [9])** *For any partition  $\mu = (0 < \mu_1 \leq \dots \leq \mu_k)$  of  $n$ , there exists an  $\mathfrak{S}_n$ -invariant ideal  $J_\mu$  of  $\mathbb{C}[X]$  such that  $\mathbb{C}[X]/J_\mu$  is isomorphic to the cohomology ring of the Springer fiber indexed by  $\mu$  and has graded Frobenius characteristic equal to the modified Hall-Littlewood symmetric function*

$$\tilde{H}_\mu(X; t) = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda \quad \text{inside } \text{Sym}[t]$$

where  $n(\mu) := \mu_{k-1} + 2\mu_{k-2} + \dots + (k-1)\mu_1$  and  $K_{\lambda\mu}(t)$  is the Kostka-Foulkes polynomial.

**Example 5.2** *Tanisaki [19] gives a construction for the ideal  $J_\mu$ . If  $\mu = (1^k, n - k)$  is a hook then  $J_{1^k, n-k}$  is generated by  $e_1, \dots, e_k$  and all monomials  $x_{i_1} \dots x_{i_{k+1}}$  with  $1 \leq i_1 < \dots < i_{k+1} \leq n$ .*

Now consider a composition  $\alpha = (\alpha_1, \dots, \alpha_\ell)$ . The major index of  $\alpha$  is  $\text{maj}(\alpha) := \sum_{i \in D(\alpha)} i$ . Viewing a partition  $\mu = (0 < \mu_1 \leq \mu_2 \leq \dots)$  as a composition one has  $\text{maj}(\mu) = n(\mu)$ . Recall that  $\overleftarrow{\alpha} := (\alpha_\ell, \dots, \alpha_1)$  and  $\alpha^c$  is the composition of  $n$  with  $D(\alpha^c) = [n - 1] \setminus D(\alpha)$ . We define  $\alpha' := \overleftarrow{\alpha^c} = (\overleftarrow{\alpha})^c$ . One can identify  $\alpha$  with a *ribbon diagram*, i.e. a connected skew Young diagram without 2 by 2 boxes, which has row lengths  $\alpha_1, \dots, \alpha_\ell$ , ordered from bottom to top. Note that a ribbon diagram is a Young diagram if and only if it is a hook. One can check that  $\alpha'$  is the transpose of  $\alpha$ ; see the example below.





Bergeron and Zabrocki [3] introduced a noncommutative modified Hall-Littlewood symmetric function

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t) := \sum_{\beta \preceq \alpha} t^{\text{maj}(\beta)} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[t] \quad (8)$$

and a  $(q, t)$ -analogue

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t) := \sum_{\beta \models n} t^{c(\alpha, \beta)} q^{c(\alpha', \bar{\beta})} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[q, t] \quad (9)$$

for every composition  $\alpha$ , where  $\mathbf{s}_\beta$  is the noncommutative ribbon Schur function indexed by  $\beta$ , and  $c(\alpha, \beta) := \sum_{i \in D(\alpha) \cap D(\beta)} i$ . In our earlier work [13] we provided a partial representation theoretic interpretation for  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$  when  $\alpha = (1^k, n - k)$  is a hook, using the  $H_n(0)$ -action on the polynomial ring  $\mathbb{F}[X]$  by the Demazure operators.

**Theorem 5.3 ([13])** *The ideal  $J_\mu$  of  $\mathbb{F}[X]$  is  $H_n(0)$ -invariant if and only if  $\mu = (1^{n-k}, k)$  is a hook, and if that holds then  $\mathbb{F}[X]/J_\mu$  becomes a graded projective  $H_n(0)$ -module with*

$$\begin{aligned} \text{ch}_t(\mathbb{F}[X]/J_\mu) &= \tilde{\mathbf{H}}_\mu(\mathbf{x}; t), \\ \text{Ch}_t(\mathbb{F}[X]/J_\mu) &= \tilde{H}_\mu(\mathbf{x}; t). \end{aligned}$$

Now we switch to the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$  and define  $I_\alpha$  to be its ideal generated by

$$\Theta_\alpha := \{\theta_i : i \in D(\alpha) \cup \{n\}\} \quad \text{and} \quad \{y_A : A \subseteq [n], |A| \notin D(\alpha) \cup \{n\}\}$$

for any composition  $\alpha$  of  $n$ . The following result is a restatement of Theorem 1.1.

**Theorem 5.4** *Let  $\alpha$  be a composition of  $n$ . Then  $\mathbb{F}[\mathcal{B}_n]/I_\alpha$  is a projective  $H_n(0)$ -module with multi-graded noncommutative characteristic equal to*

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} t^{D(\beta)} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[t_1, \dots, t_{n-1}].$$

One has  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t, t^2, \dots, t^{n-1}) = \tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ , and one obtains  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$  from  $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$  by taking  $t_i = t^i$  for all  $i \in D(\alpha)$ , and  $t_i = q^{n-i}$  for all  $i \in [n-1] \setminus D(\alpha)$ .

**Proof:** There is an  $\mathbb{F}$ -basis for  $\mathbb{F}[\mathcal{B}_n]/(\Theta_\alpha)$  given by the descent monomials  $Y_w$  defined in (3) for all  $w \in \mathfrak{S}^\alpha$ . The result follows from the  $H_n(0)$ -action on this basis and (4).  $\square$

The proof of this theorem is actually simpler than the proof of our partial interpretation for  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$  in [13]. This is because  $\bar{\pi}_i$  sends a descent monomial in  $\mathbb{F}[\mathcal{B}_n]$  to either 0 or  $\pm 1$  times a descent monomial, but sends a descent monomial in  $\mathbb{F}[X]$  to a polynomial in general (whose leading term is still a descent monomial). We view the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$  (or  $\mathbb{F}[\mathcal{B}_n]/(\emptyset)$ ) as a  $q = 0$  analogue of the polynomial ring  $\mathbb{F}[X]$ . For an odd (i.e.  $q = -1$ ) analogue, see Lauda and Russell [16].

**Remark 5.5** *If  $\alpha = (1^k, n - k)$  is a hook, one can check that the ideal  $I_{1^k, n-k}$  of  $\mathbb{F}[\mathcal{B}_n]$  has generators  $\theta_1, \dots, \theta_k$  and all  $y_A$  with  $A \subseteq [n]$  and  $|A| \notin [k]$ . By Example 5.2, the images of these generators under the transfer map  $\tau$  are the Tanisaki generators for the ideal  $J_{1^k, n-k}$  of  $\mathbb{F}[X]$ , but  $\tau(I_{1^k, n-k}) \neq J_{1^k, n-k}$ .*

For any composition  $\alpha \models n$ , one can view  $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1})$  as a modified version of

$$\mathbf{H}_\alpha = \mathbf{H}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} \underline{t}^{D(\alpha) \setminus D(\beta)} \mathbf{s}_\beta.$$

Below are some properties satisfied by  $\mathbf{H}_\alpha$ , generalizing the properties of  $\mathbf{H}_\alpha(\mathbf{x}; t)$  given in [3].

**Proposition 5.6** *Let  $\alpha$  and  $\beta$  be two compositions.*

(i)  $\mathbf{H}_\alpha(0, \dots, 0) = \mathbf{s}_\alpha$ ,  $\mathbf{H}_\alpha(1, \dots, 1) = \mathbf{h}_\alpha$ .

(ii)  $\bigcup_{n \geq 0} \{\mathbf{H}_\alpha : \alpha \models n\}$  is a basis for  $\mathbf{NSym}[t_1, t_2, \dots]$ .

(iii)  $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha, \beta^c}$  for any pair of compositions  $\alpha$  and  $\beta$ .

(iv)

$$\mathbf{H}_\alpha \cdot \mathbf{H}_\beta = \sum_{\gamma \preceq \beta} \left( \prod_{i \in D(\beta) \setminus D(\gamma)} (t_i - t_{|\alpha|+i}) \right) (\mathbf{H}_{\alpha\gamma} + (1 - t_{|\alpha|}) \mathbf{H}_{\alpha \triangleright \gamma}).$$

(v) If  $n = |\alpha|$  and  $t|n := (t_1, \dots, t_{n-1}, 1, t_1, \dots, t_{n-1}, 1, \dots)$  then

$$\mathbf{H}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) \mathbf{H}_\beta(\mathbf{x}; t|n) = \mathbf{H}_{\alpha\beta}(t|n).$$

## 6 Quasisymmetric characteristic

Now we study the quasisymmetric characteristic of  $\mathbb{F}[\mathcal{B}_n]$ . The following lemma follows easily from (4).

**Lemma 6.1** *Let  $\alpha$  be a weak composition of  $n$ . Then the  $\alpha$ -homogeneous component  $\mathbb{F}[\mathcal{B}_n]_\alpha$  of the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$  is an  $H_n(0)$ -submodule of  $\mathbb{F}[\mathcal{B}_n]$  with homogeneous multigrading  $\underline{t}^{D(\alpha)}$  and isomorphic to the cyclic module  $H_n(0)\pi_{w_0(\alpha^c)}$ .*

Since  $\mathbb{F}[\mathcal{B}_n]_\alpha$  is a cyclic multigraded  $H_n(0)$ -module, we get an  $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic

$$\text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]_\alpha) = \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} \underline{t}^{D(\alpha)} F_{D(w^{-1})} \quad (10)$$

where  $q$  keeps track of the length filtration and  $\underline{t}$  keeps track of the multigrading of  $\mathbb{F}[\mathcal{B}_n]_\alpha$ . This defines an  $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic for the Stanley-Reisner ring  $\mathbb{F}[\mathcal{B}_n]$ , which is explicitly given in Theorem 1.2 and restated below.

**Theorem 6.2** *The  $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic of  $\mathbb{F}[\mathcal{B}_n]$  is*

$$\begin{aligned} \text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \text{Com}(n, k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\text{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\text{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

**Proof:** Use the two encodings of the multichains in  $\mathcal{B}_n$  as well as the free  $\mathbb{F}[\Theta]$ -basis  $\{Y_w : w \in \mathfrak{S}_n\}$  of descent monomials for  $\mathbb{F}[\mathcal{B}_n]$  discussed in Section 3.  $\square$

Next we explain here how this theorem specializes to (1), a result of Garsia and Gessel [8, Theorem 2.2] on the multivariate generating function of the permutation statistics  $\text{inv}(w)$ ,  $\text{maj}(w)$ ,  $\text{des}(w)$ ,  $\text{maj}(w^{-1})$ , and  $\text{des}(w^{-1})$  for all  $w \in \mathfrak{S}_n$ . First recall that

$$F_\alpha = \sum_{\substack{i_1 \geq \dots \geq i_n \geq 1 \\ i \in D(\alpha) \Rightarrow i_j > i_{j+1}}} x_{i_1} \cdots x_{i_n}, \quad \forall \alpha \models n.$$

Let  $\mathbf{ps}_{q;\ell}(F_\alpha) := F_\alpha(1, q, q^2, \dots, q^{\ell-1}, 0, 0, \dots)$ , and  $(u; q)_n := (1-u)(1-qu)(1-q^2u) \cdots (1-q^nu)$ . It is not hard to check (see Gessel and Reutenauer [10, Lemma 5.2]) that

$$\sum_{\ell \geq 0} u^\ell \mathbf{ps}_{q;\ell+1}(F_\alpha) = \frac{q^{\text{maj}(\alpha)} u^{\text{des}(\alpha)}}{(u; q)_n}.$$

Then applying the linear transformation  $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$  and the specialization  $t_i = q_2^i u_2$  for all  $i = 0, 1, \dots, n$  to Theorem 6.2 we recover (1).

A further specialization of Theorem 6.2 gives a well known result which is often attributed to Carlitz [6] but actually dates back to MacMahon [17, Volume 2, Chapter 4].

**Corollary 6.3 (Carlitz-MacMahon)** *Let  $[k+1]_q := 1 + q + q^2 + \dots + q^k$ . Then*

$$\frac{\sum_{w \in \mathfrak{S}_n} q^{\text{maj}(w)} u^{\text{des}(w)}}{(u; q)_n} = \sum_{k \geq 0} ([k+1]_q)^n u^k.$$

Theorem 6.2 also implies the following result, which was obtained by Adin, Brenti, and Roichman [1] from the Hilbert series of the coinvariant algebra  $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ .

**Corollary 6.4 (Adin-Brenti-Roichman)** *Let  $\text{Par}(n)$  be the set of weak partitions  $\lambda = (\lambda_1, \dots, \lambda_n)$  with  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ , and let  $m(\lambda) = (m_0(\lambda), m_1(\lambda), \dots)$ , where  $m_j(\lambda) := \#\{1 \leq i \leq n : \lambda_i = j\}$ . Then*

$$\sum_{\lambda \in \text{Par}(n)} \binom{n}{m(\lambda)} \prod_{i=1}^n q_i^{\lambda_i} = \frac{\sum_{w \in \mathfrak{S}_n} \prod_{i \in D(w)} q_1 \cdots q_i}{(1-q_1)(1-q_1q_2) \cdots (1-q_1 \cdots q_n)}.$$

## 7 Remarks and questions for future research

### 7.1 Hecke algebra action

It is well known that the symmetric group  $\mathfrak{S}_n$  is the Coxeter group of type  $A_{n-1}$ . The Stanley-Reisner ring of  $\mathcal{B}_n$  is essentially the Stanley-Reisner ring of the Coxeter complex of  $\mathfrak{S}_n$ . The Hecke algebra  $H_W(q)$  can be defined for any finite Coxeter group  $W$ . We can generalize our action  $H_n(0)$ -action on  $\mathbb{F}[\mathcal{B}_n]$  to an  $H_W(q)$ -action on the Stanley-Reisner ring  $\mathbb{F}(q)[\Delta(W)]$  of the Coxeter complex  $\Delta(W)$  of any finite Coxeter group  $W$ . We show similar results for this  $H_W(q)$ -action.

## 7.2 Gluing the group algebra and the 0-Hecke algebra

The group algebra  $\mathbb{F}W$  of a finite Coxeter group  $W$  naturally admits both actions of  $W$  and  $H_W(0)$ . Hivert and Thiéry [12] defined the *Hecke group algebra* of  $W$  by gluing these two actions. In type  $A$ , one can also glue the usual actions of  $\mathfrak{S}_n$  and  $H_n(0)$  on the polynomial ring  $\mathbb{F}[X]$ , but the resulting algebra is different from the Hecke group algebra of  $\mathfrak{S}_n$ .

Now one has a  $W$ -action and an  $H_W(0)$ -action on the Stanley-Reisner ring  $\mathbb{F}[\Delta(W)]$ . What can we say about the algebra generated by the operators  $s_i$  and  $\bar{\pi}_i$  on  $\mathbb{F}[\Delta(W)]$ ? Is it the same as the Hecke group algebra of  $W$ ? If not, what properties (dimension, bases, presentation, simple and projective indecomposable modules, etc.) does it have?

## 7.3 Tits Building

Let  $\Delta(G)$  be the Tits building of the general linear group  $G = GL(n, \mathbb{F}_q)$  and its usual BN-pair over a finite field  $\mathbb{F}_q$ ; see e.g. Björner [4]. The Stanley-Reisner ring  $\mathbb{F}[\Delta(G)]$  is a  $q$ -analogue of  $\mathbb{F}[\mathcal{B}_n]$ . The nonzero monomials in  $\mathbb{F}[\Delta(G)]$  are indexed by multiflags of subspaces of  $\mathbb{F}_q^n$ , and there are  $q^{\text{inv}(w)}$  many multiflags corresponding to a given multichain  $M$  in  $\mathcal{B}_n$ , where  $w = \sigma(M)$ . Can one obtain the multivariate quasisymmetric function identities in Theorem 1.2 by defining a nice  $H_n(0)$ -action on  $\mathbb{F}[\Delta(G)]$ ?

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