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## Finding a subdivision of a prescribed digraph of order 4

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**Abstract:** The problem of when a given digraph contains a subdivision of a fixed digraph  $F$  is considered. Bang-Jensen et al. [2] laid out foundations for approaching this problem from the algorithmic point of view. In this paper we give further support to several open conjectures and speculations about algorithmic complexity of finding  $F$ -subdivisions. In particular, up to 5 exceptions, we completely classify for which 4-vertex digraphs  $F$ , the  $F$ -subdivision problem is polynomial-time solvable and for which it is NP-complete. While all NP-hardness proofs are made by reduction from some version of the 2-linkage problem in digraphs, some of the polynomial-time solvable cases involve relatively complicated algorithms.

**Key-words:** digraph, oriented graph, subdivision, linkage, polynomial-time algorithm, NP-completeness

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## **Trouver une subdivision d'un digraphe donné d'ordre 4.**

**Résumé :** Nous considérons le problème consistant à déterminer si un digraphe donné contient une subdivision d'un digraphe  $F$  fixé. Bang-Jensen et al. [2] ont posé les bases d'une approche algorithmique de ce problème. Dans ce rapport, nous poursuivons leur étude et donnons des indications sur la validité de certaines conjectures et spéculations. En particulier, à 5 exceptions près, nous caractérisons les digraphes à 4 sommets : ceux pour lequel le problème est NP-complet et ceux pour lequel il est polynomial. Alors que toutes les preuves de NP-complétude sont des réductions à une version particulière du problème du 2-linkage dans les digraphes, certains cas solubles en temps polynomial nécessitent des algorithmes relativement compliqués.

**Mots-clés :** digraphe, graphe orienté, subdivision, linkage, algorithme polynomial, NP-complétude

# 1 Introduction

In this paper, all digraphs are meant to be *strict*, that is without loops and without multiple arcs. In one occasion, however, multiple arcs will be allowed. In that case, we will use the term *multidigraph*. We follow standard terminology as used in [1, 4].

A *subdivision* of a digraph  $F$ , also called an  $F$ -*subdivision*, is a digraph obtained from  $F$  by replacing each arc  $ab$  of  $F$  by a directed  $(a, b)$ -path. In this paper, we consider the following problem for a fixed digraph  $F$ .

$F$ -SUBDIVISION

Input: A digraph  $D$ .

Question: Does  $D$  contain a subdivision of  $F$ ?

Bang-Jensen et al. [2] conjectured that there is a dichotomy between NP-complete and polynomial-time solvable instances.

**Conjecture 1.** For every digraph  $F$ , the  $F$ -SUBDIVISION problem is polynomial-time solvable or NP-complete.

According to this conjecture, there are only two kinds of digraphs  $F$ : *hard* digraphs  $F$ , for which  $F$ -SUBDIVISION is NP-complete, and *tractable* digraphs, for which  $F$ -SUBDIVISION is solvable in polynomial-time.

Bang-Jensen et al. [2] proved that many digraphs are hard; see Theorem 9 in Section 4. In particular, every digraph in which every vertex  $v$  is *big* (that is such that either  $d^+(v) \geq 3$ , or  $d^-(v) \geq 3$ , or  $d^-(v) = d^+(v) = 2$ ) is hard. They also give many examples of tractable digraphs. See Subsection 2.4. However, there is no clear evidence, of which graph should be tractable and which one should be hard, despite some results and conjectures give some outline.

Establishing a conjecture of Johnson et al. [9], Kawabara-yashi and Kreutzer [10] proved the Directed Grid Theorem.

**Theorem 2** (Kawabara-yashi and Kreutzer [10]). *For any positive integer  $k$ , there exists an integer  $f(k)$  such that every digraph with directed treewidth greater than  $f(k)$  contains a cylindrical grid of order  $k$  as a butterfly minor.*

Here, a *cylindrical grid* of order  $k$  consists of  $k$  concentric directed cycles and  $2k$  directed paths connecting the cycles in alternating directions. See Figure 1 for an illustration. A *butterfly minor* of a digraph  $D$  is a digraph obtained from a subgraph of  $D$  by contracting arcs which are either the only outgoing arc of their tail or the only incoming arc of their head.

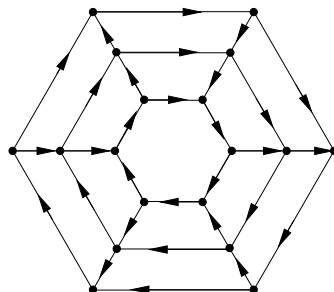
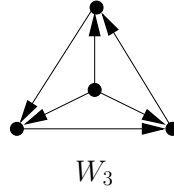


Figure 1: The cylindrical grid of order 3.

Figure 2: The 3-wheel  $W_3$ .

Because the  $k$ -LINKAGE problem (See Subsection 2.3 for definitions and details on  $k$ -LINKAGE) is polynomial-time solvable on digraph with bounded directed treewidth (see [2] for more details.), this directly implies the following :

**Corollary 3.**  *$F$ -SUBDIVISION is polynomial-time solvable when  $F$  is a planar digraph with no big vertices.*

Indeed, for every planar digraph  $F$  with no big vertices there is an  $F$ -subdivision in a cylindrical grid of order  $k$ , for some sufficiently large  $k$ . Therefore one can solve  $F$ -SUBDIVISION as follows. Given a digraph  $G$ , one estimates its directed treewidth. If it is greater than  $f(k)$  (as defined in Theorem 2), then it contains a cylindrical grid of order  $k$  and so an  $F$ -subdivision and we return ‘Yes’. Otherwise it has small treewidth, and one can solve  $F$ -subdivision in polynomial time.

Before the Directed Grid Theorem was proved, we found elementary proofs (i.e. not using this theorem) to show that digraphs of order 4 with no big vertices (all digraphs of order 4 are planar) are easy. The proofs can be easily implemented and the derived algorithms are certainly of lower complexity than the ones derived from the Directed Grid Theorem. They can be found in Appendix A.

On the other hand, Bang-Jensen et al. [2] proposed the following sort of counterpart.

**Conjecture 4** (Bang-Jensen et al. [2]).  *$F$ -SUBDIVISION is NP-complete for every non-planar digraph  $F$ .*

Bang-Jensen et al. [2] were able to classify all digraphs of order at most 3: they are all tractable except the complete symmetric digraph on three vertices, which is hard. In this paper, we aim at classifying all digraphs of order 4.

We first deal in Section 3 with oriented graphs. We prove that all oriented graphs of order 4 are tractable. In particular, we show in Subsection 3.1 that the wheel  $W_3$  is tractable. The *wheel*  $W_k$  is the graph obtained from the directed cycle on  $k$  vertices  $\vec{C}_k$  by adding a vertex, called the *centre*, dominating every vertex of  $\vec{C}_k$ . In [2], Bang-Jensen et al. proved that  $W_2$  is tractable and that  $W_k$ -SUBDIVISION is NP-complete for all  $k \geq 4$ . The case of  $W_3$  was left as an open problem.

This also completes the classification of tournaments. Bang-Jensen et al. [2] proved that every tournament of order at most 3 is tractable, and that every tournament of order at least 5 is hard (see Theorem 9). They also show that the transitive tournament of order 4 is tractable. The other tournaments of order four are  $W_3$ , its converse, and  $ST_4$ , the strong tournament of order 4, no vertex of which is big.

Next, we turn to digraphs of order 4, which are not oriented graphs. In Section 4, we show some of them to be hard, and in Section 5, we prove some of them to be tractable. Using these results, we are able to classify all digraphs of order 4 except for five of them (up to directional duality). These are the digraphs  $O_i$  for  $1 \leq i \leq 5$  depicted Figure 3.

## 1.1 Finding an $F$ -subdivision

The letters  $n$  and  $m$  will always denote the number of vertices and arcs of the input digraph  $D$  of the problem in question. By *linear time*, we mean  $O(n + m)$  time.

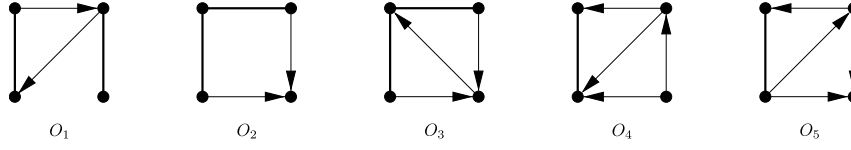


Figure 3: Digraphs on 4-vertices that are not known to be tractable or hard. Bold undirected edges represent directed 2-cycles.

**Lemma 5.** *If  $F$ -SUBDIVISION can be solved in  $f(n, m)$  time, where  $f$  is non-decreasing in  $m$ , then there is an algorithm that finds an  $F$ -subdivision (if one exists) in a digraph in  $((m + 1) \cdot f(n, m) + m)$  time.*

*Proof.* Suppose that there exists an algorithm  $F\text{-decide}(D)$  that decides in  $f(n, m)$  whether  $D$  contains an  $F$ -subdivision. We now construct an algorithm  $F\text{-find}(D)$  that finds an  $F$ -subdivision in  $D$  if there is one, and returns ‘no’ otherwise. It proceeds as follows.

Let  $a_1, \dots, a_m$  be the arcs of  $D$ . If  $F\text{-decide}(D)$  returns ‘no’, then we also return ‘no’. If not, then  $D$  contains an  $F$ -subdivision, we find it as follows: We initialize  $D_0 := D$ . For  $i = 1$  to  $m$ ,  $D_i := D_{i-1} \setminus a_i$  if  $F\text{-decide}(D_{i-1} \setminus a_i)$  returns ‘yes’, and  $D_i := D_{i-1}$  otherwise.

$F\text{-find}$  is valid because at step  $i$ , we delete the arc  $a_i$  if and only if there is an  $F$ -subdivision not containing  $i$ . Hence at each step  $i$ , we are sure that  $D_i$  contains an  $F$ -subdivision, and that any  $F$ -subdivision must contain all the arcs of  $A(D_i) \cap \{a_1, \dots, a_i\}$ .

$F\text{-find}$  runs  $(m + 1)$  times the algorithm  $F\text{-decide}$  and removes at most  $m$  times an arc. Therefore, it runs in time  $(m + 1) \cdot f(n, m) + m$ .  $\square$

Lemma 5 implies that deciding if there is an  $F$ -subdivision in a digraph is polynomial-time solvable, if and only if, finding an  $F$ -subdivision in a digraph is polynomial-time solvable. Therefore, since we are primarily interested in determining if the problems are polynomial-time solvable or NP-complete, and for sake of clarity, we only present algorithms for solving  $F$ -SUBDIVISION as a decision problem. However, the proofs of validity of all given algorithms always rely on constructive claims. Hence each algorithm can be easily transformed into a polynomial-time algorithm for finding an  $F$ -subdivision in a given digraph. Moreover, the reader can check that the additional work does not increase the time complexity. Hence, our algorithms for finding  $F$ -subdivisions have the same complexity as their decision versions.

## 2 Preliminaries

### 2.1 Definitions and notations

We rely on [1, 4] for standard notation and concepts. Let  $D$  be a digraph. The *converse* of  $D$  is the digraph  $\overline{D}$  obtained from  $D$  by reversing the orientation of all arcs. We denote by  $UG(D)$  the underlying (multi)graph of  $D$ , that is, the (multi)graph we obtain by replacing each arc by an edge. To every graph  $G$ , we can *associate* a symmetric digraph by replacing every edge  $uv$  by the two arcs  $uv$  and  $vu$ .

A *source* in  $D$  is a vertex of indegree zero and a *sink* is a vertex of outdegree zero.

An *oriented graph* is an orientation of an undirected graph. In other words, it is a digraph with no directed cycles of length 2. An *oriented path* is an orientation of an undirected path. Hence an oriented path  $P$  is a sequence  $(x_1, a_1, x_2, a_2, \dots, a_{n-1}, x_n)$ , where the  $x_i$  are distinct vertices and for all  $1 \leq j \leq n - 1$ ,  $a_j$  is either the arc  $x_j x_{j+1}$  or the arc  $x_{j+1} x_j$ . For sake of clarity, we often refer to such an oriented path  $P$  by the underlying undirected path  $x_1 x_2 \dots x_n$ . This is a slight abuse, because the oriented path  $P$  is not completely determined by this sequence as there are two possible orientations



for each edge. However, when we use this notation, either the orientation does not matter or it is clear from the context.

Let  $P = x_1x_2 \cdots x_n$  be an oriented path. We say that  $P$  is an  $(x_1, x_n)$ -path. The vertex  $x_1$  is the *initial vertex* of  $P$  and  $x_n$  its *terminal vertex*. We denote the initial vertex of  $P$  by  $s(P)$  and the terminal vertex of  $P$  by  $t(P)$ . The subpath  $x_2 \cdots x_{n-1}$  is denoted by  $P^\circ$ . If  $x_1x_2$  is an arc, then  $P$  is an *outpath*, otherwise  $P$  is an *inpath*. The path  $P$  is *directed* if no vertex is the tail of two arcs in  $P$  nor the head of two arcs. In other words, all arcs are oriented in the same direction. There are two kinds of directed paths, namely directed outpaths and directed inpaths. For convenience, a directed outpath is called a *dipath*. The *blocks* of an oriented path  $P$  are the maximal directed subpaths of  $P$ . We often enumerate them from the initial vertex to the terminal vertex of the path. The number of blocks of  $P$  is denoted by  $b(P)$ . The *opposite path* of  $P$ , denoted  $\overleftarrow{P}$ , is the path  $x_nx_{n-1} \cdots x_1$ . For  $1 \leq i \leq j \leq n$ , we denote by  $P[x_i, x_j]$  (resp.  $P]x_i, x_j[$ ,  $P[x_i, x_j[$ ,  $P]x_i, x_j]$ , the oriented subpath  $x_ix_{i+1} \cdots x_j$  (resp.  $x_{i+1}x_{i+2} \cdots x_{j-1}$ ,  $x_{i+1}x_{i+2} \cdots x_j$ ,  $x_ix_{i+1} \cdots x_{j-1}$ ).

The above definitions and notation can also be used for oriented cycles. If  $C = x_1x_2 \cdots x_nx_1$  is an oriented cycle, we shall assume that either  $C$  is a *directed cycle*, that is  $x_ix_{i+1}$  is an arc for all  $1 \leq i \leq n$ , where  $x_{n+1} = x_1$ , or both edges of  $C$  incident with  $x_1$  are directed outwards, i.e.  $x_1x_2$  and  $x_1x_n$  are arcs of  $C$ .

For a set  $X$  of vertices, the *outsection* of  $X$  in  $D$ , denoted by  $S_D^+(X)$ , is the set of vertices that are reachable from  $X$  by a dipath. The outsection of a set in a digraph can be found in linear time using the Breadth-First Search. The directional dual notion, the *insection* of  $X$  in  $D$  is denoted by  $S_D^-(X)$ .

The digraph  $D$  is *connected* (resp. *k-connected*) if  $UG(D)$  is a connected (resp. *k-connected*) graph. It is *strongly connected*, or *strong*, if for any two vertices  $u, v$ , there is a  $(u, v)$ -dipath in  $D$ . If  $D$  is strong, we use the notation  $D[u, v]$  to denote any  $(u, v)$ -dipath in  $D$ . The disjoint union of two digraphs  $D_1$  and  $D_2$  is denoted  $D_1 + D_2$ .

By *contracting* a vertex-set  $X \subseteq V(D)$  we refer to the operation of first taking the digraph  $D - X$  and then adding new vertex  $v_X$  and adding the arc  $v_Xw$  for each  $w \in V(D - X)$  with an inneighbour in  $X$  and the arc  $uv_X$  for each  $u \in V(D - X)$  with an outneighbour in  $X$ . The *contraction* of a non-strong digraph  $D$  is the digraph obtained by contracting all strong components of  $D$ .

## 2.2 Menger's Theorem

Let  $X$  and  $Y$  be two sets of vertices in a digraph  $D$ . An  $(X, Y)$ -dipath is a dipath with initial vertex in  $X$ , terminal vertex in  $Y$  and all internal vertices in  $V(D) \setminus (X \cup Y)$ . For notational clarity, for a vertex  $x$  (resp. a subdigraph  $S$  of  $D$ ), we abbreviate  $\{x\}$  to  $x$  (resp.  $V(S)$  to  $S$ ) in the notation. For example, an  $(x, S)$ -dipath is an  $(\{x\}, V(S))$ -dipath.

Let  $D$  be a digraph, and let  $x$  and  $y$  be distinct vertices of  $D$ . Two  $(x, y)$ -paths  $P$  and  $Q$  are *internally disjoint* if they have no internal vertices in common, that is if  $V(P) \cap V(Q) = \{x, y\}$ . A *k-separation* of  $(x, y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $x \in W$ ,  $y \in Z$ ,  $|S| \leq k$ , each vertex in  $W$  can be reached from  $x$  by a dipath in  $D[W]$ , and there is no arc from  $W$  to  $Z$ .

One version of the celebrated Menger's Theorem is the following.

**Theorem 6 (Menger).** *Let  $k$  be a positive integer, let  $D$  be a digraph, and let  $x$  and  $y$  be distinct vertices in  $D$  such that  $xy \notin A(D)$ . Then, in  $D$ , either there are  $k + 1$  pairwise internally disjoint  $(x, y)$ -dipaths, or there is a  $k$ -separation of  $(x, y)$ .*

For any fixed  $k$ , there exist algorithms running in linear time that, given a digraph  $D$  and two distinct vertices  $x$  and  $y$  such that  $xy \notin A(D)$ , returns either  $k + 1$  internally disjoint  $(x, y)$ -dipaths in  $D$ , or a  $k$ -separation  $(W, S, Z)$  of  $(x, y)$ . Indeed, in such a particular case, any flow algorithm like Ford–Fulkerson algorithm for example, performs at most  $k + 1$  incrementing-path searches, because it increments the flow by 1 each time, and we stop when the flow has value  $k + 1$ , or if we find a cut of size less than

$k + 1$ , which corresponds to a  $k$ -separation. Moreover each incrementing-path search consists in a search (usually Breadth-First Search) in an auxiliary digraph of the same size, and so is done in linear time. For more details, we refer the reader to the book of Ford and Fulkerson [6] or Chapter 7 of [4]. We call such an algorithm a *Menger algorithm*.

Observe that using Menger algorithms, one can decide if there are  $k$  internally disjoint  $(x, y)$ -dipaths in a digraph  $D$ . If  $xy \notin A(D)$ , then we apply a Menger algorithm directly; if  $xy \in A(D)$ , then we check whether there are  $k - 1$  internally disjoint  $(x, y)$ -dipaths in  $D \setminus xy$ .

Let  $D$  be a digraph. Let  $X$  and  $Y$  be non-empty sets of vertices in  $D$ . Two  $(X, Y)$ -paths  $P$  and  $Q$  are *disjoint* if they have no vertices in common, that is if  $V(P) \cap V(Q) = \emptyset$ . A  $k$ -separation of  $(X, Y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $X \subseteq W \cup S$ ,  $Y \subseteq Z \cup S$ ,  $|S| \leq k$ , all vertices of  $W$  can be reached from  $X \setminus S$  by dipaths in  $D[W]$ , and there is no arc from  $W$  to  $Z$ .

Let  $x$  be a vertex of  $D$  and  $Y$  be a non-empty subset of  $V(D) \setminus \{x\}$ . Two  $(x, Y)$ -paths  $P$  and  $Q$  are *independent* if  $V(P) \cap V(Q) = \{x\}$ . A  $k$ -separation of  $(x, Y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $x \in W$ ,  $Y \subseteq Z \cup S$ ,  $|S| \leq k$ , all vertices of  $W$  can be reached from  $x$  by dipaths in  $D[W]$ , and there is no arc from  $W$  to  $Z$ .

Let  $y$  be a vertex of  $D$  and  $X$  be a non-empty subset of  $V(D) \setminus \{y\}$ . Two  $(X, y)$ -paths are *independent* if  $V(P) \cap V(Q) = \{y\}$ . A  $k$ -separation of  $(X, y)$  in  $D$  is a partition  $(W, S, Z)$  of its vertex set such that  $W$  and  $Z$  are non-empty,  $X \subseteq W \cup S$ ,  $y \in Z$ ,  $|S| \leq k$ , all vertices of  $W$  can be reached from  $X \setminus S$  by dipaths in  $D[W]$ , and there are no arcs from  $W$  to  $Z$ .

Let  $W \subset V(D)$ . The digraph  $D_W$  is the one obtained from  $D$  by adding a vertex  $s_W$  and the arcs  $s_W w$  for all  $w \in W$  and the digraph  $D^W$  is the one obtained from  $D$  by adding a vertex  $t_W$  and the arcs  $wt_W$  for all  $w \in W$ .

Applying Theorem 6 to  $D_X^Y$  and  $(s_X, t_Y)$  (resp.  $D^Y$  and  $(x, t_Y)$ ,  $D_X$  and  $(s_X, y)$ ), we obtain the following version of Menger's Theorem.

**Theorem 7 (Menger).** *Let  $k$  be a positive integer, and let  $D$  be a digraph. Then the following hold.*

- (i) *If  $X$  and  $Y$  are two non-empty subsets of  $V(D)$ , then, in  $D$ , either there are  $k + 1$  pairwise disjoint  $(X, Y)$ -dipaths, or there is a  $k$ -separation of  $(X, Y)$ .*
- (ii) *If  $x$  is a vertex of  $D$  and  $Y$  is a non-empty subset of  $V(D)$ , then, in  $D$ , either there are  $k + 1$  pairwise independent  $(x, Y)$ -dipaths in  $D$ , or there is a  $k$ -separation of  $(x, Y)$ .*
- (iii) *If  $X$  is a non-empty subset of  $V(D)$  and  $y$  is a vertex of  $D$  and, then, in  $D$ , either there are  $k + 1$  pairwise independent  $(X, y)$ -dipaths in  $D$ , or there is a  $k$ -separation of  $(X, y)$ .*

Moreover, a Menger Algorithm applied to  $D_X^Y$  and  $(s_X, t_Y)$  (resp.  $D^Y$  and  $(x, t_Y)$ ,  $D_X$  and  $(s_X, Y)$ ) finds in linear time the  $k + 1$  dipaths or the separation as described in Theorem 7 (i) (resp. (ii), (iii)).

Let  $x$  and  $y$  be two vertices. An  $(x, y)$ -handle is an  $(x, y)$ -dipath if  $x \neq y$ , and a directed cycle containing  $x$  if  $x = y$ . Let  $y_1, \dots, y_p$  be  $p$  distinct vertices,  $k_1, \dots, k_p$  be positive integers and set  $k = k_1 + \dots + k_p$ . One can decide if there are  $k$  internally disjoint handles  $P_1, \dots, P_k$  such that  $k_i$  of them are  $(x, y_i)$ -handles,  $1 \leq i \leq p$ , by applying a Menger algorithm between in an auxiliary digraph  $D'$ . This digraph is obtained from  $D - (\{y_1, \dots, y_k\} \setminus \{x\})$  as follows. Add a new vertex  $y$ . For each  $i$ , create a set  $B_i$  of  $k_i$  new vertices and all arcs from  $N_D^-(y_i)$  to  $B_i$  and from  $y_i$  to  $y$ .

Similarly, suppose that  $X$  is a set of vertices,  $y_1, \dots, y_p$  be  $p$  distinct vertices not in  $X$ , and  $k = k_1 + \dots + k_p$ . One can decide if there are  $k$  internally disjoint paths  $P_1, \dots, P_k$ , all with distinct initial vertices in  $X$ , and such that  $k_i$  of them are terminating in  $y_i$ ,  $1 \leq i \leq p$ .

### 2.3 Linkage and disjoint directed cycles in digraphs

Let  $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$  be distinct vertices of a digraph  $D$ . A  $k$ -linkage from  $(x_1, x_2, \dots, x_k)$  to  $(y_1, y_2, \dots, y_k)$  in  $D$  is a system of disjoint dipaths  $P_1, P_2, \dots, P_k$  such that  $P_i$  is an  $(x_i, y_i)$ -path

in  $D$ . Fortune, Hopcroft and Wyllie [7] showed that for any  $k \geq 2$ ,  $k$ -LINKAGE is NP-complete. The problem is also NP-complete when restricted to some classes of digraphs. Recall that a vertex  $v$  is *big* if either  $d^+(v) \geq 3$ , or  $d^-(v) \geq 3$ , or  $d^-(v) = d^+(v) = 2$ .

#### RESTRICTED 2-LINKAGE

Input: A digraph  $D$  without big vertices in which  $x_1$  and  $x_2$  are sources and  $y_1$  and  $y_2$  are sinks.

Question: Is there a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ ?

**Theorem 8** (Bang-Jensen et al. [2]). *The RESTRICTED 2-LINKAGE problem is NP-complete.*

Using this theorem, Bang-Jensen et al. [2] deduced a sufficient condition for  $F$ -SUBDIVISION to be NP-complete.

For a digraph  $D$ , we denote by  $B(D)$  the set of its big vertices. A *big path* in a digraph is a directed path whose endvertices are big and whose internal vertices all have both indegree and outdegree equal to 1 in  $D$  (in particular an arc between two big vertices is a big path). Note also that two big paths with the same endvertices are necessarily internally disjoint.

The *big paths digraph* of  $D$ , denoted  $BP(D)$ , is the multidigraph with vertex set  $V(D)$  in which there are as many arcs between two vertices  $u$  and  $v$  as there are big  $(u, v)$ -paths in  $D$ .

**Theorem 9** (Bang-Jensen et al. [2]). *Let  $F$  be a digraph. If  $F$  contains two arcs  $ab$  and  $cd$  whose endvertices are big vertices and such that  $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\}$  is not isomorphic to  $BP(F)$ , then  $F$ -SUBDIVISION is NP-complete.*

**Corollary 10.** *Let  $F$  be a digraph. If  $F$  contains a directed cycle of length 2 whose vertices are big, then  $F$ -SUBDIVISION is NP-complete.*

So far, all known hard digraphs were proved hard by a reduction from RESTRICTED 2-LINKAGE. This paper is no exception: in Section 4, we show that some digraphs are NP-hard with such reductions.

## 2.4 Known results and tools for $F$ -SUBDIVISION

Let  $F$  be a digraph and  $u$  a vertex in  $F$ . In an  $F$ -subdivision  $S$ , the vertex corresponding to  $u$  is called the  *$u$ -vertex* of  $S$ . A vertex corresponding to some vertex  $u \in F$  is called an *original* vertex.

Bang-Jensen et al. [2] proved that, given a digraph  $D$  and a vertex  $z$  in  $D$ , one can decide in polynomial time if  $D$  contains a  $W_2$ -subdivision with centre  $z$ . Therefore  $W_2$ -SUBDIVISION is polynomial-time solvable. We now prove that we can also decide in polynomial time if there is a  $W_2$ -subdivision with two prescribed original vertices.

**Lemma 11.** *Let  $W_2$  be the 2-wheel with centre  $c$  and rim  $aba$ . Given a digraph  $D$  and two vertices  $b'$  and  $c'$ , one can decide in  $O(n^2(n + m))$  time if there is a  $W_2$ -subdivision in  $D$  with  $b$ -vertex  $b'$  and  $c$ -vertex  $c'$ .*

*Proof.* Let us call a  $W_2$ -subdivision with  $b$ -vertex  $b'$  and  $c$ -vertex  $c'$  a  $(b', c')$ -forced  $W_2$ -subdivision. Let  $S$  be the strong component of  $b'$  in  $D - c'$ . The key element is the following claim.

**Claim 11.1.**  *$D$  contains a  $(b', c')$ -forced  $W_2$  subdivision if and only if there exist distinct vertices  $x_1$  and  $x_2$  in  $V(S)$  such that there are two independent  $(c', \{x_1, x_2\})$ -dipaths  $P_1$  and  $P_2$  in  $D - (S \setminus \{x_1, x_2\})$  and there are two independent  $(\{x_1, x_2\}, b')$ -dipaths  $Q_1$  and  $Q_2$  in  $S$ .*

*Subproof.* Clearly, existence of two vertices  $x_1, x_2$  and four dipaths  $P_1, P_2, Q_1, Q_2$  as in the statement is a necessary condition for the existence of a  $(b', c')$ -forced  $W_2$ -subdivision. Let us now prove that it is also sufficient. Assume that such vertices  $x_1, x_2$  and dipaths  $P_1, P_2, Q_1, Q_2$  exist. Since  $S$  is strong,

it contains a dipath  $R$  from  $b'$  to  $(V(Q_1) \cup V(Q_2)) \setminus \{b'\}$ . (This set is not empty since it contains  $\{x_1, x_2\} \setminus \{b'\}$ .) Then  $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R$  is a  $(b', c')$ -forced  $W_2$ -subdivision.  $\diamond$

Our algorithm is the following. We first compute  $S$ , which can be done in linear time. Then for every pair  $\{x_1, x_2\}$  of vertices of  $S$ , we check by running twice a Menger algorithm if the dipaths  $P_1$  and  $P_2$ , and  $Q_1$  and  $Q_2$  as described in Claim 11.1 exist. If yes, we return ‘yes’, otherwise we return ‘no’. The validity of this algorithm is given by Claim 11.1. Since there are  $O(n^2)$  pairs of vertices  $\{x_1, x_2\}$ , the algorithm runs in  $O(n^2(n + m))$  time.  $\square$

A *spider* is a tree obtained from disjoint directed paths by identifying one end of each path into a single vertex. This vertex is called the *body* of the spider. Observe that if  $T$  is a spider, then every  $T$ -subdivision contains  $T$  as a subdigraph. Hence a digraph contains a  $T$ -subdivision if and only if it contains  $D$  as a subdigraph. This implies that  $T$ -SUBDIVISION can be solved in  $O(n^{|T|})$  time. It also easily implies the following.

**Lemma 12.** *Let  $F$  be a digraph and  $T$  a spider. If  $F$  is tractable, then  $F + T$  is also tractable.*

*Gluing* a spider  $T$  with body  $b$  to  $F$  at a vertex  $u \in V(F)$  consists in taking the disjoint union of  $F$  and  $T$  and identifying  $u$  and  $b$ .

**Lemma 13.** *Let  $F$  be a digraph and  $u$  a vertex of  $F$ . If given a digraph  $D$  and a vertex  $v$  of  $D$ , one can decide in polynomial time if there is an  $F$ -subdivision in  $D$  such that  $v$  is the  $u$ -vertex, then any digraph obtained from  $F$  by gluing a spider at  $u$  is tractable.*

*Proof.* Let  $T$  be a spider with body  $b$  and let  $F'$  be the digraph obtained by gluing  $T$  to  $F$  at  $u$ . Clearly, every  $F'$ -subdivision contains an  $F'$ -subdivision in which the arcs of  $T$  are not subdivided. Such an  $F'$ -subdivision is said to be *canonical*.

Consider the following algorithm. For every vertex  $v$  we repeat the following. For every set  $W$  of  $|V(T)| - 1$  vertices, we check whether  $D[W \cup \{v\}]$  contains a copy of  $T$  with body  $v$ . This can be done in constant time. Then we check if  $D - W$  contains an  $F$ -subdivision with  $u$ -vertex  $v$ . This can be done in polynomial time by our assumption.

This algorithm clearly decides in polynomial time whether a given digraph  $D$  contains a canonical  $F'$ -subdivision.  $\square$

A  $(k_1, \dots, k_p)$ -*spindle* is the union of  $p$  pairwise internally disjoint  $(a, b)$ -dipaths  $P_1, \dots, P_p$  of respective lengths  $k_1, \dots, k_p$ . The vertex  $a$  is said to be the *tail* of the spindle and  $b$  its *head*. Bang-Jensen et al. [2] proved that spindles are tractable. Their proof uses the following result.

**Theorem 14** (Bang-Jensen et al. [2]). *Let  $F$  be a spindle with tail  $a$  and head  $b$ . Given a digraph  $D$  and two vertices  $a'$  and  $b'$ , we can decide in polynomial time if  $D$  contains an  $F$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$ .*

The  $(k_1, \dots, k_p; l_1, \dots, l_q)$ -*bispindle*, denoted  $B(k_1, \dots, k_p; l_1, \dots, l_q)$ , is the digraph obtained from the disjoint union of a  $(k_1, \dots, k_p)$ -spindle with tail  $a_1$  and head  $b_1$  and an  $(l_1, \dots, l_q)$ -spindle with tail  $a_2$  and head  $b_2$  by identifying  $a_1$  with  $b_2$  into a vertex  $a$ , and  $a_2$  with  $b_1$  into a vertex  $b$ . The vertices  $a$  and  $b$  are called, respectively, the *left node* and the *right node* of the bispindle. The directed  $(a, b)$ -paths are called the *forward paths*, while the directed  $(b, a)$ -paths are called the *backward paths*. Bang-Jensen et al. [2] proved that a bispindle is hard if and only if  $p \geq 1$ ,  $q \geq 1$  and  $p + q \geq 4$ . To prove that a bispindle with two forward paths and one backward path is tractable, they provided the following theorem.

**Theorem 15** (Bang-Jensen et al. [2]). *Let  $F$  be a bispindle with two forward paths and one backward path, and let  $x$  be one of its nodes. Given a digraph  $D$  and a vertex  $a'$ , we can decide in polynomial time if  $D$  contains an  $F$ -subdivision with  $a$ -vertex  $a'$ .*

**Lemma 16.** *Let  $F$  be a digraph and let  $u_1, \dots, u_p$  be distinct vertices of  $F$ . Suppose that for every outneighbour  $v$  of  $u_1$ , replacing the arc  $u_1v$  by a dipath  $u_1wv$  of length 2, where  $w \notin V(F)$ , always results in the same digraph  $F'$ . Suppose that for every given digraph  $D$  of order  $n$  and  $p$  vertices  $x_1, \dots, x_p$  in  $D$ , one can decide in  $f(n)$  time whether there is an  $F$ -subdivision in  $D$  such that  $x_i$  is the  $u_i$ -vertex for every  $i$ . Then given a digraph  $D$  and  $p$  vertices  $x_1, \dots, x_p$ , one can decide in  $O\left(\binom{d^+(x_1)-1}{d^+(u_1)-1} \sum_{y \in N^+(x_1)} d^+(y) \cdot f(n-1)\right)$  time whether there is an  $F'$ -subdivision in  $D$  such that  $x_i$  is the  $u_i$ -vertex for every  $i$ .*

*Proof.* Set  $q = d^+(u_1)$ . For every set of  $q$  neighbours  $y_1, \dots, y_q$  of  $x_1$  and every outneighbour  $z$  of  $y_1$ , where  $z \notin \{y_2, \dots, y_q\}$ , we shall give a procedure that verifies if  $D$  contains an  $F'$ -subdivision  $S'$  such that  $x_i$  is the  $u_i$ -vertex for all  $1 \leq i \leq p$ , and  $\{x_1y_1, \dots, x_1y_q, y_1z\} \subseteq A(S')$ . Such an  $F'$ -subdivision is called *forced*.

Let  $D'$  be the digraph obtained from  $D - y_1$  by deleting all arcs leaving  $x_1$  except  $x_1y_2, \dots, x_1y_q$ , and adding the arc  $x_1z$ .

**Claim 16.1.**  *$D$  has a forced  $F'$ -subdivision if and only if  $D'$  has an  $F$ -subdivision such that  $x_i$  is the  $u_i$ -vertex for every  $i$ .*

*Subproof.* Suppose that  $S$  is an  $F$ -subdivision in  $D'$  such that  $x_i$  is the  $u_i$ -vertex for all  $i$ . Since  $x_1$  has outdegree  $q$  in  $D'$ , we have  $\{x_1y_2, \dots, x_1y_q, x_1z\} \subseteq A(S)$ . Let  $S'$  be the digraph obtained from  $S$  by replacing the arc  $x_1z$  by the dipath  $x_1y_1z$ . Because replacing the arc  $u_1v$  by a dipath of length 2 results in  $F'$  for any outneighbour  $v$  of  $u_1$ , the digraph  $S'$  is an  $F'$ -subdivision in  $D$ . Thus  $S'$  is a forced  $F'$ -subdivision in  $D$ .

Conversely, assume that  $S'$  is a forced  $F'$ -subdivision in  $D$ . Then the digraph  $S$  obtained from  $S'$  by replacing the dipath  $x_1y_1z$  by the arc  $x_1z$  is an  $F$ -subdivision in  $D'$  such that  $x_i$  is the  $u_i$ -vertex for every  $i$ .  $\diamond$

This claim implies that deciding whether  $D$  contains a forced  $F'$ -subdivision can be done by checking whether  $D'$  has an  $F$ -subdivision such that  $x_i$  is the  $u_i$ -vertex for all  $i$ . This can be done in  $f(n-1)$  time by assumption. By repeating this for every possible set  $\{y_1, \dots, y_q, z\}$  where the  $y_i$  are distinct outneighbours of  $x_1$  and  $z \notin \{y_2, \dots, y_q\}$  is an outneighbour of  $y_1$ , we obtain an algorithm to decide whether there is an  $F'$ -subdivision in  $D$  such that  $x_i$  is the  $u_i$ -vertex for all  $i$ . Since there are at most  $\binom{d^+(x_1)-1}{d^+(u_1)-1} \sum_{y \in N^+(x_1)} d^+(y)$  such sets, the running time of this algorithm is as claimed.  $\square$

### 3 Oriented graphs of order 4

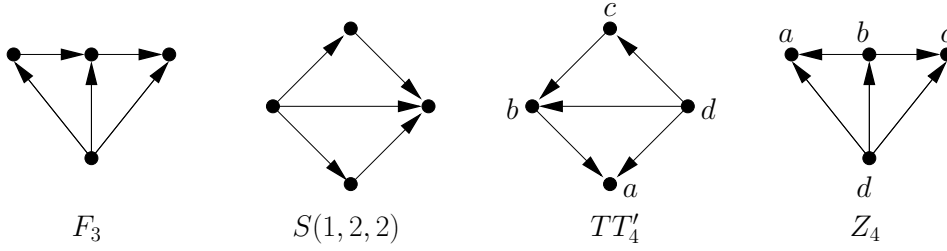
The aim of this section is to prove that every oriented graph of order 4 is tractable.

**Theorem 17.** *If  $F$  is an oriented graph of order 4, then  $F$ -SUBDIVISION is polynomial-time solvable.*

*Proof.* If  $F$  has no big vertices, then by Theorem 2,  $F$ -SUBDIVISION is polynomial-time solvable. Henceforth, we assume that  $F$  has at least one big vertex. Free to consider its converse, we may assume that  $F$  has a vertex with out-degree 3.

If  $F$  is a tournament, then it is either the transitive tournament  $TT_4$ . Bang-Jensen et al. [2] (Theorem 64) proved that  $TT_4$ -SUBDIVISION is polynomial-time solvable. We show in Subsection 3.1 that  $W_3$  is tractable.

If  $F$  is an orientation of  $K_4 \setminus e$ , the graph obtained from  $K_4$  by removing one edge, then  $F$  must be one of the oriented graphs depicted Figure 4, or the converse of one of those.  $S(1, 2, 2)$  is a spindle and  $F_3$  is the 3-fan. These digraphs have been shown tractable in [2] (Proposition 20 and Theorem 61). We prove in Subsection 3.2 that  $TT'_4$  is tractable, and in Subsection 3.3 that  $Z_4$  is tractable.

Figure 4: Some orientations of  $K_4 \setminus e$ 

If  $D$  has at most four arcs, then it is either a star or a star plus an arc. Those digraphs have been proved tractable in [2].  $\square$

### 3.1 Subdivision of the 3-wheel

**Theorem 18.**  $W_3$ -SUBDIVISION can be solved in  $O(n^6(n + m))$  time.

The proof of this theorem relies on the following notion. Let  $X$  be a set of three vertices. An  $X$ -tripod is a digraph which is the union of a directed cycle  $C$  and three disjoint dipaths  $P_1, P_2, P_3$  with initial vertices in  $X$  and terminal vertices in  $C$ . If the  $P_i$  are  $(X, C)$ -dipaths, we say that the tripod is *unfolded*. Note that the dipaths  $P_i$  may be of length 0. We shall denote the tripod described above as the 4-tuple  $(C, P_1, P_2, P_3)$ .

**Proposition 19.** Let  $X = \{x_1, x_2, x_3\}$  be a set of three distinct vertices. Any  $X$ -tripod contains an unfolded  $X$ -tripod.

We shall consider the following decision problem.

TRIPOD

Input: A strong digraph  $D$  and a set  $X$  of three distinct vertices of  $D$ .

Question: Does  $D$  contain an  $X$ -tripod?

**Lemma 20.** TRIPOD can be solved in  $O(n^2(n + m))$  time.

*Proof.* Let us describe a procedure  $\text{tripod}(D, X)$ , solving TRIPOD.

We first look for a directed cycle of length at least 3 in  $D$ . This can be done in linear time. If there is no such cycle, then we return ‘no’.

Otherwise we have a directed cycle  $C$  of length at least 3. We choose a set  $Y$  of three vertices in  $C$  and run a Menger algorithm between  $X$  and  $Y$ . If such an algorithm finds three disjoint  $(X, Y)$ -dipaths  $P_1, P_2, P_3$ , then we return the tripod  $(C, P_1, P_2, P_3)$ . Otherwise, the Menger algorithm finds a 2-separation  $(W, S, Z)$  of  $(X, Y)$ . Note that  $|S| \geq 1$  because  $D$  is strong.

Assume first that  $|S| = 1$ , say  $S = \{s\}$ . Let  $D_1$  be the digraph obtained from  $D[W \cup S]$  by adding the arc  $sw$  for every vertex  $w$  in  $W$  having an inneighbour  $z \in Z$ . We then make a recursive call to  $\text{tripod}(D_1, X)$ . This is valid by virtue of the following claim.

**Claim 20.1.** There is an  $X$ -tripod in  $D$  if and only if there is an  $X$ -tripod in  $D_1$ .

*Subproof.* Suppose first that there is an  $X$ -tripod in  $D_1$ . Then  $D_1$  contains an unfolded  $X$ -tripod  $T_1$  by Proposition 19. If  $T_1$  is contained in  $D$ , then we are done. So we may assume that it is not. Then  $T_1$  contains an arc  $sw \in A(D_1) \setminus A(D)$ . It can contain only one such arc since every vertex has outdegree

at most one in  $T_1$  and all such arcs leave  $s$ . Furthermore, the head  $w$  of this arc is in  $W$  and  $w$  has an inneighbour  $z$  in  $Z$ . Now, since  $D$  is strong, there is an  $(s, z)$ -dipath  $Q$  in  $D$ . Because there is no arc from  $W$  to  $Z$ , all internal vertices of  $Q$  are in  $Z$ . Hence the digraph  $T$  obtained from  $T_1$  by replacing the arc  $sw$  by the dipath  $Qzw$  is an  $X$ -tripod in  $D$ .

Suppose now that  $D$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T = (C, P_1, P_2, P_3)$  by Proposition 19. Since all  $(X, Z)$ -dipaths in  $D$  go through  $s$ , the terminal vertices of the  $P_i$  are in  $W \cup S$ , and  $D[Z] \cap T$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . If  $Q$  is a  $(t, z)$ -dipath, then  $T$  contains arcs  $st$  and  $zw$  for some  $w \in W$ . Then the digraph  $T_1$  obtained from  $T$  by replacing  $sQw$  by the arc  $sw$  is an  $X$ -tripod in  $D_1$ .  $\diamond$

Assume now that  $|S| = 2$ , say  $S = \{s_1, s_2\}$ . If there is no arc from  $Z$  to  $W$ , let  $D_2$  be the digraph obtained from  $D[W \cup S]$  by adding the arc  $s_1s_2$  (resp.  $s_2s_1$ ) (if the arc is not already present in  $D$ ) if there is an  $(s_1, s_2)$ -dipath (resp.  $(s_2, s_1)$ -dipath) in  $D[Z \cup S]$ . We then make a recursive call to `tripod`( $D_2, X$ ). This is valid by virtue of the following claim.

**Claim 20.2.** *There is an  $X$ -tripod in  $D$  if and only if there is an  $X$ -tripod in  $D_2$ .*

*Subproof.* Suppose first that there exists an  $X$ -tripod in  $D_2$ . Then there is an unfolded  $X$ -tripod  $T_2$  in  $D_2$ , by Proposition 19. Then either it is an  $X$ -tripod in  $D$ , or  $T_2$  contains exactly one of the arcs  $s_1s_2, s_2s_1$  and this arc is not in  $A(D)$ . Without loss of generality, we may assume that this arc is  $s_1s_2$ . Since  $s_1s_2 \in A(D_2) \setminus A(D)$ , there is an  $(s_1, s_2)$ -dipath  $Q$  in  $D[Z \cup S]$ . Hence the digraph  $T$  obtained from  $T_2$  by replacing the arc  $s_1s_2$  by the dipath  $Q$  is an  $X$ -tripod in  $D$ .

Suppose now that  $D$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T = (C, P_1, P_2, P_3)$  by Proposition 19. For  $i = 1, 2, 3$ , let  $y_i$  be the terminal vertex of  $P_i$ . Without loss of generality, we may assume that  $y_1, y_2, y_3$  appear in this order along  $C$ . Since all  $(X, Z \cup S)$ -dipaths intersect  $S$ , one of the  $y_i$ , say  $y_3$ , must be in  $W$ . The three oriented paths  $P_2, P_1C[y_1, y_2]$ , and  $\overline{C}[y_3y_2]$  are independent  $(W, y_2)$ -paths. But the graph underlying  $D$  has no edges between  $W$  and  $Z$ , by the assumption made in the current subcase. So  $y_2$  is in  $W \cup S$ . Similarly,  $y_1$  is in  $W \cup S$ . It follows that  $T \cap D[Z]$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . Moreover, the inneighbour in  $T$  of the initial vertex of  $Q$  is some vertex  $s \in S$  (because there is no arc from  $W$  to  $Z$ ) and the outneighbour in  $T$  of the terminal vertex of  $Q$  is some vertex  $s' \in S$  because there is no arc from  $Z$  to  $W$ ). Furthermore  $s \neq s'$  for otherwise  $sQs' = C$  which is impossible as since  $y_3 \in W \cap C$ . Moreover, because  $sQs'$  is an  $(s, s')$ -dipath in  $D[Z \cup S]$ ,  $ss'$  is an arc in  $D_2$ . Thus the digraph  $T_2$  obtained from  $T$  by replacing  $sQs'$  by the arc  $ss'$  is an  $X$ -tripod in  $D_2$ .  $\diamond$

Now we may assume that there is an arc  $z_1w_1$  with  $z_1 \in Z$  and  $w_1 \in W$ . Since  $D$  is strong, there is a cycle  $C'$  containing the arc  $z_1w_1$ . Necessarily, the cycle  $C'$  must go through  $S$  and it contains at least three vertices.

**Case 1:**  $S \subset V(C')$ . Set  $Y' = \{w_1, s_1, s_2\}$ . We run a Menger algorithm between  $X$  and  $Y'$ . If such an algorithm finds three disjoint  $(X, Y')$ -dipaths  $P'_1, P'_2, P'_3$ , then we return the  $X$ -tripod  $(C', P'_1, P'_2, P'_3)$ .

If not, we obtain a 2-separation  $(W', S', Z')$  of  $(X, Y')$ . We claim that  $|W'| < |W|$ . Indeed, no vertex  $z \in Z$  is in  $W'$  because every  $(X, z)$ -dipath must go through  $S$  and thus through  $S'$ . Hence  $W' \subseteq W \setminus \{w_1\}$ . Now, we replace  $C$  by  $C'$ ,  $Y$  by  $Y'$  and  $(W, S, Z)$  by  $(W', S', Z')$ , and then redo the procedure.

**Case 2:**  $|S \cap V(C')| = 1$ . Without loss of generality, we may assume  $S \cap V(C') = \{s_1\}$ . Set  $Y' = \{w_1, s_1, z_1\}$ . As in Case 1, we run a Menger algorithm between  $X$  and  $Y'$ . If such an algorithm finds three disjoint  $(X, Y')$ -dipaths  $P'_1, P'_2, P'_3$ , then we return the  $X$ -tripod  $(C', P'_1, P'_2, P'_3)$ .

If not, the Menger algorithm returns a 2-separation  $(W', S', Z')$  for  $(X, Y')$ . Observe that there is a vertex  $s'_1 \in S' \cap W$  because  $w_1$  is reachable from  $X$  in  $D[W]$ . If  $S'$  contains a vertex  $s'_2$  in  $Z$ , then one

can see that there are no  $(X, Y')$ -dipaths in  $D - \{s'_1, s_2\}$ . Thus, there is a 2-separation  $(W'', S'', Z'')$  of  $(X, Y')$  where  $S'' \subseteq \{s'_1, s_2\}$  and  $s_1 \in Z''$ . Hence, after possibly replacing the 2-separation  $(W', S', Z')$  by  $(W'', S'', Z'')$ , we may assume that  $S' \subset W \cup S$ .

If  $|W'| < |W|$ , then we set  $C := C'$ ,  $Y := Y'$ ,  $(W, S, Z) := (W', S', Z')$ , and redo the procedure.

If not, then the set  $R = Z \cap W'$  is not empty. Set  $L = Z \setminus R = Z \cap Z'$ . There is no arc from  $R$  to  $L$ , because  $(W', S', Z')$  is a 2-separation. Moreover, all  $(X, R)$ -dipaths must go through  $s_2$ . In particular,  $s_2 \in W'$ . Let  $D_3$  be the digraph obtained from  $D - L$  by adding an arc  $s_1 w$  for every  $w \in W$  having an inneighbour in  $L$ . We then make a recursive call to  $\text{tripod}(D_3, X)$ . This is valid by virtue of the following claim.

**Claim 20.3.** *There is an  $X$ -tripod in  $D$  if and only if there is an  $X$ -tripod in  $D_3$ .*

*Subproof.* Suppose first that  $D_3$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T_3$  by Proposition 19. If  $T_3$  is contained in  $D$ , then we are done. So we may assume that  $T_3$  is not contained in  $D$ . Then  $T_3$  contains an arc in  $s_1 w \in A(D_3) \setminus A(D)$ . It contains only one such arc since every vertex has outdegree at most one in  $T_3$  and all arcs of  $A(D_3) \setminus A(D)$  leave  $s_1$ . Furthermore the head  $w$  of this arc is in  $W$  and has an inneighbour  $z \in L$ . Since  $D$  is strong, there is an  $(s_1, z)$ -dipath  $Q$  in  $D$ . Moreover since  $s_2 \in W'$  all the  $(s_2, z)$ -dipaths must go through  $S'$ . But  $S' \subseteq W \cup \{s_1\}$ , so all  $(s_2, z)$ -dipaths must go through  $s_1$ . Thus  $Q$  does not go through  $s_2$ . It follows that all internal vertices of  $Q$  are in  $Z$ , because  $(W, S, Z)$  is a 2-separation, and so in  $L$  because there is no arc from  $R$  to  $L$ . Consequently, the digraph  $T$  obtained from  $T_3$  by replacing the arc  $s_1 w$  by the dipath  $Qzw$  is an  $X$ -tripod in  $D$ .

Suppose now that  $D$  contains an  $X$ -tripod. Then it contains an unfolded  $X$ -tripod  $T = (C, P_1, P_2, P_3)$  by Proposition 19. For  $i = 1, 2, 3$ , let  $y_i$  be the terminal vertex of  $P_i$ . Without loss of generality, we may assume that  $y_1, y_2, y_3$  appear in this order along  $C$ . If  $T$  is contained in  $D - L$ , then it is an  $X$ -tripod in  $D_3$ . Hence we may assume that  $T$  contains some vertices of  $L$ . Observe that the arcs entering  $L$  all leave  $s_1$ . Hence,  $y_i$  cannot be in  $L$ , since there are two  $(X, y_i)$ -dipaths in  $T$ , which are disjoint except for the common vertex  $y_i$ . Consequently, the intersection of  $T$  with  $D[L]$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . Moreover, the inneighbour in  $T$  of the initial vertex of  $Q$  is  $s_1$  and the outneighbour in  $T$  of the terminal vertex of  $Q$  is some vertex  $w \in W \cup \{s_1\}$ , because there is no arc from  $L$  to  $R \cup \{s_2\}$ . But  $w \neq s_1$  for otherwise  $s_1 Q s_1$  would be  $C$  and would contain at most one of the  $y_i$ , a contradiction. Thus the digraph  $T_3$  obtained from  $T$  by replacing  $s_1 Q w$  by the arc  $s_1 w$  is an  $X$ -tripod in  $D_3$ .  $\diamond$

Claims 20.1, 20.2 and 20.3 ensure that our algorithm is correct. Each time we do a recursive call, the number of vertices decreases. So we do at most  $n$  of them. Between two recursive calls, we first find a cycle of length at least 3 in linear time, and next run a sequence of Menger algorithms to produce a new 2-separation. At each step the size of the set  $W$  decreases. Therefore, we run at most  $n$  times the Menger algorithm between two recursive calls. Since a Menger algorithm runs in linear time, the time between two calls is at most  $O(n(n + m))$  and so  $\text{tripod}$  runs in  $O(n^2(n + m))$  time.  $\square$

With Lemma 20 in hands, we now deduce Theorem 18.

*Proof of Theorem 18.* For every vertex  $v$ , we examine whether there is a  $W_3$ -subdivision with centre  $v$  in  $D$ . Observe that such a subdivision  $S$  is the union of a directed cycle  $C$ , and three internally disjoint  $(v, C)$ -dipaths  $P_1, P_2, P_3$  with distinct terminal vertices  $y_1, y_2, y_3$ . The cycle  $C$  is contained in some strong component  $\Gamma$  of  $D - v$ . For  $i = 1, 2, 3$ , let  $x_i$  be the first vertex of  $P_i$  that belongs to  $\Gamma$ . Set  $X = \{x_1, x_2, x_3\}$ . Then the paths  $P_i[x_i, y_i]$ ,  $i = 1, 2, 3$ , and  $C$  form an  $X$ -tripod in  $\Gamma$ , and the  $P_i[v, x_i]$ ,  $i = 1, 2, 3$ , are internally disjoint  $(v, X)$ -dipaths in  $D - (\Gamma \setminus X)$ .

Hence for finding a  $W_3$ -subdivision with centre  $v$ , we use the following procedure to check whether there is a set  $X$  as above. First, we compute the strong components of  $D - v$ . Next, for every subset  $X$  of three vertices in the same strong component  $\Gamma$ , we run a Menger algorithm to check whether there are



three independent  $(v, X)$ -dipaths in  $D - (\Gamma \setminus X)$ . If yes, we check whether there is an  $X$ -tripod in  $\Gamma$ . If yes again, then we clearly have a  $W_3$ -subdivision with centre  $v$ , and we return ‘yes’. If not, there is no such subdivision, and we proceed to the next triple.

For each vertex  $v$ , there are at most  $n^3$  possible triples. And for each triple we run a Menger algorithm in time  $O(n + m)$  and possibly `tripod` in time  $O(n^2(n + m))$ . Hence the time spent on each vertex  $v$  is  $O(n^5(n + m))$ . As we examine at most  $n$  vertices, the algorithm runs in  $O(n^6(n + m))$  time.  $\square$

### 3.2 $TT'_4$ -subdivision

In this subsection, we prove that  $TT'_4$  is tractable. Our proof relies on the notion of *good triple*. A triple of distinct vertices  $(a', b', d')$  is *good* if there are a  $(b', a')$ -dipath  $Q$  in  $D - d'$  and three internally disjoint dipaths  $P_1, P_2, P_3$  with  $s(P_1) = s(P_2) = s(P_3) = d'$ ,  $t(P_1) = t(P_2) = b'$ , and  $t(P_3) = a'$ .

**Proposition 21.** *A digraph  $D$  contains a  $TT'_4$ -subdivision if and only if it has a good triple.*

*Proof.* If  $D$  contains a  $TT'_4$ -subdivision, then the triple formed by its  $a$ -vertex, its  $b$ -vertex and its  $d$ -vertex is good.

Conversely, suppose that  $D$  contains a good triple. Let  $(a', b', d')$  be a good triple that minimizes the sum of the lengths of the paths  $Q, P_1, P_2, P_3$  as named in the definition.

Assume for a contradiction that  $Q^\circ$  intersects  $P_3^\circ$ . Let  $a''$  be a vertex of  $Q^\circ \cap P_3^\circ$ . Then the triple  $(a'', b', d')$  is good because of the paths  $Q[b', a''], P_1, P_2, P_3[d', a'']$ , and contradicts the minimality of  $(a', b', d')$ . Hence  $Q^\circ$  does not intersect  $P_3^\circ$ .

Assume for a contradiction that  $Q^\circ$  intersects  $P_1^\circ \cup P_2^\circ$ . Let  $b''$  be the first vertex along  $Q^\circ$  in  $P_1^\circ \cup P_2^\circ$ . Without loss of generality, we may assume that  $b''$  is on  $P_1^\circ$ . Then the triple  $(a', b'', d')$  is good because of the paths  $Q[b'', a'], P_1[d', b''], Q[b', b'']P_2, P_3$ , and contradicts the minimality of  $(a', b', d')$ . Hence  $Q^\circ$  does not intersect  $P_1^\circ \cup P_2^\circ$ .

Therefore the paths  $Q, P_1, P_2, P_3$  are internally disjoint and the union of those dipaths is a  $TT'_4$ -subdivision.  $\square$

**Corollary 22.**  *$TT'_4$ -SUBDIVISION can be solved in  $O(n^5(n + m))$  time.*

*Proof.* According to Proposition 21,  $TT'_4$ -SUBDIVISION is equivalent to deciding if  $D$  has a good triple.

Now one can decide if a triple  $(a', b', d')$ -triple in  $O(n^2(n + m))$  time as follows. We check if there is an  $(a', b')$ -dipath  $Q$  in  $D - d'$ , and for every pair  $s_1, s_2$  of distinct outneighbours of  $b'$  in  $D - a'$ , we check if there are three independent  $(\{s_1, s_2, a\}, d')$ -dipath in  $D - b'$  by a Menger algorithm.

Doing this procedure for the  $O(n^3)$  triple of distinct vertices of  $D$ , one decides in  $O(n^5(n + m))$  time whether  $D$  has a good triple.  $\square$

### 3.3 $Z_4$ -subdivision

In this subsection, we show that  $Z_4$  is tractable. The proof relies on the following lemma.

**Lemma 23.** *Let  $D$  be a digraph. There is a  $Z_4$ -subdivision in  $D$  if and only if there exists four distinct vertices  $a', b', c'$  and  $d'$  in  $D$  such that the following hold.*

- (i) *There are three independent  $(d', \{a', b', c'\})$ -dipaths.*
- (ii) *There are two independent  $(b', \{a', c'\})$ -dipaths.*

*Proof.* If  $D$  contains a  $Z_4$ -subdivision  $S$ , then the vertices  $a', b', c', d'$  corresponding to  $a, b, c, d$  (as indicated on Figure 4) clearly satisfy conditions (i) and (ii).

Conversely, suppose that  $D$  contains four vertices  $a', b', c', d'$  satisfying conditions (i) and (ii). Let  $P_1, P_2, P_3$  be three independent  $(d', \{a', b', c'\})$ -dipaths with  $t(P_1) = a', t(P_2) = b'$  and  $t(P_3) = c'$ ; let  $Q_1, Q_2$  be two independent  $(b', \{a', c'\})$ -dipaths with  $t(Q_1) = a'$  and  $t(Q_2) = c'$ .

We consider such vertices  $a', b', c', d'$  and dipaths such that the sum of the lengths of  $P_1, P_2, P_3, Q_1$  and  $Q_2$  is minimized.

**Claim 23.1.**  $V(Q_1) \cap V(P_1) = \{a'\}$  and  $V(Q_2) \cap V(P_3) = \{c'\}$ .

*Subproof.* Suppose  $V(Q_1) \cap V(P_1) \neq \{a'\}$ . Then there is a vertex  $a''$  distinct from  $a'$  in  $V(Q_1) \cap V(P_1)$ . The vertices  $a'', b', c', d'$  satisfy condition (i) with  $P_1[d', a''], P_2, P_3$  and condition (ii) with  $Q_1[b', a''], Q_2$ . This contradicts our choice of  $a', b', c', d'$  and the corresponding paths, and so  $V(Q_1) \cap V(P_1) = \{a'\}$ .

The conclusion that  $V(Q_2) \cap V(P_3) = \{c'\}$  is proved in the same way; the details are omitted.  $\diamond$

**Claim 23.2.**  $(V(Q_1) \cup V(Q_2)) \cap V(P_2) = \{b'\}$ .

*Subproof.* Suppose not. Then let  $b''$  be the last vertex distinct from  $b'$  along  $P_2$  which is in  $V(Q_1) \cup V(Q_2)$ . By symmetry, we may assume that  $b'' \in V(Q_1)$ . But the four vertices  $a', b'', c', d'$  satisfy condition (i) with  $P_1, P_2[d', b''], P_3$  and condition (ii) with  $Q_1[b'', a'], P_2[b'', b']Q_2$ . This contradicts our choice of  $a', b', c', d'$  and proves our claim.  $\diamond$

**Claim 23.3.**  $V(Q_1) \cap V(P_3) = \emptyset$  and  $V(Q_2) \cap V(P_1) = \emptyset$ .

*Subproof.* Suppose not. Then  $V(Q_1) \cap V(P_3)$  or  $V(Q_2) \cap V(P_1)$  is not empty.

Assume first that these two sets are both non-empty. Let  $a''$  be a vertex in  $V(Q_2) \cap V(P_1)$  and  $c''$  be a vertex in  $V(Q_1) \cap V(P_3)$ . Then the four vertices  $a'', b', c'', d'$  satisfy condition (i) with  $P_1[d', a''], P_3[d', c''], P_2$  and condition (ii) with  $Q_2[b', a''], Q_1[b', c'']$ . This contradicts our choice of  $a', b', c', d'$ .

Hence, exactly one of the two sets is empty. By symmetry, we may assume that  $V(Q_1) \cap V(P_3) \neq \emptyset$ . Let  $b''$  be a vertex in  $V(Q_1) \cap V(P_3)$ . Now the four vertices  $a', b'', c', d'$  satisfy condition (i) with  $P_1, P_3[d', b''], P_2Q_2$  and condition (ii) with  $Q_1[b'', a'], P_3[b'', c']$ . This contradicts our choice of  $a', b', c', d'$  and proves our claim.  $\diamond$

Claims 23.1, 23.2 and 23.3 imply that  $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2$  is a  $Z_4$ -subdivision.  $\square$

**Theorem 24.**  $Z_4$ -SUBDIVISION can be solved in  $O(n^4(n + m))$  time.

*Proof.* By Lemma 23,  $Z_4$ -SUBDIVISION is equivalent to deciding whether there are four vertices satisfying the condition (i) and (ii) of the lemma. But given four vertices  $a', b', c', d'$ , one can check in linear time if conditions (i) and (ii) hold by running two Menger algorithms. Since there are  $O(n^4)$  sets of four vertices in  $D$ ,  $Z_4$ -SUBDIVISION can be solved in  $O(n^4(n + m))$  time.  $\square$

## 4 Some hard digraphs

Theorem 9 and more specifically Corollary 10 imply that many digraphs on 4 vertices are hard. We now prove that some additional digraphs that are not covered by Theorem 9 are also hard. These graphs are depicted in Figure 5, where each of the bold edges without indicated direction represents a pair of oppositely directed arcs.

**Proposition 25.** For each digraph  $N_i$ ,  $1 \leq i \leq 9$ , depicted Figure 5,  $N_i$ -SUBDIVISION is NP-complete.

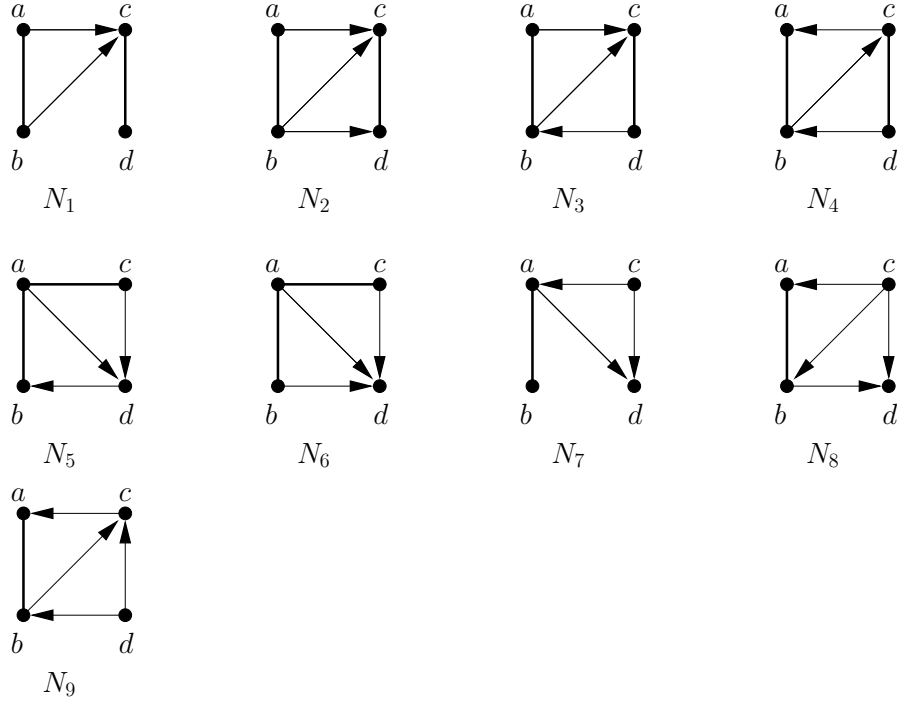


Figure 5: Some hard digraphs on 4-vertices. Bold undirected edges represent directed 2-cycles.

*Proof.* In each case, the problem is proved to be NP-complete by reduction from RESTRICTED 2-LINKAGE. Let  $D$ ,  $x_1$ ,  $x_2$ ,  $y_1$  and  $y_2$  be an instance of this problem. We construct a digraph  $D_i$  by putting  $D$  on two arcs  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  of  $N_i$  (that will be specified later), that is by taking the disjoint union of  $D$  and  $N_i$ , by removing the arcs  $e_1$  and  $e_2$  and adding the arcs  $u_1x_1$ ,  $y_1v_1$ ,  $u_2x_2$  and  $y_2v_2$ . We then show that  $D_i$  contains an  $N_i$ -subdivision if and only if there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ . This implies that  $N_i$ -SUBDIVISION is NP-complete.

Clearly, by construction of  $D_i$ , if there is a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ , then  $D_i$  contains an  $N_i$ -subdivision. We now prove the converse for each  $i$ . In each case we shall assume that  $D_i$  contains an  $N_i$ -subdivision  $S$ , and we shall denote by  $a', b', c', d'$  the vertices in  $S$  corresponding to  $a, b, c, d$ , respectively.

$i = 1$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $c' = c$ . Because  $d_{D_1}^-(c) = 3$ , the arcs  $ac$ ,  $bc$  and  $dc$  are in  $S$ . Moreover, the arc  $ba$  is in  $S$ , because every vertex has indegree at least 1 in  $S$ . Thus  $d_S^+(b) \geq 2$ , and so either  $b = b'$  or  $b = a'$ . By symmetry between  $a$  and  $b$  in  $N_1$ , we may assume that  $b = b'$ . Then, necessarily,  $a = a'$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i \in \{2, 3, 4\}$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $\{b, c\} = \{b', c'\}$ . Therefore, the arc  $bc$  is contained in  $S$ , and this shows that  $b' = b$  and  $c' = c$ . Now for degree reasons, all arcs incident to  $b$  and  $c$  must be in  $S$ . It follows that  $a' = a$  and  $d' = d$ . (This is clear for  $N_3$  and  $N_4$ . For  $N_2$ , we first conclude that  $\{a', d'\} = \{a, d\}$  and then consider degrees of  $a$  and  $d$  to obtain the same conclusion.) Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induced a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 5$ : We choose  $e_1 = ba$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $a' = a$ . Hence all

the arcs incident to  $a$  are in  $A(S)$ . Therefore  $c$  is either  $b'$  or  $c'$ . But  $d^-(c) = 1$ , so  $c$  cannot be  $b'$ , and thus  $c = c'$ . All vertices have outdegree at least 1 in  $S$ , so  $db \in A(S)$ . Now there are two internally disjoint  $(a', b)$ -dipaths in  $S - c'$ , so necessarily,  $b = b'$ . Moreover,  $d'$  must be in one of those dipaths, so  $d = d'$ . Therefore, in  $S$ , there are internally disjoint  $(b, a)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 6$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $a' = a$  and  $d' = d$ . Hence all arcs incident to those two vertices are in  $S$ . Therefore  $\{b', c'\} = \{b, c\}$ . By symmetry of  $N_6$ , we may assume that  $b' = b$  and  $c' = c$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 7$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $a' = a$ . Hence all arcs incident to  $a$  are in  $S$ . So  $c$  and  $d$  are in  $V(S)$ . Since  $d_{D_7}^+(d) = 0$ , we have  $d = d'$ ; since  $d_{D_7}^-(c) = 0$ , we have  $c = c'$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 8$ : We choose  $e_1 = ab$  and  $e_2 = cd$ . Since  $D$  contains no big vertices, we have  $b' = b$  and  $c' = c$ . Hence all arcs incident to those two vertices are in  $S$ . So  $d \in V(S)$ . Since  $d_{D_8}^+(d) = 0$ , it follows that  $d = d'$ . The arcs  $ba$  and  $ca$  show that  $d_S^-(a) \geq 2$ . Thus  $a = a'$ . Therefore, in  $S$ , there are disjoint  $(a, b)$ - and  $(c, d)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .

$i = 9$ : We choose  $e_1 = ab$  and  $e_2 = dc$ . Since  $D$  contains no big vertices, we have  $b' = b$ . Hence all arcs incident to  $b$  are in  $S$ . In particular  $c, d \in V(S)$ . Since  $d_{D_9}^-(d) = 0$ , we have  $d' = d$ . Since  $d_S^+(c) \geq 1$ , the arc  $ca$  is in  $A(S)$ , so  $d_S^-(a) = 2$ , and thus  $a \in \{a', c'\}$ . Since  $a'$  and  $c'$  are both in the outsection of  $d$  in  $N_9 - b$ ,  $S$  contains a  $(d, a)$ -dipath disjoint from  $b$ . This dipath must pass through  $c$  and therefore the arc  $y_2c$  lies in  $S$ . This implies that  $d_S^-(c) \geq 2$ , so  $c = c'$  and then we have  $a = a'$ . Consequently, in  $S$ , there are disjoint  $(a, b)$ - and  $(d, c)$ -dipaths. These two paths induce a 2-linkage from  $(x_1, x_2)$  to  $(y_1, y_2)$  in  $D$ .  $\square$

## 5 Some tractable digraphs

### 5.1 Easier cases

A *symmetric star* is a symmetric digraph associated to a star. The *centre* of a symmetric star is the centre of the star to which it is associated. A *superstar* is a digraph obtained from a symmetric star by adding an arc joining two non-central vertices. The *centre* of a superstar is the centre of the star from which it is derived. The symmetric star of order  $k + 1$  is denoted by  $SS_k$  and the superstar of order  $k + 1$  is denoted by  $SS_k^*$ . An  $SS_k$ -subdivision with centre  $a$  is the union of  $k$  internally disjoint  $(a, a)$ -handle. Therefore, one can decide if there is an  $SS_k$ -subdivision with centre  $a$  in linear time using a Menger algorithm. Bang-Jensen et al. [2] showed that  $SS_3^*$ -SUBDIVISION is polynomial-time solvable. Their result can be extended to all superstars.

**Theorem 26.** *Let  $k$  be a positive integer. Given digraph  $D$  and a vertex  $v$  of  $D$ , one can decide in  $O(n^{2k}(n + m))$ -time whether  $D$  contains an  $SS_k^*$ -subdivision with centre  $v$ .*

*Proof.* We describe a procedure that given  $v$ , a set  $X = \{x_1, \dots, x_k\}$  of  $k$  distinct outneighbours of  $v$  and a set  $Y = \{y_1, \dots, y_k\}$  of  $k$  distinct inneighbours of  $v$  checks if there is an  $SS_k^*$ -subdivision  $S$  with centre  $v$  such that  $\{vx_1, \dots, vx_k\} \cup \{y_1v, \dots, y_kv\} \in A(S)$ . (Observe that it is allowed that  $X \cap Y \neq \emptyset$ .) Such a subdivision will be called  $(v, X, Y)$ -forced.

Applying a Menger algorithm, check whether in  $D - v$  there are  $k$  disjoint dipaths  $P_1, \dots, P_k$  from  $X$  to  $Y$ . If not, then  $D$  certainly does not contain any  $(v, X, Y)$ -forced  $SS_k$ -subdivision. If yes, then check whether there is a dipath  $Q$  from some  $P_i$  to a different  $P_j$  whose internal vertices are not in

$\{v\} \cup \bigcup_{i=1}^k P_i$ . This can be done in linear time by running a search on the digraph obtained from  $D - v$  by contracting each path  $P_i$  into a single vertex. If such a  $Q$  exists, then  $P_1, \dots, P_k$  and  $Q$  together with  $v$  and the arcs from  $v$  to  $X$  and from  $Y$  to  $v$  form a  $(v, X, Y)$ -forced  $SS_k^*$ -subdivision. If not, then no  $(v, X, Y)$ -forced  $SS_k^*$ -subdivision using the chosen arcs exists, because there is no vertex  $x \in X$  with two vertices of  $Y$  in its outsection in  $D - v$ .

Applying this linear-time procedure, for every possible pair  $(X, Y)$ , we can decide in  $O(n^{2k}(n+m))$ -time whether  $D$  contains an  $SS_k^*$ -subdivision with centre  $v$ .  $\square$

**Corollary 27.** *For every positive integer  $k$ ,  $SS_k^*$ -SUBDIVISION can be solved in  $O(n^{2k+1}(n+m))$ -time.*

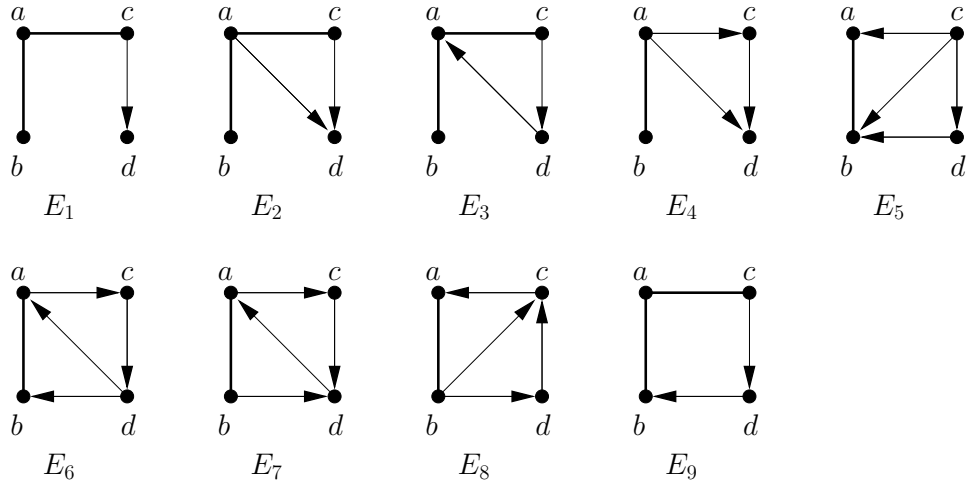


Figure 6: Some digraphs on 4-vertices, that are tractable. Bold undirected edges represent directed 2-cycles.

**Proposition 28.** *For  $1 \leq i \leq 8$ , the digraph  $E_i$  depicted in Figure 6 is tractable.*

*Proof.*  $i = 1$ : Let us describe a procedure that, given two distinct vertices  $a'$  and  $d'$  in  $D$  and two outneighbours  $s_1, s_2$  of  $a'$  distinct from  $d'$ , decides whether there is an  $E_1$ -subdivision with  $a$ -vertex  $a'$  and  $d$ -vertex  $d'$  such that  $a's_1$  and  $a's_2$  are arcs of  $S$ . Such a subdivision is said to be  $(a's_1, a's_2, d)$ -forced.

We check whether there is a dipath  $Q$  from  $\{s_1, s_2\}$  to  $d'$  in  $D - a'$ , and with a Menger algorithm we check whether there are two independent  $(\{s_1, s_2\}, a')$ -dipaths  $P_1$  and  $P_2$  in  $D - d'$ . If these three dipaths do not exist, then  $D$  contains no  $(a's_1, a's_2, d)$ -forced  $E_1$ -subdivision, and we return ‘no’. If the three paths  $Q, P_1, P_2$  exist, then we return ‘yes’. Indeed, denoting by  $c'$  the last vertex along  $Q$  in  $P_1 \cup P_2$ , the digraph  $a's_1 \cup P_1 \cup a's_2 \cup P_2 \cup Q[c', d']$  is an  $(a's_1, a's_2, d)$ -forced  $E_1$ -subdivision.

Applying the above procedure for all possible triples  $(a's_1, a's_2, d')$ , one solves  $E_1$ -SUBDIVISION in  $O(n^4(n+m))$  time.

$i = 2$ : Let us describe a procedure that given two distinct vertices  $a'$  and  $d'$  in  $D$ , a set  $U = \{u_1, u_2, u_3\}$  of three outneighbours of  $a'$ , returns ‘yes’ if it finds an  $E_2$ -subdivision and returns ‘no’ only if there is no  $E_2$ -subdivision with  $a$ -vertex  $a'$  and  $d$ -vertex  $d'$  such that  $\{a'u_1, a'u_2, a'u_3\} \subseteq A(S)$ . Such a subdivision is said to be  $(a', d', U)$ -forced.

We check with a Menger algorithm whether  $|S_{D-a'}^-(d') \cap U| \geq 2$  and whether there are three internally disjoint dipaths  $P_1, P_2, P_3$  with distinct initial vertices in  $U$  and with  $t(P_1) = t(P_2) = a'$  and  $t(P_3) = d'$ . If these two conditions are not both fulfilled, then  $D$  contains no  $(a', d', U)$ -forced  $E_2$ -subdivision, and we return ‘no’. If these conditions are fulfilled, then we return ‘yes’. Indeed consider three dipaths  $P_1, P_2, P_3$  as above. Without loss of generality,  $s(P_i) = u_i$  for  $1 \leq i \leq 3$ . Since  $|S_{D-a'}^-(d') \cap U| \geq 2$ , there exists a  $(P_1 \cup P_2, P_3)$ -dipath in  $D - a'$ . Let us denote its terminal vertex by  $d''$ . Then the union of the directed cycles  $a'u_1P_1$ ,  $a'u_2P_2$ , and the dipaths  $a'u_3P_3[u_3, d'']$ , and  $Q$  is an  $E_2$ -subdivision.

Applying the above procedure for all possible triples  $(a', d', U)$ , one solves  $E_2$ -SUBDIVISION in  $O(n^5(n+m))$  time.

$i = 3$ : Let us describe a procedure that given two distinct vertices  $a'$  and  $d'$  in  $D$  and two outneighbours  $s_1, s_2$  of  $a'$  distinct from  $d'$ , returns ‘yes’ when it finds an  $E_3$ -subdivision and returns ‘no’ only if there is no  $E_3$ -subdivision with  $a$ -vertex  $a'$  and  $d$ -vertex  $d'$  such that  $\{a's_1, a's_2\} \subseteq S$ . Such a subdivision is said to be  $(a's_1, a's_2, d')$ -forced.

We check whether there is an  $(\{s_1, s_2\}, d')$  dipath  $Q$  in  $D - a'$  and whether there are three independent  $(\{s_1, s_2, d'\}, a')$ -dipaths  $P_1, P_2, P_3$  in  $D$ . If these two conditions are not both fulfilled, then  $D$  contains no  $(a's_1, a's_2, d')$ -forced  $E_3$ -subdivision, and we return ‘no’. If these conditions are fulfilled then we return ‘yes’.

Indeed, suppose there are four such dipaths  $Q, P_1, P_2, P_3$ . We may assume without loss of generality that  $s(P_3) = d'$ . Denote by  $c'$  the last vertex along  $Q$  in  $P_1 \cup P_2$ , and by  $d''$  the first vertex in  $Q[c', d']$  which is on  $P_3$ . Then the union of the two directed cycles  $a's_1P_1a', a's_2P_2a'$  and the dipaths  $Q[c', d'']$  and  $P_3[d'', a']$  is an  $E_3$ -subdivision.

Applying the above procedure for all possible triples  $(a's_1, a's_2, d')$ , one solves  $E_3$ -SUBDIVISION in  $O(n^4(n+m))$  time.

$i = 4$ : Let us describe a procedure that, given an arc  $sa'$  and a vertex  $d' \notin \{s, a'\}$ , checks whether there is an  $E_4$ -subdivision  $S$  with  $a$ -vertex  $a'$ ,  $d$ -vertex  $d'$ , and such that  $sa' \in A(S)$ . Such a subdivision is said to be  $(sa', d')$ -forced.

We check with a Menger algorithm whether there are three independent  $(a', \{s, d'\})$ -dipaths, where two of the paths end up at  $d'$  and one at  $s$ . If three such dipaths do not exist, then there is clearly no  $(sa', d')$ -forced  $E_4$ -subdivision, and we return ‘no’. If three such dipaths exist, then their union together with the arcs  $sa'$  form an  $(sa', d')$ -forced  $E_4$ -subdivision.

Applying the above procedure for all possible pairs  $(sa', d')$ , one solves  $E_4$ -SUBDIVISION in  $O(mn(n+m))$  time.

$i = 5$ : Let us describe a procedure that, given two distinct vertices  $b', c'$  and a set  $S = \{s_1, s_2, s_3\}$  of three distinct inneighbours of  $b'$  checks whether there is an  $E_5$ -subdivision  $S'$  with  $b$ -vertex  $b'$ ,  $c$ -vertex  $c'$ , and such that  $\{s_1b', s_2b', s_3b'\} \subset A(S')$ . Such a subdivision is said to be  $(b', c', S)$ -forced.

We check with a Menger algorithm, if there are three independent  $(c', S)$ -dipaths  $P_1, P_2, P_3$ , and we check whether there is a  $(b', S \setminus \{c'\})$ -dipath  $Q$  in  $D - c'$ . If four such dipaths do not exist, then we return ‘no’ because there is no  $(b', c', S)$ -forced  $E_5$ -subdivision. If such dipaths  $P_1, P_2, P_3$  and  $Q$  exist, then let  $x$  be the first vertex of  $Q$  in  $P_1 \cup P_2 \cup P_3$ . Then the union of  $P_1, P_2, P_3, Q[b', x]$  and the three arcs  $s_1b', s_2b', s_3b'$  form a  $(b', c', S)$ -forced  $E_5$ -subdivision.

Applying the above procedure for all possible triples  $(a', b', S)$ , one solves  $E_5$ -SUBDIVISION in  $O(n^5(n+m))$  time.

$i = 6$ : Observe that every  $E_6$ -subdivision may be seen as an  $E_6$ -subdivision in which the arc  $cd$  is not subdivided. Henceforth, by an  $E_6$ -subdivision, we mean such a subdivision.

Let us describe a procedure that, given two disjoint arcs,  $sb' d'c'$ , returns ‘yes’ if it finds an  $E_6$ -subdivision and returns ‘no’ only if there is no  $E_6$ -subdivision  $S$  with  $b$ -vertex  $b'$ ,  $c$ -vertex  $c'$ ,  $d$ -vertex  $d'$  and such that  $\{sb', d'c'\} \subseteq A(S)$ . Such a subdivision is called  $(sb', d'c')$ -forced.

Applying a Menger algorithm, we check whether in  $D$  there are three independent  $(b', \{s, c', d'\})$ -dipaths  $P_1, P_2, P_3$  with  $t(P_1) = s$  and applying a search we check whether there is a  $(c', s)$ -dipath  $Q$  in  $D - \{b', d'\}$ . Clearly, if four such dipaths do not exist, then  $D$  contains no  $(sb', d'c')$ -forced  $E_6$ -subdivision, so we return ‘no’. Conversely, if these dipaths exist, then  $Q$  contains a  $(c', P_1)$ -subdipath  $R$ . Let  $c''$  be the last vertex along  $R$  in  $V(P_2 \cup P_3)$ . Now in  $P_2 \cup P_3 \cup R[c', c''] \cup d'c'$ , there are two internally disjoint  $(b', c'')$ -dipaths  $P'_2, P'_3$ . Thus  $P_1 \cup sb' \cup P'_2 \cup P'_3 \cup R[c'', t(R)]$  is an  $E_6$ -subdivision, and we return ‘yes’.

Doing this for every possible pair  $(sb', d'c')$ , one decides in  $O(m^2(n+m))$  time whether  $D$  contains an  $E_6$ -subdivision.

$i = 7$ : We proceed in two stages. We first check whether there is an  $E_7$ -subdivision in which the arc  $ab$  is not subdivided. Next we check whether there is an  $E_7$ -subdivision in which the arc  $ab$  is subdivided.

In the first stage we decide whether there is an  $E_7$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$  for some arc  $a'b'$ . To do so, for every dipath  $a'uv$  in  $D - b'$ , we check whether there is an  $E_7$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$ , and which contains the arcs of  $\{a'u, uv, a'b'\}$ . Such a subdivision is said to be  $(a'uv, a'b')$ -forced.

We proceed as follows. Applying a Menger algorithm, we check whether in  $D - u$  there are independent  $(\{v, b'\}, a')$ -dipaths  $P_1$  and  $P_2$  with  $s(P_1) = v$ , and applying a search we check whether there is a  $(v, b')$ -dipath  $Q$  in  $D - a' - u$ . Clearly, if three such dipaths do not exist, then  $D$  contains no  $(a'uv, a'b')$ -forced  $E_7$ -subdivision, so we return ‘no’. Conversely, if these dipaths exist, then  $Q$  contains a  $(P_1, P_2)$ -subdipath  $R$ . Then the union of  $P_1, P_2, R, a'uv$ , and  $a'b'$  is an  $E_7$ -subdivision, and we return ‘yes’. Doing this for every possible pair  $(a'uv, a'b')$ , one decides in  $O(m^2(n+m))$  time that either  $D$  contains an  $E_7$ -subdivision, or that  $D$  contains no  $E_7$ -subdivision in which the arc  $ab$  is not subdivided.

Let  $G_7$  be the digraph obtained from  $E_7$  by subdividing the arc  $ab$  into a dipath  $awb$  of length 2. The second stage consists in deciding whether  $D$  contains an  $G_7$ -subdivision. We use a procedure similar to the one for detecting superstar subdivision. Given a pair  $\{a'w_1x_1, a'w_2x_2\}$  of dipaths that are disjoint except for their initial vertex  $a'$ , and two distinct inneighbours  $y_1, y_2$  of  $a'$  that are not in  $\{w_1, w_2\}$  (allowing the possibility that  $\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$ ), the procedure returns ‘yes’ if it finds an  $G_7$ -subdivision and returns ‘no’ only if there is no  $G_7$ -subdivision with  $a$ -vertex  $a'$  containing all arcs in  $A' = \{a'w_1, w_1x_1, a'w_2, w_2x_2, y_1a', y_2a'\}$ . Such a subdivision is called  $A'$ -forced.

The procedure proceeds as follows. With a Menger algorithm, we first check whether in  $D - \{a', w_1, w_2\}$  there are two disjoint dipaths  $P_1, P_2$  from  $\{x_1, x_2\}$  to  $\{y_1, y_2\}$ . If not, then  $D$  certainly does not contain any  $A'$ -forced  $G_7$ -subdivision. If yes, then check whether there is a  $(P_1, P_2)$ -dipath  $Q$  in  $D - \{a', w_1, w_2\}$ . If such a dipath exists, then the union of the paths  $P_1, P_2, Q, a'w_1x_1, a'w_2x_2$  and the arcs  $y_1a'$  and  $y_2a'$  is an  $G_7$ -subdivision and we return ‘yes’. Next, we check if there is a  $(P_2, P_1)$ -dipath  $Q$  in  $D - \{a', w_1, w_2\}$ . If  $Q$  exists, we return ‘yes’. If not, then no  $A'$ -forced  $G_7$ -subdivision exists, because there is no vertex  $x \in \{x_1, x_2\}$  with two vertices of  $\{y_1, y_2\}$  in its outsection in  $D - \{a', w_1, w_2\}$ . So we return ‘no’.

This procedure runs in linear time. Thus, running it for every possible set  $A'$ , one decides in  $O(m^2n^3(n+m))$  time whether  $D$  contains an  $G_7$ -subdivision, which is nothing but an  $E_7$ -subdivision in which the arc  $ab$  is subdivided.

Doing the two stages one after another, we obtain an  $O(m^2n^3(n+m))$ -time algorithm for solving  $E_7$ -SUBDIVISION.

$i = 8$ : Similarly to the case  $i = 7$ , we proceed in two stages. We first check whether there is an  $E_8$ -subdivision in which the arc  $ab$  is not subdivided. Next we check whether there is an  $E_8$ -subdivision in which the arc  $ab$  is subdivided.

The first stage is the following. For every vertex  $a'$ , every two distinct outneighbours  $b', u$ , and every inneighbour  $t'$  of  $a'$  distinct from  $b'$  and  $u$ , we run a procedure that returns ‘yes’ if it finds an  $E_8$ -subdivision, and return ‘no’ if there is no  $E_8$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$  and whose arc

set includes  $\{t'a', a'b', a'u\}$ . Such a subdivision is called  $(t'a', a'b', a'u)$ -forced. The procedure is the following. With a Menger algorithm, we check whether in  $D - u$  there are two independent  $(b', \{a', t'\})$ -dipaths  $P_1, P_2$  and whether there is a  $(u, t')$ -dipath  $Q$  in  $D - \{a', b'\}$ . If three such paths do not exist, then  $D$  certainly contains no  $(t'a', a'b', a'u)$ -forced  $E_8$ -subdivision and we return ‘no’. If these three paths exist, we then we return ‘yes’. Indeed let  $d'$  be the first vertex along  $Q$  in  $P_1 \cup P_2$ . Now the union of  $P_1, P_2, Q[u, d'], a'b', t'a'$  and  $a'u$  is an  $E_8$ -subdivision with  $a$ -vertex  $a'$  and  $b$ -vertex  $b'$ .

Doing this for every possible triple  $(t'a', a'b', a'u)$ , one can decide in time  $O(n^2m(n+m))$  whether there is an  $E_8$ -subdivision in which the arc  $ab$  is not subdivided.

Observe that an  $E_8$ -subdivision in which  $ab$  is subdivided is an  $G_7$ -subdivision. Hence the second phase is exactly the same as the one for  $E_7$ .

Doing the two stages one after another, we obtain an  $O(m^2n^3(n+m))$ -time algorithm for solving  $E_8$ -SUBDIVISION.  $\square$

## 5.2 $E_9$ is tractable

**Theorem 29.**  $E_9$ -SUBDIVISION can be solved in  $O(n^7(n+m))$  time.

The proof relies on the following notion. A *shunt* is a digraph composed of three dipaths  $P, Q$  and  $R$  such that  $R$  has length at least 2,  $s(R) \in P, t(R) \in Q$  and  $P, Q, R^0$  are disjoint. We frequently refer to a shunt by the triple  $(P, Q, R)$ . An  $(S, T)$ -shunt is a shunt  $(P, Q, R)$  such that  $\{s(P), s(Q)\} = S$  and  $\{t(P), t(Q)\} = T$ .

We consider the following decision problem.

SHUNT

**Input:** A digraph  $D$  and four distinct vertices  $s_1, s_2, t_1, t_2$ .

**Question:** Does  $D$  contain an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt?

Assume that there are two disjoint dipaths  $P, Q$  from  $\{s_1, s_2\}$  to  $\{t_1, t_2\}$  in  $D$ . We now give some necessary and sufficient conditions considering  $P$  and  $Q$  for  $D$  to have an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.

For any vertex  $x$  in  $V(P)$ , an  $x$ -bypass is a dipath  $B$  internally disjoint from  $P$  and  $Q$  with initial vertex in  $P[s(P), x[$  and terminal vertex in  $P]x, t(P)]$ . Similarly, for any vertex  $x$  in  $V(Q)$ , an  $x$ -bypass is a dipath  $B$  internally disjoint from  $P$  and  $Q$  with initial vertex in  $Q[s(Q), x[$  and terminal vertex in  $Q]x, t(Q)]$ . If  $x$  is the end-vertex of an arc between  $P$  and  $Q$ , then every  $x$ -bypass is said to be an *arc bypass*. A *crossing* (with respect to  $P$  and  $Q$ ) is a pair of arcs  $\{uv, u'v'\}$  such that  $u$  is before  $v'$  along  $P$  and  $u'$  is before  $v$  along  $Q$ . If  $uv'$  is an arc of  $P$  and  $u'v$  is an arc of  $Q$ , then the crossing is *tight*. Otherwise it is *loose*.

Let  $C = \{uv, u'v'\}$  be a tight crossing. A  $C$ -forward path is a dipath internally disjoint from  $P$  and  $Q$  either with initial vertex  $u$  and terminal vertex  $v'$ , or with initial vertex  $u'$  and terminal vertex  $v$ . A  $C$ -backward path is a dipath internally disjoint from  $P$  and  $Q$  either with initial vertex in  $P[v', t(P)]$  and terminal vertex in  $P[s(P), u]$ , or with initial vertex in  $Q[v, t(Q)]$  and terminal vertex in  $Q[s(Q), u']$ . A  $C$ -backward arc is an arc that forms a  $C$ -backward path of length 1. A  $C$ -bypass is an  $x$ -bypass  $B$ , where  $x$  is an endvertex of a  $C$ -backward arc and if  $x \in P[s(P), u]$  (resp.  $Q[s(Q), u']$ ),  $t(B)$  is also in  $P[s(P), u]$  (resp.  $Q[s(Q), u']$ ), or if  $x \in P[v', t(P)]$  (resp.  $Q[v, t(Q)]$ ),  $s(B)$  is also in  $P[v', t(P)]$  (resp.  $Q[v, t(Q)]$ ).

**Lemma 30.** Let  $D$  be a digraph, and let  $P$  and  $Q$  be two disjoint dipaths from  $\{s_1, s_2\}$  to  $\{t_1, t_2\}$ .  $D$  has an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt if and only if one of the following holds :

- (a) there is a  $(P, Q)$ -dipath or a  $(Q, P)$ -dipath  $R$  of length  $\geq 2$ ;
- (b) there is an arc bypass for some arc  $uv$  between  $P$  and  $Q$ ;



- (c) there is a loose crossing;
- (d) there is a tight crossing  $C$  with a  $C$ -forward path, a backward path of length at least 2 or a crossing bypass.

*Proof.* Let us first prove that if one of (a)–(e) holds, then  $D$  has an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.

- (a) If such a dipath  $R$  exists, then  $(P, Q, R)$  or  $(Q, P, R)$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.
- (b) If  $B$  is a  $u$ -bypass and  $u \in V(P)$ , then  $(P[s(P), s(B)] \cup B \cup P[t(B), t(P)], Q, P[s(B), u] \cup uv)$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. There is a shunt constructed analogously if  $u \in V(Q)$  and also when  $B$  is a  $v$ -bypass.
- (c) Let  $\{uv, u'v'\}$  be a loose crossing. By symmetry, we may assume that  $uv'$  is not an arc. Then  $(P[s(P), u] \cup uv \cup Q[v, t(Q)], Q[s(Q), u'] \cup u'v' \cup P[v', t(P)], P[u, v'])$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.
- (d) Let  $C = \{uv, u'v'\}$  be a tight crossing.

If there is a  $C$ -forward path, then replacing the arc  $uv'$  on  $P$  or the arc  $u'v$  on  $Q$  by this  $C$ -forward path, we obtain two dipaths with a loose crossing, so we are done by (c).

If there is a  $C$ -backward path  $R$  of length at least 2, then  $P[s(P), u] \cup uv \cup Q[v, t(Q)], Q[s(Q), u'] \cup u'v' \cup P[v', t(P)]$  and  $R$  form an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.

Suppose now that  $B$  is a  $C$ -bypass. By symmetry and directional duality, we may assume that  $B$  is an  $x$ -bypass with  $t(B) \in P[s(P), u]$ . Let  $a = xw$  be the corresponding  $C$ -backward arc  $a$ , where  $w \in P[v', t(P)]$ . Then  $(Q[s(Q), u'] \cup u'v' \cup P[v', t(P)], P[s(P), s(B)] \cup B \cup P[t(B), u] \cup uv \cup Q[v, t(Q)], wx \cup P[x, t(B)])$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt.

Let us now prove the reciprocal by the contrapositive. Suppose for a contradiction none of (i)–(iv) holds, but  $D$  contains an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt  $(P', Q', R')$ . Without loss of generality, we may assume that this shunt maximizes  $|(A(P) \cup A(Q)) \cap (A(P') \cup A(Q'))|$ . Free to swap the names of  $P$  and  $Q$ , we may assume that  $s(P) = s(P')$ .

Let  $u$  be the farthest vertex along  $P'$  such that  $P'[s(P'), u]$  does not intersect  $Q$ . Necessarily  $u \in V(P)$  for otherwise there would be a dipath of length at least 2 from  $P$  to  $Q$ . In addition, for the same reason, if  $u \neq t(P)$ , then the out-neighbour  $v$  of  $u$  in  $P'$  must be in  $Q$ . Hence all vertices of  $P'[s(P'), u] \cap P$  are in  $P[s(P), u]$ , for otherwise there would be a  $u$ -bypass in  $P$ , which would be an arc bypass for  $uv$ . Note also that, for every vertex  $x$  in  $P[s(P), u] - P'$ , there is a subdipath of  $P'$  which is an  $x$ -bypass. So  $Q' \cap P[s(P), u] = \emptyset$ , for otherwise in  $Q'$  there would be a dipath from  $Q$  to  $P[s(P), u]$  which is either has length at least 2 or is an arc with an arc bypass in  $P'$ . Let  $R''$  be the shortest subdipath of  $P'$  with initial vertex in  $V(P)$  and terminal vertex  $s(R')$  if  $s(R') \in P'[s(P'), u]$ , and let  $R''$  be the path of length 0 ( $s(R')$ ) otherwise. Now,  $(P'', Q'', R) = (P[s(P), u] \cup P'[u, t(P)], Q', R'' \cup R')$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. Moreover if  $P'[s(P'), u] \neq P[s(P), u]$ , then  $P''$  and  $Q''$  have more arcs in common with  $P$  and  $Q$  than  $P'$  and  $Q'$ , which contradicts our choice of  $(P', Q', R')$ . Therefore  $P'[s(P'), u] = P[s(P), u]$ .

Let  $u'$  be the farthest vertex along  $Q'$  such that  $Q'[s(Q'), u']$  does not intersect  $P$ . As above, one shows that  $Q'[s(Q'), u'] = Q[s(Q), u']$ .

If  $u = t(P)$ , then  $P' = P$  and necessarily  $Q = Q'$ . Thus  $R'$  is a dipath of length at least 2 from  $P$  to  $Q$  as  $(P', Q', R')$  is a shunt, which is a contradiction. Therefore, we may assume that  $u \neq t(P)$  and similarly  $u' \neq t(Q)$ . Furthermore the out-neighbour  $v$  of  $u$  in  $P'$  is in  $V(Q)$  and the out-neighbour  $v'$  of  $u'$  is in  $V(P)$ . Since  $P'$  and  $Q'$  are disjoint,  $P'[s(P'), u] = P[s(P), u]$  and  $Q'[s(Q'), u'] = Q[s(Q), u']$ , it follows that  $C = \{uv, u'v'\}$  is a crossing with respect to  $P$  and  $Q$ , and thus a tight crossing.

Consider the dipath  $R'$ .

- Assume first that  $s(R') \in P'[s(P'), u]$ . Let  $S$  be the shortest subdipath of  $R' \cup Q'[t(R'), t(Q')]$  such that  $s(S) = s(R')$  and  $t(S) \in V(P) \cup V(Q)$ . Vertex  $t(S)$  cannot be in  $Q[s(Q), u']$  for otherwise  $S = R'$  and it would be a dipath of length at least 2 between  $P$  and  $Q$ . Furthermore,  $\{s(R')t(S), u'v'\}$  is a loose crossing, since the distance between  $u'$  and  $t(S)$  in  $Q$  is at least 2 ( $u$  is between  $s(R')$  and  $v$ , and  $v$  is between  $u'$  and  $t(S)$ ). Therefore  $t(S) \in V(P)$  and so  $t(S)$  is on  $P[v', t(P)]$ . But then  $S$  is a forward path or an arc bypass in  $P$ , a contradiction.

- Assume now that  $s(R') \in P'[v, t(P')]$ .

Set  $P^* = Q[s(Q), u'] \cup u'v \cup P'[v, t(P')]$  and  $Q^* = P[s(P), u] \cup uv' \cup Q'[v', t(Q')]$ . If  $t(R') \in Q'[v', t(Q')]$ , then  $(P^*, Q^*, R')$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. But  $P^*$  and  $Q^*$  have more arcs in common with  $P$  and  $Q$  than  $P'$  and  $Q'$ , which contradicts our choice of  $(P', Q', R')$ . Therefore  $t(R') \in Q'[s(Q'), u']$ .

Let  $S$  be the shortest subdipath of  $P'[v, s(R')] \cup R'$  such that  $t(S) = t(R')$  and  $s(S) \in V(P) \cup V(Q)$ .

Assume first that  $s(S) \in V(Q)$ . Then  $S$  is a  $C$ -backward path. Hence it must have length 1. Therefore  $s(S) \notin V(P') \cup V(Q')$  because  $R'$  has length at least 2. Let  $u_1$  be the farthest vertex on  $P'[v, t(P')]$  that is in  $V(Q)$  and such that  $P'[v, u_1]$  does not intersect  $P$ . Observe that  $u_1$  appears before  $s(S)$  in  $Q$ , for otherwise there would be a  $C$ -bypass in  $P'$ , as  $s(S) \notin P'$ . In particular,  $u_1$  is not the terminal vertex of  $P'$ . Let  $v_1$  be the first vertex after  $u_1$  along  $P'$  which is on  $P \cup Q$ . It must be in  $V(P)$  by the choice of  $u_1$ . Therefore  $u_1v_1$  is an arc because there is no dipath of length at least 2 between  $Q$  and  $P$ . Let  $u_2$  be the farthest vertex on  $Q'[v', t(Q')]$   $\cap P$  such that  $Q'[v', u_2]$  does not intersect  $Q$ . Then  $v_1$  is after  $u_2$  along  $P$ , for otherwise there would be an arc bypass in  $P$  for  $u_1v_1$ . Thus  $u_2$  is not the terminal vertex of  $Q'$ . Let  $v_2$  be the first vertex after  $u_2$  along  $Q'$  which is on  $P \cup Q$ . It must be in  $V(Q)$  by the choice of  $u_2$ . Hence  $u_2v_2$  is an arc because there is no dipath of length at least 2 between  $P$  and  $Q$ . Moreover, observe that for every vertex  $x$  in  $Q[v, u_1] - P'$  there is a subdipath of  $P'$  which is an  $x$ -bypass. Therefore  $v_2$  must be in  $Q[u_1, t(Q)]$  for otherwise it would be an arc bypass. Hence  $\{u_2v_2, u_1v_1\}$  is a crossing for  $P \cup Q$ , and so it must be tight. This implies in particular that  $s(S) \in Q[v_2, t(Q)]$ .

Set  $P^+ = P'[s(P'), u] \cup u, v' \cup Q[v', u_2] \cup u_2v_1 \cup P'[v_1, t(P')]$  and  $Q^+ = Q'[s(Q), u'] \cup u'v \cup P'[v, u_1] \cup u_1v_2 \cup Q'[v_2, t(Q')]$ . If  $s(R') \in P'[v_1, t(P')]$ , then  $(P^+, Q^+, R')$  is an  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt. But  $P^+$  and  $Q^+$  have more arcs in common with  $P$  and  $Q$  than  $P'$  and  $Q'$ , which contradicts our choice of  $(P', Q', R')$ . Therefore  $s(R') \in P'[(v, u_1)]$ . Now  $P'[v, s(R')] \cup R'$  contains a subdipath  $T$  that is internally disjoint from  $P$  and  $Q$  and has initial vertex in  $Q[v, u_1]$  and terminal vertex in  $P \cup Q[v_2, t(Q)]$ . Necessarily,  $t(T) \in V(P)$  for otherwise  $T$  is an arc bypass. Hence  $T$  is an arc. Furthermore,  $t(T)$  could not be in  $P[v', u_2]$  for otherwise  $Q'$  would contain a  $t(T)$ -bypass, which would be an arc bypass. Hence  $t(T) \in P[v_1, t(Q)]$  and  $\{u_2v_2, T\}$  is a loose crossing, a contradiction.

Assume now that  $s(S) \in V(P)$ . Then it must be in  $P[v', t(P)]$ . Since there is no dipath of length at least 2 from  $P$  to  $Q$ ,  $S$  has length 1. Moreover, since  $R'$  has length at least 2,  $s(S)$  is an internal vertex of  $R'$ , so it is not in  $V(P' \cup Q')$ . Let  $u_2$  be the farthest vertex on  $Q'[v', t(Q')]$  that is in  $V(P)$  and such that  $Q'[v', u_2]$  does not intersect  $Q$ . Then  $u_2$  appears before  $s(S)$  on  $P$ , for otherwise there would be an arc bypass for  $s(S)t(S)$  in  $P$  and so  $u_2$  is not the terminal vertex of  $Q'$ . Let  $v_2$  be the first vertex after  $u_2$  along  $Q'$  which is on  $P \cup Q$ . It must be in  $V(Q)$  by the choice of  $u_2$ , and so on  $Q[v, t(Q)]$ .  $u_2v_2$  is an arc for otherwise there would be a dipath of length 2 from  $P$  to  $Q$ . Let  $u_1$  be the farthest vertex on  $P'[v, t(P')]$  that is also in  $V(Q)$  such that  $P'[v, u_1]$  does not intersect  $P$ . Vertex  $u_1$  appears before  $v_2$  in  $Q$ , for otherwise there would be an arc bypass for  $u_2v_2$  in  $Q$ , and so  $u_1$  is not the terminal vertex of  $P'$ . Let  $v_1$  be the first vertex after  $u_1$  along

$P'$  which is on  $P \cup Q$ . It must be in  $V(Q)$  by the choice of  $u_1$ . Hence  $u_1v_1$  is an arc because there is no dipath of length at least 2 between  $Q$  and  $P$ . Moreover, observe that for every vertex  $x$  in  $P[v', u_2] - Q'$  there is a subdipath of  $P'$  which is an  $x$ -bypass. Therefore  $v_1$  must be in  $P]u_2, t(P)]$  for otherwise it would be an arc bypass. Hence  $\{u_2v_2, u_1v_1\}$  is crossing for  $P \cup Q$ , and so it must be tight. This implies in particular that  $s(S) \in P[v_1, t(P)]$ .

We then find a contradiction as in the previous case by considering  $P^+$  and  $Q^+$ .

This finishes the proof of the Lemma.  $\square$

**Theorem 31.** SHUNT can be solved in  $O(n^2(n + m))$  time.

*Proof.* We describe a procedure  $\text{shunt}(D, s_1, s_2, t_1, t_2)$ , solving SHUNT and estimate its time complexity. The procedure then check, by a Menger algorithm, if there are two disjoint dipaths  $P, Q$  from  $\{s_1, s_2\}$  to  $\{t_1, t_2\}$ , which runs in  $O(n + m)$  time. Observe that the arcs  $s_1s_2$  and  $s_2s_1$  are useless, so we remove them from  $D$  if they exist. Then we should check if there are paths of length at least 2, arc bypasses, loose crossings,  $C$ -forward paths, backward paths of length at least 2 or crossing bypasses with respect to  $P$  and  $Q$ , according to Lemma 30. For every vertex  $u \in P$  (and any vertex in  $Q$ , similarly), we do the following: if  $u$  has a neighbour in  $Q$ , we test if there is a path from  $P[s(P), u[$  to  $P]u, t(P)]$ , which would be an arc bypass. Let  $v'$  be the last vertex of  $Q$  such that  $uv'$  is an arc (and such that  $v'u$  is an arc, similarly). Then, for a vertex  $v$  in  $P]u, t(P)]$ , we check if there is a vertex  $u'$  in  $Q[s(Q), v'[$  such that  $u'v$  ( $vu'$ ) is an arc. Then if  $u, v$  and  $u'v'$  have distance at least 2 in  $P$  and  $Q$  respectively, it would be a loose crossing. Otherwise, if such edges exists there is a tight crossing  $C = \{uv', u'v\}$  containing  $u$ . We then run a Menger algorithm one more time, to test if there is a dipath from  $u$  to  $v$  in  $D - P - Q$ , which would be a forward path. So far, the running time of the algorithm is bounded by  $O(n^2(n + m))$ : the complexity of calculating the  $P$  and  $Q$  initially plus the complexity of, for each vertex in  $P \cup Q$ , look for an arc bypass, plus the running time of analysing if each pair of vertices in  $P$  or  $Q$  are part of a loose crossing and finally plus the time of looking for a forward path. Then, still considering the same tight crossing  $C$ , for every vertex  $x$  in  $P[v, t(P)]$ , we check if there is a dipath to some  $y$  in  $P[v, t(P)]$ . If it is the case and  $xy$  is an arc, we then look for dipaths from  $P[s(P), y[$  to  $P]y, u]$  and from  $P[v, x[$  to  $P]x, t(P)]$ . This can be done in  $O(n^2(n + m))$ : for every pair of vertices  $u$  and  $x$ , we uses Menger algorithm possibly three times to compute the dipaths above. So,  $\text{shunt}(D, s_1, s_2, t_1, t_2)$  runs in  $O(n^2(n + m))$  time in total.  $\square$

With Theorem 31 in hands, we now deduce Theorem 29. We believe that it could also be used to prove some other digraphs  $F$  to be tractable.

*Proof of Theorem 29.* For every vertex  $v$  of  $D$  and for every set of two outneighbours  $s_1, s_2$  and two inneighbours  $t_1, t_2$  of  $v$ , we check if there is a  $(\{s_1, s_2\}, \{t_1, t_2\})$ -shunt in  $D$ . Observe that there is an  $E_9$ -SUBDIVISION in  $D$  in which  $v$  is the  $a$ -vertex if and only if there is a shunt for a pair of outneighbours and a pair of inneighbours of  $v$ . So, since there are  $n^5$  possible choices for vertex  $v$  and its neighbours, and for each of them we apply the procedure  $\text{shunt}$  that runs in  $O(n^2(n + m))$  time, our algorithm decides whether there is an  $E_9$ -SUBDIVISION in  $D$  in  $O(n^7(n + m))$  time.  $\square$

## 6 Towards a full classification of digraphs of order 4

In this section, we review all digraphs  $D$  of order 4, and determine if they are tractable or hard or if their status is unknown.

For a digraph  $D$ , its 2-cycle graph  $G_D$  is the graph with the same vertex set in which two vertices are linked by an edge if they are in a directed 2-cycle in  $D$ . Thus, the 2-cycle graph of an oriented graph is an empty graph. We denote by  $A'(D)$  be the set of arcs of  $D$  which are not in directed 2-cycles.

Let  $F$  be a digraph of order 4. By Corollary 10, if  $F$  contains a directed 2-cycle whose vertices are big, then  $F$  is hard. So we may assume that  $F$  contains no such 2-cycles. In particular, it implies that  $G_F$  has at most one vertex of degree at least two. So  $G_F$  has at most three edges.

Case 0:  $G_F$  has no edges. Then  $F$  is tractable by Theorem 17.

Case 1:  $G_F$  has three edges. Then necessarily,  $G_F$  is the star of order 3. Hence  $F$  is either the symmetric star or the superstar of order 4. In both cases,  $F$  is tractable, see Subsection 5.1.

Case 2:  $G_F$  has exactly two edges which are non-adjacent.

If  $|A'(F)| \leq 1$ , then  $F$  has no big vertex, so by Corollary 3  $F$  is tractable.

If  $|A'(F)| \geq 2$ , then  $F$  is either one of  $N_1, N_2, N_3, N_4, O_1$  and their converses, or  $F$  has no big vertex. In the later case,  $F$  is tractable by Corollary 3. If  $F = N_i$  for  $i \in \{1, 2, 3, 4\}$ , then  $F$  is hard by Proposition 25. We do not know the complexity of  $O_1$ -subdivision.

Case 3:  $G_F$  has exactly two edges which are adjacent.

If  $A'(F)$  is empty, then  $F = SS_2 + K_1$ , where  $K_1$  is the digraph on one vertex. As discussed in Subsection 5,  $SS_2$  is tractable. Thus, by Lemma 12,  $F$  is tractable.

If  $|A'(F)| = 1$ , then  $F$  either is  $SS_2^* + K_1$ , or  $E_1$  or the converse of  $E_1$ , or is obtained from  $SS_2^*$  by gluing an arc on its centre. Now  $SS_2^* + K_1$  is tractable by Corollary 27 and Lemma 12;  $E_1$  (and thus its converse) is tractable by Proposition 28; if  $F$  is obtained from  $SS_2^*$  by gluing an arc on its centre, then it is tractable by Theorem 26 and by Lemma 13.

If  $|A'(F)| = 2$ , then  $F$  is either  $E_2, E_3, E_9, O_2$  or one of their converses. If  $F \in \{E_2, E_3, E_9\}$ , then it is tractable by Proposition 28. If  $F = O_2$ , then we do not know.

If  $|A'(F)| = 3$ , then  $F$  is either  $N_5, N_6, O_3$  or one of their converses. If  $F \in \{N_5, N_6\}$ , then it is hard by Proposition 25. The complexity of  $O_3$ -SUBDIVISION is still unknown.

Case 4:  $G_F$  has exactly one edge.

If  $F$  has no big vertices, then, by Corollary 3,  $F$  is tractable. Henceforth, we may assume that  $F$  has a big vertex, i.e. a vertex with in-degree or out-degree at least 3. Observe that it implies that  $F$  is connected and  $|A'(F)| \geq 2$ .

$|A'(F)| = 2$ , then either  $F$  obtained from  $\vec{C}_2$  by gluing a spider on one its vertices, or it is one of the  $E_i$  for  $i = 4, 5$  or one of their converses. In each of these cases,  $F$  is tractable by Lemma 13 or by Proposition 28.

If  $|A'(F)| = 3$ , then we distinguish several subcases according to the position of the arcs of  $A'(F)$  relatively to the directed 2-cycle  $C$  of  $F$ .

- $A'(F)$  induces an orientation of a star. Then  $F$  is obtained from  $W_2$  or its converse by gluing an arc on its centre. Thus  $F$  is tractable by Lemma 13.
- $A'(F)$  induces an oriented path whose first vertex is a vertex of  $C$  and whose third vertex is the other vertex of  $C$ . Then  $F$  is obtained either from the bispindle  $B(2, 1; 1)$  by gluing an arc on one of its nodes, or from  $W_2$  or its converse by gluing an arc on one of its vertices. In both cases,  $F$  is tractable by Lemma 13 and Theorem 15 and Lemma 11.
- $A'(F)$  induces an oriented 3-cycle. If this cycle is directed, then  $F$  is a *windmill*, that is a subdivision of a symmetric stars. Bang-Jensen et al. [2] proved that windmills are tractable, so  $F$  is tractable. If this cycle is not directed, then  $F$  is either  $E_4$  or its converse, or  $N_7$ . If  $F$  is  $E_4$  or its converse, then it is tractable by Proposition 28. If  $F = N_7$ , then it is hard by Proposition 25.

If  $|A'(F)| = 4$ , then it is either  $N_8$ ,  $N_9$ ,  $E_5$ ,  $E_6$ ,  $E_7$ ,  $E_8$ ,  $O_4$ ,  $O_5$ , or one of their converses. If  $F$  is  $N_8$  or  $N_9$ , then it is hard by Proposition 25. If  $F$  is  $E_5$ ,  $E_6$ ,  $E_7$ , or  $E_8$ , then it is tractable by Proposition 28. The complexity of  $O_4$ -SUBDIVISION and  $O_5$ -SUBDIVISION is still open.

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## A Solving $F$ -subdivision without using the Directed Grid Theorem

The tractability of some digraphs is given by Corollary ?? and so derives from the Directed Grid Theorem. In this appendix, we give elementary algorithms (i.e. not requiring the Directed Grid Theorem) digraphs of order 4 with no big vertices are tractable. Such algorithms were already given for some digraphs in [2]. We present here elementary algorithms for the remaining ones. We first start by oriented graphs in Subsection ?.?. Then, after giving other tools, we give the algorithms for the other digraphs of order 4 with no big vertices.

### A.1 Oriented graphs

There are four oriented of order 4 with no big vertices that were not proved to be tractable in [2]. They are depicted in Figure 7.

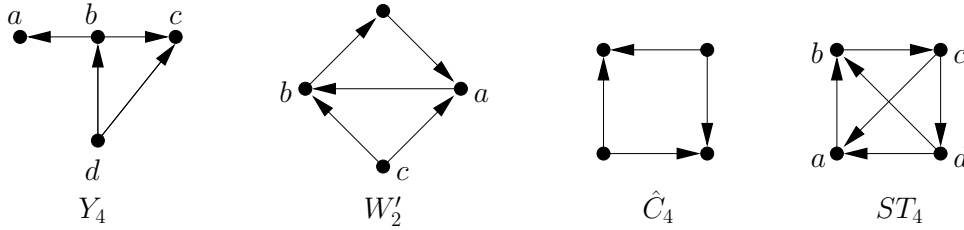


Figure 7: The oriented graphs  $Y_4$ ,  $W'_2$ ,  $\hat{C}_4$ , and  $ST_4$

#### A.1.1 $Y_4$ -subdivision

**Theorem 32.**  $Y_4$ -SUBDIVISION can be solved in  $O(n^5(n+m))$  time.

*Proof.* Let us describe a procedure that given three distinct vertices  $a'$ ,  $c'$ ,  $d'$ , and two distinct arcs  $d'u_1$  and  $d'u_2$  in  $D - a'$  decides whether a digraph  $D$  contains a  $Y_4$ -subdivision  $S$  with  $a$ -vertex  $a'$ ,  $c$ -vertex  $c'$ ,  $d$ -vertex  $d'$  such that  $\{d'u_1, d'u_2\} \subseteq A(S)$ . Such a subdivision is said to be  $(a', c', d'u_1, d'u_2)$ -forced.

We check whether there are two independent  $(\{u_1, u_2\}, c')$ -dipaths  $P_1, P_2$  in  $D - \{a', d'\}$  and a  $(\{u_1, u_2\}, a')$ -dipath  $Q$  in  $D - \{c', d'\}$ . This can be done in linear time using a Menger algorithm for each of the tasks. The existence of  $P_1, P_2, Q$  is clearly a necessary condition to contain an  $(a', c', d'u_1, d'u_2)$ -forced  $Y_4$ -subdivision. So if we do not find such dipaths, we return ‘no’. If we have such dipaths, then we return ‘yes’. Indeed the union of the dipaths  $d'u_1, d'u_2, P_1, P_2$ , and  $R$ , where  $R$  is the  $(P_1 \cup P_2, a')$ -subdipath of  $Q$ , is an  $(a', c', d'u_1, d'u_2)$ -forced  $Y_4$ -subdivision.

Doing this for every 5-tuple  $(a', c', d', u_1, u_2)$  of vertices, we obtain an algorithm solving  $Y_4$ -SUBDIVISION in  $O(n^5(n+m))$  time.  $\square$

#### A.1.2 Subdivision of the antidirected cycle of length 4

**Theorem 33.**  $\hat{C}_4$ -SUBDIVISION can be solved in  $O(n^3(n+m))$  time.

*Proof.* Observe that a  $\hat{C}_4$ -subdivision is an oriented cycle with four blocks.

We shall describe a polynomial-time procedure  $\hat{C}_4\text{-Subdivision}(a, b, D)$  that, given two vertices  $a, b$ , either finds a  $\hat{C}_4$ -subdivision (not necessarily with sources  $a$  and  $b$ ) and in this case returns ‘yes’, or verifies that there is no  $\hat{C}_4$ -subdivision in  $D$  with  $a$  and  $b$  as sources and returns ‘no’. Since a  $\hat{C}_4$ -subdivision has two sources, running this procedure for every pair  $\{a, b\}$  of vertices yields an algorithm

to decide whether  $D$  contains a  $\hat{C}_4$ -subdivision; in addition the algorithm runs in  $O(n^3(n+m))$  time, because the procedure  $\hat{C}_4\text{-Subdivision}(a, b, D)$  only needs  $O(n(n+m))$  time.

First, we determine the outsections  $S_a = S_{D-b}^+(a)$  and  $S_b = S_{D-a}^+(b)$ . If there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ , then its two sinks must be in  $X = S_a \cap S_b$ . Thus if  $|X| \leq 1$ , we return ‘no’. Henceforth, we assume that  $|X| \geq 2$ .

Let  $A$  (resp.  $B$ ) be the set of vertices  $x \in X$  such that there is an  $(a, x)$ -dipath in  $D - b$  (resp.  $(b, x)$ -dipath in  $D - a$ ) whose internal vertices are not in  $X$ . If there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ , then  $A$  and  $B$  must both be of size at least 2. Thus if  $|A| \leq 1$  or  $|B| \leq 1$ , we return ‘no’. Henceforth, we assume that  $|A| \geq 2$  and  $|B| \geq 2$ .

**Claim 33.1.** *Let  $D'$  be the digraph obtained from  $D[X]$  by adding  $a, b$  and all arcs from  $a$  to  $A$  and from  $b$  to  $B$ . Then*

- (i) *if  $D$  has a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$ , then so does  $D'$ ;*
- (ii) *if  $D'$  has a  $\hat{C}_4$ -subdivision, then so does  $D$ .*

*Subproof.* (i) Assume that  $D$  contains a  $\hat{C}_4$ -subdivision  $S$  with sources  $a$  and  $b$ , and let  $c$  and  $d$  be the sinks of  $S$ . Let  $P_1$  (resp.  $P_2, Q_1$ , and  $Q_2$ ) be the  $(a, c)$ -dipath, (resp.  $(a, d)$ -dipath,  $(b, c)$ -dipath, and  $(b, d)$ -dipath) in  $S$  and let  $a_1$  (resp.  $a_2, b_1, b_2$ ) be the last vertex of  $A$  (resp.  $A, B$ , and  $B$ ) on this path. Observe that  $V(P_1[a_1, c]) \subseteq X$  and that a similar property holds for each of the paths  $P_2[a_2, d]$ ,  $Q_1[b_1, c]$ , and  $Q_2[b_2, d]$ . This shows that the digraph which is the union of the four dipaths  $aa_1P_1[a_1, c]$ ,  $aa_2P_2[a_2, d]$ ,  $bb_1Q_1[b_1, c]$  and  $bb_2Q_2[b_2, d]$  is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D'$ .

(ii) Suppose that  $D'$  has a  $\hat{C}_4$ -subdivision  $S'$ . If  $a \in V(S')$ , let  $a_1$  and  $a_2$  be the two outneighbours of  $a$  in  $S'$ . Clearly,  $a_1, a_2 \in A$ . Therefore in  $D - b$ , there exist an  $(a, a_1)$ -dipath  $P_1$  and an  $(a, a_2)$ -dipath whose internal vertices are not in  $X$ . Let  $a'$  be the last vertex in  $P_1 \cap P_2$  on  $P_1$ . We set  $P' = \overleftarrow{P_2}[a_1, a']P_2[a', a_2]$ .

Similarly, if  $b \in V(S')$ , denoting  $b_1$  and  $b_2$  the two outneighbours of  $b$  in  $S'$ , one can find a  $(b_1, b_2)$ -inpath with two blocks whose internal vertices are not in  $X$ . Call this path  $Q'$ .

Now replacing in  $S'$  the oriented path  $a_1aa_2$  by  $P'$  if  $a \in V(S')$  and the oriented path  $b_1bb_2$  by  $Q'$  if  $b \in V(S')$  results in a  $\hat{C}_4$ -subdivision in  $D$ .  $\diamond$

By Claim 33.1, we can replace  $D$  by  $D'$ , i.e. we may assume henceforth that  $D = D'$ ,  $X = V(D) \setminus \{a, b\}$ ,  $A = N^+(a)$  and  $B = N^+(b)$ . Moreover, we will assume that  $N^-(a) = N^-(b) = \emptyset$ .

If  $|A \cap B| \geq 2$ , then we return ‘yes’. Indeed, for any two distinct vertices  $c$  and  $d$  in  $A \cap B$ , the cycle  $acbdba$  is isomorphic to  $\hat{C}_4$ . Therefore, we may assume that  $|A \cap B| \leq 1$ .

If  $|A \cap B| = 1$ , say  $A \cap B = \{d\}$ , then we check with a Menger algorithm for each vertex  $c \in V(D) \setminus \{a, b, d\}$ , whether there are independent  $(\{a, b\}, c)$ -dipaths. If there is a vertex  $c$  with two such dipaths  $P$  and  $Q$ , then we return ‘yes’. Otherwise, then we return ‘no’. This is valid by the following claim.

**Claim 33.2.** *If  $A \cap B = \{d\}$ , then  $D$  contains a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  if and only if there is a vertex  $c \in V(D) \setminus \{a, b, d\}$  such that  $D - d$  contains two independent  $(\{a, b\}, c)$ -dipaths.*

*Subproof.* If  $D$  contains a  $\hat{C}_4$ -subdivision  $S$  with sources  $a$  and  $b$ , then one of two oriented  $(a, b)$ -paths, say  $R$ , forming  $S$  does not contain  $d$ . Thus the sink in  $R$  is the desired vertex  $c$ .

If there is a vertex  $c$  as described above, then let  $P$  and  $Q$  be two independent  $(\{a, b\}, c)$ -dipaths with respective initial vertex  $a$  and  $b$ . Then  $P\overleftarrow{Q}bda$  is a  $\hat{C}_4$ -subdivision.  $\diamond$

Assume now that  $A \cap B = \emptyset$ . We take a shortest  $(a, B)$ -dipath  $P_a$  (this can be in done in linear time by Breadth-First Search). Such a path exists because  $X$  is the outsection of  $a$  in  $D - b$ . Let  $c$  be the

terminal vertex of  $P_a$ . We then search for a shortest  $(a, B \setminus \{c\})$ -dipath in  $D - c$ . If we find such a path  $Q_a$  with terminal vertex  $d$ , then we return ‘yes’. Indeed denoting by  $a'$  the last vertex in  $P_a \cap Q_a$  on  $Q_a$ , the oriented cycle  $P_a[a', c]cb\overleftarrow{Q_a}[d, a']$  is a  $\hat{C}_4$ -subdivision.

Hence we may assume that every  $(a, B)$ -dipath goes through  $c$ . Let  $b'$  be a vertex in  $B \setminus \{c\}$ , and let  $D^*$  be the digraph obtained by contracting  $\{b, b'\}$  into a vertex  $b^*$  and removing all arcs entering  $b^*$ . We return  $\hat{C}_4\text{-Subdivision}(a, b^*, D^*)$ . This is valid by Claim 33.3.

**Claim 33.3.** (i) *If there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ , then there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b^*$  in  $D^*$ .*

(ii) *If there is a  $\hat{C}_4$ -subdivision in  $D^*$ , then there is a  $\hat{C}_4$ -subdivision in  $D$ .*

*Subproof.* (i) Assume there is a  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$  in  $D$ . Let  $S$  be such a subdivision with minimum number of vertices. Let  $b_1$  and  $b_2$  be the two outneighbours of  $b$  in  $S$ .

If  $b' \notin V(S)$ , then the digraph obtained from  $S$  by replacing the vertex  $b$  and the arcs  $bb_1$  and  $bb_2$  by the vertex  $b^*$  and the arcs  $b^*b_1, b^*b_2$  is a  $\hat{C}_4$ -subdivision in  $D^*$ .

Suppose now that  $b' \in V(S)$ . Then  $bb'$  is an arc of  $S$ . Indeed if it were not, then replacing the  $(b, b')$ -path in  $S$  not containing  $a$  by the arc  $bb'$ , we would obtain a smaller  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$ . Thus, we may assume that  $b' = b_1$ .

Now  $b'$  is not a sink in  $S$ . Indeed suppose it were. Let  $Q$  be the  $(a, b')$ -dipath in  $S$ . Necessarily,  $Q$  goes through  $c$ . Thus, the digraph obtained from  $S$  by replacing  $Q$  by  $Q[a, c]$  and  $bb'$  with  $bc$  is a smaller  $\hat{C}_4$ -subdivision with sources  $a$  and  $b$ , a contradiction.

Hence,  $b'$  has an outneighbour  $b''$  in  $S$ . Then the digraph obtained from  $S$  by replacing the vertices  $b$  and  $b'$  and the arcs  $bb', b'b''$  and  $bb_2$  by the vertex  $b^*$  and the arcs  $b^*b'', b^*b_2$  is a  $\hat{C}_4$ -subdivision in  $D^*$  with sources  $a$  and  $b^*$ .

(ii) Assume that  $S^*$  is a  $\hat{C}_4$ -subdivision in  $D^*$ . If  $b^*$  is not a vertex of  $S^*$ , then  $S^*$  is contained in  $D$  and we have the result. If  $b^*$  is a vertex in  $S^*$ , then it is a source since its indegree in  $D^*$  is zero. Let  $s$  and  $t$  be its two outneighbours in  $S^*$ . By definition of  $D^*$ ,  $s$  and  $t$  are both in  $N_D^+(b) \cup N_D^+(b')$ . If  $s$  and  $t$  are both in  $N_D^+(b)$  (resp.  $N_D^+(b')$ ), then the digraph obtained from  $S^*$  by replacing the vertex  $b^*$  and the arcs  $b^*s$  and  $b^*t$  by the vertex  $b$  (resp.  $b'$ ) and the arcs  $bs$  and  $bt$  (resp.  $b's$  and  $b't$ ) respectively, is a  $\hat{C}_4$ -subdivision in  $D$ . If  $s \in N_D^+(b)$  and  $t \in N_D^+(b')$ , then the digraph obtained from  $S^*$  by replacing the vertex  $b^*$  and the arcs  $b^*s$  and  $b^*t$  by the vertices  $b, b'$  and the arcs  $bs, bb'$  and  $b't$  is a  $\hat{C}_4$ -subdivision in  $D$ .  $\diamond$

Let us now estimate the time complexity of  $\hat{C}_4\text{-Subdivision}$ . It first computes two outsections, which can be done in linear time. Then either it leads a recursive call or it does not because it stops. In the preparation of a recursive call, it possibly computes a dipath (in the case  $A \cap B = \emptyset$ ). Moreover, the order of digraph decreases by one in the call. In the second case, either it stops for some easy reason in  $O(1)$  steps, or it stops after using a Menger algorithm which runs in linear time. Let  $r$  be the number of recursive calls made by the  $\hat{C}_4\text{-Subdivision}$ . Clearly  $r \leq n$  and the procedure runs in  $O(r \times (n + m) + (n + m))$  time, that is in  $O(n(n + m))$  time.  $\square$

### A.1.3 Subdivision of $W'_2$

In this subsection, we consider  $W'_2\text{-SUBDIVISION}$ . Using Lemmas 11 and 16, one can easily give a polynomial-time algorithm to solve it.

**Theorem 34.**  $W'_2\text{-SUBDIVISION}$  can be solved in  $O(n^5(n + m))$  time.

*Proof.* By Lemma 11, applied to every vertex  $c'$  in  $D$ , for every vertex  $a'$  of  $D$ , one can decide in  $O(n^3(n + m))$  if there is a  $W_2$ -subdivision with  $b$ -vertex  $b'$ . Observe that  $W'_2$  is obtained from  $W_2$  by



subdividing once the arc  $ba$ , which is the only arc leaving  $b$ . Therefore, Lemma 16 applies. Thus for every vertex  $b'$  one can decide in  $O\left(\sum_{y \in N^+(b')} d^+(y) n^3(n+m)\right)$  if there is a  $W'_2$ -subdivision with  $b$ -vertex  $b'$ . Hence, one can decide in  $O(n^5(n+m))$  time whether there is a  $W'_2$ -subdivision in  $D$ .  $\square$

We now give a more complicated but faster algorithm based on an algorithm deciding if there is a  $W'_2$ -subdivision with prescribed  $c$ -vertex. This proof uses in a simpler way the technique that we use in Subsection 3.1 for proving that  $W_3$  is tractable.

**Theorem 35.** *Given a digraph  $D$  and a vertex  $v$  of  $D$ , one can decide in  $O(n^3(n+m))$  time if  $D$  contains a  $W'_2$ -subdivision with centre  $v$ . So  $W'_2$ -SUBDIVISION can be solved in  $O(n^4(n+m))$  time.*

The proof of this theorem relies on the following notion. Let  $X$  be a set of two vertices. An  $X$ -bipod is a digraph which is the union of a directed cycle  $C$  of length at least 3 and two disjoint dipaths  $P_1$  and  $P_2$  with initial vertices in  $X$  and terminal vertices in  $C$ . If the  $P_i$  are  $(X, C)$ -dipaths, we say that the bipod is *unfolded*. Note that the dipaths  $P_i$  may be of length 0. We often denote a bipod by the triple  $(C, P_1, P_2)$  described above.

**Proposition 36.** *Let  $X = \{x_1, x_2\}$  be a set of two distinct vertices. Any  $X$ -bipod contains an unfolded  $X$ -bipod.*

*Proof.* Let  $(C, P_1, P_2)$  be an  $X$ -bipod, where  $P_i$  has initial vertex  $x_i$ , for  $i = 1, 2$ . Let  $y_i$  be the first vertex on  $C$  along  $P_i$ . Then  $(C, P_1[x_1, y_1], P_2[x_2, y_2])$  is an unfolded  $X$ -bipod.  $\square$

We shall consider the following decision problem.

**BIPOD**

Input: A strong digraph  $D$  and a set  $X$  of two distinct vertices of  $D$ .

Question: Does  $D$  contain an  $X$ -bipod?

**Lemma 37.** *BIPOD can be solved in  $O(n(n+m))$  time.*

*Proof.* Let us describe a procedure  $\text{bipod}(D, X)$ , solving BIPOD.

We first look for a directed cycle of length at least 3 in  $D$ . This can be done in linear time. If there is no such cycle, then we return ‘no’.

Otherwise we have a directed cycle  $C$  of length at least 3. We choose a set  $Y$  of two vertices in  $C$  and run a Menger algorithm between  $X$  and  $Y$ . If this algorithm finds two disjoint  $(X, Y)$ -dipaths  $P_1, P_2$ , then we return the bipod  $(C, P_1, P_2)$ . Otherwise, the Menger algorithm finds a 1-separation  $(W, S, Z)$  of  $(X, Y)$ . Note that  $|S| = 1$  because  $D$  is strong. Set  $S = \{s\}$ .

Let  $D'$  be the digraph obtained from  $D$  by contracting  $Z$  into a vertex  $t$ . Note that  $D'$  is strong. We now make a recursive call to  $\text{bipod}(D', X)$ . This is valid by virtue of the following claim.

**Claim 37.1.** *There is an  $X$ -bipod in  $D$  if and only if there is an  $X$ -bipod in  $D'$ .*

*Subproof.* Suppose first that there is an  $X$ -bipod in  $D'$ . Then  $D'$  contains an unfolded  $X$ -bipod  $B'$  by Proposition 36. If  $B'$  is contained in  $D$ , then we are done. So we may assume that it is not. Then  $B'$  contains a dipath  $stw$  for some  $w \in W$ . It contains only one such dipath since every vertex has outdegree at most one in  $B'$ . Moreover,  $t$  has indegree 1 in  $D'$ , so it has indegree 1 also in  $B'$ . Since  $t$  was obtained by contraction of  $Z$ ,  $w$  has an inneighbour  $z \in Z$ . Now, since  $D$  is strong, there is an  $(s, z)$ -dipath  $Q$  in  $D$ . Because there is no arc from  $W$  to  $Z$ , all the internal vertices of  $Q$  are in  $Z$ . Hence the digraph  $B$  obtained from  $B'$  by replacing the dipath  $stw$  by the dipath  $Qzw$  is an  $X$ -bipod in  $D$ .

Suppose now that  $D$  contains an  $X$ -bipod. Then it contains an unfolded  $X$ -bipod  $B = (C, P_1, P_2)$  by Proposition 36. Since all  $(X, Z)$ -dipaths in  $D$  go through  $s$ , the terminal vertices of the  $P_i$  are in  $W \cup S$ , and  $D[Z] \cap B$  is a dipath  $Q$  which is a subpath of one of the  $P_i$  or  $C$ . If  $Q$  is a  $(u, z)$ -dipath, then  $B$

contains arcs  $su$  and  $zw$  for some  $w \in W$ . Then the digraph  $B'$  obtained from  $B$  by replacing  $suQzw$  by the dipath  $stw$  is an  $X$ -bipod in  $D'$ . Indeed, if  $Q$  was a subpath of  $C$ , then the directed cycle in  $B'$  has length at least 3, as it contains the three vertices  $s$ ,  $t$  and  $w$ .  $\diamond$

Each time we do a recursive call, the number of vertices decreases. So we do at most  $n$  of them. Between two recursive calls, we search for a directed cycle of length at least 3 and run a Menger algorithm. Both can be done in linear time. So `bipod` runs in  $O(n(n+m))$  time.  $\square$

With Lemma 37 in hands, we now deduce Theorem 35.

*Proof of Theorem 35.* Let  $v$  be a vertex of  $D$ . Let us describe an algorithm that decides whether there is a  $W'_2$ -subdivision with centre  $v$  in  $D$ . Observe that such a subdivision  $S$  is the union of a directed cycle  $C$  of length at least 3, and two internally disjoint  $(v, C)$ -dipaths  $P_1, P_2$  with distinct terminal vertices  $y_1, y_2$ . Since it is strong, the cycle  $C$  is contained in some strong component  $\Gamma$  of  $D - v$ . For  $i = 1, 2$  let  $x_i$  be the first vertex of  $P_i$  in  $\Gamma$ . Set  $X = \{x_1, x_2\}$ . Then the paths  $P_i[x_i, y_i]$ ,  $i = 1, 2$ , and  $C$  form an  $X$ -bipod in  $\Gamma$ , and the  $P_i[v, x_i]$ ,  $i = 1, 2$ , are independent  $(v, X)$ -dipaths in  $D - (\Gamma \setminus X)$ . Hence for finding a  $W'_2$ -subdivision with centre  $v$ , the following procedure checks whether there is a set  $X$  as above. First, we compute the strong components of  $D - v$ . Next, for every subset  $X$  of two vertices in the same strong component  $\Gamma$ , we run a Menger algorithm to check whether there are two independent  $(v, X)$ -dipaths in  $D - (\Gamma \setminus X)$ . If yes, we check using `bipod` whether there is an  $X$ -bipod in  $\Gamma$ . If yes again, then we clearly have a  $W'_2$ -subdivision with centre  $v$ , and we return ‘yes’. Otherwise, there is no such subdivision, and we proceed to the next pair.

There are at most  $n^2$  possible pairs  $X$ . And for each pair we run a Menger algorithm in  $O(n+m)$  time and possibly `bipod` in  $O(n(n+m))$  time. Hence our algorithm decides whether there is  $W'_2$ -subdivision with centre  $v$  in  $D$  in  $O(n^3(n+m))$  time.

To solve  $W'_2$ -subdivision, we check for every vertex  $v$  in turn if there is a  $W'_2$ -subdivision with centre  $v$ . As we examine at most  $n$  vertices, this algorithm runs in  $O(n^4(n+m))$  time.  $\square$

#### A.1.4 Subdivision of $ST_4$

The aim of this subsection is to prove that  $ST_4$  is tractable. We shall need the following two lemmas.

**Lemma 38.** *Let  $D$  be a digraph,  $C$  a directed cycle in  $D$ , and  $x$  a vertex in  $V(D) \setminus V(C)$ . If there are two  $(x, C)$ -dipaths  $P_1$  and  $P_2$  and a  $(C, x)$ -dipath  $Q$  such that  $s(Q)$ ,  $t(P_1)$  and  $t(P_2)$  are distinct, then  $D$  contains an  $ST_4$ -subdivision.*

*Proof.* Assume first that  $P_1$  and  $P_2$  are independent. If  $Q \cap (P_1^\circ \cup P_2^\circ) = \emptyset$ , then  $C \cup P_1 \cup P_2 \cup Q$  is an  $ST_4$ -subdivision. If  $Q$  intersects  $P_1^\circ \cup P_2^\circ$ , then without loss of generality, we may assume that the first vertex  $y$  along  $Q$  in  $P_1^\circ \cup P_2^\circ$  is on  $P_1$ . Let  $z_2$  be the first vertex in  $V(Q) \cap V(P_2)$  along  $Q$ . Such a vertex exists because  $x \in V(Q) \cap V(P_2)$ . Let  $z_1$  be the last vertex on  $Q[y, z_2]$  which is on  $P_1[y, t(P_1)]$ . Now  $Q[z_1, z_2] \cup P_1[y, t(P_1)] \cup Q[s(Q), y] \cup P_2[z_2, t(P_2)] \cup C$  is an  $ST_4$ -subdivision.

Assume now that  $P_1$  and  $P_2$  are not independent. Let  $x'$  be the last vertex in  $P_1 \cap P_2$  along  $P_1$ . Set  $P'_1 = P_1[x', t(P_1)]$ ,  $P'_2 = P_2[x', t(P_2)]$ , and let  $Q'$  be the  $(s(Q), x')$ -dipath contained in the walk  $QP_2[x, x']$ . Then  $P'_1$  and  $P'_2$  are independent  $(x', C)$ -dipaths. Hence by the previous case,  $D$  contains an  $ST_4$ -subdivision.  $\square$

**Lemma 39.** *Let  $D$  be a strong digraph,  $C$  a directed cycle in  $D$ ,  $x$  a vertex in  $V(D - C)$ . If there are three  $(x, C)$ -dipaths with distinct terminal vertices, then  $D$  contains an  $ST_4$ -subdivision.*

*Subproof.* Suppose that there are three  $(x, C)$ -dipaths  $P_1, P_2, P_3$  such that  $t(P_1)$ ,  $t(P_2)$ , and  $t(P_3)$  are distinct. Since  $D$  is strong, there is a  $(C, x)$ -dipath  $Q$ . Without loss of generality, we may assume that  $s(Q) \notin \{t(P_1), t(P_2)\}$ . Thus by Lemma 38,  $D$  contains an  $ST_4$ -subdivision.  $\diamond$

**Theorem 40.**  $ST_4$ -SUBDIVISION can be solved in  $O(n^5(n+m))$  time.

*Proof.* Since  $ST_4$  is strong, its subdivisions are also strong. So we only need to prove the result for a strong input digraph  $D$ ; if the digraph is non-strong, it suffices to check whether one of its strong components contains an  $ST_4$ -subdivision.

We shall describe a procedure  $ST_4\text{-Subdivision}(D, d')$ , that, given a strong digraph  $D$  and a vertex  $d'$ , returns ‘no’ only if there is no  $ST_4$ -subdivision in  $D$  with  $d$ -vertex  $d'$  and returns ‘yes’ when it finds an  $ST_4$ -subdivision (not necessarily with  $d$ -vertex  $d'$ ). Running this procedure for every vertex  $d'$  yields an algorithm to decide whether  $D$  contains an  $ST_4$ -subdivision; in addition, the algorithm runs in  $O(n^5(n+m))$  time, because the procedure  $ST_4\text{-Subdivision}(D, d')$  only needs  $O(n^4(n+m))$  time.

First, we check whether  $d'$  is the centre of a  $W'_2$ -subdivision. This can be done in  $O(n^3(n+m))$  time, according to Theorem 35. If not, then we return ‘no’ since every  $ST_4$ -subdivision with  $d$ -vertex  $d'$  contains a  $W'_2$ -subdivision with centre  $d'$ .

If there is a  $W'_2$ -subdivision with centre  $d'$ , let us denote by  $C$  its directed cycle, and by  $P_1$  and  $P_2$  the two  $(d', C)$ -dipaths in it. For  $i = 1, 2$ , let  $x_i$  be the terminal vertex of  $P_i$ .

Let  $S^-$  and  $S^+$  be the insection and outsection, respectively, of  $d'$  in  $D - \{x_1, x_2\}$ . We compute  $S^-$  and  $S^+$ . If  $S^-$  contains a vertex in  $V(C) \setminus \{x_1, x_2\}$ , then there is a  $(C, d')$ -dipath  $Q$  with initial vertex  $x_3 \notin \{x_1, x_2\}$ . So by Lemma 38, there is an  $ST_4$ -subdivision in  $D$ , and we return ‘yes’. Similarly, because of Lemma 39, we return ‘yes’ if  $S^+$  contains a vertex in  $V(C) \setminus \{x_1, x_2\}$ .

Assume now that  $(S^- \cup S^+) \cap (V(C) \setminus \{x_1, x_2\}) = \emptyset$ . By the definition of outsection, no arc is leaving  $S^+$  in  $D - \{x_1, x_2\}$ , so in  $D$  every arc leaving  $S^+$  has its head in  $\{x_1, x_2\}$ . Similarly, all arcs entering  $S^-$  have tail in  $\{x_1, x_2\}$ . Moreover, because  $D$  is strong, for every vertex  $s \in S^+$ , there is an  $(s, \{x_1, x_2\})$ -dipath in  $D[S^+ \cup \{x_1, x_2\}]$ .

Since  $D$  is strong, there is a directed  $(C, d')$ -dipath in  $D$ . Its first arc goes from  $\{x_1, x_2\}$  to  $S^-$ . Hence at least one vertex of  $\{x_1, x_2\}$  has an outneighbour in  $S^-$ .

**Claim 40.1.** Suppose both  $x_1$  and  $x_2$  have an outneighbour in  $S^-$ . If there is a  $(C, S^+)$ -dipath  $R$  with  $s(R) \notin \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.

*Subproof.* There is a  $(t(R), \{x_1, x_2\})$ -dipath  $P$  with internal vertices in  $S^+$ . Without loss of generality, we may assume that  $t(P) = x_1$ . Since  $x_2$  has an outneighbour in  $S^-$ , there is an  $(x_2, t(R))$ -dipath  $Q$  whose internal vertices are in  $D - C$ . Hence by the directional dual of Lemma 38 ( $ST_4$  is isomorphic to its converse),  $D$  contains an  $ST_4$ -subdivision.  $\diamond$

Each  $x_i$  has an inneighbour in  $P_i$ , and so an inneighbour in  $S^+$ . Hence a similar reasoning as the proof of Claim 40.1 gives the following.

**Claim 40.2.** If there is an  $(S^-, C)$ -dipath with terminal vertex  $x_3 \notin \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.

For  $i = 1, 2$ , let  $S_i^+$  be the set of vertices  $s$  of  $S^+$  for which there is an  $(s, x_i)$ -dipath with internal vertices in  $V(D - C)$ . In the very same way as Claim 40.1, one can prove the following claim.

**Claim 40.3.** Suppose  $x_i$  has no outneighbour in  $S^-$ . If there is a  $(C, S_i^+)$ -dipath with initial vertex  $x_3 \notin \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.

Case 1: Assume first that both  $x_1$  and  $x_2$  have an outneighbour in  $S^-$ .

Let  $T^+$  be the outsection of  $S^-$  in  $D - \{x_1, x_2\}$ ,  $T^-$  the insection of  $S^+$  in  $D - \{x_1, x_2\}$ . The definition of  $T$  implies the following property:

(T<sub>1</sub>) If  $u \in V(D) \setminus (T \cup \{x_1, x_2\})$  and  $Q$  is a  $(u, d')$ -path in  $D$  with at most two blocks, then  $Q$  contains a vertex in  $\{x_1, x_2\}$ .

Now, we compute  $T = T^- \cup T^+$ . If  $T$  contains a vertex of  $V(C) \setminus \{x_1, x_2\}$ , then we return ‘yes’, since  $D$  contains an  $ST_4$ -subdivision by Claim 40.1 or 40.2. If not, then  $T \cap (V(C) \setminus \{x_1, x_2\}) = \emptyset$ . Let  $D'$  be the digraph obtained from  $D[T \cup \{x_1, x_2\}]$  by adding the arcs  $x_1x_2$  and  $x_2x_1$  if they were not in  $A(D)$ . Observe that  $D'$  has fewer vertices than  $D$ , because the vertices of  $V(C) \setminus \{x_1, x_2\}$  are not in  $V(D')$  and this set is not empty because  $C$  has length at least 3. We then return  $ST_4\text{-Subdivision}(D', d')$ . The validity of this recursive call is established by the following claim.

**Claim 40.4.**  *$D$  contains an  $ST_4$ -subdivision with  $d$ -vertex  $d'$  if and only if  $D'$  does.*

*Subproof.* From every  $ST_4$ -subdivision in  $D'$  with  $d$ -vertex  $d'$ , one can obtain an  $ST_4$ -subdivision in  $D$  with  $d$ -vertex  $d'$  by replacing the arc  $x_1x_2$  (resp.  $x_2x_1$ ) by  $C[x_1, x_2]$  (resp.  $C[x_2, x_1]$ ).

Assume now that  $D$  contains an  $ST_4$ -subdivision  $S$  with  $d$ -vertex  $d'$ . Let  $a', b'$ , and  $c'$  be the vertices in  $S$  corresponding to  $a, b$ , and  $c$ , respectively.

Our first goal is to prove that  $a', b', c' \in V(D')$ . Let  $u \in \{a', b', c'\}$  be one of these three vertices. Note that there are three internally disjoint paths in  $S$  joining  $u$  with  $d'$ , and each of these paths has at most two blocks. If  $u \notin V(D')$ , then Property  $(T_1)$  stated above implies that each of these paths contains  $x_1$  or  $x_2$  as one of its internal vertices. Since the three paths are internally disjoint, this is not possible, and we conclude that  $u \in V(D')$ .

Hence,  $a', b', c', d'$  all belong to  $V(D')$ . Therefore, the intersection of  $S$  with  $V(D) \setminus T$  is a dipath  $P$  whose initial vertex is dominated by  $x \in \{x_1, x_2\}$  and whose terminal vertex dominates the vertex  $x'$  of  $\{x_1, x_2\} \setminus \{x\}$ . Hence,  $D'$  contains the  $ST_4$ -subdivision obtained from  $S$  by replacing  $xPx'$  by  $xx'$ .  $\diamond$

Case 2: Assume that one vertex in  $\{x_1, x_2\}$ , say  $x_1$ , has no outneighbour in  $S^-$ .

Let  $T^+$  be the outsection of  $S^-$  in  $D - \{x_1, x_2\}$ ,  $T_1^-$  the insection of  $S_1^+$  in  $D - \{x_1, x_2\}$  and  $T = T^+ \cup T_1^-$ . Observe that  $S^+ \subseteq T$  because  $d' \in S^-$ . The definition of  $T$  implies the following property:

- $(T_2)$  If  $u \in V(D) \setminus (T \cup \{x_1, x_2\})$  and  $Q$  is a  $(u, d')$ -path with at most two blocks, then either  $Q$  contains a vertex in  $\{x_1, x_2\}$ , or  $Q$  has two blocks and there is a vertex  $v \in S_2^+ \setminus (S_1^+ \cup S^-)$  such that  $Q$  is composed of a  $(u, v)$ -dipath  $R_1$  and a  $(d', v)$ -dipath  $R_2$ .

An  $ST_4$ -subdivision  $S$  is *special* if its  $d$ -vertex is  $d'$ , its  $c$ -vertex is  $x_2$ , its  $a$ -vertex is not in  $T \cup \{x_1, x_2\}$ , and  $x_1 \in V(S)$ .

We check if  $D$  contains a special  $ST_4$ -subdivision. To do so, we check for every vertex  $a'$  in  $V(D) \setminus (T \cup \{x_1, x_2\})$ , if there are two independent  $(\{x_1, x_2\}, a')$ -dipaths  $Q_1$  and  $Q_2$  in  $D$  and an  $(a', S^+)$ -dipath  $R$  in  $D - \{x_1, x_2\}$ . If we find a vertex  $a' \in V(D) \setminus (T \cup \{x_1, x_2\})$  such that three such dipaths exist, we return ‘yes’. This is valid by the following claim.

**Claim 40.5.** *Let  $a' \in V(D) \setminus (T \cup \{x_1, x_2\})$ . If there are two independent  $(\{x_1, x_2\}, a')$ -dipaths  $Q_1$  and  $Q_2$  in  $D$  and an  $(a', S^+)$ -dipath  $R$  in  $D - \{x_1, x_2\}$ , then  $D$  contains an  $ST_4$ -subdivision.*

*Subproof.* The vertex  $t(R)$  is in  $S_2^+ \setminus S_1^+$  because  $a' \notin T_1^-$ . Thus, there is a  $(t(R), x_2)$ -dipath  $R_1$  with internal vertices in  $S^+$ . Let  $y_2$  be an outneighbour of  $x_2$  in  $S^-$ . Since  $a' \notin S^-$ , the vertex  $y_2$  is not on  $R_1$ . By definition of  $S^+$  and  $S^-$ , there is a  $(y_2, R_1)$ -dipath  $R_2$  in  $D[S^+ \cup S^-]$ .

Let  $C'$  be the directed cycle  $x_2y_2 \cup R_2 \cup R_1[t(R_2), x_2]$ . Since  $y_2 \in S^-$ , there is a directed  $(y_2, x_1)$ -dipath  $R_3$  in  $D[S^- \cup S^+ \cup \{x_1\}]$ . This dipath does not intersect  $R_1$  because  $V(R_1) \setminus \{x_2\} \subseteq S_2^+ \setminus S_1^+$ . Let  $z_2$  be the last vertex along  $R_3$  that lies in  $C'$ . The three vertices  $x_2, z_2, t(R_2)$  are distinct. Moreover, the two dipaths  $Q_1$  and  $Q_2$  do not intersect  $C'$  for otherwise there would be a  $(y_2, a')$ -dipath in  $D - \{x_1, x_2\}$  and  $a'$  would be in  $T^+$ . Thus  $R_3 \cup Q_1$  contains a  $(y_3, a')$ -dipath  $R_3^*$  which is a  $(C', a')$ -dipath. Hence, we have two  $(C', a')$ -dipaths  $R_3^*$  and  $Q_2$  and the  $(a', C')$ -dipath  $R'$  contained in  $R \cup R_1[s(R_1), t(R_2)]$  whose vertices  $s(R_3^*), s(Q_2) = x_2$  and  $t(R')$  on  $C'$  are pairwise distinct. Hence, by Lemma 38,  $D$  contains an  $ST_4$ -subdivision.  $\diamond$

If for every  $a'$  in  $V(D) \setminus (T \cup \{x_1, x_2\})$ , three dipaths  $Q_1, Q_2, R$  as used above do not exist, then  $D$  has no special  $ST_4$ -subdivision, which we will assume henceforth. Let  $D'$  be the digraph obtained from  $D[T \cup \{x_1, x_2\}]$  by adding the arcs  $x_1x_2$  and  $x_2x_1$  if they were not in  $A(D)$ . Observe that  $D'$  has fewer vertices than  $D$ , because the vertices of  $V(D) \setminus \{x_1, x_2\}$  are not in  $V(D')$ . We then return  $ST_4\text{-Subdivision}(D', d')$ . The validity of this recursive call is established by the following claim.

**Claim 40.6.**  *$D$  contains an  $ST_4$ -subdivision with  $d$ -vertex  $d'$  if and only if  $D'$  does.*

*Subproof.* From every  $ST_4$ -subdivision  $S'$  with  $d$ -vertex  $d'$  in  $D'$ , one can obtain an  $ST_4$ -subdivision with  $d$ -vertex  $d'$  in  $D$  by replacing the arc  $x_1x_2$  (resp.  $x_2x_1$ ) by  $C[x_1, x_2]$  (resp.  $C[x_2, x_1]$ ).

Assume now that  $D$  contains an  $ST_4$ -subdivision  $S$  with  $d$ -vertex  $d'$ . Let  $a', b'$ , and  $c'$  be the vertices in  $S$  corresponding to  $a, b$ , and  $c$ , respectively. Each arc in  $ST_4$  corresponds to a dipath in  $S$ . We will denote these dipaths by  $S[a', b']$ ,  $S[b', c']$ , etc.

Observe that in  $S$ , there are three internally disjoint directed paths (in both directions) between  $b'$  and  $d'$ . So  $b' \in V(D')$ , because directed paths between  $V(D) \setminus V(D')$  and  $d'$  must go through  $\{x_1, x_2\}$  by Property (T<sub>2</sub>).

Next, we claim that  $a' \in V(D')$ . Suppose for a contradiction that  $a' \notin V(D')$ . Then both paths  $S[d', a']$  and  $S[c', a'] \cup S[c', d']$  must go through  $\{x_1, x_2\}$  by Property (T<sub>2</sub>). The path  $S[a', b'] \cup S[d', b']$  is thus disjoint from  $\{x_1, x_2\}$ , and by (T<sub>2</sub>) we have that  $b' \in S_2^+ \setminus (S_1^+ \cup S^-)$ . The path  $S[b', c'] \cup S[c', d']$  must go through  $x_2$  since  $b' \notin S^- \cup S_1^+$ . Thus,  $x_2$  lies on  $S[c', d']$ . Since there is no special  $ST_4$ -subdivision in  $D$ ,  $c' \neq x_2$ . Hence,  $S[c', a']$  does not meet  $\{x_1, x_2\}$ , and the path  $S[d', b'] \cup S[b', c'] \cup S[c', a']$  shows that  $a' \in S^+$ , a contradiction.

Let us prove that  $c' \in V(D')$ . Suppose for a contradiction that  $c' \notin V(D')$ . Then  $c' \notin \{x_1, x_2\}$  and both,  $S[d', b'] \cup S[b', c']$  and  $S[c', d']$  must go through  $\{x_1, x_2\}$ . Moreover,  $x_2$  is in  $S[c', d']$  because  $x_1$  has no outneighbour in  $S^-$ . Since  $x_2$  is also on  $S[d', a'] \cup S[a', b'] \cup S[b', c']$ , we conclude that  $x_2 \in S[b', c']$ . Now, the path  $S[d', a'] \cup S[c', a']$  gives a contradiction to the property (T<sub>2</sub>).

We have shown that  $\{a', b', c', d'\} \subseteq V(D')$ . Therefore, the part of  $S$  outside  $D'$  is a directed path  $P$  whose initial vertex is dominated by  $x \in \{x_1, x_2\}$  and whose terminal vertex dominates the vertex  $x'$  of  $\{x_1, x_2\} \setminus \{x\}$ . Hence,  $D'$  contains the  $ST_4$ -subdivision obtained from  $S$  by replacing  $xPx'$  by  $xx'$ .  $\diamond$

Each time we do a recursive call, the number of vertices decreases. So we do at most  $n$  of them. Between two recursive calls, we search for a  $W'_2$ -subdivision, which can be done in  $O(n^3(n + m))$  by Theorem 35, and we compute some outsections and insection, which can be done in linear time. The only part that may need more time is in Case 2, when we check for every  $a'$  in  $V(D) \setminus (T \cup \{x_1, x_2\})$  if  $D$  contains a special  $ST_4$ -subdivision. Each such test needs linear time by Claim 40.5. During this procedure, we either discover an  $ST_4$ -subdivision or not. If yes, we have spent at most  $O(n(n + m))$  time for completing this task. Otherwise we spend linear time per vertex  $a'$ , which is henceforth omitted when we proceed with the recursive call. This shows that  $ST_4\text{-Subdivision}$  runs in  $O(n^4(n + m))$  time.  $\square$

## A.2 Others digraphs of order 4

There are also some digraphs with no big vertices and with order 4 that are not oriented graphs and that have not been proved to be tractable without the Directed Grid Theorem. They are depicted in Figure 8. In this subsection, we give some elementary algorithms showing that those digraphs are tractable.

### A.2.1 More tools

Some digraphs can be proved tractable using linkage, because  $k$ -LINKAGE is polynomial-time solvable when restricted to some classes of digraphs. This for example the case for acyclic digraphs, as shown

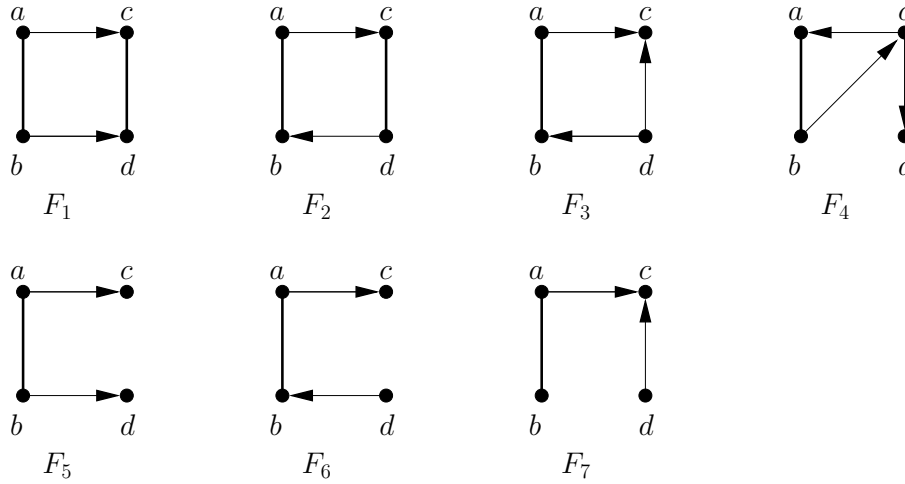


Figure 8: Some digraphs on 4-vertices, that are tractable. Bold undirected edges represent directed 2-cycles.

Fortune, Hopcroft and Wyllie [7].

**Theorem 41** (Fortune, Hopcroft and Wyllie [7]). *For every fixed  $k$  the  $k$ -LINKAGE problem for acyclic digraphs can be solved in time  $O(n^k(n + m))$ .*

For every fixed  $k$ ,  $k$ -LINKAGE is also polynomial-time solvable on digraphs of bounded DAG-width [3], on digraphs of bounded directed tree-width [9], or on tournaments [5].

A *feedback vertex set* or *cycle transversal* in a digraph  $D$  is a set of vertices  $S$  such that  $D - S$  is acyclic. The minimum number of vertices in a cycle transversal of  $D$  is the *cycle-transversal number* and is denoted by  $\tau(D)$ . The maximum number of disjoint directed cycles in a digraph  $D$  is called the *cycle-packing number* and is denoted by  $\nu(D)$ . Clearly,  $\nu(D) \leq \tau(D)$ . Conversely, proving the so-called Gallai-Younger Conjecture, Reed et al. [12] proved that there exists a minimum function  $f$  such that  $\tau(D) \leq f(\nu(D))$ . It is obvious that  $f(1) = 1$  and McCuaig [11] proved that  $f(2) = 3$ . Using this function  $f$ , Reed et al. [12] gave a polynomial-time algorithm to decide for every fixed  $k$  whether a digraph  $D$  contains  $k$  disjoint directed cycles. Basically, it tests all possible sets  $T$  of  $f(k)$  vertices. If none of them is a cycle transversal, then it returns ‘yes’. If one of them is a cycle transversal, it reduces the problem to a finite number (but depending on  $k$ ) of  $f(k)$ -linkage problem in  $D - T$ . See also [8] for details.

**Theorem 42** (Reed et al. [12]). *Let  $k$  be a fixed integer. There is an algorithm running in time  $O(n^{f(k)}(n + m))$  that decides whether there are  $k$  disjoint directed cycles in a digraph.*

Theorem 42 and Lemma 5 directly imply the following.

**Corollary 43.** *Let  $k$  be a fixed integer. There is an algorithm running in time  $O(n^{f(k)}(n + m)m)$  that finds  $k$  disjoint directed cycles in a digraph if they exist, and returns ‘no’ otherwise.*

In fact, Reed et al. proved the following stronger statement than Theorem 42.

**Theorem 44** (Reed et al. [12]). *For any digraph  $F$ ,  $F$ -SUBDIVISION is polynomial-time solvable when restricted to the class of digraphs with bounded cycle-transversal number.*

Note that this results is implied by the one of Berwanger et al. [3] stating that for every fixed  $k$ ,  $k$ -LINKAGE is polynomial-time solvable on digraphs of bounded DAG-width.

### A.2.2 $F_1$ is tractable

**Theorem 45.**  $F_1$ -SUBDIVISION can be solved in  $O(n^4m(n+m))$ -time.

*Proof.* Let  $D$  be a digraph and let  $x$  be a vertex of  $D$ . An  $F_1$ -subdivision is  $x$ -suitable, if  $x$  is on the subdivision of the directed cycle  $aba$ .

We shall present a procedure  $F_1\text{-Subdivision}(D, x)$ , that given a digraph  $D$  and a vertex  $x$  returns ‘no’ only if there is no  $x$ -suitable  $F_1$ -subdivision, and returns ‘yes’ when it finds an  $F_1$ -subdivision (not necessarily  $x$ -suitable). Moreover, this procedure runs in  $O(n^3m(n+m))$  time. Hence running  $F_1\text{-Subdivision}(D, x)$  for every vertex  $x \in V(D)$ , one solves  $F_1$ -SUBDIVISION in  $O(n^4m(n+m))$  time.

$F_1\text{-Subdivision}(D, x)$  uses a subprocedure  $\text{Reduction}(D, x, \mathcal{S})$  that, given a 1-separation  $\mathcal{S} = (W_1, T, W_2)$  in  $D$  such that  $x \in W_1$  and  $W_2 \neq \emptyset$ , reduces the problem to two smaller instances of  $F_1\text{-Subdivision}$ .  $\text{Reduction}(D, x, \mathcal{S})$  proceeds as follows.

Let  $y$  be a vertex in  $W_2$ . We run a Menger algorithm that finds a 1-separation  $(W'_1, T', W'_2)$  of  $(x, y)$ . The set  $W'_1$  is the set of vertices reachable from  $x$  in  $D - T'$ . We then replace  $\mathcal{S}$  by  $\mathcal{S}'$ , that is, we set  $W_1 := W'_1$ ,  $T := T'$ , and  $W_2 := W'_2$ . So now every vertex in  $W_1$  can be reached from  $x$ .

If  $T = \emptyset$ , then we return  $F_1\text{-Subdivision}(D[W_1], x)$ . This is clearly valid since all the vertices of an  $x$ -suitable  $F_1$ -subdivision are in the outsection of  $x$  and thus cannot be in  $W_2$  because there are no arcs from  $W_1$  to  $W_2$ .

Suppose now that  $|T| = 1$ , say  $T = \{t\}$ . A vertex  $w_1$  of  $W_1$  is  $W_2$ -reachable if in  $D$  there exists a  $(t, w_1)$ -dipath whose internal vertices are all in  $W_2$ , and a vertex  $w_2$  of  $W_2$  is  $W_1$ -reaching if in  $D$  there exists a  $(w_2, t)$ -dipath whose internal vertices are all in  $W_1$ . Let  $D_1$  be the digraph obtained from  $D[W_1 \cup \{t\}]$  by adding the arc  $tw_1$  (if it is not already in  $A(D)$ ) for every  $W_2$ -reachable vertex  $w_1 \in W_1$ ; let  $D_2$  be the digraph obtained from  $D[W_2 \cup \{t\}]$  by adding the arc  $w_2t$  (if it is not already in  $A(D)$ ) for every  $W_1$ -reaching vertex  $w_2$  of  $W_2$ .

$\text{Reduction}(D, x, \mathcal{S})$  returns  $(F_1\text{-Subdivision}(D_1, x)$  or  $F_1\text{-Subdivision}(D_2, t)$ ).

The validity of the subprocedure  $\text{Reduction}$  is justified by the following claim.

**Claim 45.1.** (i) If  $D$  contains an  $x$ -suitable  $F_1$ -subdivision, then either  $D_1$  contains an  $x$ -suitable  $F_1$ -subdivision or  $D_2$  contains a  $t$ -suitable  $F_1$ -subdivision.

(ii) For any  $i = 1, 2$ , if  $D_i$  contains an  $F_1$ -subdivision, then  $D$  contains an  $F_1$ -subdivision.

*Subproof.* (i) Assume that  $D$  contains an  $x$ -suitable  $F_1$ -subdivision  $S$ . Let  $C_1$  and  $C_2$  be the directed cycles in  $S$  corresponding to  $aba$  and  $cdc$ , respectively, and let  $P_1$  and  $P_2$  be the two disjoint  $(V(C_1), V(C_2))$ -dipaths in  $S$ . By definition,  $x \in V(C_1)$ .

We distinguish several cases according to the position of  $C_1$  and  $C_2$ .

Assume first that  $C_1$  is contained in  $D[W_1 \cup \{t\}]$ . Since all dipaths from  $W_1$  to  $W_2$  go through  $t$ , one of the  $P_i$ , say  $P_1$ , is in  $D[W_1]$ , and  $V(C_2) \cap W_1 \neq \emptyset$ .

- Suppose that  $C_2$  is in  $D[W_1]$ . If  $V(P_2) \cap W_2 = \emptyset$ , then  $S$  is an  $x$ -suitable  $F_1$ -subdivision in  $D_1$ . If  $V(P_2) \cap W_2 \neq \emptyset$ , then there is a vertex  $w_1$  of  $W_1 \cap V(P_2)$  such that  $P_2 \cap D[W_2] = P_2]t, w_1[$ . Therefore the digraph  $S_1$  obtained from  $S$  by replacing  $P_2[t, w_1]$  by the arc  $tw_1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D_1$ .
- Suppose now that  $V(C_2) \cap W_2 \neq \emptyset$ . Then necessarily  $t \in V(C_2)$ , and  $P_1$  and  $P_2$  are in  $D[W_1 \cup \{t\}]$ . Moreover, there is a vertex  $w_1$  of  $W_1 \cap V(C_2)$  such that  $C_2 \cap D[W_2] = C_2]t, w_1[$ . Therefore the digraph  $S_1$  obtained from  $S$  by replacing  $C_2[t, w_1]$  by the arc  $tw_1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D_1$ .

Suppose now that  $V(C_1) \cap W_2 \neq \emptyset$ . Since  $x \in V(C_1)$ ,  $C_1$  necessarily contains  $t$ , because there is no arcs from  $W_1$  to  $W_2$ . Moreover, there exist two vertices  $w_1 \in W_1 \cap V(C_1)$  and  $w_2 \in W_2 \cap V(C_1)$  such that  $C_1 = C_1[t, w_2]w_2w_1C_1[w_1, t]$ ,  $D[W_1] \cap C_1 = C_1[w_1, t]$  and  $D[W_2] \cap C_1 = C_1[t, w_2]$ . Now,  $C_2$  is contained in  $D[W_1 \cup W_2]$ , and so  $C_2$  is contained either in  $D[W_1]$  or in  $D[W_2]$ .

- Assume that  $C_2$  is in  $D[W_1]$ . Let  $w_1^i$  be the first vertex along  $P_i$  in  $W_1$ . Since there is no arc from  $W_1$  to  $W_2$ , all vertices in  $P_i[s(P_i), w_1^i]$  are in  $W_2$  and all vertices in  $P_i[w_1^i, t(P_i)]$  are in  $W_1$ .
  - a) If  $s(P_1) = w_1^1$  and  $s(P_2) = w_1^2$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[t, w_1]$  by the arc  $tw_1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D_1$ .
  - b) If  $s(P_1) \neq w_1^1$  and  $s(P_2) = w_1^2$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[t, w_1]$  and  $P_1[s(P_1), w_1^1]$  by the arcs  $tw_1$  and  $tw_1^1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D_1$ .
  - c) Assume finally that  $s(P_1) \neq w_1^1$  and  $s(P_2) \neq w_1^2$ . Then both  $s(P_1)$  and  $s(P_2)$  are in  $W_2 \cup \{t\}$ . Since every vertex of  $W_1$  is reachable from  $x$  in  $D[W_1]$ , there is a  $(V(C_1) \cap W_1, V(C_2))$ -dipath  $Q$  in  $D[W_1]$ . Observe that  $s(Q)$  is distinct from  $s(P_1)$  and  $s(P_2)$  because it is in  $W_1$ . If  $Q$  does not intersect  $P_1 \cup P_2$ , set  $Q' := Q$ . Otherwise, without loss of generality, the first vertex  $z$  along  $Q$  in  $P_1 \cup P_2$  is in  $P_2$ . In this case, set  $Q' = Q[s(Q), z]P_2[z, t(P_2)]$ . In both cases, the subdigraph  $S'$  obtained from  $S$  by replacing  $P_2$  by  $Q'$  is an  $F_1$ -subdivision. Now Subcase (b) applies to  $S'$ , so  $D_1$  contains an  $x$ -suitable  $F_1$ -subdivision in  $D_1$ .
- Assume that  $C_2$  is in  $D[W_2]$ . Then  $P_1$  and  $P_2$  are in  $D[W_2 \cup \{t\}]$ , and so  $C_1 \cap D[W_1] = C_1[w_1, t]$ . Hence the digraph  $S_2$  obtained from  $S$  by replacing  $C_1[w_2, t]$  by the arc  $w_2t$  is a  $t$ -suitable  $F_1$ -subdivision in  $D_2$ .

(ii) Suppose that  $S_1$  is an  $F_1$ -subdivision in  $D_1$ . By construction of  $D_1$ , all arcs of  $A(S_1) \setminus A(D)$  are joining  $t$  to some  $W_2$ -reachable vertex. Since each vertex in  $F_1$  has outdegree at most 2, there are at most two arcs in  $A(S_1) \setminus A(D)$ .

If there is no arc in  $A(S_1) \setminus A(D)$ , then  $S_1$  is an  $F_1$ -subdivision in  $D$ . If there is a unique arc  $tw_1$  in  $A(S_1) \setminus A(D)$ , then the digraph  $S$  obtained from  $S_1$  by replacing the arc  $tw_1$  by a  $(t, w_1)$ -dipath with internal vertices in  $W_2$  is an  $F_1$ -subdivision contained in  $D$ . Assume finally that  $A(S_1) \setminus A(D)$  contains two arcs,  $tw_1$  and  $tw_1'$ . Note that  $t$  has indegree 1 and outdegree 2 in  $S_1$ . Let  $P$  (resp.  $P'$ ) be a  $(t, w_1)$ -dipath (resp.  $(t, w_1')$ -dipath) with all internal vertices in  $W_2$ . Let  $t'$  be the last vertex along  $P'$  which is in  $V(P) \cap V(P')$ . Now the digraph  $S$  obtained from  $S_1$  by replacing  $tw_1$  and  $tw_1'$  by the union of  $P$  and  $P'[t', w_1']$  is an  $F_1$ -subdivision contained in  $D$ .

A similar argument shows that if  $D_2$  contains an  $F_1$ -subdivision, then  $D$  contains an  $F_1$ -subdivision.

◇

Using Reduction, we construct another procedure `cleaning( $C_1, C_2, x$ )` that given two disjoint directed cycles  $C_1$  and  $C_2$  and the vertex  $x$ , either reduces the problem or finds a pair of disjoint directed cycles  $(C'_1, C'_2)$  such that  $x \in V(C'_1)$ . This procedure proceeds as follows.

If  $C_1$  contains  $x$ , then we set  $(C'_1, C'_2) := (C_1, C_2)$ . If  $C_2$  contains  $x$ , then we set  $(C'_1, C'_2) := (C_2, C_1)$ .

Assume now that  $x$  is not in  $V(C_1 \cup C_2)$ . We first check whether there is a cycle  $C$  containing  $x$ . If not, then we return 'no' because  $D$  does certainly not contain any  $x$ -suitable  $F_1$ -subdivision. If  $C$  does not intersect  $C_1$ , then we set  $(C'_1, C'_2) := (C, C_1)$ . If  $C$  does not intersect  $C_2$ , then we set  $(C'_1, C'_2) := (C, C_2)$ . Henceforth,  $C$  intersects both  $C_1$  and  $C_2$ . Let  $y$  be the first vertex after  $x$  in  $C$  that is in  $V(C_1 \cup C_2)$ , and let  $z$  be the last vertex before  $x$  in  $C$  that is in  $V(C_1) \cup V(C_2)$ . Free to permute the indices of  $C_1$  and  $C_2$ , we may assume that  $y \in V(C_1)$ . Moreover, if  $z \in V(C_1)$ , then we set  $(C'_1, C'_2) := (C[x, y]C_1[y, z]C[z, x], C_2)$ . So we may assume that  $z \in V(C_2)$ . Using a Menger algorithm, we check whether there are two disjoint  $(x, C_2)$ -dipaths. If not, then we obtain



a 1-separation  $\mathcal{S} = (W_1, T, W_2)$  in  $D$  such that  $x \in W_1$  and  $V(C_2) \subseteq T \cup W_2$ . In that case, we return  $\text{Reduction}(D, x, \mathcal{S})$ . Suppose now that there are two independent  $(x, C_2)$ -dipaths  $Q_1$  and  $Q_2$ . If  $Q_i$  does not intersect  $C_1$ , then the closed walk  $Q_i C_2[t(Q_i), y] C[y, x]$  contains a cycle through  $x$ . We return this cycle and  $C_1$  as  $(C'_1, C'_2)$ . If  $Q_1$  and  $Q_2$  both intersect  $C_1$ , then there are two disjoint  $(C_1, C_2)$ -dipaths, whose union with  $C_1$  and  $C_2$  is an  $F_1$ -subdivision. So we return ‘yes’. This finishes the subprocedure  $\text{cleaning}(C_1, C_2, x)$ .

Finally, let us describe  $F_1\text{-Subdivision}(D, x)$ .

We first check whether there are two disjoint directed cycles in  $D$ . If not, then we return ‘no’ because  $D$  cannot contain an  $F_1$ -subdivision in this case. Henceforth, we may assume that there are two disjoint directed cycles  $\Gamma_1$  and  $\Gamma_2$ .

We then run  $\text{cleaning}(\Gamma_1, \Gamma_2, x)$ . If the instance was not reduced by this procedure, we get two disjoint directed cycles  $(\Gamma'_1, \Gamma'_2)$  such that  $x \in V(\Gamma'_1)$ .

We run a Menger algorithm to check whether there are two disjoint  $(V(\Gamma'_1), V(\Gamma'_2))$ -dipaths. If two such dipaths  $P_1$  and  $P_2$  exist, then  $\Gamma'_1 \cup \Gamma'_2 \cup P_1 \cup P_2$  is an  $x$ -suitable  $F_1$ -subdivision, and we return ‘yes’. If not, then the Menger algorithm returns a 1-separation  $\mathcal{S} = (W_1, T, W_2)$  of  $(V(\Gamma'_1), V(\Gamma'_2))$ . If  $x \in W_1$ , then we return  $\text{Reduction}(D, x, \mathcal{S})$ . If  $x \notin W_1$ , then  $T = \{x\}$ . In this case we proceed as described below.

From now on, we may assume that  $D$  contains no  $x$ -suitable  $F_1$ -subdivision in which the cycle through  $x$  lies in  $D[W_2 \cup \{x\}]$  and the other cycle lies in  $D[W_1]$ . This is guaranteed either by nonexistence of  $C_1$  or  $C_2$ , or by the outcome of the previous step.

Let  $D'_1$  be the digraph obtained from  $D$  by contracting  $W_2$  into a vertex  $u$  and let  $D_2$  be the digraph obtained from  $D[W_2 \cup \{x\}]$  by adding the arc  $w_2 x$  (if it is not already in  $A(D)$ ) for every  $W_1$ -reaching vertex  $w_2$  of  $W_2$ . We return  $(F_1\text{-Subdivision}(D'_1, x)$  or  $F_1\text{-Subdivision}(D_2, x))$ .

This is valid by the following claim whose proof is very similar to the one of Claim 45.1.

**Claim 45.2.** (i) *If  $D$  contains an  $x$ -suitable  $F_1$ -subdivision, then either  $D'_1$  contains an  $x$ -suitable  $F_1$ -subdivision or  $D_2$  contains a  $x$ -suitable  $F_1$ -subdivision.*

(ii) *If  $D'_1$  or  $D_2$  contains an  $F_1$ -subdivision, then  $D$  contains an  $F_1$ -subdivision.*

*Subproof.* (i) Assume that  $D$  contains an  $x$ -suitable  $F_1$ -subdivision  $S$ . Let  $C_1$  and  $C_2$  be the directed cycles in  $S$  corresponding to  $aba$  and  $cdc$ , respectively, and let  $P_1$  and  $P_2$  be the two disjoint  $(V(C_1), V(C_2))$ -dipaths in  $S$ . By definition,  $x \in V(C_1)$ .

We distinguish several cases according to the positions of  $C_1$  and  $C_2$ . Suppose first that  $C_1$  is contained in  $D[W_1 \cup \{x\}]$ . Since all dipaths from  $W_1$  to  $W_2$  go through  $x$ , one of the  $P_i$ , say  $P_1$ , is in  $D[W_1]$ , and  $V(C_2) \cap W_1 \neq \emptyset$ . Hence  $C_2$  is in  $D[W_1]$ . If  $V(P_2) \cap W_2 = \emptyset$ , then  $S$  is an  $x$ -suitable  $F_1$ -subdivision in  $D'_1$ . If  $V(P_2) \cap W_2 \neq \emptyset$ , then  $P_2$  contains  $x$  and there is a vertex  $w_1$  of  $W_1 \cap V(P_2)$  such that  $P_2 \cap D[W_2] = P_2[x, w_1]$ . Therefore the digraph  $S_1$  obtained from  $S$  by replacing  $P_2[x, w_1]$  by the dipath  $xuw_1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D'_1$ .

The second possibility is that  $C_1 \subseteq D[W_2 \cup \{x\}]$ . Since  $C_1$  contains  $x$ ,  $C_2$  is contained in  $D[W_1 \cup W_2]$ , and so  $C_2$  is contained either in  $D[W_1]$  or in  $D[W_2]$ . If  $C_2$  lies in  $D[W_1]$ , then the digraph  $S'$  obtained from  $S$  by contracting all the vertices of  $W_2$  into  $u$  is an  $x$ -suitable  $F_1$ -subdivision in  $D'_1$ . If  $C_2 \subseteq D[W_2]$ , then  $S$  is an  $x$ -suitable  $F_1$ -subdivision in  $D_2$ .

Finally, suppose that  $V(C_1) \cap W_2 \neq \emptyset$  and  $V(C_1) \cap W_1 \neq \emptyset$ . Then there exist two vertices  $w_1 \in W_1 \cap V(C_1)$  and  $w_2 \in W_2 \cap V(C_1)$  such that  $C_1 = C_1[x, w_2]w_2w_1C_1[w_1, x]$ ,  $D[W_1] \cap C_1 = C_1[w_1, x]$ , and  $D[W_2] \cap C_1 = C_1[x, w_2]$ . Now  $C_2$  is contained in  $D[W_1 \cup W_2]$ , and so  $C_2$  is contained either in  $D[W_1]$  or in  $D[W_2]$ .

- Suppose first that  $C_2 \subseteq D[W_1]$ . Let  $w_1^i$  be the first vertex along  $P_i$  in  $W_1$ . Since there is no arc from  $W_1$  to  $W_2$ , all vertices of  $P_i[s(P_i), w_1^i]$  are in  $W_2$  and all vertices of  $P_i[w_1^i, t(P_i)]$  are in  $W_1$ .

- a) If both  $P_1$  and  $P_2$  are contained in  $D[W_1 \cup \{x\}]$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[x, w_1]$  by the dipath  $xuw_1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D'_1$ .
  - b) If one of the two  $P_i$ 's, say  $P_1$  is contained in  $D[W_1 \cup \{x\}]$ , then the digraph  $S_1$  obtained from  $S$  by replacing  $C_1[x, w_1]$  and  $P_1[s(P_1), w_1^1]$  by the dipaths  $xuw_1$  and  $uw_1^1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D'_1$ .
  - c) Assume now that  $P_1$  and  $P_2$  both intersect  $W_2$ . Both  $s(P_1)$  and  $s(P_2)$  are in  $W_2 \cup \{x\}$ . Observe that every vertex of  $W_1$  is reachable from  $\Gamma'_1 \setminus \{x\}$  in  $D_1$ , so there is an  $(x, C_2)$ -dipath  $Q$  in  $D[W_1 \cup \{x\}]$ . If  $Q$  does not intersect  $P_1 \cup P_2$ , then set  $Q' := Q$ . Otherwise, we may assume that the first vertex  $z$  in  $V(P_1 \cup P_2)$  along  $Q - x$  is in  $P_2$ . Set  $Q' := Q[x, z]P_2[z, t(P_2)]$ . Observe that  $Q'$  and  $P_1$  are disjoint except possibly in  $x$ . Now the digraph  $S_1$  obtained from  $C_1 \cup C_2 \cup P_1 \cup Q'$  by replacing  $C_1[x, w_1]$  and  $P_1[s(P_1), w_1^1]$  by the dipaths  $xuw_1$  and  $uw_1^1$  is an  $x$ -suitable  $F_1$ -subdivision in  $D'_1$ .
- The second possibility is that  $C_2 \subseteq D[W_2]$ . Then  $P_1$  and  $P_2$  are in  $D[W_2 \cup \{x\}]$ , and so  $C_1 \cap D[W_1] = C_1[w_1, x]$ . Hence the digraph  $S_2$  obtained from  $S$  by replacing  $C_1[w_2, x]$  by the arc  $w_2x$  is an  $x$ -suitable  $F_1$ -subdivision in  $D_2$ .

(ii) We already showed in Claim 45.1 that if  $D_2$  contains an  $F_1$ -subdivision, then  $D$  contains an  $F_1$ -subdivision. Let us now prove that if  $D'_1$  contains an  $F_1$ -subdivision, then  $D$  contains an  $F_1$ -subdivision.

Suppose that  $S_1$  is an  $F_1$ -subdivision in  $D'_1$ . If  $u \notin V(S_1)$ , then  $S_1$  is an  $F_1$ -subdivision in  $D$ . If  $u \in V(S_1)$ , then  $S_1$  contains a dipath  $(x, u, w_1)$  for some  $w_1 \in W_1$ , and possibly one other arc  $uw_1^1$ . By definition of  $D'_1$ , there are a  $(t, w_1)$ -dipath  $P$  and a  $(t, w_1^1)$ -dipath  $P'$  with internal vertices in  $W_2$ . Let  $t'$  be the last vertex along  $P'$  which is in  $V(P) \cap V(P')$ . Then the digraph  $S$  obtained from  $S_1$  by replacing  $tw_1$  by  $P$  and  $tw_1^1$  (if it exists) by  $P'[t', w_1^1]$  is an  $F_1$ -subdivision contained in  $D$ .  $\diamond$

Let us now estimate the complexity of  $F_1\text{-Subdivision}(D, x)$ . This procedure first finds two disjoint directed cycles and then runs a few Menger algorithms and either returns an answer or make a recursive call on two smaller instances, which are either  $D_1$  and  $D_2$ , or  $D'_1$  and  $D_2$ . Two disjoint directed cycles can be found in  $O(n^3m(n+m))$  by Corollary 43.

The smaller instances  $D_1$  and  $D'_1$  can be constructed in linear time: indeed a vertex  $w_1 \in W_1$  is  $W_2$ -reachable if and only if it has an inneighbour in the outsection of  $x$  in  $D[W_2 \cup \{x\}]$ , and so all  $W_2$ -reachable vertices can be found in linear time. Similarly, the set of  $W_1$ -reaching vertices in  $W_2$  can be determined in linear time, and thus  $D_2$  can be constructed in linear time. Hence  $F_1\text{-Subdivision}(D, x)$  makes at most  $cn^2$  operations before calling recursively, for some absolute constant  $c$ .

Let us denote by  $T(n)$  the maximum time for  $F_1\text{-Subdivision}(D, x)$  on a digraph with  $n$  vertices. Since  $|V(D_1)| + |V(D_2)| = |V(D)| + 1$  and  $|V(D'_1)| + |V(D_2)| = |V(D)| + 2$ , we have

$$T(n) \leq \max\{T(n_1) + T(n_2) + cn^2 \mid n_1, n_2 < n \text{ and } n_1 + n_2 \in \{n+1, n+2\}\}.$$

This implies that  $T(n) \leq O(n^3)$ .

Therefore  $F_1\text{-Subdivision}$  runs in time  $O(n^3m(n+m))$ .  $\square$

### A.2.3 $F_2$ is tractable

The aim of this subsection is to prove that the digraph  $F_2$  depicted on Figure 8 is tractable.

**Theorem 46.**  $F_2\text{-SUBDIVISION}$  can be solved in  $O(n^4m(n+m))$  time.

In order to prove Theorem 46, we need some preliminary results.

Let  $F$  be a subdigraph of a digraph  $D$ . An *ear* of  $F$  in  $D$  is an oriented path in  $D$  containing at least one edge, whose endvertices lie in  $F$  but whose edges and internal vertices do not belong to  $F$ . A

*directed ear* of  $F$  is an ear of  $F$  that is a directed path. A digraph is said to be *robust* if it is strong and 2-connected. The following lemma is well-known; it is very similar to Proposition 5.11 of [4].

**Lemma 47.** *Let  $F$  be a non-trivial strong subdigraph of a robust digraph  $D$ . Then  $F$  has a directed ear in  $D$ . Moreover such a directed ear can be found in time  $O(n(n + m))$ .*

*Proof.* Because  $D$  is 2-connected,  $F$  has an oriented ear in  $D$ . Among all such ears, we choose one in which the number of reverse arcs (those directed towards its initial vertex) is as small as possible. We show that this path  $P$  is in fact a directed ear.

Assume the contrary, and let  $uv$  be a backward arc of  $P$ . Because  $D$  is strong, there exist in  $D$  an  $(F, u)$ -dipath  $Q$  and a  $(v, F)$ -dipath  $R$  (one of which might be of length zero). The initial vertex of  $Q$  and the terminal vertex of  $R$  must be one and the same vertex, for otherwise the directed walk  $QuvR$  would contain a directed ear of  $F$ , contradicting the choice of  $P$  and our assumption that  $P$  is not a directed ear. Let this common vertex be  $z$ . We may assume that  $z \neq s(P)$  (the case  $z \neq t(P)$  being analogous). Then the  $(s(P), z)$ -walk  $P[s(P), v]Rz$  contains an oriented  $(s(P), z)$ -path that contradicts the choice of  $P$ . Thus  $P$  is indeed a directed ear of  $F$ .

A directed ear of  $F$  may be found by running a search from each vertex of  $F$  in  $D \setminus A(F)$ . Hence it can be found in  $O(n(n + m))$  steps.  $\square$

Let  $D_1$  and  $D_2$  be two subdigraphs in  $D$ . Two dipaths are  $(D_1, D_2)$ -*opposite* if they are disjoint and one of them is a  $(D_1, D_2)$ -dipath and the other is a  $(D_2, D_1)$ -dipath. Opposite dipaths play an important role in detecting  $F_2$ -subdivisions because of the following easy lemma.

**Lemma 48.** *Let  $D$  be a digraph and  $D_1$  and  $D_2$  two disjoint non-trivial strong subdigraphs of  $D$ . If there are  $(D_1, D_2)$ -opposite dipaths in  $D$ , then  $D$  contains an  $F_2$ -subdivision.*

*Proof.* Let  $P_1$  and  $P_2$  be two  $(D_1, D_2)$ -opposite dipaths, with  $P_1$  a  $(D_1, D_2)$ -dipath and  $P_2$  a  $(D_2, D_1)$ -dipath. Since  $D_i$  is strong, there is an  $(t(P_{3-i}), s(P_i))$ -dipath  $Q_i$  in  $D_i$ . For the same reason, there is a  $(Q_i - s(Q_i), s(Q_i))$ -dipath  $R_i$  in  $D_i$ . Now  $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R_1 \cup R_2$  is an  $F_2$ -subdivision in  $D$ .  $\square$

*Proof of Theorem 46.* We shall present a recursive procedure  $F_2\text{-Subdivision}(D)$ , that given a digraph  $D$  decides whether it contains an  $F_2$ -subdivision or not.

This procedure proceeds as follows. We first check whether  $D$  is robust. If not, then we solve the problem for each robust component separately. Henceforth, we may assume that  $D$  is robust.

We next check whether there are two disjoint directed cycles. If not, then we return ‘no’ since  $F_2$  contains two disjoint directed cycles. If two such cycles  $C_1$  and  $C_2$  exist, then we compute the strong component  $D_1$  of  $C_1$  in  $D - C_2$ , and next the strong component  $D_2$  of  $C_2$  in  $D - D_1$ . Hence  $D_1$  and  $D_2$  are two disjoint non-trivial strong subdigraphs in  $D$ . Moreover, they satisfy the following property.

**Claim 48.1.** *If  $P$  is a  $(D_1, D_2)$ -dipath and  $Q$  is a  $(D_2, D_1)$ -dipath, then  $P$  and  $Q$  are internally disjoint.*

*Subproof.* Suppose, by way of contradiction, that  $P$  and  $Q$  have a common internal vertex  $x$ . The vertex  $x$  is in the strong component of  $D_1$  in  $D - D_2$ . Hence it is in the strong component of  $C_1$  in  $D - C_2$ . So  $x \in D_1$ , a contradiction.  $\diamond$

We check whether there are  $(D_1, D_2)$ -opposite dipaths in  $D$ . By using Claim 48.1 this task reduces to finding a  $(D_1, D_2)$ -dipath and a  $(D_2, D_1)$ -dipath whose endvertices are disjoint. If there are such paths, then, by Lemma 48,  $D$  contains an  $F_2$ -subdivision and we return ‘yes’. Henceforth we may assume that there are no  $(D_1, D_2)$ -opposite dipaths in  $D$ .

By Lemma 47, there is a directed ear  $P_1$  of  $D_1$ . Since  $D_1 \cup P_1$  is strong,  $P_1$  must intersect  $D_2$ . Furthermore, the intersection of  $P_1$  and  $D_2$  is reduced to a single vertex, because there are no  $(D_1, D_2)$ -opposite dipaths. Let  $u_1$  be the initial vertex of  $P_1$ ,  $v_1$  the terminal vertex of  $P_1$ , and let  $u_2$  be the vertex

of  $P_1 \cap D_2$ . By Lemma 47, there is a directed ear  $P_2$  of  $D_2$ . If the terminal vertex of  $P_2$  is  $u_2$ , then we consider the converse of  $D, D_1, D_2, P_1$  and  $P_2$ . (This is valid since  $F_2$  is its own converse.) Hence, we may assume that the terminal vertex  $v_2$  of  $P_2$  is different from  $u_2$ . Similarly to  $P_1$ , the directed ear  $P_2$  intersects  $D_1$  in a single vertex  $w_1$ . Necessarily,  $w_1 = v_1$  for otherwise  $P_1[u_2, v_1]$  and  $P_2[w_1, v_2]$  are  $(D_1, D_2)$ -opposite, by Claim 48.1. Also, the initial vertex of  $P_2$  is  $u_2$ , and thus we may assume that both ears have common segment  $P_1[u_2, v_1] = P_2[u_2, v_1]$ . Furthermore,  $P_1[u_1, u_2]$  and  $P_2[v_1, v_2]$  are disjoint for otherwise, there are two  $(D_1, D_2)$ -opposite dipaths.

Set  $P = P_1 P_2[v_1, v_2]$ . We check whether  $D - (D_1 \cup D_2)$  contains a non-trivial strong component  $D_3$ . If  $D_3$  exists and intersects both  $P[u_1, v_1]$  and  $P[u_2, v_2]$ , then we return ‘yes’. This is valid by the following claim.

**Claim 48.2.** *If  $D_3$  intersects both  $P[u_1, v_1]$  and  $P[u_2, v_2]$ , then  $D$  contains an  $F_2$ -subdivision.*

*Subproof.* Suppose first that  $D_3$  intersects  $P[u_2, v_1]$ . If  $D_3$  also intersects  $P[u_1, u_2]$ , then there are two  $(D_1, D_3)$ -opposite dipaths, and so by Lemma 48,  $D$  contains an  $F_2$ -subdivision. Similarly, if  $D_3$  also intersects  $P[v_1, v_2]$ , then there are two  $(D_2, D_3)$ -opposite dipaths, and so  $D$  contains an  $F_2$ -subdivision. Hence we may assume that  $D_3$  does not intersect  $P[u_1, u_2] \cup P[v_1, v_2]$ . If  $D_3$  intersects  $P[u_2, v_1]$  in more than one vertex, then let  $u_3$  (resp.  $v_3$ ) be the first (resp. last) vertex of  $D_3$  along  $P[u_2, v_1]$ . The two dipaths  $P[u_1, u_3]$  and  $P[v_3, v_1]$  are  $(D_1, D_3)$ -opposite. Hence, by Lemma 48,  $D$  contains an  $F_2$ -subdivision.

So  $D_3$  intersects  $P[u_2, v_1]$  in a unique vertex, say  $w_3$ . By Lemma 47, there is a directed ear  $P_3$  of  $D_3$ . By definition of  $D_1$  and  $D_2$ ,  $P_3$  intersects both  $D_1$  and  $D_2$ . Now one of the two endvertices of  $P_3$ , say  $u_3$ , is distinct from  $w_3$ .

If  $u_3$  is the initial vertex of  $P_3$ , then consider the first vertex  $v_3$  along  $P_3$  in  $D_1 \cup D_2$ . By definition of  $D_2$ ,  $v_3 \in V(D_1)$ . Now  $P_3[u_3, v_3]$  is disjoint from  $P[u_2, w_3]$  since  $D_3$  is a strong component of  $D - D_1$ . Furthermore  $P_3[u_3, v_3]$  is internally disjoint from  $P[u_1, u_2]$  because the dipaths  $P[u_1, u_2]$  and  $P[u_2, w_3] D_3[w_3, u_3] P_3[u_3, v_3]$  are internally disjoint by Claim 48.1. If  $v_3 \neq u_1$ , then  $P[u_1, w_3]$  and  $P_3[u_3, v_3]$  are  $(D_1, D_3)$ -opposite, and so by Lemma 48,  $D$  contains an  $F_2$ -subdivision. Finally, if  $v_3 = u_1$ , then  $P[v_1, v_2]$  is disjoint from  $P_3[u_3, v_3]$ . In this case,  $P_3[u_3, v_3]$  and  $P[v_1, v_2] D_2[v_2, u_2] P[u_2, w_3]$  are  $(D_1, D_3)$ -opposite paths giving an  $F_2$ -subdivision.

If  $u_3$  is the terminal vertex of  $P_3$ , then we get the result analogously.

Suppose now that  $D_3$  does not intersect  $P[u_2, v_1]$ . Then it must intersect both  $P[u_1, u_2]$  and  $P[v_1, v_2]$ . Let  $u_3$  be the first vertex of  $D_3$  along  $P[u_1, u_2]$  and  $v_3$  be the last vertex of  $D_3$  along  $P[v_1, v_2]$ . Now  $P[u_1, u_3]$  and  $P[v_3, v_2] D_2[v_2, u_2] P[u_2, v_1]$  are two  $(D_1, D_3)$ -opposite dipaths. Thus, by Lemma 48,  $D$  contains an  $F_2$ -subdivision.  $\diamond$

If  $D_3$  exists, we are either done by Claim 48.2, or  $D_3$  is disjoint from one of the paths,  $P[u_1, v_1]$  or  $P[u_2, v_2]$ . Now, we replace  $D_1$  by the strong component of  $D - D_3$  containing  $D_1 \cup D_2$ . Observe that this makes the order of  $D_1$  increase. Further, we replace  $D_2$  by  $D_3$  and replace  $C_2$  by a cycle in this new strong digraph. By doing this, Claim 48.1 remains valid. By repeating the process as long as possible, we reach the situation where all strong components of  $D - (D_1 \cup D_2)$  are trivial, that is  $D - (D_1 \cup D_2)$  is acyclic. We check whether  $D_2 - u_2$  contains a directed cycle. If it contains such a cycle  $C'_2$ , then let  $D'_1$  be the strong component of  $C_1$  in  $D_2 - C'_2$  and  $D'_2$  the strong component of  $C'_2$  in  $D - D'_1$ . Clearly,  $D'_1$  is a superdigraph of  $D_1 \cup P_1$ , so  $|D'_1| > |D_1|$ . Hence, we replace  $D_1, D_2$  by  $D'_1$  and  $D'_2$ , respectively, and repeat the procedure for the new pair  $D_1, D_2$ . So we may assume that there is no cycle in  $D_2 - u_2$ .

Moreover, if there is a  $(v_2, u_1)$ -dipath  $Q$  whose internal vertices are not in  $V(D_1 \cup D_2)$ , we also check whether there is a cycle in  $D_2 - v_2$ . If yes, then as above we find new pair of non-trivial strong digraphs  $(D'_1, D'_2)$  with  $|D'_1| > |D_1|$ . Hence in that case, we may also assume that there is no cycle in  $D_2 - v_2$ .

Let  $D^*$  be the digraph obtained from  $D$  by contracting  $D_2$  into a single vertex  $w^*$ . We return  $F_2$ -Subdivision( $D^*$ ). The following claim shows that this recursive call is valid.

**Claim 48.3.**  *$D$  contains an  $F_2$ -subdivision if and only if  $D^*$  contains an  $F_2$ -subdivision.*

*Subproof.* Suppose that  $D$  contains an  $F_2$ -subdivision  $S$ . Let  $C_1$  and  $C_2$  be the two disjoint directed cycles in  $S$  corresponding to the subdivision of  $aba$  and  $cdc$ . Observe that each  $C_i$  intersects  $D_1 \cup D_2$ , because there is no strong component in  $D - (D_1 \cup D_2)$ .

$C_1$  and  $C_2$  cannot be in both in  $D_2$  for otherwise one of the two avoids  $u_2$ , which is impossible. Moreover, one of the cycles cannot be in  $D_1$  while the other one is in  $D_2$  for otherwise the  $(C_1, C_2)$ - and  $(C_2, C_1)$ -dipaths in  $S$  would contain two  $(D_1, D_2)$ -opposite dipaths in  $D$ , which is impossible. If  $C_1$  and  $C_2$  are both contained in  $D_1$ , then either  $S$  is contained in  $D_1$ , in which case it is also in  $D^*$ , or the arcs of  $S$  which are not in  $A(D_1)$  induce a directed ear  $R$  which intersects with  $D_2$  in a single vertex  $w_2$ , because there are no  $(D_1, D_2)$ -opposite dipaths. Hence the digraph  $S^*$  obtained from  $D$  by replacing the vertex  $w_2$  by  $w^*$  is an  $F_2$ -subdivision in  $D^*$ . So we may assume that one of the cycles intersects  $D_1$  and  $D_2$ .

Case 1: There is no  $(v_2, u_1)$ -dipath whose internal vertices are not in  $V(D_1 \cup D_2)$ . In that case, all  $(D_2, D_1)$ -dipaths are  $(u_2, v_1)$ -dipaths. Therefore, the two cycles  $C_1$  and  $C_2$  cannot both intersect  $D_1$  and  $D_2$ . Thus one of them, say  $C_1$ , does not intersect both, and thus must be contained in  $D_1$ . Consequently,  $C_2$  intersects both  $D_1$  and  $D_2$ . Thus  $C_2$  contains a  $(u_2, v_1)$ -dipath. Therefore, the  $(C_2, C_1)$ -dipath in  $S$  must be in  $D_1$ , and the  $(C_1, C_2)$ -dipath intersects  $D_2$  in  $u_2$ . Therefore the digraph  $S^*$  obtained from  $S$  by contracting the vertices of  $V(S) \cap V(D_2)$  into  $w^*$  is an  $F_2$ -subdivision in  $D^*$ .

Case 2: There is a  $(v_2, u_1)$ -dipath  $Q$  whose internal vertices are not in  $V(D_1 \cup D_2)$ . In this case, all  $(D_1, D_2)$ -dipaths are  $(u_1, u_2)$ - or  $(v_1, v_2)$ -dipaths because there are no  $(D_1, D_2)$ -opposite dipaths in  $D$ . For the same reason, all  $(D_2, D_1)$ -dipaths are  $(u_2, v_1)$ - or  $(v_2, u_1)$ -dipaths. Therefore, the two cycles  $C_1$  and  $C_2$  cannot both intersect  $D_1$  and  $D_2$ . Thus one of them, say  $C_1$ , does not intersect both, and thus must be contained in  $D_1$ . Consequently,  $C_2$  intersects both  $D_1$  and  $D_2$ .

We are in one of the three following cases:  $C_2 = P[u_1, v_1]D_1[v_1, u_1]$ ,  $C_2 = P[u_2, v_2]D_2[v_2, u_2]$ , or  $C = PQ$ . In each of these cases, one can see that the digraph  $S^*$  obtained from  $S$  by contracting the vertices of  $V(S) \cap V(D_2)$  in  $w^*$  contains an  $F_2$ -subdivision in  $D^*$ .

Conversely, suppose that  $D^*$  contains an  $F_2$ -subdivision  $S^*$ . If  $S^*$  does not contain  $w^*$ , then it is contained in  $D$ . So we may assume that  $S^*$  contains  $w^*$ .

Suppose  $w^*$  has indegree and outdegree 1 in  $S^*$ . Let  $u$  (resp.  $v$ ) be the inneighbour (resp. outneighbour) of  $w^*$  in  $S^*$ . By definition of  $D^*$ , the vertex  $u$  has an outneighbour  $u'_2$  in  $D_2$  and the vertex  $v$  has an inneighbour  $v'_2$  in  $D_2$ . Hence the digraph  $S$  obtained from  $S^*$  by replacing the dipath  $uw^*v$  by the dipath  $uD[u'_2, v'_2]v$  is an  $F_2$ -subdivision in  $D$ .

Suppose  $w^*$  has indegree 1 and outdegree 2 in  $S^*$ . Let  $u$  be the inneighbour of  $w^*$  in  $S^*$  and let  $v$  and  $v'$  be the outneighbours of  $w^*$  in  $S$ . By definition of  $D^*$ , the vertex  $u$  has an outneighbour  $u'_2$  in  $D_2$  and the vertex  $v$  (resp.  $v'$ ) has an inneighbour  $v'_2$  (resp.  $v''_2$ ) in  $D_2$ . Let  $P$  be a  $(u'_2, v'_2)$ -dipath in  $D_2$  and  $Q$  be a  $(P, v''_2)$ -dipath in  $D$ . The digraph  $S$  obtained from  $S^*$  by replacing the vertex  $w^*$  by  $P \cup Q$  is an  $F_2$ -subdivision in  $D$ .

If  $w^*$  has indegree 2 and outdegree 1, we find an  $F_2$ -subdivision in  $D$  in a similar way.  $\diamond$

Let us now estimate the time complexity of  $F_2$ -Subdivision. The procedure first constructs the digraphs  $D_1$  and  $D_2$ . It requires to find two disjoint directed cycles and then to compute two strong components. By Corollary 43 this can be done in time  $O(n^3m(n+m))$ . Next, the algorithm checks a few times for opposite paths, and for directed cycles, before either increasing the order of  $D_1$  or making a recursive call. Checking if there are  $(D_1, D_2)$ -opposite paths can be done in  $O(n(n+m))$  time by running searches in  $D \setminus A(D_1 \cup D_2)$  from each vertex, and finding if there is a directed cycle in a digraph can be done in  $O(n(n+m))$  time by checking for each vertex  $v$  if there is a  $(v, v)$ -handle. Thus, since the order of  $D_1$  increases at most  $O(n)$  times, there are at most  $O(nm(n+m))$  such operations between two recursive calls. Hence the time between two recursive calls is at most  $O(n^3m(n+m))$ . At each

call, the order of the instance digraph decreases. Hence the time complexity of  $F_2$ -Subdivision is  $O(n^4 m(n + m))$ .  $\square$

#### A.2.4 $F_3$ is tractable

**Theorem 49.**  $F_3$  is tractable.

Given two vertices  $c'$  and  $d'$  in  $D$ , a  $(c', d')$ -forced  $F_3$ -subdivision is an  $F_3$ -subdivision in  $D$  with  $c$ -vertex  $c'$  and  $d$ -vertex  $d'$ .

To prove Theorem 49, we shall describe a polynomial-time algorithm to solve  $F_3$ -SUBDIVISION. The key ingredient of our algorithm is a polynomial-time procedure  $F_3$ -STRONG $+(D, \{c_1, c_2\}, \{d_1, d_2\})$  whose input is a strong digraph  $D$  and two sets of two vertices  $\{c_1, c_2\}, \{d_1, d_2\}$  ( $c_1 \neq c_2$  and  $d_1 \neq d_2$ ). Let  $\hat{D}(\{c_1, c_2\}, \{d_1, d_2\})$  be the digraph obtained from  $D$  by adding two new vertices  $c'', d''$  and the four arcs  $c_1 c'', c_2 c'', d'' d_1, d'' d_2$ . The procedure  $F_3$ -STRONG $+(D, \{c_1, c_2\}, \{d_1, d_2\})$  returns ‘yes’ if it finds an  $F_3$ -subdivision in  $\hat{D}$ , and returns ‘no’ if  $\hat{D}$  has no  $(c'', d'')$ -forced  $F_3$ -subdivision.

Before describing the procedure  $F_3$ -STRONG $+$ , let us describe the algorithm for  $F_3$ -SUBDIVISION, assuming we have such a procedure.

**The algorithm** For every vertex  $c'$  and  $d'$ , we run a procedure  $F_3$ -FORCED $(c', d')$  that returns ‘yes’ if it finds an  $F_3$ -subdivision, and return ‘no’ if it finds evidence that there is no  $(c', d')$ -forced  $F_3$ -subdivision in  $D$ . Since there are  $O(n^2)$  possible choices of  $c'$  and  $d'$ , if  $F_3$ -FORCED runs in polynomial time, the overall algorithm will also run in polynomial time.

$F_3$ -FORCED $(c', d')$  proceeds as follows. We first compute the strong components  $G_1, \dots, G_p$  of  $D - \{c', d'\}$ . Observe that the directed cycle (corresponding to  $aba$ ) in an  $F_3$ -subdivision must be contained in one of the strong components. For each strong component  $G_i$ , we run a procedure  $F_3$ -SUITABLE $(c', d', G_i)$  that returns ‘yes’ if it finds an  $F_3$ -subdivision and returns ‘no’ only if there is no  $(c', d')$ -forced  $F_3$ -subdivision whose directed cycle is in  $G_i$ . Such a subdivision is called  $(c', d', G_i)$ -suitable.

We first test if there is a  $(d', c')$ -dipath in  $D - G_i$ . We then run two separate procedures depending on whether or not such a path exists.

**Case 1:** Assume there is a  $(d', c')$ -dipath  $P$  in  $D - G_i$ . Let  $X$  be the set of vertices  $x \in V(G_i)$  that are terminal vertices of a  $(d', G_i)$ -dipath in  $D$ . The set  $X$  can be computed in linear time by running a search from  $d'$  in the digraph obtained from  $D$  by deleting all the arcs having their tail in  $G_i$ . Let  $Y$  be the set of vertices  $y \in V(G_i)$  that are initial vertices of a  $(G_i, c')$ -dipath in  $D$ . Similarly to  $X$ , the set  $Y$  can be determined in linear time.

If there are no two distinct vertices  $x \in X$  and  $y \in Y$ , then we return ‘no’. Otherwise we return ‘yes’. This is valid according to the following claim.

**Claim 49.1.** (i) If there are no two distinct vertices  $x \in X$  and  $y \in Y$ , then  $D$  contains no  $(c', d', G_i)$ -suitable  $F_3$ -subdivision.

(ii) If there are two distinct vertices  $x \in X$  and  $y \in Y$ , then  $D$  contains an  $F_3$ -subdivision.

*Subproof.* (i) If  $D$  contains a  $(c', d', G_i)$ -suitable  $F_3$ -subdivision  $S$ , then consider the directed cycle  $C$  in  $S$ . This cycle is in  $G_i$ . Moreover, in  $S$ , there are two disjoint  $(d', C)$ - and  $(C, c')$ -dipaths  $Q$  and  $Q'$ , respectively. Then the first vertex  $x$  in  $G_i$  along  $Q$  is in  $X$  and the last vertex  $y$  in  $G_i$  along  $Q'$  is in  $Y$ . Since  $Q$  and  $Q'$  are disjoint,  $x$  and  $y$  are distinct.

(ii) Suppose that there are vertices  $x \in X$  and  $y \in Y$ , where  $x \neq y$ . Let  $Q$  be a  $(d', G_i)$ -dipath with terminal vertex  $x$  and let  $Q'$  be a  $(G_i, c')$ -dipath with initial vertex  $y$ . Observe that  $Q$  and  $Q'$  do not intersect because  $G_i$  is a strong component of  $D - \{c', d'\}$ . Since  $G_i$  is strong, there are an  $(x, y)$ -dipath  $R$  and a  $(y, R - y)$ -dipath  $R'$  in  $G_i$ . Now  $P$  contains a  $(Q, Q')$ -dipath  $P'$ . Thus  $P' \cup Q[s(P'), x] \cup Q'[y, t(P')] \cup R \cup R'$  is an  $F_3$ -subdivision.  $\diamond$

Case 2: Assume now there is no  $(d', c')$ -dipath in  $D - G_i$ .

If  $D$  contains a  $(c', d', G_i)$ -suitable  $F_3$ -subdivision  $S$ , then the two independent  $(d', c')$ -dipaths in  $S$  intersect  $G_i$ . Since  $G_i$  is a strong component of  $D - \{c', d'\}$ , each of these two paths consists of three segments: a  $(d', G_i)$ -dipath, followed by a dipath in  $G_i$ , and ending with a  $(G_i, c')$ -dipath. The idea is to guess which are the first vertices  $d_1, d_2$  and last vertices  $c_1, c_2$  in  $G_i$  along these dipaths.

Hence, consider every pair of sets of two vertices,  $\{c_1, c_2\}, \{d_1, d_2\}$ , where  $c_1 \neq c_2$  and  $d_1 \neq d_2$ . Observe that there must be two independent  $(d', \{d_1, d_2\})$ -dipaths whose internal vertices are not in  $G_i$  and two independent  $(\{c_1, c_2\}, c')$ -dipaths whose internal vertices are not in  $G_i$ . We can check for the existence of two independent  $(d', \{d_1, d_2\})$ -dipaths with internal vertices not in  $G_i$  by running a Menger algorithm in the digraph obtained from  $D$  by deleting all the arcs with tail in  $G_i$ . Similarly, we also check the existence of two independent  $(\{c_1, c_2\}, c')$ -dipaths with internal vertices not in  $G_i$ . If one of these pairs of dipaths do not exist, then we proceed to the next pair. If the two pairs of dipaths exist, they are internally disjoint from each other since  $G_i$  is a strong component. In that case, one can easily see that  $D$  contains a  $(c', d', G_i)$ -suitable  $F_3$ -subdivision  $S$  such that  $d_1, d_2$  (resp.  $c_1, c_2$ ) are the first (resp. last) vertices in  $G_i$  along the two independent  $(d', c')$ -dipaths in  $S$  if and only if the digraph  $\hat{G}_i(\{c_1, c_2\}, \{d_1, d_2\})$  has no  $(c'', d'')$ -forced  $F_3$ -subdivision. Henceforth, we run  $F_3\text{-Strong}+(G_i, \{c_1, c_2\}, \{d_1, d_2\})$ . If this procedure returns ‘yes’, we also return ‘yes’. If it returns ‘no’, we proceed to the next pair of sets  $\{c_1, c_2\}, \{d_1, d_2\}$ .

If all the pairs have been considered without returning ‘yes’, we return ‘no’. This procedure is clearly valid provided that we have  $F_3\text{-Strong}+$  subroutine.

Hence our algorithm is valid and runs in polynomial time provided that the procedure  $F_3\text{-Strong}+$  is valid and runs in polynomial time. We now describe this subprocedure.

**Detecting  $F_3$  in strong digraphs** We now present procedure  $F_3\text{-Strong}+(D, \{c_1, c_2\}, \{d_1, d_2\})$ . Recall that procedure  $F_3\text{-Strong}+(D, \{c_1, c_2\}, \{d_1, d_2\})$  returns ‘yes’ if it finds an  $F_3$ -subdivision in  $\hat{D}$ , and should return ‘no’ if  $\hat{D}$  has no  $(c'', d'')$ -forced  $F_3$ -subdivision. The assumption is that the input digraph  $D$  is strongly connected.

In the first phase, we treat the case when  $D$  is not 2-connected and reduce to the case when it is. Suppose that  $D$  has a cutvertex  $x$ . Let  $X_1, \dots, X_p$  be the connected components of  $D - x$ , and for  $1 \leq i \leq p$ , let  $D_i = D[X_i \cup \{x\}]$ . Observe that each  $D_i$  is strong because  $D$  is strong.

Suppose first that  $c_1$  and  $c_2$  lie in different connected components of  $D - x$ , say  $X_1$  and  $X_2$  (respectively). Let  $P_1$  be an  $(x, c_1)$ -dipath in  $D_1$ ,  $P'_1$  a  $(c_1, P_1 - c_1)$ -dipath, and  $P_2$  a  $(c_2, x)$ -dipath in  $D_2$ . The digraph  $P_1 \cup P'_1 \cup P_2 \cup c_1 c'' \cup c_2 c''$  is an  $F_3$ -subdivision in  $\hat{D}$ , and we return ‘yes’. Similarly, if  $d_1$  and  $d_2$  are in different connected components of  $D - x$ ,  $\hat{D}$  contains an  $F_3$ -subdivision, and we return ‘yes’.

Henceforth, we may assume that there is  $i, j \in \{1, \dots, p\}$  such that  $\{c_1, c_2\} \subseteq V(D_i)$  and  $\{d_1, d_2\} \subseteq V(D_j)$ . If  $i \neq j$ , then in  $D$ , there cannot be two internally disjoint  $(d'', c'')$ -dipaths, and thus there is no  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}$ . Therefore, we return ‘no’. If  $i = j$ , then since  $F_3$  is 2-connected, there is a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}$  if and only if there is a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}_i(\{c_1, c_2\}, \{d_1, d_2\})$ . Hence we return  $F_3\text{-Strong}+(D_i, \{c_1, c_2\}, \{d_1, d_2\})$ .

Assume now that  $D$  is 2-connected, and so  $D$  is robust. The procedure uses a similar approach as the procedure  $F_2\text{-Subdivision}(D)$  to decide whether a digraph  $D$  contains an  $F_2$ -subdivision, and a key notion is the one of opposite dipaths. Recall that two dipaths are  $(D_1, D_2)$ -opposite if they are disjoint and one of them is a  $(D_1, D_2)$ -dipath and the other is a  $(D_2, D_1)$ -dipath. Since an  $F_2$ -subdivision contains an  $F_3$ -subdivision, Lemma 48 implies directly the following one.

**Lemma 50.** *Let  $D$  be a digraph and  $D_1$  and  $D_2$  disjoint non-trivial strong subdigraphs of  $D$ . If there are  $(D_1, D_2)$ -opposite paths in  $D$ , then  $D$  contains an  $F_3$ -subdivision.*

**Lemma 51.** *Suppose that  $D'$  is a strong subdigraph of  $D$  and  $R$  is a path in  $D$  with its endvertices in  $D'$  and with its internal vertices in  $D - D'$ . If the path  $R$  has three blocks, then  $D$  contains an  $F_3$ -subdivision.*

*Proof.* Let  $s = s(R)$ ,  $t = t(R)$ . Let  $Q$  be a  $(t, s)$ -dipath in  $D'$  and let  $Q'$  be an  $(s, Q - s)$ -dipath in  $D'$ . Then  $Q \cup Q' \cup R$  is an  $F_3$ -subdivision in  $D$ .  $\square$

Returning to the algorithm description, we first check if there are two disjoint directed cycles in  $D$ . If not, then one can solve the problem in polynomial time according to Theorem 44.

If two such cycles  $C_1$  and  $C_2$  exist, then we first compute the strong component  $D_1$  of  $C_1$  in  $D - C_2$ , and next we compute the strong component  $D_2$  of  $C_2$  in  $D - D_1$ . Hence  $D_1$  and  $D_2$  are two disjoint non-trivial strong subdigraphs in  $D$ . Moreover they satisfy the following property (Claim 48.1).

**Claim 51.1.** *If  $P$  is a  $(D_1, D_2)$ -dipath and  $Q$  is a  $(D_2, D_1)$ -dipath, then  $P$  and  $Q$  are internally disjoint.*

We check if there are  $(D_1, D_2)$ -opposite paths in  $D$ . If there are, then by Lemma 50,  $D$  contains an  $F_3$ -subdivision and we return ‘yes’. Henceforth we may assume that there are no  $(D_1, D_2)$ -opposite paths in  $D$ .

By Lemma 47, there is a directed ear  $P_1$  of  $D_1$ . Since  $D_1 \cup P_1$  is strong,  $P_1$  must intersect  $D_2$ . Furthermore, the intersection of  $P_1$  and  $D_2$  is reduced to a single vertex, because there are no  $(D_1, D_2)$ -opposite paths. Let  $u_1$  be the initial vertex of  $P_1$ ,  $v_1$  the terminal vertex of  $P_1$ , and let  $u_2$  be the vertex of  $P_1 \cap D_2$ . By Lemma 47, there is a directed ear  $P_2$  of  $D_2$ . If the terminal vertex of  $P_2$  is  $u_2$ , then we consider the converse of  $\hat{D}$ ,  $P_1$  and  $P_2$  and exchange the roles of  $c''$  and  $d''$  (i.e.  $(c'', d'') := (d'', c'')$ ) and their neighbours  $((\{c_1, c_2\}, \{d_1, d_2\}) := (\{d_1, d_2\}, \{c_1, c_2\}))$ . (This is valid since  $F_3$  is self-converse.) Hence, we may assume that the terminal vertex  $v_2$  of  $P_2$  is different from  $u_2$ . Similarly to  $P_1$ , the dipath  $P_2$  intersects  $D_1$  in a single vertex  $w_1$ . Clearly,  $w_1 = v_1$  for otherwise  $P_1[u_2, v_1]$  and  $P_2[w_1, v_2]$  are  $(D_1, D_2)$ -opposite paths. Furthermore,  $P_1[u_1, u_2]$  and  $P_2[v_1, v_2]$  are disjoint for otherwise, there are two  $(D_1, D_2)$ -opposite dipaths.

Set  $P = P_1 P_2[v_1, v_2]$ . We check whether  $D - (D_1 \cup D_2)$  contains a non-trivial strong component  $D_3$ . If  $D_3$  does not intersect  $P[u_1, v_1]$ , then we replace  $D_1$  and  $D_2$  by two disjoint non-trivial strong digraphs,  $D_1 \cup D_2 \cup P[u_1, v_1]$  and  $D_3$ , respectively. Similarly, if  $D_3$  does not intersect  $P[u_2, v_2]$ , then we replace  $D_1, D_2$  by disjoint non-trivial strong digraphs  $D_1 \cup D_2 \cup P[u_2, v_2]$  and  $D_3$ . In either case, we extend the first digraph to a strong component of  $D - D_3$ , while  $D_3$  is already a strong component in the complement of the first digraph. Thus, Claim 51.1 remains valid. Observe that this change makes the order of  $D_1$  increase. We also redefine  $u_1, v_1, u_2, v_2$  and the path  $P$  if the change occurred.

If  $D_3$  intersects both  $P[u_1, v_1]$  and  $P[u_2, v_2]$ , then we return ‘yes’. This is valid because in this case, by Claim 48.2,  $D$  contains an  $F_2$ -subdivision and so an  $F_3$ -subdivision.

Henceforth, we may assume that all strong components of  $D - (D_1 \cup D_2)$  are trivial, that is  $D - (D_1 \cup D_2)$  is acyclic.

Let  $F_{12}$  be the set of pairs  $(x, y) \in V(D_1) \times V(D_2)$  such that there exists a  $(D_1, D_2)$ -dipath  $R$  with  $s(R) = x$  and  $t(R) = y$ . Similarly, let  $F_{21}$  be the set of pairs  $(y, x) \in V(D_2) \times V(D_1)$  such that there exists a  $(D_2, D_1)$ -dipath  $R$  with  $s(R) = y$  and  $t(R) = x$ . By Claim 51.1 and because there are no  $(D_1, D_2)$ -opposite dipaths, we have one of the following two possible outcomes:

**Case (A):**  $F_{21} = \{(u_2, v_1)\}$ . In this case,  $F_{12} = (U_1 \times \{u_2\}) \cup (\{v_1\} \times V_2)$ , where  $\{u_1\} \subseteq U_1 \subseteq V(D_1)$  and  $\{v_2\} \subseteq V_2 \subseteq V(D_2)$ .

**Case (B):**  $F_{21} = \{(u_2, v_1), (v_2, u_1)\}$ . In this case,  $F_{12} = \{(u_1, u_2), (v_1, v_2)\}$ . By setting  $U_1 = \{u_1\}$  and  $V_2 = \{v_2\}$ , the set  $F_{12}$  can be written in the same way as in Case (A).

For each vertex  $x \in V(D) \setminus V(D_1 \cup D_2)$ , there is an  $(x, D_1 \cup D_2)$ -dipath and a  $(D_1 \cup D_2, x)$ -dipath. Since  $D - (D_1 \cup D_2)$  has only trivial strong components, these two paths are internally disjoint and form a  $(D_1, D_2)$ -path  $R_x$ . We define  $Z(U_1, u_2)$  as the set of all vertices  $x \in V(D) \setminus V(D_1 \cup D_2)$ , whose path  $R_x$  is a  $(U_1, u_2)$ -dipath. In the same way we define vertex-sets  $Z(u_2, v_1)$ ,  $Z(v_1, V_2)$ , and  $Z(v_2, u_1)$ .



Note that the latter set may be non-empty only when we have Case (B) and that these four sets partition  $V(D) \setminus V(D_1 \cup D_2)$ .

Next, we derive a sufficient condition for existence of  $F_3$ -subdivisions in  $\hat{D}$ .

**Claim 51.2.** *If there is a  $(D_1, D_2)$ -path with two or three blocks in  $\hat{D}$ , then  $\hat{D}$  contains an  $F_3$ -subdivision.*

*Subproof.* Let  $R$  be a  $(D_1, D_2)$ -inpath with two blocks and let  $y$  be the vertex of outdegree 2 in  $R$ . Let  $w_1$  be a vertex in  $\{u_1, v_1\} \setminus \{s(R)\}$  and let  $w_2 = u_2$  if  $w_1 = u_1$  and  $w_2 = v_2$  if  $w_1 = v_1$ . If  $P[w_1, w_2]$  is disjoint from  $R$ , then the path  $R \cup P[w_1, w_2] \cup D_2[w_2, t(R)]$  has three blocks and by Lemma 51,  $\hat{D}$  contains an  $F_3$ -subdivision. On the other hand, if  $P[w_1, w_2]$  intersects  $R$ , let  $z$  be the first vertex on  $P[w_1, w_2]$  that lies on  $R$ . Since  $D_1$  is a strong component of  $D - D_2$ ,  $z \in R[y, t(R)]$  and  $z \neq y$ . Therefore,  $R[s(R), z]$  and  $P[w_1, z]$  form a path with three blocks and we are done by Lemma 51.

Similarly, by directional duality, if there is a  $(D_1, D_2)$ -outpath with two blocks, then  $\hat{D}$  contains an  $F_3$ -subdivision.

Suppose now that there is a  $(D_1, D_2)$ -outpath  $R$  in  $\hat{D}$  with three blocks. Let  $s = s(R)$ ,  $t = t(R)$ , and let  $x$  and  $y$  be vertices on  $R$  whose indegree and outdegree (respectively) is equal to 2.  $R[s, x] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$  because  $D_1$  is a strong component of  $D - D_2$ . If  $R[x, t] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$ , then there is a  $(D_1, D_2)$ -path with two blocks and we have the result by the above case.

Hence, we may assume that  $R$  does not intersect  $(Z(u_2, v_1) \cup Z(v_2, u_1))$ . In particular,  $R$  and  $P[u_2, v_1]$  are internally disjoint. If  $s \neq v_1$ , then  $RD[t, u_2]P[u_2, v_1]$  is an ear of  $D_1$  with three blocks and by Lemma 51  $D$  contains an  $F_3$ -subdivision. Similarly, if  $t \neq u_2$ , then  $D$  contains an  $F_3$ -subdivision. Henceforth we may assume that  $s = v_1$  and  $t = u_2$ .

If  $R$  is internally disjoint from  $P[u_1, u_2]$ , then  $P[u_1, v_1] \cup D_1[v_1, u_1] \cup R$  is an  $F_3$ -subdivision in  $\hat{D}$ , and if  $R$  is internally disjoint from  $P[v_1, v_2]$ , then  $P[u_2, v_2] \cup D_2[v_2, u_2] \cup R$  is an  $F_3$ -subdivision in  $\hat{D}$ . Thus, we may assume that  $R$  intersects both  $P[u_1, u_2]$  and  $P[v_1, v_2]$ .

Let  $z$  be the first vertex on  $P$  that belongs to  $R$ . If  $z \in R[s, y]$ , then  $P[u_1, z]R[z, t]$  is a  $(D_1, D_2)$ -dipath with three blocks, and we get the result as above because its initial vertex is not  $v_1$ . If  $z \in R[y, t]$ , then  $P[u_1, z] \cup R \cup P[u_2, v_1] \cup D[v_1, u_1]$  is an  $F_3$ -subdivision. Therefore, we may assume that  $z = y$ .

Analogously, we may assume that the last vertex on  $P[v_1, v_2]$  that belongs to  $R$  is  $x$ . Now  $P[u_1, y]R[y, x]P[x, v_2]$  contains a  $(D_1, D_2)$ -dipath with initial vertex  $u_1$  and terminal vertex  $v_2$ . By Claim 51.1, this dipath and  $P[u_2, v_1]$  are  $(D_1, D_2)$ -opposite dipaths, a contradiction.

Similarly, by directional duality, if there is a  $(D_1, D_2)$ -inpath with three blocks, then  $\hat{D}$  contains an  $F_3$ -subdivision. This completes the proof.  $\diamond$

**Claim 51.3.** (i) *For  $i = 1, 2$ , if  $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$  and  $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$ , then  $\hat{D}$  contains an  $F_3$ -subdivision.*

(ii) *For  $i = 1, 2$ , if  $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$  and  $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$ , then  $\hat{D}$  contains an  $F_3$ -subdivision.*

(iii) *If  $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$  or  $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ , then  $\hat{D}$  contains an  $F_3$ -subdivision.*

*Subproof.* (i) Suppose  $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$  and  $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$ . Without loss of generality, we may assume that  $i = 1$  and that  $c_1 \in V(D_1)$  and  $c_2 \notin V(D_1)$ . Since  $D$  is strong, there is a  $(D_1 \cup D_2, c_2)$ -dipath  $Q$  and a  $(c_2, D_1 \cup D_2)$ -dipath  $R$  in  $D$ . If  $s(Q) \in D_2$ , then  $c_1 c'' c_2 \overleftarrow{Q}$  is a  $(D_1, D_2)$ -path in  $\hat{D}$  with two blocks, so by Claim 51.2,  $\hat{D}$  contains an  $F_3$ -subdivision. If  $s(Q) \in D_1$ , then  $t(R) \in D_1$ , because  $D_1$  is a strong component in  $D - D_2$ . Hence  $c_1 c'' c_2 R$  is a  $(D_1, D_2)$ -path with two blocks, so by Claim 51.2,  $\hat{D}$  contains an  $F_3$ -subdivision.

(ii) This claim is proved analogously to (i).

(iii) Assume that  $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ . (The case when  $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$  is proved in the same way.) Since  $D$  is strong, there exist a  $(D_1 \cup D_2, c_1)$ -dipath  $Q_1$ ,

and a  $(D_1 \cup D_2, c_2)$ -dipath  $Q_2$ . If  $s(Q_1)$  and  $s(Q_2)$  do not lie in the same  $D_i$ , then there is a  $(D_1, D_2)$ -path (which is either contained in  $Q_1 \cup Q_2$  if  $Q_1$  and  $Q_2$  intersect, or passes through  $c''$  if they are disjoint) in  $\hat{D}$  having two blocks, so by Claim 51.2,  $\hat{D}$  contains an  $F_3$ -subdivision. Henceforth, we may assume that  $s(Q_1)$  and  $s(Q_2)$  are in the same  $D_i$ , say  $D_1$ .

Since  $D$  is strong, for  $i = 1, 2$ , there exists a  $(c_i, D_1 \cup D_2, )$ -dipath  $R_i$ . Its endvertex  $t(R_i)$  cannot be in  $D_1$ , because  $D_1$  is a strong component of  $D - D_2$ . Thus  $t(R_i) \in V(D_2)$ . If  $R_1$  intersects  $Q_2$  and  $R_2$  intersects  $Q_2$ , then  $c_1$  and  $c_2$  are in the same strong component of  $D - (D_1 \cup D_2)$ , which contradicts one of our previous assumptions. Therefore, without loss of generality, we may assume that  $R_2$  does not intersect  $Q_1$ . Now  $Q_1 \cup c_1 c'' \cup c'' c_2 \cup R_2$  is a  $(D_1, D_2)$ -path with three blocks. Thus by Claim 51.2,  $\hat{D}$  contains an  $F_3$ -subdivision.  $\diamond$

In view of Claim 51.3, if  $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$  and  $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$  or  $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$  and  $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$  for some  $i \in \{1, 2\}$ , then we return ‘yes’. The same holds if  $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$  or  $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ . Thus, we may assume henceforth that there are indices  $i_c, i_d \in \{1, 2\}$  such that  $\{c_1, c_2\} \subseteq V(D_{i_c})$  and  $\{d_1, d_2\} \subseteq V(D_{i_d})$ .

We now run a procedure `2or3blocks`( $D, D_1, D_2$ ) for finding a  $(D_1, D_2)$ -path with two or three blocks in  $D$ . If such a path is found, we stop the main procedure  `$F_3$ -strong+` by returning ‘yes’ since in this case we have an  $F_3$ -subdivision by Claim 51.2.

The procedure `2or3blocks`( $D, D_1, D_2$ ) proceeds as follows. Let  $S_1^-$  be the insection of  $D_1$  in  $D - D_2$  and  $S_2^-$  the insection of  $D_2$  in  $D - D_1$ . It is easy to see that there is a  $(D_1, D_2)$ -inpath with two blocks, if and only if  $S_1^- \cap S_2^-$  contains a vertex of  $D - (D_1 \cup D_2)$ . Therefore we compute  $S_1^- \cap S_2^-$  and return ‘yes’ if this set contains a vertex in  $D - (D_1 \cup D_2)$ .

Similarly, to detect if there is a  $(D_1, D_2)$ -outpath with two blocks, we compute the outsection  $S_1^+$  of  $D_1$  in  $D - D_2$  and the outsection  $S_2^+$  of  $D_2$  in  $D - D_1$ , and if  $S_1^+ \cap S_2^+$  contains a vertex of  $D - (D_1 \cup D_2)$ , we return ‘yes’.

Let us now describe how to discover paths with three blocks. Let  $tz$  be an arc and  $y$  be a vertex in  $D$  such that  $t \in V(D - D_2)$ ,  $y, z \in V(D - (D_1 \cup D_2))$ ,  $y \notin \{t, z\}$ . Arc  $tz$  and vertex  $y$  are said to be in *3-block-position* if there are a  $(y, z)$ -dipath and a  $(y, D_2)$ -dipath in  $D - (V(D_1) \cup \{t\})$  which are independent, and a  $(D_1, t)$ -dipath in  $D - (D_2 \cup \{y, z\})$ .

**Claim 51.4.** *There is a  $(D_1, D_2)$ -outpath in  $D$  with three blocks if and only if there are an arc  $tz$  and a vertex  $y$  in 3-block-position.*

*Subproof.* Trivially, if there is a  $(D_1, D_2)$ -outpath with three blocks, then there are an arc  $tz$  and a vertex  $y$  in 3-block-position.

Let us now prove the converse. Assume that  $tz$  and  $y$  are in 3-block-position. Let  $Q_1$  and  $Q_2$  be the two independent paths from  $y$  to  $z$  and  $D_2$ , respectively, and let  $R$  be the  $(D_1, t)$ -dipath in  $D - (D_2 \cup \{y, z\})$ .

If  $R$  does not intersect  $Q_1 \cup Q_2$ , then  $R \cup tz \cup \overleftarrow{Q_1} \cup Q_2$  is a  $(D_1, D_2)$ -path with three blocks.

Assume now that  $R$  intersects  $Q_1 \cup Q_2$ . Let  $x$  be the first vertex along  $R$  in  $V(Q_1 \cup Q_2)$ . Note that  $x \neq y$  by definition of  $R$ . If  $x \in V(Q_1)$ , then  $R[s(R), x] \cup \overleftarrow{Q_1}[x, y] \cup Q_2$  is a  $(D_1, D_2)$ -path with three blocks. If  $x \in V(Q_2)$ , then consider a  $(z, D_1 \cup D_2)$ -dipath  $R'$  in  $D$ . Because  $D_1$  is a strong component of  $D - D_2$  and  $RtzR'$  is a dipath,  $t(R') \in D_2$ . Furthermore  $R'$  does not meet  $R \cup Q_1 \cup Q_2[y, x]$ , because  $D - (D_1 \cup D_2)$  is acyclic. Hence  $R[s(R), x] \cup \overleftarrow{Q_2}[x, y] \cup Q_1 \cup R'$  is  $(D_1, D_2)$ -path with three blocks.  $\diamond$

Therefore, for every possible arc  $tz$  and vertex  $y$  such that  $t \in V(D - D_2)$ ,  $y, z \in V(D - (D_1 \cup D_2))$ ,  $y \notin \{t, z\}$ , we check if they are in 3-block-position. This can be done by running Menger algorithm. If we find an arc and a vertex in 3-block-position, then we return ‘yes’ because there is an  $F_3$ -subdivision by Claims 51.4 and 51.2.

We deal similarly with the  $(D_1, D_2)$ -inpaths with three blocks. This ends the procedure `2or3blocks`- $(D, D_1, D_2)$ . After it, there is no  $(D_1, D_2)$ -path in  $\hat{D}$  with two blocks and no  $(D_1, D_2)$ -path in  $D$  with three blocks.

We now show that we can reduce  $D$  to a digraph with vertex set  $V(D_1) \cup V(D_2)$ .

Let  $D^*$  be the digraph obtained from  $D_1 \cup D_2$  by adding all arcs in  $F_{12} \cup F_{21}$ . In other words, we add all arcs  $x_1x_2$  with  $x_1 \in V(D_1)$  and  $x_2 \in V(D_2)$  such that there is a  $(D_1, D_2)$ -dipath with initial vertex  $x_1$  and terminal vertex  $x_2$ , and adding all arcs  $x_2x_1$  with  $x_1 \in V(D_1)$  and  $x_2 \in V(D_2)$  for which there is a  $(D_2, D_1)$ -dipath with initial vertex  $x_2$  and terminal vertex  $x_1$ . Set  $\hat{D}^* = \hat{D}^*(\{c_1, c_2\}, \{d_1, d_2\})$ .

**Claim 51.5.**  $\hat{D}$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision if and only if  $\hat{D}^*$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision.

*Subproof.* As mentioned above,  $D^*$  either contains the four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$  (Case (B)), or contains the arcs  $uu_2$  ( $u \in U_1$ ),  $u_2v_1$ , and  $v_1v$  ( $v \in V_2$ ), which is Case (A). For each of these arcs  $uv$ , there is a corresponding directed path  $R_{uv}$  in  $D$ . One can transform a  $(c'', d'')$ -forced  $F_3$ -subdivision  $S^*$  in  $\hat{D}^*$  into an  $F_3$ -subdivision  $S$  of  $\hat{D}$  by replacing each arc  $uv$  in  $S^*$  between  $D_1$  and  $D_2$  by the path  $R_{uv}$ . If all added paths  $R_{uv}$  are pairwise internally disjoint, this clearly gives rise to an  $F_3$ -subdivision in  $\hat{D}$ . The only possibility that two of such paths may not be internally disjoint (cf. Lemma 50) is that we have two paths  $R_{uu_2}$  and  $R_{u'u_2}$  (where  $u, u' \in U_1$  and  $u \neq u'$ ) or two paths  $R_{v_1v}$  and  $R_{v_1v'}$  (where  $v, v' \in V_2$  and  $v \neq v'$ ). However, since every vertex in  $F_3$  has in- and outdegrees at most 2, there are at most two such paths entering  $u_2$  and at most two leaving  $v_1$ . For two of them, we can always achieve that their intersection is a common subpath, and in that case, the resulting digraph is again an  $F_3$ -subdivision. Clearly, the resulting  $F_3$ -subdivision in  $\hat{D}$  is  $(c'', d'')$ -forced.

Suppose now that  $\hat{D}$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision  $S$ . Let  $a', b'$ , be the vertices of  $S$  corresponding to  $a, b$ , respectively, and let  $C$  be the directed cycle in  $S$ . If  $a', b' \in V(D_1) \cup V(D_2)$ , then the arcs in  $S$  that are not in  $D_1 \cup D_2$  form a collection of internally disjoint  $(D_1, D_2)$ - and  $(D_2, D_1)$ -dipaths. By replacing each of these dipaths by the corresponding arc in  $D^*$ , we obtain an  $F_3$ -subdivision in  $\hat{D}^*$ .

Assume now that  $a' \notin V(D_1) \cup V(D_2)$ . The cycle  $C$  must intersect both  $D_1$  and  $D_2$ , and thus  $C$  contains a  $(D_2, D_1)$ -dipath. Without loss of generality, we may assume that the initial vertex of this dipath is  $u_2$  and its terminal vertex is  $v_1$ .

Now let  $z_2$  be the first vertex in  $V(D_1 \cup D_2)$  along the  $(a', c'')$ -dipath in  $S$ . This vertex exists because  $\{c_1, c_2\} \subseteq V(D_1 \cup D_2)$ . Now since there are no  $(D_1, D_2)$ -opposite paths, and by definition of the  $D_i$ ,  $z_2 \in V(D_2)$ , so  $a'$  does not lie in  $C[u_2, v_1]$ . Let  $y_2$  be the first vertex after  $a'$  along  $C$  in  $V(D_1 \cup D_2)$ . For the same reason,  $y_2 \in V(D_2)$  and so  $v_1$  is the unique vertex in  $C \cap D_1$ , for otherwise, there would be  $(D_1, D_2)$ -opposite paths. Note that  $y_2, z_2 \in V_2$ .

If  $\{c_1, c_2\} \subseteq V(D_1)$ , then the  $(z_2, c'')$ -dipath in  $S$  contains a  $(D_2, D_1)$ -dipath that together with  $C[v_1, y_2]$  gives  $(D_1, D_2)$ -opposite paths.

This shows that  $\{c_1, c_2\} \subseteq V(D_2)$ . The  $(d'', c'')$ -dipath in  $S - E(C)$  must have all its internal vertices in  $D_2$ , because every  $(D_1, D_2)$ -dipath meets  $\{u_2, v_1\}$ . Therefore the digraph obtained from  $S \cap (D_1 \cup D_2)$  by adding the arcs  $u_2v_1, v_1y_2$ , and  $v_1z_2$  is an  $F_3$ -subdivision in  $\hat{D}^*$ .

If  $b' \notin V(D_1) \cup V(D_2)$ , we get the result analogously.  $\diamond$

In view of Claim 51.5, we replace  $D$  by  $D^*$ . Henceforth, now  $V(D) = V(D_1) \cup V(D_2)$ . Moreover, there are at most two arcs with tail in  $D_2$  and head in  $D_1$ , namely  $u_2v_1$  and possibly  $v_2u_1$ .

Let  $D'_1$  be the digraph obtained from  $D$  by contracting  $D_2$  into a vertex  $z_2$ . If all four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$  are present in  $D$  (Case (B)), then we also add into  $D'_1$  the arcs  $u_1v_1$  and  $v_1u_1$  if they are not already contained in  $D_1$ . Similarly, we let  $D'_2$  be the digraph obtained from  $D$  by contracting  $D_1$  into a vertex  $z_1$  and adding the arcs  $u_2v_2$  and  $v_2u_2$  if  $D$  contains all four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ . Observe that  $D'_1$  and  $D'_2$  are both strong, and contain fewer vertices than  $D$ .

If  $i_c = i_d$ , then we return  $F_3\text{-Strong+}(D'_{i_c}, \{c_1, c_2\}, \{d_1, d_2\})$ . This is valid by the following claim.

**Claim 51.6.** *If  $i_c = i_d$ , then  $\hat{D}$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision if and only if  $\hat{D}_{i_c}$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision.*

*Subproof.* We shall assume that  $i_c = 1$ . (The case when  $i_c = 2$  is proved in the same way.) Suppose first that  $\hat{D}$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision  $S$ . If  $S$  does not intersect  $D_2$ , then  $S$  is a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$ . Henceforth we assume that  $S$  intersects  $D_2$ . Observe that vertices  $a'$  and  $b'$  corresponding to  $a$  and  $b$  in  $S$  belong to a  $(d'', c'')$ -dipath in  $S$ . Therefore,  $a'$  and  $b'$  belong to  $D_1$ , since there are no  $(D_1, D_2)$ -opposite dipaths in  $D$ . Consequently, every vertex in  $S$  lies on a  $(d'', c'')$ -dipath or on an  $(a', b')$ -dipath in  $S$ , and each such path intersects  $D_2$  in at most one vertex. If  $S$  contains only one vertex  $v$  in  $D_2$ , the digraph  $S'$  obtained from  $S$  by replacing  $v$  by  $z_2$ , we obtain a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$ . If  $S$  contains two vertices in  $D_2$ , then  $S$  contains all four arcs  $u_1u_2, u_2v_1, v_1v_2, v_2u_1$  and hence  $D'_1$  contains the arcs  $u_1v_1$  and  $v_1u_1$ . Moreover,  $\{a', b'\} = \{u_1, v_1\}$  and the cycle in  $S$  is the 4-cycle  $u_1u_2v_1v_2u_1$ . Therefore, the arcs  $u_1v_1$  and  $v_1u_1$  are not both in  $S$ . Then we replace the cycle in  $S$  by the cycle  $u_1v_1z_2u_1$  in  $D'_1$  and obtain a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$ .

Suppose now that  $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision  $S'$ . Let us first assume that  $S'$  does not contain  $z_2$ . If  $S'$  contains an arc that is not in  $A(D)$ , then this is either the arc  $e = u_1v_1$  or  $e' = v_1u_1$ . This arc was added to  $D'_1$  only in Case (B). Thus, we can replace  $e$  in  $S'$  by the path  $u_1u_2v_1$  and  $e'$  by the path  $v_1v_2u_1$ . By making these changes (if needed), we obtain a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}$ . Henceforth we assume that  $S'$  contains  $z_2$ .

If  $S'$  contains an arc  $e \in \{v_1u_1, u_1, v_1\}$  and  $e \notin A(D)$ , then we have Case (B). In  $S'$ , the vertex  $z_2$  has in- and outdegree equal to 1, while each of  $u_1$  and  $v_1$  has either in- or outdegree equal to 2. Therefore, the arc in  $\{v_1u_1, u_1, v_1\} \setminus \{e\}$  cannot be in  $S'$ . By replacing the path in  $S'$  joining  $u_1$  and  $v_1$  through  $z_2$  by that arc, we obtain an  $F_3$ -subdivision that does not contain  $z_2$ , and we are done in the same way as above. Thus, we may assume that  $S'$  contains no edge in  $\{v_1u_1, u_1, v_1\} \setminus A(D)$ .

The vertex  $z_2$  has an inneighbour  $x_1$  and an outneighbour  $y_1$  in  $S'$ , and possibly has a third neighbour  $z_1$ . If  $z_1$  exists, then we assume that the arcs  $x_1z_2$  and  $z_2y_1$  lie on the cycle in  $S'$ . By definition of contraction,  $x_1$  has an outneighbour  $x_2$  in  $V(D_2)$  and  $y_1$  has an inneighbour  $y_2$  in  $V(D_2)$ . Moreover if  $z_1$  exists, let  $w_2$  be one of its outneighbours (resp. inneighbours) in  $V(D_2)$  corresponding to the arc joining  $z_1$  and  $z_2$  in  $D'_1$ . Let  $Q$  be an  $(x_2, y_2)$ -dipath in  $D_2$  and  $Q'$  be a  $(w_2, Q)$ -dipath (resp.  $(Q, w_2)$ -dipath) in  $D_2$  if  $z_2z_1 \in A(S')$  (resp.  $z_1z_2 \in A(S')$ ). Now the digraph obtained from  $S'$  by replacing  $z_2$  and the arcs incident to it by the paths  $Q$  and  $Q'$ , and the arcs  $x_1x_2, y_1y_2$  and  $z_1w_2$  or  $w_2z_1$ , is a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}$ .  $\diamond$

Henceforth, we have  $i_c \neq i_d$ . If  $i_d = 2$  and  $i_c = 1$ , then a  $(c'', d'')$ -forced  $F_3$ -subdivision contains two disjoint arcs from  $D_2$  to  $D_1$ . Thus, necessarily  $v_2u_1$  is an arc, because there are no  $(D_1, D_2)$ -opposite paths. In this case, we consider exchanging the roles of  $D_1$  and  $D_2$ . Thus, we may assume henceforth that we are in the case when  $i_d = 1$  and  $i_c = 2$ .

Let  $D_1^* = D[V(D_1) \cup \{u_2\}]$  and  $D_2^* := D[V(D_2) \cup \{v_1\}]$ . Observe that  $D_1^*$  and  $D_2^*$  are both strong.

A  $(c'', d'')$ -forced  $F_3$ -subdivision contains two internally disjoint  $(d'', c'')$ -dipaths. Therefore, using a Menger algorithm, we check if two such dipaths exist in  $\hat{D}$ . If two such dipaths do not exist, then we return 'no'. Otherwise there are two internally disjoint  $(d'', c'')$ -dipaths. Because there are no  $(D_1, D_2)$ -opposite paths, one of them say  $P_1^*$  must go through  $v_1$  and the other, say  $P_2^*$ , through  $u_2$ . We return  $(F_3\text{-Strong+}(D_1^*, \{v_1, u_2\}, \{d_1, d_2\}))$  or  $F_3\text{-Strong+}(D_2^*, \{c_1, c_2\}, \{v_1, u_2\})$ . This is valid by the following claim.

**Claim 51.7.**  *$\hat{D}$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision if and only if either  $\hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$  or  $\hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision.*

*Subproof.* Set  $\hat{D}_1^* := \hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$  and  $\hat{D}_2^* := \hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$ .

A  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}_1^*$  (or one in  $\hat{D}_2^*$ ) can easily be transformed into a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}$  by replacing the arcs  $v_1 c''$  and  $u_2 c''$  by  $P_1^*[v_1, c'']$  and  $P_2^*[u_2, c'']$ .

Suppose now that  $\hat{D}$  contains a  $(c'', d'')$ -forced  $F_3$ -subdivision  $S$ . Let  $C$  be the directed cycle in  $S$  and let  $Q_1$  and  $Q_2$  be the two internally disjoint  $(d'', c'')$ -dipaths in  $S$ . Because there are no  $(D_1, D_2)$ -opposite paths, one of these dipaths, say  $Q_1$  goes through  $v_1$  and the other goes through  $u_2$ . Moreover,  $C$  intersects either  $D_1$  or  $D_2$  in at most one vertex. If  $D$  does not contain the arc  $v_2 u_1$  (Case (A)), then if  $C$  intersects  $D_1$  (resp.  $D_2$ ) in one vertex, this vertex must be  $v_1$  (resp.  $u_2$ ). We may assume that the same holds in Case (B) after possibly exchanging the roles of  $u_1$  and  $v_1$  and of  $u_2$  and  $v_2$ . Hence the digraph obtained from  $S$  by replacing  $Q_1[d'', v_1]$  and  $Q_2[d'', u_2]$  (resp.  $Q_1[v_1, c'']$  and  $Q_2[u_2, c'']$ ) by the arcs  $d'' v_1$  and  $d'' u_2$  (resp.  $v_1 c''$  and  $u_2 c''$ ) is a  $(c'', d'')$ -forced  $F_3$ -subdivision in  $\hat{D}_2^*$  (resp.  $\hat{D}_1^*$ ).  $\diamond$

This completes the procedure  $F_3\text{-Strong+}$ . Let us now examine its time complexity. Let  $T(n)$  be the maximum running time on a digraph with at most  $n$  vertices. Clearly, the running time between two recursive calls is bounded by a polynomial  $P(n)$ . When treating a graph  $D$  on  $n$  vertices, it then makes a recursive call either to a smaller digraph, or to two smaller digraphs  $D_1^*$  and  $D_2^*$  such that  $|D_1^*| + |D_2^*| \leq n + 2$ . Hence  $T(n)$  satisfies the inequality

$$T(n) \leq P(n) + \max \{T(n-1); \max\{T(n_1) + T(n_2) \mid n_1 + n_2 \leq n + 2, n_1 < n, n_2 < n\}\}.$$

This implies that  $T(n)$  is bounded above by a polynomial value in  $n$ .

### A.2.5 Detecting $F_4$

**Proposition 52.** *The digraph  $F_4$  depicted in Figure 8 is tractable. More precisely,  $F_4\text{-SUBDIVISION}$  can be solved in time  $O(n^2 m(n + m))$ .*

*Proof.* Let  $D$  be a digraph. Observe that every  $F_4$ -subdivision contains an  $F_4$ -subdivision in which the arc  $cd$  is not subdivided. Henceforth by  $F_4$ -subdivision, we mean such a subdivision.

Given four distinct vertices  $a', b', c', d'$  such that  $c'd'$  is an arc, we say that an  $F_4$ -subdivision is  $(a', b', c'd')$ -forced if  $a'$  is its  $a$ -vertex,  $b'$  its  $b$ -vertex,  $c'$  its  $c$ -vertex, and  $d'$  its  $d$ -vertex.

We shall present a procedure  $F_4\text{-Subdivision}(D, a', b', c'd')$ , that returns ‘no’ only if there is no  $(a', b', c'd')$ -forced  $F_4$ -subdivision in  $D$ , and returns ‘yes’ if it finds an  $F_4$ -subdivision in  $D$  (not necessarily one that is  $(a', b', c'd')$ -forced). We proceed as follows.

Suppose first that  $a'b'$  is an arc. Using a Menger algorithm, we check whether there are two independent  $(b', \{a', c'\})$ -dipaths in  $D - d'$ , and using a search, we check whether there exists a  $(c', a')$ -dipath in  $D' - \{b', d'\}$ . If three such dipaths do not exist, then there is no  $(a', b', c'd')$ -forced  $F_4$ -subdivision in  $D$ , and we return ‘no’. If three such dipaths exist, then we return ‘yes’. This is valid by virtue of the following claim.

**Claim 52.1.** *If there are a  $(c', a')$ -dipath  $R$  in  $D' - \{b', d'\}$  and two independent  $(b', \{a', c'\})$ -dipaths  $P_1, P_2$  in  $D - d'$ , then  $D$  contains an  $F_4$ -subdivision.*

*Subproof.* Without loss of generality, we may assume that  $t(P_1) = a'$  and  $t(P_2) = c'$ . The dipath  $R$  contains a subdipath  $R'$  with initial vertex  $s$  in  $P_2$  and terminal vertex in  $P_1[b', a']$ . Let  $s^+$  be the outneighbour of  $s$  in  $P_2 \cup c'd'$ . Then  $a'b' \cup P_1 \cup P_2[b', s] \cup R' \cup ss^+$  is an  $F_4$ -subdivision.  $\diamond$

Henceforth, we assume that  $a'b'$  is not an arc in  $D$ .

If  $d_{D-\{c', d'\}}^+(a') = 0$ , then there is no  $(a', b')$ -dipath in  $D - c' - d'$ , and thus no  $(a', b', c'd')$ -forced  $F_4$ -subdivision. Hence we return ‘no’.

If  $d_{D-\{c',d'\}}^+(a') = 1$ , then denote by  $a''$  the unique outneighbour of  $a'$  in  $D - \{c',d'\}$ . By our assumption,  $a'' \neq b'$ . Let  $D^*$  be the digraph obtained from  $D$  by first removing all arcs entering  $a''$  and then identifying  $a'$  and  $a''$  into a single vertex  $a^*$ . Note that  $a^*$  is dominated by the inneighbours of  $a'$  in  $D$  and dominates the outneighbours of  $a''$  in  $D$ . We then return  $F_4\text{-Subdivision}(D^*, a^*, b', c'd')$ . The validity of this recursive call is shown by the following claim.

**Claim 52.2.** *If  $d_{D-\{c',d'\}}^+(a') = 1$ , then  $D$  contains an  $(a', b', c'd')$ -forced  $F_4$ -subdivision if and only if  $D^*$  contains an  $(a^*, b', c'd')$ -forced  $F_4$ -subdivision.*

*Subproof.* Assume that  $S$  is an  $(a', b', c'd')$ -forced  $F_4$ -subdivision in  $D$ . Since  $d_{D-\{c',d'\}}^+(a') = 1$ ,  $S$  contains the arc  $a'a''$  since  $a''$  is the unique outneighbour of  $a'$  in  $D - \{c',d'\}$ . Now the digraph  $S^*$  obtained from  $S$  by replacing  $a'$  and  $a''$  and the four arcs  $ua', va', a'a'', a''w$  by the vertex  $a^*$  and the three arcs  $ua^*, va^*, a^*w$  is an  $(a^*, b', c'd')$ -forced  $F_4$ -subdivision in  $D^*$ , because  $a'' \neq b'$ .

Conversely, if  $S^*$  is an  $(a^*, b', c'd')$ -forced  $F_4$ -subdivision in  $D^*$ , then the digraph  $S$  obtained from  $S^*$  by replacing  $a^*$  and its three incident arcs  $ua^*, va^*, a^*w$  by vertices  $a'$  and  $a''$  and the four arcs  $ua', va', a'a'', a''w$  is clearly an  $(a', b', c'd')$ -forced  $F_4$ -subdivision in  $D$ .  $\diamond$

Henceforth, we may assume that  $d_{D-\{c',d'\}}^+(a') \geq 2$ . Using a Menger algorithm, we check whether there are two independent  $(b', \{a', c'\})$ -dipaths in  $D - d'$ , and using a search we check whether there exists an  $(a', b')$ -dipath in  $D - \{c', d'\}$ , and whether there exists a  $(c', a')$ -dipath in  $D - \{b', d'\}$ . If four such dipaths do not exist, then there is no  $(a', b', c'd')$ -forced  $F_4$ -subdivision in  $D$ , and we return ‘no’. If four such dipaths exist, then we return ‘yes’ by virtue of the following claim.

**Claim 52.3.** *If there are two independent  $(b', \{a', c'\})$ -dipaths  $P_1$  and  $P_2$  in  $D - d'$ , an  $(a', b')$ -dipath  $Q$  in  $D - \{c', d'\}$ , and a  $(c', a')$ -dipath  $R$  in  $D - \{b', d'\}$ , then  $D$  contains an  $F_4$ -subdivision.*

*Subproof.* Without loss of generality, we may assume that  $t(P_1) = a'$  and  $t(P_2) = c'$ . Let  $v$  be the last vertex along  $Q - b'$  that is in  $P_1 \cup P_2$ . We distinguish two cases according to whether  $v$  is on  $P_1$  or  $P_2$ .

Case 1:  $v \in V(P_1)$ . Note that this is in particular the case when  $Q$  is internally disjoint from  $P_1$  and  $P_2$ . Let  $C$  be the directed cycle formed by the union of  $P_1[b'v]$  and  $Q[v, b']$ , let  $Q'$  be the  $(a', C)$ -subdipath in  $Q$ , and let  $R' = RQ'$ . The directed walk  $R'$  contains a subdipath  $R''$  with initial vertex  $s$  in  $P_2[b', c']$  and terminal vertex  $t$  in  $C$ . Let  $s^+$  be the outneighbour of  $s$  in  $P_2 \cup c'd'$ . Then  $C \cup P_2[b', s] \cup R'' \cup ss^+$  is an  $F_4$ -subdivision.

Case 2:  $v \in V(P_2)$ . Let  $v^+$  be the outneighbour of  $v$  in  $P_2 \cup c'd'$ . The dipath  $Q[a', v]$  contains a subdipath  $Q'$  with initial vertex  $u$  in  $P_1$  and terminal vertex  $w$  in  $P_2[b', v]$  whose internal vertices are not in  $P_1 \cup P_2[b', v]$ . Let  $C'$  be the directed cycle formed by the union of  $P_2[b', v]$  and  $Q[v, b']$ . If  $u \neq a'$ , let  $u^+$  be the outneighbour of  $u$  in  $P_1$ . Then  $C' \cup P_1[b', u^+] \cup Q'$  is an  $F_4$ -subdivision. Henceforth, we may assume that  $u = a'$ .

Let  $u'$  be the outneighbour of  $a'$  in  $Q'$ . Now  $d_{D-\{c',d'\}}^+(a') \geq 2$  and  $a'b'$  is not an arc. Hence,  $a'$  has an outneighbour  $z$  distinct from  $b', c', d'$ , and  $u'$ .

- If  $z \notin V(C' \cup P_1 \cup Q')$ , then  $C' \cup P_1 \cup Q' \cup a'z$  is an  $F_4$ -subdivision.
- If  $z \in V(Q')$ , then  $C' \cup P_1 \cup a'z \cup Q'[z, w] \cup a'u'$  is an  $F_4$ -subdivision.
- Assume  $z \in V(P_1)$ . If  $v^+ \notin V(Q')$ , then  $P_1 \cup a'z \cup Q' \cup P_2[w, v] \cup Q[v, b'] \cup vv^+$  is an  $F_4$ -subdivision. If  $v^+ \in V(Q')$ , then  $v^+ \neq d'$  and so  $P_2[v^+, c']$  is not an empty dipath. Denote by  $C''$  the directed cycle  $P_1[z, a'] \cup a'z$ . The dipath  $Q[v^+, b'] \cup P_1[b', z]$  contains a  $(P_2[v^+, c'], C'' - a')$ -dipath  $Q''$ . Let  $s^+$  be the outneighbour of  $s(Q'')$  in  $P_2 \cup c'd'$ . Now  $C'' \cup Q[a', v^+] \cup P_2[v^+, s(Q'')] \cup Q'' \cup s(Q'')s^+$  is an  $F_4$ -subdivision.

- Assume  $z \in V(Q[v, b'])$ . Then one can replace the  $(a', b')$ -dipath  $Q$  by  $a'zQ[z, b']$ . This dipath is internally disjoint from  $P_1$  and  $P_2$ , and we get the result by Case 1.
- Assume finally that  $z \in V(P_2[b', v])$ . If  $u' \neq w$ , then  $C' \cup P_1 \cup a'z \cup a'u'$  is an  $F_4$ -subdivision. Henceforth we assume that  $u' = w$ , so  $a'w$  is an arc. Without loss of generality, we may assume that  $z$  precedes  $w$  along  $P_2$ . For  $i = 1, 2$ , let  $b_i^+$  be the outneighbour of  $b'$  in  $P_i$ . By the previous assumption,  $b_2^+ \neq w$ . Let  $t$  be the last vertex along  $R - a'$  in  $V(P_1 \cup P_2 \cup Q[v, b'])$ .

If  $t \in V(P_1)$ , then one of the two dipaths  $P_1[t, a']$  and  $R[t, a']$  has length at least 2. Let  $t^+$  be the outneighbour of  $t$  in this dipath, and let  $T$  be the other dipath. Now  $C' \cup P_1[b', t] \cup T \cup a'w \cup tt^+$  is an  $F_4$ -subdivision.

If  $t \in V(Q[v, b'] \cup P_2[w, v])$ , then  $R[t, a'] \cup a'w \cup P_2[w, v] \cup Q[v, b'] \cup P_1 \cup b'b_2^+$  is an  $F_4$ -subdivision.

If  $t \in V(P_2[v, c'])$ , then  $R[t, a'] \cup a'w \cup P_2[w, t] \cup Q[v, b'] \cup P_1 \cup b'b_2^+$  is an  $F_4$ -subdivision.

If  $t \in V(P_2[z, w])$ , then  $R[t, a'] \cup a'z \cup P_2[z, v] \cup Q[v, b'] \cup vv^+ \cup P_1$  is an  $F_4$ -subdivision.

It remains to consider the case when  $t \in V(P_2[b', z])$ . Let  $t^+$  be the outneighbour of  $t$  on  $P_2$ . Then  $P_1 \cup P_2[b', t^+] \cup R[t, a'] \cup a'w \cup P_2[w, v] \cup Q[v, b']$  is an  $F_4$ -subdivision in  $D$ .

◇

One can easily see that the procedure  $F_4\text{-Subdivision}(D, a', b', c'd')$  runs in linear time as it either reduces the problem in constant time (when  $d^+(a') = 1$ ) or runs a Menger algorithm and at most two searches, which can be done in linear time. Running this procedure for the  $O(n^2m)$  possible choices of  $(a', b', c'd')$ , we obtain an algorithm with running time  $O(n^2m(n+m))$  that solves  $F_4\text{-SUBDIVISION}$ . □

**Proposition 53.** *For every  $i \in \{5, 6, 7\}$ , the digraph  $F_i$  depicted in Figure ?? is tractable.*

*Proof.*  $i = 5$ : Observe first that if a digraph contains an  $F_5$ -subdivision, then it contains such an  $F_5$ -subdivision in which  $ac$  and  $bd$  are not subdivided. Henceforth, by  $F_5$ -subdivision, we mean an  $F_5$ -subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs  $a'c'$  and  $b'd'$ , returns ‘yes’ if it finds an  $F_5$ -subdivision and returns ‘no’ only if there is no  $F_5$ -subdivision with  $a$ -vertex  $a'$ ,  $b$ -vertex  $b'$ ,  $c$ -vertex  $c'$  and  $d$ -vertex  $d'$ . Such a subdivision is said to be  $(a'c', b'd')$ -forced.

We check whether, in  $D - \{c', d'\}$ , there exists an  $(a', b')$ -dipath  $P$  and a  $(b', a')$ -dipath  $Q$ . If two such dipaths do not exist, then there is clearly no  $(a'c', b'd')$ -forced  $F_5$ -subdivision, and we return ‘no’. If two such paths  $P$  and  $Q$  exist, then we return ‘yes’. Indeed let  $b''$  be the last vertex on  $Q - a'$  that is in  $P \cap Q$ . If  $b'' = b'$ , then set  $d'' = d'$ , otherwise let  $d''$  be the successor of  $b''$  on  $P$ . Then the union of the directed cycle  $P[a', b'']Q[b'', a']$  and the two arcs  $a'c'$  and  $b''d''$  form an  $F_5$ -subdivision.

Applying the above procedure for all possible pairs of distinct arcs  $(a'c', b'd')$ , one solves  $F_5\text{-SUBDIVISION}$  in  $O(m^2(n+m))$  time.

$i = 6$ : The proof is similar to the case  $i = 7$ , and we leave details to the reader.

$i = 7$ : Observe first that if a digraph contains an  $F_7$ -subdivision, then it contains an  $F_7$ -subdivision in which  $dc$  is not subdivided. Henceforth, by  $F_7$ -subdivision, we mean an  $F_7$ -subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs  $sa'$  and  $d'c'$ , checks whether there is an  $F_7$ -subdivision  $S$  with  $a$ -vertex  $a'$ ,  $c$ -vertex  $c'$  and  $d$ -vertex  $d'$ , and such that  $sa' \in A(S)$ . Such a subdivision is said to be  $(sa', d'c')$ -forced.

With a Menger algorithm, we check whether  $D - d'$  contains two independent  $(a', \{s, c'\})$ -dipaths. If two such dipaths do not exist, then there is clearly no  $(a'c', b'd')$ -forced  $F_7$ -subdivision, and we return

‘no’. If two such dipaths  $P$  and  $Q$  exist, then without loss generality  $t(P) = s$  and  $t(Q) = c$ . The union of the directed cycle  $Psa'$ , the dipath  $Q$  and the arc  $d'c'$  is an  $(sa', d'c')$ -forced  $F_7$ -subdivision.

Applying the above procedure for all possible pairs of distinct arcs  $(sa', d'c')$ , one solves  $F_7$ -SUBDIVISION in  $O(m^2(n + m))$  time.  $\square$





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