



Layer-averaged Euler and Navier-Stokes equations

Marie-Odile Bristeau, Bernard Di Martino, Cindy Guichard, Jacques
Sainte-Marie

► To cite this version:

Marie-Odile Bristeau, Bernard Di Martino, Cindy Guichard, Jacques Sainte-Marie. Layer-averaged Euler and Navier-Stokes equations. 2015. hal-01202042v2

HAL Id: hal-01202042

<https://inria.hal.science/hal-01202042v2>

Preprint submitted on 21 Sep 2015 (v2), last revised 27 Jun 2016 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Layer-averaged Euler and Navier-Stokes equations

M.-O. Bristeau, C. Guichard, B. Di Martino, J. Sainte-Marie

September 21, 2015

Abstract

In this paper we propose a strategy to approximate incompressible free surface Euler and Navier-Stokes models. The main advantage of the proposed models is that the water depth is a dynamical variable of the system and hence the model is formulated over a fixed domain.

The proposed strategy extends previous works approximating the Euler and Navier-Stokes systems using a multilayer description. Here, the needed closure relations are obtained using an energy-based optimality criterion instead of an asymptotic expansion. Moreover, the layer-averaged description is successfully applied to the Navier-Stokes system with a general form of the Cauchy stress tensor.

Keywords : Incompressible Navier-Stokes equations, incompressible Euler equations, free surface flows, newtonian fluids, complex rheology

Contents

1	Introduction	2
2	The Navier-Stokes system	3
2.1	The hydrostatic Navier-Stokes system	3
2.2	Boundary conditions	4
2.2.1	Free surface conditions	4
2.2.2	Bottom conditions	5
2.3	Other witting	5
2.4	Energy balance	5
3	Depth-averaged solutions of the Euler system	6
3.1	Vertical discretization of the fluid domain	6
3.2	Layer-averaging of the Euler solution	7
3.3	Closure relations	10

4	The proposed layer-averaged Euler system	13
4.1	Formulation	13
4.2	The vertical velocity	15
5	The Navier-Stokes system	16
5.1	Layers averaged Navier-Stokes system	17
5.2	Newtonian fluids	22
5.3	An extended Saint-Venant system	24
6	Conclusion	25

1 Introduction

Due to computational issues associated with the free surface Navier-Stokes or Euler equations, the simulations of geophysical flows are often carried out with shallow water type models of reduced complexity. Indeed, for vertically averaged models such as the Saint-Venant system [7], efficient and robust numerical techniques (relaxation schemes [9], kinetic schemes [23, 2],...) are available and avoid to deal with moving meshes.

In order to describe and simulate complex flows where the velocity field cannot be approximated by its vertical mean, multilayer models have been developped [1, 3, 4, 8, 13, 12]. Unfortunately these models are physically relevant for non miscible fluids.

In [14, 6, 5, 24], some authors have proposed a simpler and more general formulation for multilayer model with mass exchanges between the layers. The obtained model has the form of a conservation law with source terms, its hyperbolicity remains an open question. Notice that in [5] the hydrostatic Navier-Stokes equations with variable density is tackled and in [24] the approximation of the non-hydrostatic terms in the multilayer context is studied.

The objective of the paper is twofold. First we want to present another derivation of the models proposed in [6, 5, 24], no more based on an asymptotic expansion but on an energy-based optimality criterion. Such a strategy is widely used in the kinetic framework to obtain kinetic descriptions e.g. of conservations laws [18, 23]. Second, we intend to obtain a multilayer formulation of the Navier-Stokes system with a rheology more complex than the one arising when considering newtonian fluids.

The paper is organized as follows. In Section 2 we recall the incompressible hydrostatic Navier-Stokes equations with free surface with the associated boundary conditions. In Section 3 we detail the layer averaging process for the Euler system and obtained the required closure relations. The proposed layer-averaged Euler system is given in Section 4 and its extension to the Navier-Stokes system with a general rheology is presented in Section 5.

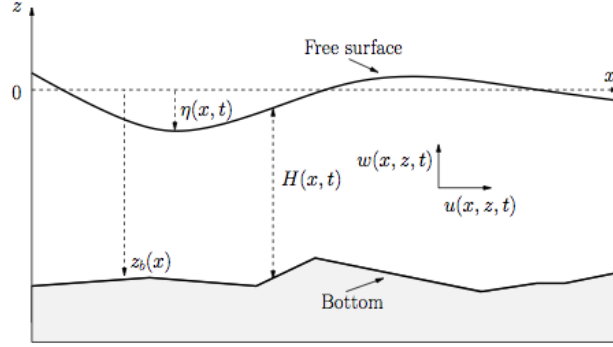


Figure 1: Flow domain with water height $H(x, t)$, free surface $\eta(x, t)$ and bottom $z_b(x, t)$.

2 The Navier-Stokes system

We consider the two-dimensional hydrostatic Navier-Stokes system [19] describing a free surface gravitational flow moving over a bottom topography $z_b(x)$. For free surface flows, the hydrostatic assumption consists in neglecting the vertical acceleration, see [10, 16, 21] for justifications of the obtained models.

2.1 The hydrostatic Navier-Stokes system

We denote with x and z the horizontal and vertical directions, respectively. The system has the form:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z}, \quad (2)$$

$$\frac{\partial p}{\partial z} = -g + \frac{\partial \Sigma_{zx}}{\partial x} + \frac{\partial \Sigma_{zz}}{\partial z}, \quad (3)$$

and we consider solutions of the equations for

$$t > t_0, \quad x \in \mathbb{R}, \quad z_b(x) \leq z \leq \eta(x, t),$$

where $\eta(x, t)$ represents the free surface elevation, $\mathbf{u} = (u, w)^T$ the velocity vector, p the fluid pressure and g the gravity acceleration. The flow height is $H = \eta - z_b$, see Fig. 1. The Cauchy stress tensor Σ_T is defined by $\Sigma_T = -pI_d + \Sigma$ with

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xz} \\ \Sigma_{zx} & \Sigma_{zz} \end{pmatrix},$$

and Σ represents the fluid rheology.

As in Ref. [15], we introduce the indicator function for the fluid region

$$\varphi(x, z, t) = \begin{cases} 1 & \text{for } (x, z) \in \Omega = \{(x, z) \mid z_b \leq z \leq \eta\}, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

The fluid region is advected by the flow, which can be expressed, thanks to the incompressibility condition, by the relation

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} = 0. \quad (5)$$

The solution φ of this equation takes the values 0 and 1 only but it needs not be of the form (4) at all times. The analysis below is limited to the conditions where this form is preserved. For a more complete presentation of the Navier-Stokes system and its closure, the reader can refer to [19].

Remark 2.1 *Notice that in the fluid domain, Eq. (5) reduces to the divergence free condition whereas across the upper and lower boundaries it gives the kinematic boundary conditions defined in the following.*

2.2 Boundary conditions

The system (1)-(3) is completed with boundary conditions. The outward unit normal vector to the free surface \mathbf{n}_s and the upward unit normal vector to the bottom \mathbf{n}_b are given by

$$\mathbf{n}_s = \frac{1}{\sqrt{1 + \left(\frac{\partial \eta}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial \eta}{\partial x} \\ 1 \end{pmatrix}, \quad \mathbf{n}_b = \frac{1}{\sqrt{1 + \left(\frac{\partial z_b}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial z_b}{\partial x} \\ 1 \end{pmatrix},$$

respectively.

2.2.1 Free surface conditions

At the free surface we have the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s = 0, \quad (6)$$

where the subscript s indicates the value of the considered quantity at the free surface.

Assuming negligible the air viscosity, the continuity of stresses at the free boundary imposes

$$\Sigma_T \mathbf{n}_s = -p^a \mathbf{n}_s, \quad (7)$$

where $p^a = p^a(x, t)$ is a given function corresponding to the atmospheric pressure. Within this paper, we consider $p^a = 0$.

2.2.2 Bottom conditions

The kinematic boundary condition at the bottom consists in a classical no-penetration condition:

$$\mathbf{u}_b \cdot \mathbf{n}_b = 0, \quad \text{or} \quad u_b \frac{\partial z_b}{\partial x} - w_b = 0. \quad (8)$$

For the stresses at the bottom we consider a wall law under the form

$$\mathbf{t}_b \cdot \Sigma_T \mathbf{n}_b = \kappa u_b, \quad (9)$$

where \mathbf{t}_b is a unit vector satisfying $\mathbf{t}_b \cdot \mathbf{n}_b = 0$. If $\kappa(\mathbf{u}_b, H)$ is constant then we recover a Navier friction condition as in [15]. Introducing a laminar friction k_l and a turbulent friction k_t , we use the expression

$$\kappa(\mathbf{u}_b, H) = k_l + k_t H |\mathbf{u}_b|,$$

corresponding to the boundary condition used in [20]. Another form of $\kappa(\mathbf{u}_b, H)$ is used in [9], and for other wall laws the reader can also refer to [22]. Due to thermo-mechanical considerations, in the sequel we will suppose $\kappa(\mathbf{u}_b, H) \geq 0$, and $\kappa(\mathbf{u}_b, H)$ will be often simply denoted by κ .

2.3 Other writting

For reasons that will appear later, we rewrite (1)-(3) under the form

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (10)$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + g \frac{\partial \eta}{\partial x} = \frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z} + \frac{\partial^2}{\partial x^2} \int_z^\eta \Sigma_{zx} dz_1 - \frac{\partial \Sigma_{zz}}{\partial x}, \quad (11)$$

where Eq. (11) has been obtained as follows. Integrating Eq. (3) from z to η and taking into account the boundary condition (7) gives

$$p = g(\eta - z) - \frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 + \Sigma_{zz}. \quad (12)$$

Inserting the previous expression for p in Eq. (2) gives Eq. (11).

2.4 Energy balance

Lemma 2.2 *We recall the fondamentale stability property related to the fact that the hydrostatic Navier-Stokes system admits an energy that can be written under the form*

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{z_b}^\eta E \, dz + \frac{\partial}{\partial x} \int_{z_b}^\eta \left[u(E + g(\eta - z)) - u(\Sigma_{xx} - \Sigma_{zz}) - w \Sigma_{zx} - u \frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 \right] dz \\ &= - \int_{z_b}^\eta \Sigma_{xx} \frac{\partial u}{\partial x} dz - \int_{z_b}^\eta \Sigma_{xz} \frac{\partial u}{\partial z} dz - \int_{z_b}^\eta \frac{\partial w}{\partial x} \Sigma_{zx} dz - \int_{z_b}^\eta \Sigma_{zz} \frac{\partial w}{\partial z} dz - \kappa u_b, \end{aligned} \quad (13)$$

with E defined by (20).

Proof of lemma 2.2 *The way the energy balance (13) is obtained is classical. Considering smooth solutions, first we multiply Eq. (2) by u and Eq. (3) by w then we sum the two obtained equations. After simple manipulations and using the kinematic and dynamic boundary conditions (6)-(9), we obtain the relation*

$$\begin{aligned} \frac{\partial}{\partial t} \int_{z_b}^{\eta} E \, dz + \frac{\partial}{\partial x} \int_{z_b}^{\eta} \left[u(E + p) - u\Sigma_{xx} - w\Sigma_{zx} \right] dz \\ = - \int_{z_b}^{\eta} \Sigma_{xx} \frac{\partial u}{\partial x} dz - \int_{z_b}^{\eta} \Sigma_{xz} \frac{\partial u}{\partial z} dz - \int_{z_b}^{\eta} \frac{\partial w}{\partial x} \Sigma_{zx} dz - \int_{z_b}^{\eta} \Sigma_{zz} \frac{\partial w}{\partial z} dz - \kappa u_b. \end{aligned}$$

Replacing p by its expression given by (12) in the previous relation gives the result. \blacksquare

3 Depth-averaged solutions of the Euler system

In this section, neglecting the viscous effects in Eqs. (1)-(3), we consider the free surface hydrostatic Euler equations written in a conservative form

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} = 0, \quad (14)$$

$$\frac{\partial \varphi u}{\partial t} + \frac{\partial \varphi u^2}{\partial x} + \frac{\partial \varphi uw}{\partial z} + \frac{\partial p}{\partial x} = 0, \quad (15)$$

$$\frac{\partial p}{\partial z} = -\varphi g, \quad (16)$$

with φ defined by (4). This system is completed with the boundary conditions (6),(8) and (7) that reduces to

$$p_s = 0. \quad (17)$$

From Eqs. (16),(17), we get

$$p = \varphi g(\eta - z). \quad (18)$$

The energy balance associated with the hydrostatic Euler system is given by

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} E \, dz + \frac{\partial}{\partial x} \int_{z_b}^{\eta} u(E + p) \, dz = 0, \quad (19)$$

with

$$E = \frac{u^2}{2} + gz. \quad (20)$$

3.1 Vertical discretization of the fluid domain

The interval $[z_b, \eta]$ is divided into N layers $\{L_\alpha\}_{\alpha \in \{1, \dots, N\}}$ of thickness $l_\alpha H(x, t)$ where each layer L_α corresponds to the points satisfying $z \in L_\alpha(x, t) = [z_{\alpha-1/2}, z_{\alpha+1/2}]$ with

$$\begin{cases} z_{\alpha+1/2}(x, t) = z_b(x, t) + \sum_{j=1}^{\alpha} l_j H(x, t), \\ h_\alpha(x, t) = z_{\alpha+1/2}(x, t) - z_{\alpha-1/2}(x, t) = l_\alpha H(x, t), \quad \alpha \in \{1, \dots, N\}, \end{cases} \quad (21)$$

with $l_j > 0$, $\sum_{j=1}^N l_j = 1$, see Fig. 2.

We also define

$$z_\alpha = \frac{z_{\alpha+1/2} + z_{\alpha-1/2}}{2} = z_{\alpha-1/2} + \frac{h_\alpha}{2}, \quad \alpha = \{1, \dots, N\}. \quad (22)$$

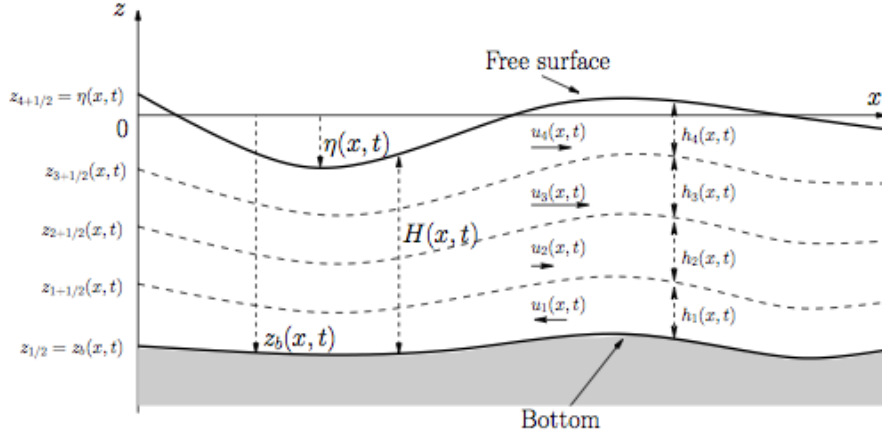


Figure 2: Notations for the multilayer approach.

3.2 Layer-averaging of the Euler solution

In this section we take the vertical average of the Euler system and study the necessary closure relations for this system.

Let us denote $\langle f \rangle_\alpha$ the integral along the vertical axis in the layer α of the quantity $f = f(z)$ i.e.

$$\langle f \rangle_\alpha(x, t) = \int_{\mathbb{R}} f(x, z, t) \mathbf{1}_{z \in L_\alpha(x, t)} dz. \quad (23)$$

The goal is to propose a new derivation of the so-called *multilayer model with mass exchanges* [6, 5] using the entropy-based moment closures proposed by Levermore in [17] for kinetic equations. This method has already been successfully used by some of the authors in [11].

Taking into account the kinematic boundary conditions (6) and (8), the layer-averaged form of the Euler system (14)–(16) writes

$$\frac{\partial}{\partial t} \langle \varphi \rangle_\alpha + \frac{\partial}{\partial x} \langle \varphi u \rangle_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2}, \quad (24)$$

$$\frac{\partial}{\partial t} \langle \varphi u \rangle_\alpha + \frac{\partial}{\partial x} \langle \varphi u^2 \rangle_\alpha + \left\langle \frac{\partial p}{\partial x} \right\rangle_\alpha = u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}, \quad (25)$$

$$\left\langle \frac{\partial p}{\partial z} \right\rangle_\alpha = -\langle \varphi g \rangle_\alpha, \quad (26)$$

$$\frac{\partial}{\partial t} \langle \varphi z \rangle_\alpha + \frac{\partial}{\partial x} \langle \varphi z u \rangle_\alpha = \langle \varphi w \rangle_\alpha + z_{\alpha+1/2} G_{\alpha+1/2} - z_{\alpha-1/2} G_{\alpha-1/2}, \quad (27)$$

for $\alpha \in \{1, \dots, N\}$ where p is defined by Eq. (18). $G_{\alpha+1/2}$ is defined by

$$G_{\alpha+1/2} = \varphi_{\alpha+1/2} \left(\frac{\partial z_{\alpha+1/2}}{\partial t} + u_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} - w_{\alpha+1/2} \right), \quad (28)$$

and corresponds to the mass flux leaving/entering the layer α through the interface $z_{\alpha+1/2}$. Notice that the kinematic boundary conditions (6) and (8) can be written

$$G_{1/2} = 0, \quad G_{N+1/2} = 0. \quad (29)$$

These equations just express that there is no loss/supply of mass through the bottom and the free surface. Taking into account the condition (29), the sum for $j = 1, \dots, \alpha$ of the relations (24) gives

$$G_{\alpha+1/2} = \frac{\partial}{\partial t} \sum_{j=1}^{\alpha} \langle \varphi \rangle_j + \frac{\partial}{\partial x} \sum_{j=1}^{\alpha} \langle \varphi u \rangle_j. \quad (30)$$

The quantities

$$u_{\alpha+1/2} = u(x, z_{\alpha+1/2}, t), \quad (31)$$

corresponding to the velocities values on the interfaces will be defined later. Notice that when using the expression (30), the velocities $w_{\alpha+1/2}$ no more appear in Eqs. (24)-(27) and thus need not be defined.

Equation (27) is a rewriting of

$$\left\langle \int_{z_{\alpha-1/2}}^z \left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} \right) dz \right\rangle_\alpha = \left\langle z \left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} \right) \right\rangle_\alpha = 0,$$

using again the kinematic boundary conditions. Notice also that because of the hydrostatic assumption, Eq. (27) is not a kinematic constraint over the velocity field but the definition of the vertical velocity $\langle \varphi w \rangle_\alpha$. The form of Eq. (27) is useful to derive energy balances but other equivalent writtings can be used, see paragraph 4.2.

Simple manipulations allow to obtain the system (24)-(28) from the Euler system (14)-(16) with (6) and (8) e.g. for Eq. (24), starting from (14) we write

$$\left\langle \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} \right\rangle_\alpha = 0,$$

and using the Leibniz rule to permute the derivative and the integral directly gives (24). Likewise, the Leibniz rule written for the pressure p gives

$$\left\langle \frac{\partial p}{\partial x} \right\rangle_\alpha = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\partial p}{\partial x} dz = \frac{\partial}{\partial x} \langle p \rangle_\alpha - p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} + p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial x},$$

and from (26),(17), we get

$$p_{\alpha+1/2} = p(x, z_{\alpha+1/2}, t) = \sum_{j=\alpha+1}^N \langle \varphi g \rangle_j. \quad (32)$$

From Eq. (18), we also have

$$\left\langle \frac{\partial p}{\partial x} \right\rangle_\alpha = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} g \frac{\partial}{\partial x} (\varphi(\eta - z)) dz = \frac{\partial}{\partial x} \left(\frac{g}{2} \langle \varphi \rangle_\alpha H \right) + g \langle \varphi \rangle_\alpha \frac{\partial z_b}{\partial x}.$$

Relation (18) also leads to

$$p = p_{\alpha+1/2} + g\varphi(z_{\alpha+1/2} - z) = p_{\alpha-1/2} + g\varphi(z_{\alpha-1/2} - z),$$

and hence

$$\langle p \rangle_\alpha = \langle \varphi \rangle_\alpha \frac{p_{\alpha+1/2} + p_{\alpha-1/2}}{2} = \langle \varphi \rangle_\alpha p_{\alpha+1/2} + \frac{g}{2} \langle \varphi \rangle_\alpha^2. \quad (33)$$

Therefore, the system (24)-(28) can be rewritten under the form

$$\frac{\partial}{\partial t} \langle \varphi \rangle_\alpha + \frac{\partial}{\partial x} \langle \varphi u \rangle_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2}, \quad (34)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \varphi u \rangle_\alpha + \frac{\partial}{\partial x} (\langle \varphi u^2 \rangle_\alpha + \langle p \rangle_\alpha) &= u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} \\ &\quad + p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} - p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial x}, \end{aligned} \quad (35)$$

$$\frac{\partial}{\partial t} \langle \varphi z \rangle_\alpha + \frac{\partial}{\partial x} \langle \varphi z u \rangle_\alpha = \langle \varphi w \rangle_\alpha + z_{\alpha+1/2} G_{\alpha+1/2} - z_{\alpha-1/2} G_{\alpha-1/2}, \quad (36)$$

with (32),(33) and completed with relations (30).

Considering smooth solutions, multiplying (15) by u and integrating it over the layer α gives, after simple manipulations, the energy balance

$$\begin{aligned} \frac{\partial}{\partial t} \langle E \rangle_\alpha + \frac{\partial}{\partial x} \langle u(E + p) \rangle_\alpha &= \left(\frac{u_{\alpha+1/2}^2}{2} + p_{\alpha+1/2} + g z_{\alpha+1/2} \right) G_{\alpha+1/2} \\ &\quad - \left(\frac{u_{\alpha-1/2}^2}{2} + p_{\alpha-1/2} + g z_{\alpha-1/2} \right) G_{\alpha-1/2} - p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial t} + p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial t}, \end{aligned} \quad (37)$$

where $E = E(z; u)$ is defined by (20). The sum for $\alpha = 1, \dots, N$ of the relations (37) gives

$$\frac{\partial}{\partial t} \sum_{\alpha=1}^N \langle E \rangle_\alpha + \frac{\partial}{\partial x} \sum_{\alpha=1}^N \langle u(E + p) \rangle_\alpha = 0.$$

Therefore the system (34)-(36) completed (30), (32) and (33) has three equations with three unknowns, namely $\langle \varphi \rangle_\alpha$, $\langle \varphi u \rangle_\alpha$ and $\langle \varphi w \rangle_\alpha$ and closure relations are needed to define $\langle \varphi u^2 \rangle_\alpha$, $\langle \varphi z u \rangle_\alpha$ and $u(x, z_{\alpha+1/2}, t)$.

3.3 Closure relations

If u'_α is defined as the deviation of u with respect to its layer-average over the layer α , then it comes for $z \in L_\alpha$

$$\varphi u = \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} + \varphi u'_\alpha, \quad (38)$$

with $\langle \varphi u'_\alpha \rangle = 0$. Following the moment closure proposed by Levermore [17], we study the minimization problem

$$\min_{u'_\alpha} \langle \{ \varphi E(z; u) \} \rangle_\alpha. \quad (39)$$

The energy $E(z; u)$ being quadratic with respect to u we notice that

$$\begin{aligned} \langle \varphi u^2 \rangle_\alpha &= \frac{\langle \varphi u \rangle_\alpha^2}{\langle \varphi \rangle_\alpha} + \frac{2\langle \varphi u u' \rangle_\alpha}{\langle \varphi \rangle_\alpha} + \langle \varphi (u'_\alpha)^2 \rangle_\alpha \\ &= \frac{\langle \varphi u \rangle_\alpha^2}{\langle \varphi \rangle_\alpha} + \langle \varphi (u'_\alpha)^2 \rangle_\alpha \\ &\geq \frac{\langle \varphi u \rangle_\alpha^2}{\langle \varphi \rangle_\alpha}. \end{aligned} \quad (40)$$

Equation (40) means that the solution of the minimization problem (39) is given by

$$\langle \varphi E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) \rangle_\alpha = \min_{u'_\alpha} \langle \{ \varphi E(z; u) \} \rangle_\alpha. \quad (41)$$

and

$$\langle \varphi E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) \rangle_\alpha = \frac{\langle \varphi u \rangle_\alpha^2}{2\langle \varphi \rangle_\alpha} + g \langle \varphi z \rangle_\alpha. \quad (42)$$

Since the only choice leading to an equality in relation (40) corresponds to

$$\varphi u = \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha}, \quad \text{for } z \in L_\alpha, \quad (43)$$

this allows to precise the closure relation associated to a minimal energy, namely

$$\langle \varphi u^2 \rangle_\alpha = \frac{\langle \varphi u \rangle_\alpha^2}{\langle \varphi \rangle_\alpha}, \quad (44)$$

$$\langle \varphi z u \rangle_\alpha = \langle \varphi z \rangle_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha}. \quad (45)$$

It remains to define the quantities $u_{\alpha+1/2}$. We adopt the definition

$$u_{\alpha+1/2} = \begin{cases} \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} & \text{if } G_{\alpha+1/2} \leq 0 \\ \frac{\langle \varphi u \rangle_{\alpha+1}}{\langle \varphi \rangle_{\alpha+1}} & \text{if } G_{\alpha+1/2} > 0 \end{cases} \quad (46)$$

corresponding to an upwind depending on the mass exchange sign between the layers α and $\alpha + 1$. This choice is justified by the form of energy balance in the following proposition.

Proposition 3.1 *The solutions of the Euler system (14)-(16) with (6),(8) satisfying the closure relations (44)-(46) are also solutions of the system*

$$\frac{\partial}{\partial t} \langle \varphi \rangle_\alpha + \frac{\partial}{\partial x} \langle \varphi u \rangle_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2}, \quad (47)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle \varphi u \rangle_\alpha + \frac{\partial}{\partial x} \left(\frac{\langle \varphi u \rangle_\alpha^2}{\langle \varphi \rangle_\alpha} + \langle p \rangle_\alpha \right) &= u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} \\ &\quad + p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} - p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial x}, \end{aligned} \quad (48)$$

$$\frac{\partial}{\partial t} \langle \varphi z \rangle_\alpha + \frac{\partial}{\partial x} \left(\langle \varphi z \rangle_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) = \langle \varphi w \rangle_\alpha + z_{\alpha+1/2} G_{\alpha+1/2} - z_{\alpha-1/2} G_{\alpha-1/2}, \quad (49)$$

completed with relation (30). The quantities $\langle p \rangle_\alpha$ and $p_{\alpha+1/2}$ are defined by (32) and (33).

This system is a layer-averaged approximation of the Euler system and admits – for smooth solutions – an energy equality under the form

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{\alpha=1}^N \left\langle E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) \right\rangle_\alpha + \frac{\partial}{\partial x} \sum_{\alpha=1}^N \left\langle \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \left(E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) + \langle p \rangle_\alpha \right) \right\rangle_\alpha = \\ - \frac{1}{2} \sum_{\alpha=1}^N \left(\frac{\langle \varphi u \rangle_{\alpha+1}}{\langle \varphi \rangle_{\alpha+1}} - \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right)^2 |G_{\alpha+1/2}|. \end{aligned} \quad (50)$$

Remark 3.2 *Instead of (46), the definition*

$$u_{\alpha+1/2} = \frac{1}{2} \left(\frac{\langle \varphi u \rangle_{\alpha+1}}{\langle \varphi \rangle_{\alpha+1}} + \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right), \quad (51)$$

is also possible and gives a vanishing right hand side in (50). But such a choice does not allow to obtain an energy balance in the variable density case and does not give a maximum principle, at the discrete level, see [5]. Simple calculations show that any other choice than (46) or (51) leads to a non negative r.h.s. in (50), see Eq. (52) in the proof of prop. 3.1.

Remark 3.3 *It is important to notice that whereas the solution H, u, w, p of the Euler system (14)-(17),(6),(8) also satisfies the system (34)-(36), only the solutions H, u, w, p of the Euler system (14)-(17),(6),(8) satisfying the closure relations (44)-(45),(46) are also solutions of the system (47)-(50). On the contrary, any solutions $\langle \varphi \rangle_\alpha$, $\langle \varphi u \rangle_\alpha$, $\langle \varphi w \rangle_\alpha$ and $\langle p \rangle_\alpha$ of (47)-(49) with (46) are also solutions of (34)-(37).*

Proof of prop. 3.1 *Only the manipulations allowing to obtain (50) have to be detailed. For that purpose, we multiply (48) by $\frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha}$ giving*

$$\left(\frac{\partial}{\partial t} \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} + \frac{\partial}{\partial x} \left(\frac{\langle \varphi u \rangle_\alpha^2}{\langle \varphi \rangle_\alpha} + \langle p \rangle_\alpha \right) \right) \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} = \left(u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} \right.$$

$$+ \frac{\partial z_{\alpha+1/2}}{\partial x} p_{\alpha+1/2} - \frac{\partial z_{\alpha-1/2}}{\partial x} p_{\alpha-1/2} \bigg) \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha},$$

and we rewrite each of the obtained terms.

Considering first the left hand side of the preceding equation excluding the pressure terms, we denote

$$I_{u,\alpha} = \left(\frac{\partial}{\partial t} \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} + \frac{\partial}{\partial x} \left(\frac{\langle \varphi u \rangle_\alpha^2}{\langle \varphi \rangle_\alpha} \right) \right) \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha},$$

and using (24) we have

$$I_{u,\alpha} = \frac{\partial}{\partial t} \left(\frac{\langle \varphi u \rangle_\alpha^2}{2 \langle \varphi \rangle_\alpha} \right) + \frac{\partial}{\partial x} \left(\frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\langle \varphi u \rangle_\alpha^2}{2 \langle \varphi \rangle_\alpha} \right) + \frac{\langle \varphi u \rangle_\alpha^2}{2 \langle \varphi \rangle_\alpha^2} (G_{\alpha+1/2} - G_{\alpha-1/2}).$$

Now we consider the contribution of the pressure terms over the energy balance i.e.

$$I_{p,\alpha} = \left(\frac{\partial \langle p \rangle_\alpha}{\partial x} - p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} + p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial x} \right) \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha}.$$

Using (33) we get the equality

$$p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} - p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial x} = \frac{\langle p \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\partial h_\alpha}{\partial x} - \langle g \varphi \rangle_\alpha \frac{\partial z_\alpha}{\partial x},$$

holds, it comes

$$\begin{aligned} I_{p,\alpha} &= \frac{\partial}{\partial x} \left(\langle p \rangle_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) - \langle p \rangle_\alpha \frac{\partial}{\partial x} \left(\frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) - \frac{\langle p \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\partial \langle \varphi \rangle_\alpha}{\partial x} + g \langle \varphi u \rangle_\alpha \frac{\partial z_\alpha}{\partial x} \\ &= \frac{\partial}{\partial x} \left(\langle p \rangle_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) - \frac{\langle p \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\partial \langle \varphi u \rangle_\alpha}{\partial x} + \frac{\partial}{\partial x} \left(g h_\alpha z_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) - z_\alpha \frac{\partial}{\partial x} (g \langle \varphi u \rangle_\alpha) \\ &= \frac{\partial}{\partial x} \left(\langle p \rangle_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) - \frac{\langle p \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\partial \langle \varphi u \rangle_\alpha}{\partial x} + \frac{\partial}{\partial x} \left(g h_\alpha z_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) + z_\alpha \frac{\partial}{\partial t} (\langle g \varphi \rangle_\alpha) \\ &\quad - g z_\alpha (G_{\alpha+1/2} - G_{\alpha-1/2}) \\ &= \frac{\partial}{\partial x} \left(\langle p \rangle_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) - \frac{\langle p \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\partial \langle \varphi u \rangle_\alpha}{\partial x} + \frac{\partial}{\partial x} \left(g h_\alpha z_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) + \frac{\partial}{\partial t} (g h_\alpha z_\alpha) \\ &\quad - g h_\alpha \frac{\partial z_\alpha}{\partial t} - g z_\alpha (G_{\alpha+1/2} - G_{\alpha-1/2}). \end{aligned}$$

Let us rewrite $I_{p,\alpha}$ under the form

$$\begin{aligned} I_{p,\alpha} &= \frac{\partial}{\partial x} \left(\langle p \rangle_\alpha \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) + \frac{\partial}{\partial t} (g h_\alpha z_\alpha) + g \frac{\partial}{\partial x} \left(h_\alpha z_\alpha \frac{\langle u \rangle_\alpha}{h_\alpha} \right) \\ &\quad - g (z_{\alpha+1/2} G_{\alpha+1/2} - z_{\alpha-1/2} G_{\alpha-1/2}) + J_{p,\alpha}, \end{aligned}$$

with

$$J_{p,\alpha} = - \frac{\langle p \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\partial \langle \varphi u \rangle_\alpha}{\partial x} - g h_\alpha \frac{\partial z_\alpha}{\partial t} + g \frac{h_\alpha}{2} (G_{\alpha+1/2} + G_{\alpha-1/2}).$$

Since we have

$$\frac{\langle p \rangle_\alpha}{\langle \varphi \rangle_\alpha} \frac{\partial \langle \varphi u \rangle_\alpha}{\partial x} = \frac{\langle \varphi \rangle_\alpha}{\langle \varphi \rangle_\alpha} \left(G_{\alpha+1/2} - G_{\alpha-1/2} - \frac{\partial h_\alpha}{\partial t} \right),$$

we obtain

$$J_{p,\alpha} = p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial t} - p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial t} - p_{\alpha+1/2} G_{\alpha+1/2} + p_{\alpha-1/2} G_{\alpha-1/2}.$$

Then summing $I_{u,\alpha}$ and $I_{p,\alpha}$ gives

$$\begin{aligned} \frac{\partial}{\partial t} \langle E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) \rangle_\alpha + \frac{\partial}{\partial x} \langle \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \left(E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) + \langle p \rangle_\alpha \right) \rangle_\alpha = \\ \left(u_{\alpha+1/2} \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} - \frac{1}{2} \left(\frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right)^2 \right) G_{\alpha+1/2} - \left(u_{\alpha-1/2} \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} - \frac{1}{2} \left(\frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right)^2 \right) G_{\alpha-1/2}. \end{aligned}$$

Finally, the sum of the preceding relations for $\alpha = 1, \dots, N$

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{\alpha=1}^N \langle E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) \rangle_\alpha + \frac{\partial}{\partial x} \sum_{\alpha=1}^N \langle \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \left(E \left(z; \frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right) + \langle p \rangle_\alpha \right) \rangle_\alpha = \\ \sum_{\alpha=1}^N \left(u_{\alpha+1/2} \left(\frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} - \frac{\langle \varphi u \rangle_{\alpha+1}}{\langle \varphi \rangle_{\alpha+1}} \right) - \frac{1}{2} \left(\frac{\langle \varphi u \rangle_\alpha}{\langle \varphi \rangle_\alpha} \right)^2 + \frac{1}{2} \left(\frac{\langle \varphi u \rangle_{\alpha+1}}{\langle \varphi \rangle_{\alpha+1}} \right)^2 \right) G_{\alpha+1/2}, \quad (52) \end{aligned}$$

and the definition (46) gives relation (50) that completes the proof. Notice that any other choice than (46) or (51) leads to a non negative r.h.s. in (52), see remark 3.2. ■

4 The proposed layer-averaged Euler system

4.1 Formulation

The closure relations (44)-(45) motivate the definition of piecewise constant approximation of the variables u and w .

Let us consider the space $\mathbb{P}_{0,H}^{N,t}$ of piecewise constant functions defined by

$$\mathbb{P}_{0,H}^{N,t} = \{ \mathbf{1}_{z \in L_\alpha(x,t)}(z), \quad \alpha \in \{1, \dots, N\} \},$$

where $\mathbf{1}_{z \in L_\alpha(x,t)}(z)$ is the characteristic function of the layer α . Using this formalism, the projection of u and w on $\mathbb{P}_{0,H}^{N,t}$ is a piecewise constant function defined by

$$X^N(x, z, \{z_\alpha\}, t) = \sum_{\alpha=1}^N \mathbf{1}_{[z_{\alpha-1/2}, z_{\alpha+1/2}]}(z) X_\alpha(x, t), \quad (53)$$

for $X \in (u, w)$. In the following, we no more handle variables corresponding to vertical means of the solution of the Euler equations (14)-(16) and we adopt notations inherited from (53).

By analogy with (47)-(49) we consider the following model

$$\sum_{\alpha=1}^N \frac{\partial h_{\alpha}}{\partial t} + \sum_{\alpha=1}^N \frac{\partial(h_{\alpha}u_{\alpha})}{\partial x} = 0, \quad (54)$$

$$\begin{aligned} \frac{\partial h_{\alpha}u_{\alpha}}{\partial t} + \frac{\partial}{\partial x} (h_{\alpha}u_{\alpha}^2 + h_{\alpha}p_{\alpha}) &= u_{\alpha+1/2}G_{\alpha+1/2} - u_{\alpha-1/2}G_{\alpha-1/2} \\ &\quad + \frac{\partial z_{\alpha+1/2}}{\partial x}p_{\alpha+1/2} - \frac{\partial z_{\alpha-1/2}}{\partial x}p_{\alpha-1/2}, \end{aligned} \quad (55)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \right) + \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} u_{\alpha} \right) &= h_{\alpha}w_{\alpha} \\ &\quad + z_{\alpha+1/2}G_{\alpha+1/2} - z_{\alpha-1/2}G_{\alpha-1/2}, \end{aligned} \quad (56)$$

by analogy with (30)

$$G_{\alpha+1/2} = \frac{\partial}{\partial t} \sum_{j=1}^{\alpha} h_j + \frac{\partial}{\partial x} \sum_{j=1}^{\alpha} (h_j u_j), \quad (57)$$

and we have p_{α} , $p_{\alpha+1/2}$ given by

$$p_{\alpha} = g \left(\frac{h_{\alpha}}{2} + \sum_{j=\alpha+1}^N h_j \right) \quad \text{and} \quad p_{\alpha+1/2} = g \sum_{j=\alpha+1}^N h_j. \quad (58)$$

The definition of $u_{\alpha+1/2}$ is equivalent (46) i.e.

$$u_{\alpha+1/2} = \begin{cases} u_{\alpha} & \text{if } G_{\alpha+1/2} \leq 0 \\ u_{\alpha+1} & \text{if } G_{\alpha+1/2} > 0 \end{cases}$$

The smooth solutions of (54)-(56) satisfies the energy balance

$$\begin{aligned} \frac{\partial}{\partial t} E_{\alpha}^N + \frac{\partial}{\partial x} (u_{\alpha} (E_{\alpha}^N + h_{\alpha}p_{\alpha})) &= \left(u_{\alpha+1/2}u_{\alpha} - \frac{u_{\alpha}^2}{2} + p_{\alpha+1/2} + gz_{\alpha+1/2} \right) G_{\alpha+1/2} \\ &\quad - \left(u_{\alpha-1/2}u_{\alpha} - \frac{u_{\alpha}^2}{2} + p_{\alpha-1/2} + gz_{\alpha-1/2} \right) G_{\alpha-1/2} \\ &\quad - p_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial t} + p_{\alpha-1/2} \frac{\partial z_{\alpha-1/2}}{\partial t}, \end{aligned} \quad (59)$$

with

$$E_{\alpha}^N = \frac{h_{\alpha}u_{\alpha}^2}{2} + \frac{g}{2}(z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2).$$

Adding the preceding relations for $\alpha = 1, \dots, N$, we obtain the global equality

$$\frac{\partial}{\partial t} \left(\sum_{\alpha=1}^N E_{\alpha}^N \right) + \frac{\partial}{\partial x} \left(\sum_{\alpha=1}^N u_{\alpha} (E_{\alpha}^N + h_{\alpha}p_{\alpha}) \right) = - \sum_{\alpha=1}^N \frac{1}{2} (u_{\alpha+1/2} - u_{\alpha})^2 |G_{\alpha+1/2}|. \quad (60)$$

Using (58), the pressure terms in (55) can be rewritten under the form

$$\frac{\partial}{\partial x} (h_{\alpha}p_{\alpha}) - \frac{\partial z_{\alpha+1/2}}{\partial x} p_{\alpha+1/2} + \frac{\partial z_{\alpha-1/2}}{\partial x} p_{\alpha-1/2} = \frac{\partial}{\partial x} \left(\frac{g}{2} H h_{\alpha} \right) + g h_{\alpha} \frac{\partial z_b}{\partial x}. \quad (61)$$

4.2 The vertical velocity

Equation (56) is a definition of the vertical velocities $\{w_\alpha\}_{1 \leq \alpha \leq N}$. The quantities w_α are not unknowns of the problem but only output variables. Indeed, once H and $\{u_\alpha\}_{1 \leq \alpha \leq N}$ have been calculated solving (54),(55) with (57), the velocities w_α can be determined using (56).

Using simple manipulations, Eq. (56) can be rewritten under several forms. In particular, the following proposition holds

Proposition 4.1 *Let us introduce $\hat{w} = \hat{w}(x, z, t)$ defined for $z \in L_\alpha$, $\alpha = 1, \dots, N$ by*

$$\hat{w} = k_\alpha - z \frac{\partial u_\alpha}{\partial x}, \quad (62)$$

with $k_\alpha = k_\alpha(x, t)$ recursively defined by

$$\begin{aligned} k_1 &= \frac{\partial(z_b u_1)}{\partial x}, \\ k_{\alpha+1} &= k_\alpha + \frac{\partial}{\partial x}(z_{\alpha+1/2}(u_{\alpha+1} - u_\alpha)). \end{aligned}$$

The quantity \hat{w} is affine in z and discontinuous at each interface $z_{\alpha+1/2}$. It obviously satisfies

$$\frac{\partial u_\alpha}{\partial x} + \frac{\partial \hat{w}}{\partial z} = 0,$$

but also for $z \in L_\alpha$

$$w(x, z, t) = \hat{w}(x, z, t).$$

Therefore we have

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} w dz = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \hat{w} dz = h_\alpha w_\alpha, \quad (63)$$

meaning the quantities \hat{w} is a natural and consistent affine extension of the layer-averaged quantities w_α defined by (56). Using (63), an integration along the layer α of (62) gives

$$h_\alpha w_\alpha = h_\alpha k_\alpha - \frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \frac{\partial u_\alpha}{\partial x}. \quad (64)$$

Proof of prop. 4.1 *A simple integration along z of the divergence free condition (1) using (8) gives*

$$w = -\frac{\partial}{\partial x} \int_{z_b}^z u \, dz, \quad (65)$$

and therefore, for $z \in L_1$ we get

$$w = -\frac{\partial}{\partial x} \int_{z_b}^z u_1 \, dz = -\frac{\partial}{\partial x}((z - z_b)u_1),$$

i.e.

$$w = \hat{w} = \frac{\partial}{\partial x}(z_b u_1) - z \frac{\partial u_1}{\partial x}.$$

For $z \in L_\alpha$, relation (65) gives

$$w = - \sum_{j=1}^{\alpha-1} \frac{\partial}{\partial x}(h_j u_j) - \frac{\partial}{\partial x}((z - z_{\alpha-1/2})u_\alpha), \quad (66)$$

and we easily obtain

$$w = \hat{w} = k_\alpha - z \frac{\partial u_\alpha}{\partial x}.$$

Now we intend to prove (63).

Using the definition (22), relation (56) also writes

$$h_\alpha w_\alpha = \frac{\partial}{\partial x}(z_\alpha h_\alpha u_\alpha) - z_{\alpha+1/2} \sum_{j=1}^{\alpha} \frac{\partial(h_j u_j)}{\partial x} + z_{\alpha-1/2} \sum_{j=1}^{\alpha-1} \frac{\partial(h_j u_j)}{\partial x},$$

leading to a new expression governing w_α under the form

$$h_\alpha w_\alpha = -\frac{h_\alpha}{2} \frac{\partial(h_\alpha u_\alpha)}{\partial x} - h_\alpha \sum_{j=1}^{\alpha-1} \frac{\partial(h_j u_j)}{\partial x} + h_\alpha u_\alpha \frac{\partial z_\alpha}{\partial x}. \quad (67)$$

And from (66), we get

$$\begin{aligned} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \hat{w} dz &= -h_\alpha \sum_{j=1}^{\alpha-1} \frac{\partial}{\partial x}(h_j u_j) + h_\alpha \frac{\partial}{\partial x}(z_{\alpha-1/2} u_\alpha) - h_\alpha z_\alpha \frac{\partial u_\alpha}{\partial x} \\ &= -h_\alpha \sum_{j=1}^{\alpha-1} \frac{\partial}{\partial x}(h_j u_j) - \frac{h_\alpha}{2} \frac{\partial}{\partial x}(h_\alpha u_\alpha) + h_\alpha u_\alpha \frac{\partial z_\alpha}{\partial x}, \end{aligned}$$

corresponding to (67) and proving the result. ■

Using also (61), we are able to rewrite the system (54)-(56) under the form

$$\sum_{\alpha=1}^N \frac{\partial h_\alpha}{\partial t} + \sum_{\alpha=1}^N \frac{\partial(h_\alpha u_\alpha)}{\partial x} = 0, \quad (68)$$

$$\frac{\partial h_\alpha u_\alpha}{\partial t} + \frac{\partial}{\partial x} \left(h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha H \right) = -g h_\alpha \frac{\partial z_b}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}, \quad (69)$$

$$w_\alpha = -\frac{1}{2} \frac{\partial(h_\alpha u_\alpha)}{\partial x} - \sum_{j=1}^{\alpha-1} \frac{\partial(h_j u_j)}{\partial x} + u_\alpha \frac{\partial z_\alpha}{\partial x}. \quad (70)$$

5 The Navier-Stokes system

Instead of considering the Euler system, we can also depart from the Navier-Stokes equations to derive a layer-averaged model.

5.1 Layers averaged Navier-Stokes system

The model derivation is similar to what has been done in Section 3 for the Euler system, in particular the minimization principle (39) allows to define the needed closure relations. It remains to find a layer averaged formulation for the r.h.s. of Eq. (11).

In this paragraph, the components of the Cauchy stress tensor Σ are not specified. Since the rheology terms often require high order derivatives, their discretization requires quadrature formula that are not inherited from the layers average discretization. In particular, we adopt the following definition

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \Sigma_{a,b} dz = h_{\alpha} \Sigma_{a,b,\alpha}, \quad (71)$$

with $(a, b) \in (x, z)^2$. For the terms having the form

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} z \Sigma_{a,b} dz,$$

a closure relation is needed and we choose the approximation

$$\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} z \Sigma_{a,b} dz \approx \frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{a,b,\alpha}. \quad (72)$$

Since the rheology terms are more related to elliptic than hyperbolic type behavior, we also use the centered approximation

$$\Sigma_{xz,\alpha+1/2} = \frac{\Sigma_{xz,\alpha+1} + \Sigma_{xz,\alpha}}{2}. \quad (73)$$

We have the following proposition.

Proposition 5.1 *Using formula (71)-(73), the layer-averaging applied to the Navier-Stokes system (10)-(11) completed with the boundary conditions (6)-(9) and compatibility conditions defined hereafter leads to the system*

$$\frac{\partial}{\partial t} \sum_{j=1}^N h_j + \frac{\partial}{\partial x} \sum_{j=1}^N h_j u_j = 0, \quad (74)$$

$$\begin{aligned} \frac{\partial}{\partial t} (h_{\alpha} u_{\alpha}) + \frac{\partial}{\partial x} \left(h_{\alpha} u_{\alpha}^2 + \frac{g}{2} h_{\alpha} H \right) &= -g h_{\alpha} \frac{\partial z_b}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} \\ &+ \frac{\partial}{\partial x} \left(h_{\alpha} \Sigma_{xx,\alpha} - h_{\alpha} \Sigma_{zz,\alpha} + \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{zx,\alpha} \right) \right) \\ &+ z_{\alpha+1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} - z_{\alpha-1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha}^N h_j \Sigma_{zx,j} \\ &+ \Sigma_{xz,\alpha+1/2} - \Sigma_{xz,\alpha-1/2} - \kappa u_{\alpha} \mathbf{1}_{\alpha=1}, \end{aligned} \quad (75)$$

$$w_\alpha = -\frac{1}{2} \frac{\partial(h_\alpha u_\alpha)}{\partial x} - \sum_{j=1}^{\alpha-1} \frac{\partial(h_j u_j)}{\partial x} + u_\alpha \frac{\partial z_\alpha}{\partial x}, \quad (76)$$

with the exchange terms $G_{\alpha+1/2}$ given by (57).

For smooth solutions, we obtain the balance

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_{\alpha=1}^N E_\alpha^N \right) + \frac{\partial}{\partial x} \left(\sum_{\alpha=1}^N u_\alpha \left(E_\alpha^N + \frac{g}{2} h_\alpha H - h_\alpha (\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}) \right. \right. \\ \left. \left. - \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{xz,\alpha} \right) \right) - \sum_{\alpha=1}^N h_\alpha w_\alpha \Sigma_{zx,\alpha} \right) = \\ - \sum_{\alpha=1}^N \left(\frac{\partial u_\alpha}{\partial x} h_\alpha (\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}) \right. \\ \left. + h_\alpha \frac{\partial w_\alpha}{\partial x} \Sigma_{zx,\alpha} + \frac{\Sigma_{xz,\alpha}}{2} (u_{\alpha+1} - u_{\alpha-1}) \right) - \kappa u_1^2, \end{aligned} \quad (77)$$

with $E_\alpha^N = \frac{h_\alpha u_\alpha^2}{2} + \frac{g(z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2)}{2}$.

In (75) the convention

$$\Sigma_{xz,1/2} = \Sigma_{xz,N+1/2} = 0, \quad (78)$$

is adopted whereas in (77), we use the convention

$$u_0 = u_1, \quad u_{N+1} = u_N. \quad (79)$$

Since the expression of the components of the Cauchy stress tensor are not precised, we are not able to precise all the terms in Eq. (77) and we only intend to demonstrate that the energy balance (77) is consistent with (13). In particular we do not prove that the right hand side of (77) is non positive.

The discretization by layer given by (53) imposes to define boundary conditions for the rheology terms at the interface between each layer. More precisely, a closure relation has to be defined for the quantities

$$\begin{aligned} \mathbf{t}_{\alpha+1/2} \cdot \Sigma_{T,\alpha+1/2} \mathbf{n}_{\alpha+1/2} = \frac{1}{1 + \left(\frac{\partial z_{\alpha+1/2}}{\partial x} \right)^2} \left(\Sigma_{xz,\alpha+1/2} \right. \\ \left. - \frac{\partial z_{\alpha+1/2}}{\partial x} \left(\Sigma_{xx,\alpha+1/2} + \frac{\partial z_{\alpha+1/2}}{\partial x} \Sigma_{zx,\alpha+1/2} - \Sigma_{zz,\alpha+1/2} \right) \right), \end{aligned} \quad (80)$$

$\alpha = 1, \dots, N-1$. $\Sigma_T = \Sigma - pI_d$ is the total stress tensor and the unit normal vector $\mathbf{n}_{\alpha+1/2}$ is given by

$$\mathbf{n}_{\alpha+1/2} = \frac{1}{\sqrt{1 + \left(\frac{\partial z_{\alpha+1/2}}{\partial x} \right)^2}} \begin{pmatrix} -\frac{\partial z_{\alpha+1/2}}{\partial x} \\ 1 \end{pmatrix},$$

$\mathbf{t}_{\alpha+1/2}$ is the unit vector orthogonal to $\mathbf{n}_{\alpha+1/2}$. Notice that for $\alpha = 1, N$, the quantities (80) coincide with the boundary conditions and hence are given. More precisely

$$\mathbf{t}_{1/2} \cdot \Sigma_{T,1/2} \mathbf{n}_{1/2} = -\kappa u_1,$$

and

$$\mathbf{t}_{N+1/2} \cdot \Sigma_{T,N+1/2} \mathbf{n}_{N+1/2} = 0.$$

For $1 < \alpha < N$, the quantity $\mathbf{t}_{\alpha+1/2} \cdot \Sigma_{T,\alpha+1/2} \mathbf{n}_{\alpha+1/2}$ represents the tangential component of the stress tensors at the interface $z_{\alpha+1/2}$. For $1 < \alpha < N$, we assume that $\mathbf{t}_{\alpha+1/2} \cdot \Sigma_{T,\alpha+1/2} \mathbf{n}_{\alpha+1/2}$ reduces to

$$\mathbf{t}_{\alpha+1/2} \cdot \Sigma_{T,\alpha+1/2} \mathbf{n}_{\alpha+1/2} = \frac{\Sigma_{xz,\alpha+1/2}}{1 + \left(\frac{\partial z_{\alpha+1/2}}{\partial x}\right)^2}, \quad (81)$$

this means we consider

$$-\frac{\partial z_{\alpha+1/2}}{\partial x} \left(\Sigma_{xx,\alpha+1/2} + \frac{\partial z_{\alpha+1/2}}{\partial x} \Sigma_{zx,\alpha+1/2} - \Sigma_{zz,\alpha+1/2} \right) = 0.$$

Remark 5.2 *The assumption (81) obviously holds when*

$$\frac{\partial z_{\alpha+1/2}}{\partial x} = 0, \quad \alpha = \{1, \dots, N-1\},$$

corresponding to a flat bottom and free surface. In more complex situations, the assumption (81) is the necessary counterpart of the layer averaged description and of the closure relation (72). We will see in prop. (5.5) that any other choice than (81) leads to complementary terms in the energy balance that can neither be considered as conservative nor dissipative.

Remark 5.3 *Notice that until now, the chosen form of the quantities*

$$\Sigma_{xx,\alpha}, \quad \Sigma_{zz,\alpha} \quad \text{and} \quad \Sigma_{zx,\alpha},$$

for $\alpha = 1, \dots, N$ – appearing in the results given in prop. 5.1 – have not been precised.

Remark 5.4 *Comparing the two energy balances (13) and (77), it appears that the first two terms in the right hand side of (77) account for a layer-averaging of*

$$-\int_{z_b}^{\eta} \Sigma_{xx} \frac{\partial u}{\partial x} dz - \int_{z_b}^{\eta} \Sigma_{zz} \frac{\partial w}{\partial z} dz,$$

appearing in the right hand side of (13). Likewise

$$-\int_{z_b}^{\eta} \Sigma_{xz} \frac{\partial u}{\partial z} dz - \int_{z_b}^{\eta} \frac{\partial w}{\partial x} \Sigma_{zx} dz, \quad (82)$$

in the right hand side of (13) is discretized by

$$-\sum_{\alpha=1}^N \left(\frac{\Sigma_{xz,\alpha}}{2} (u_{\alpha+1} - u_{\alpha-1}) + h_{\alpha} \frac{\partial w_{\alpha}}{\partial x} \Sigma_{zx,\alpha} \right), \quad (83)$$

in the layer-average context of Eq. (77).

Proof of proposition 5.1 *The derivation of Eqs. (74) and (76) is similar to what has been done to obtain the layers-averaged Euler system (68)-(70). Only the treatment of the viscous terms has to be precised.*

We calculate the layer-average of Eq. (11). Using (69), it remains to find an expression for

$$V_\alpha = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\frac{\partial \Sigma_{xx}}{\partial x} + \frac{\partial \Sigma_{xz}}{\partial z} + \frac{\partial^2}{\partial x^2} \int_z^\eta \Sigma_{zx} dz_1 - \frac{\partial \Sigma_{zz}}{\partial x} \right) dz.$$

We have

$$\begin{aligned} V_\alpha &= \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\Sigma_{xx} + \frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 - \Sigma_{zz} \right) dz \\ &\quad + \Sigma_{xz, \alpha+1/2} - \frac{\partial z_{\alpha+1/2}}{\partial x} \left(\Sigma_{xx} + \frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 - \Sigma_{zz} \right) \Big|_{z_{\alpha+1/2}} \\ &\quad - \Sigma_{xz, \alpha-1/2} + \frac{\partial z_{\alpha-1/2}}{\partial x} \left(\Sigma_{xx} + \frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 - \Sigma_{zz} \right) \Big|_{z_{\alpha-1/2}}. \end{aligned}$$

Noticing that in the expression V_α we have the term

$$\begin{aligned} &\frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left(\frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 \right) dz \\ &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \int_z^\eta \Sigma_{zx} dz_1 dz - \frac{\partial z_{\alpha+1/2}}{\partial x} \int_{z_{\alpha+1/2}}^\eta \Sigma_{zx} dz \right. \\ &\quad \left. + \frac{\partial z_{\alpha-1/2}}{\partial x} \int_{z_{\alpha-1/2}}^\eta \Sigma_{zx} dz \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} z \Sigma_{zx} dz + z_{\alpha+1/2} \frac{\partial}{\partial x} \int_{z_{\alpha+1/2}}^\eta \Sigma_{zx} dz \right. \\ &\quad \left. - z_{\alpha-1/2} \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^\eta \Sigma_{zx} dz \right), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial z_{\alpha+1/2}}{\partial x} \left(\frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 \right) \Big|_{z_{\alpha+1/2}} &= \frac{\partial z_{\alpha+1/2}}{\partial x} \frac{\partial}{\partial x} \int_{z_{\alpha+1/2}}^\eta \Sigma_{zx} dz + \left(\frac{\partial z_{\alpha+1/2}}{\partial x} \right)^2 \Sigma_{zx, \alpha+1/2}, \\ \frac{\partial z_{\alpha-1/2}}{\partial x} \left(\frac{\partial}{\partial x} \int_z^\eta \Sigma_{zx} dz_1 \right) \Big|_{z_{\alpha-1/2}} &= \frac{\partial z_{\alpha-1/2}}{\partial x} \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^\eta \Sigma_{zx} dz + \left(\frac{\partial z_{\alpha-1/2}}{\partial x} \right)^2 \Sigma_{zx, \alpha-1/2}. \end{aligned}$$

Using the condition (81) and the notation (71),(73), it comes

$$V_\alpha = \frac{\partial}{\partial x} \left(h_\alpha \Sigma_{xx, \alpha} - h_\alpha \Sigma_{zz, \alpha} + \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} z \Sigma_{zx} dz \right)$$

$$\begin{aligned}
& + z_{\alpha+1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} - z_{\alpha-1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha}^N h_j \Sigma_{zx,j} \\
& + \Sigma_{xz,\alpha+1/2} - \Sigma_{xz,\alpha-1/2}.
\end{aligned}$$

The approximation (72) gives

$$\begin{aligned}
V_\alpha & \approx R_\alpha + \Sigma_{xz,\alpha+1/2} - \Sigma_{xz,\alpha-1/2} - \kappa u_\alpha \mathbf{1}_{\alpha=1} \\
& = \frac{\partial}{\partial x} \left(h_\alpha \Sigma_{xx,\alpha} - h_\alpha \Sigma_{zz,\alpha} + \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{zx,\alpha} \right) \right) \\
& + z_{\alpha+1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} - z_{\alpha-1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha}^N h_j \Sigma_{zx,j} \\
& + \Sigma_{xz,\alpha+1/2} - \Sigma_{xz,\alpha-1/2} - \kappa u_\alpha \mathbf{1}_{\alpha=1}.
\end{aligned}$$

For the energy balance we write

$$\begin{aligned}
R_\alpha u_\alpha & = \frac{\partial}{\partial x} \left(u_\alpha h_\alpha (\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}) + u_\alpha \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{zx,\alpha} \right) \right. \\
& + z_{\alpha+1/2} u_\alpha \frac{\partial}{\partial x} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} - z_{\alpha-1/2} u_\alpha \frac{\partial}{\partial x} \sum_{j=\alpha}^N h_j \Sigma_{zx,j} \Big) \\
& - h_\alpha (\Sigma_{xx,\alpha} - \Sigma_{zz,\alpha}) \frac{\partial u_\alpha}{\partial x} - \frac{\partial u_\alpha}{\partial x} \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{zx,\alpha} \right) \\
& - \frac{\partial}{\partial x} (z_{\alpha+1/2} u_\alpha) \frac{\partial}{\partial x} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} + \frac{\partial}{\partial x} (z_{\alpha-1/2} u_\alpha) \frac{\partial}{\partial x} \sum_{j=\alpha}^N h_j \Sigma_{zx,j}. \quad (84)
\end{aligned}$$

Notice that, using an integration by part, it comes that the three terms

$$\begin{aligned}
& \frac{\partial}{\partial x} \left(u_\alpha \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{zx,\alpha} \right) + z_{\alpha+1/2} u_\alpha \frac{\partial}{\partial x} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} \right. \\
& \quad \left. - z_{\alpha-1/2} u_\alpha \frac{\partial}{\partial x} \sum_{j=\alpha}^N h_j \Sigma_{zx,j} \right),
\end{aligned}$$

appearing in Eq. (84) are a discretization of the quantity

$$\frac{\partial}{\partial x} \left(u_\alpha \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{\partial}{\partial x} \int_z^\eta \Sigma_{xz} dz_1 dz \right),$$

in the energy balance Eq. (77).

Denoting $\tilde{R}_\alpha u_\alpha$ the last three terms in Eq. (84), we write

$$\tilde{R}_\alpha u_\alpha = - \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \frac{\partial u_\alpha}{\partial x} \Sigma_{zx,\alpha} \right) + \frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \frac{\partial^2 u_\alpha}{\partial x^2} \Sigma_{zx,\alpha}$$

$$\begin{aligned}
& -\frac{\partial}{\partial x}(z_{\alpha+1/2}u_\alpha)\frac{\partial}{\partial x}\sum_{j=\alpha+1}^N h_j\Sigma_{zx,j} + \frac{\partial}{\partial x}(z_{\alpha-1/2}u_\alpha)\frac{\partial}{\partial x}\sum_{j=\alpha}^N h_j\Sigma_{zx,j} \\
& = \frac{\partial}{\partial x}\left((h_\alpha w_\alpha - h_\alpha k_\alpha)\Sigma_{zx,\alpha}\right) + \frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2}\frac{\partial^2 u_\alpha}{\partial x^2}\Sigma_{zx,\alpha} \\
& -\frac{\partial}{\partial x}(z_{\alpha+1/2}u_\alpha)\frac{\partial}{\partial x}\sum_{j=\alpha+1}^N h_j\Sigma_{zx,j} + \frac{\partial}{\partial x}(z_{\alpha-1/2}u_\alpha)\frac{\partial}{\partial x}\sum_{j=\alpha}^N h_j\Sigma_{zx,j},
\end{aligned}$$

where (64) has been used. And simple manipulations give

$$\begin{aligned}
\tilde{R}_\alpha u_\alpha & = \frac{\partial}{\partial x}\left(h_\alpha w_\alpha \Sigma_{zx,\alpha}\right) - h_\alpha \frac{\partial w_\alpha}{\partial x}\Sigma_{zx,\alpha} - k_\alpha \frac{\partial}{\partial x}(h_\alpha \Sigma_{zx,\alpha}) \\
& -\frac{\partial}{\partial x}(z_{\alpha+1/2}u_\alpha)\frac{\partial}{\partial x}\sum_{j=\alpha+1}^N h_j\Sigma_{zx,j} + \frac{\partial}{\partial x}(z_{\alpha-1/2}u_\alpha)\frac{\partial}{\partial x}\sum_{j=\alpha}^N h_j\Sigma_{zx,j} \\
& = \frac{\partial}{\partial x}\left(h_\alpha w_\alpha \Sigma_{zx,\alpha}\right) - h_\alpha \frac{\partial w_\alpha}{\partial x}\Sigma_{zx,\alpha} \\
& + \hat{w}_{\alpha+1/2}\frac{\partial}{\partial x}\sum_{j=\alpha+1}^N h_j\Sigma_{zx,j} - \hat{w}_{\alpha-1/2}\frac{\partial}{\partial x}\sum_{j=\alpha}^N h_j\Sigma_{zx,j},
\end{aligned}$$

with $\hat{w}_{\alpha+1/2}$ defined by

$$\hat{w}_{\alpha+1/2} = k_\alpha - \frac{\partial(z_{\alpha+1/2}u_\alpha)}{\partial x} = k_{\alpha+1} - \frac{\partial(z_{\alpha+1/2}u_{\alpha+1})}{\partial x},$$

and satisfying $\hat{w}_{1/2} = \hat{w}_{N+1/2} = 0$. Finally, the quantity

$$\sum_{\alpha=1}^N V_\alpha u_\alpha,$$

gives the expression involving of the terms related to the Cauchy stress tensor in (77) proving the result. ■

5.2 Newtonian fluids

When considering a Newtonian fluid, the chosen form of the viscosity tensor is

$$\Sigma_{xx} = 2\mu\frac{\partial u}{\partial x}, \quad \Sigma_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad (85)$$

$$\Sigma_{zz} = 2\mu\frac{\partial w}{\partial z}, \quad \Sigma_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad (86)$$

where μ is a dynamic viscosity coefficient.

When considering the fluid rheology is given by (85)-(86), prop. 5.1 becomes

Proposition 5.5 *The layer-averaging applied to the Navier-Stokes system for a newtonian fluid gives*

$$\frac{\partial}{\partial t} \sum_{j=1}^N h_j + \frac{\partial}{\partial x} \sum_{j=1}^N h_j u_j = 0, \quad (87)$$

$$\begin{aligned} \frac{\partial}{\partial t} (h_\alpha u_\alpha) + \frac{\partial}{\partial x} \left(h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha H \right) &= -g h_\alpha \frac{\partial z_b}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} \\ &+ \frac{\partial}{\partial x} \left(4\mu h_\alpha \frac{\partial u_\alpha}{\partial x} + \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{zx,\alpha} \right) \right) \\ &+ z_{\alpha+1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} - z_{\alpha-1/2} \frac{\partial^2}{\partial x^2} \sum_{j=\alpha}^N h_j \Sigma_{zx,j} \\ &+ \Sigma_{xz,\alpha+1/2} - \Sigma_{xz,\alpha-1/2} - \kappa u_\alpha \mathbf{1}_{\alpha=1}, \end{aligned} \quad (88)$$

$$w_\alpha = -\frac{1}{2} \frac{\partial (h_\alpha u_\alpha)}{\partial x} - \sum_{j=1}^{\alpha-1} \frac{\partial (h_j u_j)}{\partial x} + u_\alpha \frac{\partial z_\alpha}{\partial x}, \quad (89)$$

with the expression

$$h_\alpha \Sigma_{zx,\alpha} = h_\alpha \Sigma_{xz,\alpha} = \mu \left(h_\alpha \frac{\partial w_\alpha}{\partial x} + \frac{u_{\alpha+1} - u_{\alpha-1}}{2} \right), \quad (90)$$

and $\Sigma_{xz,\alpha+1/2}$ is still defined by (73). In (88), the conventions (78)-(79) are adopted. The exchange terms $G_{\alpha+1/2}$ are given by (57).

For smooth solutions, we obtain the balance

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_{\alpha=1}^N E_\alpha^N \right) + \frac{\partial}{\partial x} \left(\sum_{\alpha=1}^N u_\alpha \left(E_\alpha^N + \frac{g}{2} h_\alpha H - 4\mu h_\alpha \frac{\partial u_\alpha}{\partial x} - \frac{\partial}{\partial x} \left(\frac{z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2}{2} \Sigma_{zx,\alpha} \right) \right. \right. \\ \left. \left. + z_{\alpha+1/2} \frac{\partial}{\partial x} \sum_{j=\alpha+1}^N h_j \Sigma_{zx,j} - z_{\alpha-1/2} \frac{\partial}{\partial x} \sum_{j=\alpha}^N h_j \Sigma_{zx,j} \right) \right. \\ \left. - \sum_{\alpha=1}^N h_\alpha w_\alpha \Sigma_{zx,\alpha} \right) = -\kappa u_1^2 - \sum_{\alpha=1}^N \left(4\mu h_\alpha \left(\frac{\partial u_\alpha}{\partial x} \right)^2 + \frac{h_\alpha}{\mu} \Sigma_{zx,\alpha}^2 \right), \end{aligned} \quad (91)$$

where $\Sigma_{xz,\alpha} = \Sigma_{zx,\alpha}$ is defined by (90) and with $E_\alpha^N = \frac{h_\alpha u_\alpha^2}{2} + \frac{g(z_{\alpha+1/2}^2 - z_{\alpha-1/2}^2)}{2}$.

Remark 5.6 Using the fomula (90), the r.h.s. of Eq. (91) writes

$$- \sum_{\alpha=1}^N \left(4\mu h_\alpha \left(\frac{\partial u_\alpha}{\partial x} \right)^2 + \mu h_\alpha \left(\frac{\partial w_\alpha}{\partial x} + \frac{u_{\alpha+1} - u_{\alpha-1}}{2h_\alpha} \right)^2 \right),$$

that is a consistent discretization of (13). This reinforces the choice (81). We do not claim that the closure relation (81) is not an approximation but it is mandatory in the context of the layer averaged formulation.

Proof. of prop. 5.5 *The results given in this proposition are only a particular case of those obtained in prop. 5.1 and hence have already been proved in a more general setting.*

■

5.3 An extended Saint-Venant system

In the simplified case of a single layer, the model given in prop. 5.1 corresponds to the classical Saint-Venant system but completed with rheology terms. It writes

Proposition 5.7 *The classical Saint-Venant corresponds to the single-layer version of the layer-averaged Navier-Stokes system. With obvious notations, it is given by*

$$\begin{aligned} \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) &= 0, \\ \frac{\partial(H\bar{u})}{\partial t} + \frac{\partial}{\partial x} \left(H\bar{u}^2 + \frac{g}{2}H^2 \right) &= -gH \frac{\partial z_b}{\partial x} \\ &+ \frac{\partial}{\partial x} \left(H\bar{\Sigma}_{xx} - H\bar{\Sigma}_{zz} + \frac{\partial}{\partial x} \left(\frac{(H+z_b)^2 - z_b^2}{2} \bar{\Sigma}_{zx} \right) \right) - z_b \frac{\partial^2}{\partial x^2} (H\bar{\Sigma}_{zx}) - \kappa\bar{u}, \\ \bar{w} &= -\frac{1}{2} \frac{\partial(H\bar{u})}{\partial x} + \bar{u} \frac{\partial}{\partial x} \left(\frac{H+2z_b}{2} \right). \end{aligned}$$

For smooth solutions, we obtain the balance

$$\begin{aligned} \frac{\partial E}{\partial t} + \frac{\partial}{\partial x} \left(\bar{u} \left(E + \frac{g}{2}H^2 - H(\bar{\Sigma}_{xx} - \bar{\Sigma}_{zz}) - \frac{\partial}{\partial x} \left(\frac{(H+z_b)^2 - z_b^2}{2} \bar{\Sigma}_{zx} \right) \right) \right. \\ \left. - H\bar{w}\bar{\Sigma}_{zx} \right) = -H \frac{\partial \bar{u}}{\partial x} (\bar{\Sigma}_{xx} - \bar{\Sigma}_{zz}) - H \frac{\partial \bar{w}}{\partial x} \bar{\Sigma}_{zx} - \kappa\bar{u}^2, \end{aligned}$$

with $E = \frac{H\bar{u}^2}{2} + \frac{g}{2} \left((H+z_b)^2 - z_b^2 \right)$.

In the particular case of a newtonian fluid, the Saint-Venant system given in prop. 5.7 is given by

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0, \tag{92}$$

$$\begin{aligned} \frac{\partial(H\bar{u})}{\partial t} + \frac{\partial}{\partial x} \left(H\bar{u}^2 + \frac{g}{2}H^2 \right) &= -gH \frac{\partial z_b}{\partial x} \\ &+ \frac{\partial}{\partial x} \left(4\mu H \frac{\partial \bar{u}}{\partial x} + \frac{\partial}{\partial x} \mu \left(\frac{(H+z_b)^2 - z_b^2}{2} \frac{\partial \bar{w}}{\partial x} \right) \right) - z_b \mu \frac{\partial^2}{\partial x^2} \left(H \frac{\partial \bar{w}}{\partial x} \right) - \kappa\bar{u}, \end{aligned} \tag{93}$$

$$\bar{w} = -\frac{1}{2} \frac{\partial(H\bar{u})}{\partial x} + \bar{u} \frac{\partial}{\partial x} \left(\frac{H+2z_b}{2} \right). \tag{94}$$

For smooth solutions, we obtain the energy balance

$$\begin{aligned} \frac{\partial E}{\partial t} + \frac{\partial}{\partial x} \left(\bar{u} \left(E + \frac{g}{2} H^2 - 2\mu H \frac{\partial \bar{u}}{\partial x} \right) - \frac{\partial}{\partial x} \left(\mu \frac{(H + z_b)^2 - z_b^2}{2} \frac{\partial \bar{w}}{\partial x} \right) \right) \\ - \mu \frac{H}{2} \frac{\partial \bar{w}^2}{\partial x} \Bigg) = -2\mu H \left(\frac{\partial \bar{u}}{\partial x} \right)^2 - \mu H \left(\frac{\partial \bar{w}}{\partial x} \right)^2 - \kappa \bar{u}^2. \quad (95) \end{aligned}$$

Remark 5.8 Notice that, compared to the classical viscous Saint-Venant system [15], the model (92)-(95) has complementary terms.

6 Conclusion

We have proposed a layer averaged discretization for the approximation of the incompressible free surface Euler and Navier-Stokes equations. The obtained models do not rely on any asymptotic expansion but on a criterion of minimal kinetic energy. Notice also that the layer averaging for the Navier-Stokes system has been carried out for a fluid with a general rheology.

Since these models are formulated over a fixed domain, it is possible to derive efficient numerical techniques for their approximation. For the approximation of the proposed models, a finite volume strategy – relying on a kinetic interpretation and satisfying stability properties such as a fully discrete entropy inequality – will be published in a forthcoming paper.

References

- [1] E. Audusse, *A multilayer Saint-Venant model : Derivation and numerical validation*, Discrete Contin. Dyn. Syst. Ser. B **5** (2005), no. 2, 189–214.
- [2] E. Audusse, F. Bouchut, M.-O. Bristeau, and J. Sainte-Marie, *Kinetic entropy inequality and hydrostatic reconstruction scheme for the Saint-Venant system.*, (Accepted for publication in Math. Comp.) http://hal.inria.fr/hal-01063577/PDF/kin_hydrost.pdf, May 2015.
- [3] E. Audusse and M.-O. Bristeau, *Finite-volume solvers for a multilayer Saint-Venant system*, Int. J. Appl. Math. Comput. Sci. **17** (2007), no. 3, 311–319.
- [4] E. Audusse, M.-O. Bristeau, and A. Decoene, *Numerical simulations of 3d free surface flows by a multilayer Saint-Venant model*, Internat. J. Numer. Methods Fluids **56** (2008), no. 3, 331–350.
- [5] E. Audusse, M.-O. Bristeau, M. Pelanti, and J. Sainte-Marie, *Approximation of the hydrostatic Navier-Stokes system for density stratified flows by a multilayer model. Kinetic interpretation and numerical validation.*, J. Comp. Phys. **230** (2011), 3453–3478.

- [6] E. Audusse, M.-O. Bristeau, B. Perthame, and J. Sainte-Marie, *A multilayer Saint-Venant system with mass exchanges for Shallow Water flows. Derivation and numerical validation*, ESAIM: M2AN **45** (2011), 169–200.
- [7] A.-J.-C. Barré de Saint-Venant, *Théorie du mouvement non permanent des eaux avec applications aux crues des rivières et à l'introduction des marées dans leur lit*, C. R. Acad. Sci. Paris **73** (1871), 147–154.
- [8] F. Bouchut and T. Morales de Luna, *An entropy satisfying scheme for two-layer shallow water equations with uncoupled treatment*, M2AN Math. Model. Numer. Anal. **42** (2008), 683–698.
- [9] F. Bouchut and M. Westdickenberg, *Gravity driven shallow water models for arbitrary topography*, Comm. in Math. Sci. **2** (2004), 359–389.
- [10] Y. Brenier, *Homogeneous hydrostatic flows with convex velocity profiles*, Nonlinearity **12** (1999), no. 3, 495–512.
- [11] M. O. Bristeau, A. Mangeney, J. Sainte-Marie, and N. Seguin, *An energy-consistent depth-averaged Euler system: derivation and properties*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), no. 4, 961–988.
- [12] M.-J. Castro, J. Macías, and C. Parés, *A q-scheme for a class of systems of coupled conservation laws with source term. application to a two-layer 1-D shallow water system*, M2AN Math. Model. Numer. Anal. **35** (2001), no. 1, 107–127.
- [13] M.J. Castro, J.A. García-Rodríguez, J.M. González-Vida, J. Macías, C. Parés, and M.E. Vázquez-Cendón, *Numerical simulation of two-layer shallow water flows through channels with irregular geometry*, J. Comput. Phys. **195** (2004), no. 1, 202–235.
- [14] E.D. Fernández-Nieto, E.H. Koné, and T. Chacón Rebollo, *A multilayer method for the hydrostatic Navier-Stokes equations: a particular weak solution*, Journal of Scientific Computing **60** (2014), no. 2, 408–437 (English).
- [15] J.-F. Gerbeau and B. Perthame, *Derivation of Viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation*, Discrete Contin. Dyn. Syst. Ser. B **1** (2001), no. 1, 89–102.
- [16] E. Grenier, *On the derivation of homogeneous hydrostatic equations*, ESAIM: M2AN **33** (1999), no. 5, 965–970.
- [17] C. D. Levermore, *Entropy-based moment closures for kinetic equations*, Proceedings of the International Conference on Latest Developments and Fundamental Advances in Radiative Transfer (Los Angeles, CA, 1996), vol. 26, 1997, pp. 591–606. MR 1481496

- [18] C.D. Levermore and M. Sammartino, *A shallow water model with eddy viscosity for basins with varying bottom topography*, Nonlinearity **14** (2001), no. 6, 1493–1515.
- [19] P.-L. Lions, *Mathematical Topics in Fluid Mechanics. Vol. 1: Incompressible models.*, Oxford University Press, 1996.
- [20] F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects*, European Journal of Mechanic /B **26** (2007), 49–63.
- [21] N. Masmoudi and T. Wong, *On the Hs theory of hydrostatic Euler equations*, Archive for Rational Mechanics and Analysis **204** (2012), no. 1, 231–271 (English).
- [22] B. Mohammadi, O. Pironneau, and F. Valentin, *Rough boundaries and wall laws*, Internat. J. Numer. Methods Fluids **27** (1998), no. 1-4, 169–177.
- [23] B. Perthame, *Kinetic formulation of conservation laws*, Oxford University Press, 2002.
- [24] J. Sainte-Marie, *Vertically averaged models for the free surface Euler system. Derivation and kinetic interpretation*, Math. Models Methods Appl. Sci. (M3AS) **21** (2011), no. 3, 459–490.